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A different approach to fixed-time stability for a wide class of time-varying neural networks

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Abstract. We present sufficient conditions for fixed-time stability for a wide class of neural networks described by a system of differential equations with right-hand side satisfying the Carathéodory conditions. In contrast to the results given in the literature, where the settling-time function is estimated by an unknown Lyapunov function, we estimate the settling-time by a known function. In addition, the settling-time function does not depend on the initial values. We also give numerical examples, which confirm the theoretical results.

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Keywords: fixed-time stability, finite-time stability, neural networks, differential equations, Lyapunov function

1 Introduction

In many practical applications of neural networks, it is especially important to know not only whether a neural network is stable, but also whether is finite-time stable. Finite-time stability means that we can construct a settling-time function that estimates the time when the investigated system is stable. Many authors have considered this issue; see, for example, [2, 4, 6, 11, 16], where the settling-time function depends on the initial conditions (in [16], it additionally depends on the impulsive sequence). However, the initial conditions in many practical applications such as physical models, vehicle monitoring, or robotics may be very difficult or even impossible to get. This leads to a poor estimation of the settling-time. Therefore there was a new concept introduced, called the *fixed-time stability*. Polyakov [13] was the first who gave sufficient conditions for fixed-time stability of the origin for a nonautonomous system and gave a formula for the upper bounded estimate for the settling-time function. He starts from defining the settling-time function, which is known a priori, regardless of initial conditions and then takes a special controller, which guarantees the fixed-time stability of the origin. It was possible thanks to the new condition on the Lyapunov function. This condition became very popular and is used in many papers (see, e.g., [1, 3, 5, 7, 8, 9, 10]). It allows proving the fixed-time stability for various kinds of dynamical systems described by differential equations. In [3, 5, 8, 9, 10], autonomous systems are investigated. In [1] and [7] the authors investigate the fixed-time stability for nonautonomous systems. In all papers mentioned a continuous settling-time function is constructed by the Lyapunov function, which is unknown a priori. In [5] the settling-time estimation is more accurate than in other papers (see references above) but still requires information about the Lyapunov function.

In [14, 15] and [17] the authors uses a different approach: they estimate the settling-time function without knowledge of the Lyapunov function. To show the fixed-time stability for dynamical systems described by differential inclusions, they need a C-regular Lyapunov function, which can be differentiable only for almost all t. The upper bound of the derivative of the Lyapunov function cannot be negative. In our paper, we prove the fixed-time stability for a wide class of neural networks described by a system of differential equations with right-hand side satisfying Carathéodory conditions. To do this, we use the condition first introduced in [13], but similar to those in [14, 15] and [17], and we estimate the settling-time function by a known function. In our case, this function belongs to a new class of functions $\bar{\mathcal{P}}$ (this class is a slight modification of the class $\mathcal{P}$ that was first defined in [11]). This class of functions allows only the measurability of functions, which can take zero value even on sets of positive measure. Theoretical results show that this estimation is more accurate than estimation that uses a Lyapunov function (see examples later in this paper). We also give a new assumption, which allow us estimate the settling-time regardless of initial conditions.

2 Preliminaries

Consider the differential equation

$$x' = f(t, x), \tag{2.1}$$

where $f : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory function, which means that it is measurable with respect to t and continuous with respect to x.

In our previous work [11], we proved the global finite-time stability of the origin for system given by (2.1).

For convenience of the readers, we give the most important assumptions and definitions from [11]. Let f be the right-hand side of (2.1), and let $G \subseteq \mathbb{R}^n$ be an open set containing zero. We assume that the function $t \mapsto f(t, x)$ is measurable in $[0, \infty)$ for all $x \in G$, the function $x \mapsto f(t, x)$ is continuous in G for a.a. $t \in [0, \infty)$, and $f(t, 0) = 0$ for all $t \in [0, \infty)$. In addition, there exists a locally bounded function $m \in L^{\infty}_{\mathrm{loc}}([0, \infty))$ such that $\|f(t, x)\| \leqslant m(t)$ for a.a. $t \geqslant 0$ and all $x \in G$.

We use a Lyapunov function to prove the global finite-time stability for system (2.1) in [11]. We assume that $V : [0, \infty) \times G \to [0, \infty)$ is a continuous function such that for a continuous increasing function $K : [0, \infty) \to [0, \infty)$ such that $K(0) = 0$ and $K(r) \to \infty$ as $r \to \infty$ and for a continuous nonpositive function $\kappa : [0, \infty) \to (-\infty, 0]$, the following conditions are satisfied:

$$\inf_{t \geqslant 0} V(t, x) \geqslant K\big(\|x\|\big) > 0 \quad \text{for } x \in G \setminus \{0\}, \tag{2.2}$$

$$V(t, 0) = K(0) = 0 \quad \text{for } t \in [0, \infty), \tag{2.3}$$

and there is a set $\Gamma \subseteq [0, \infty)$ of measure zero such that

$$\dot{V}(t, x) \leqslant \kappa\big(\|x\|\big) \quad \text{for } t \in [0, \infty) \setminus \Gamma \text{ and } x \in G \setminus \{0\}. \tag{2.4}$$

In addition, there exists an at most countable set $C \subseteq [0, \infty)$ such that for all $t \in (0, \infty) \setminus C$ and $x \in G \setminus \{0\}$, there exist $\varepsilon_{tx} \in (0, t)$ and $P_{tx} > 0$ such that for $s \in (t - \varepsilon_{tx}, t + \varepsilon_{tx})$ and $z \in B(x, \varepsilon_{tx})$, where B is an open ball centered at x and with radius $\varepsilon_{tx} > 0$, we have $\hat{\partial} V(s, z) \neq \emptyset$ and

$$\sup_{\substack{s \in (t - \varepsilon_{tx}, t + \varepsilon_{tx}) \\ z \in B(x, \varepsilon_{tx})}} \sup_{v^* \in \hat{\partial} V(s, z)} \|v^*\| \leqslant P_{tx}, \tag{2.5}$$

where $\hat{\partial} V$ denotes a presubdifferential of a function V (see, e.g., [12, p. 90]).

We denote by $\mathcal{P}$ the class of nonnegative functions $c : [0, \infty) \to [0, \infty)$ that are measurable and upper-bounded on each compact subinterval $[0, \infty)$ and such that $\int_{t_0}^{\infty} c(\tau)\, \mathrm{d}\tau = \infty$ for some $t_0 \geqslant 0$.

Let $\mathcal{S}_{t_0,x_0}$ be the set of all right-maximal defined solutions φ to the differential equation (2.1) with initial condition $\varphi(t_0) = x_0$. Then for $t_0 \geqslant 0$, $x_0 \in G \setminus \{0\}$, and $\varphi \in \mathcal{S}_{t_0,x_0}$, denote by $c_\varphi(t_0, x_0)$ a finite number (if it exists) belonging to the domain of φ and satisfying the following conditions:

1. $\varphi(t) \in G \setminus \{0\}$ for $t \in (t_0, c_\varphi(t_0, x_0))$.
2. $\lim_{t\to c_\varphi(t_0,x_0)^-} \varphi(t) = 0$.

Denote

$$\tau_\varphi(t_0, x_0) = \begin{cases} c_\varphi(t_0, x_0) & \text{if it exists,} \\ \infty & \text{otherwise.} \end{cases}$$

DEFINITION 1. By a settling-time function we mean any function $T : [0, \infty) \times G \to \mathbb{R}^+ \cup \{\infty\}$ satisfying the following conditions:

1. $T(t_0, 0) = t_0$ for $t_0 \geqslant 0$.
2. $T(t_0, x_0) = \sup\{\tau_\varphi(t_0, x_0),\ \varphi \in \mathcal{S}_{t_0,x_0}\}$ for $t_0 \geqslant 0$ and $x_0 \in G \setminus \{0\}$.

Let us take any function $c \in \mathcal{P}$, $t \in [0, \infty)$, $w \in \mathbb{R}$, and $\alpha \in (0, 1)$. Then we easily see that the function $s \mapsto \int_t^s c(\tau)\,\mathrm{d}\tau$, $s \in [0, \infty)$ is nondecreasing and absolutely continuous on any compact subset of $[0, \infty)$ and that $\int_t^\infty c(\tau)\,\mathrm{d}\tau = \infty$. Hence for any $w \in \mathbb{R}$ and $\alpha \in (0, 1)$, there exists $\bar{t} \geqslant t$ such that $\int_t^{\bar{t}} c(\tau)\,\mathrm{d}\tau = |w|^{1-\alpha}/(1-\alpha)$.

Let

$$t_{c,w} = \inf\left\{ \bar{t} \geqslant t\colon \int_t^{\bar{t}} c(\tau)\,\mathrm{d}\tau = \frac{|w|^{1-\alpha}}{1-\alpha} \right\}. \tag{2.6}$$

For convenience of the readers, we give the definition of the global finite-time stability.

DEFINITION 2. We say that the origin is globally stable for the differential equation (2.1) in finite-time if it is stable and the settling-time function has only finite values.

Theorem 1. *Let $V : [0, \infty) \times \mathbb{R}^n \to [0, \infty)$ with $G = \mathbb{R}^n$ for the differential equation* (2.1) *be a continuous function satisfying conditions* (2.2), (2.3), *and* (2.5), *let a function $c : [0, \infty) \to [0, \infty)$ be of class $\mathcal{P}$, and let $\alpha \in (0, 1)$ be such that*

$$\dot{V}(t, x) + c(t)\big(V(t, x)\big)^\alpha \leqslant 0 \quad \text{for } t \in [0, \infty) \setminus \Gamma \text{ and } x \in \mathbb{R}^n \setminus \{0\},$$

where $\Gamma \subseteq [0, \infty)$ is a set of measure zero. Then the origin is globally finite-time stable for the differential equation (2.1).

Proof. See [11]. □

3 Finite-time and fixed-time stability

In Theorem 1, system (2.1) is globally stable in finite-time, and we estimate this time by a Lyapunov function. Suppose a Lyapunov function is not known a priori. In this paper, we estimate the settling-time function using a known function $c(t)$ that belongs to a new class $\bar{\mathcal{P}}$, a subclass of $\mathcal{P}$.

DEFINITION 3. We denote by $\bar{\mathcal{P}}$ the class of nonnegative functions $c : [0, \infty) \to [0, \infty)$ that are measurable and upperbounded on each compact subinterval $[0, \infty)$ and such that $\int_{t_0}^\infty c(\tau)\,\mathrm{d}\tau = \infty$ for some $t_0 \geqslant 0$ and, in addition, $\int_{t_0}^\infty 1/c(\tau)\mathrm{d}\tau < \infty$.

Remark 1. The last inequality in the definition given will be very important in the proof of the fixed-time stability theorem. This assumption allows us estimate the settling-time regardless of initial conditions.

Theorem 2. *Assume that the origin is globally stable in finite-time for the differential equation* (2.1). *Then for any solution to* (2.1) *with initial conditions* (t_0, x_0), $t_0 \geqslant 0$ *and* $x_0 \in \mathbb{R}^n$, *the settling-time*

$$\bar{t} = {c_p}^{-1}\left(\frac{\|x_0\|^{1+\alpha}}{1-\alpha} + c_p(t_0)\right),$$

where c_p *is the antiderivative of* c.

Proof. Consider any solution to (2.1) with initial conditions (t_0, x_0), $t_0 \geqslant 0$, and $x_0 \in \mathbb{R}^n$. Because the origin of (2.1) is globally stable in finite-time, there exists $\bar{t} > 0$ such that $\bar{t} = t_{c,w}$. Because $c \in \bar{\mathcal{P}}$, there exists an antiderivative c_p such that $c_p'(t_0) = c(t_0)$. Hence from (2.6) we get

$$c_p(\bar{t}) = \frac{|w|^{1-\alpha}}{1-\alpha} + c_p(t_0).$$

Because the function c_p is strictly increasing, there exists an inverse function ${c_p}^{-1}$ such that for $w = \|x_0\|$,

$$\bar{t} = {c_p}^{-1}\left(\frac{\|x_0\|^{1-\alpha}}{1-\alpha} + c_p(t_0)\right). \tag{3.1}$$

Hence we have $\sup_{x_0 \in G} T(t_0, x_0) < \infty$, where $G \subseteq \mathbb{R}^n$, which means that the settling-time estimation is independent of the initial conditions. □

Example 1. Let us consider the following system of differential equations:

$$x' = g(t, x), \tag{3.2}$$

where

$$g(t,x) = \big(g_1(t,x_1), \dots, g_n(t,x_n)\big), \quad g_i(t,x_i) = \frac{-\operatorname{sign}(x_i)|x_i|^{1/2}(t+1) - x_i}{t+1}$$

for $t \in [0, \infty)$, $x \in \mathbb{R}^n$, $n \in \mathbb{N}$. It is obvious that $g(t,x)$ is continuous in $\mathbb{R}^n$, $g(t,0) = 0$, and $\|g(t,x)\| \leqslant \sqrt{kn} + kn$ in some ball $\bar{B}(0,k)$, $k \in \mathbb{N}$.

Take the Lyapunov candidate function $V(x) = \|x\|^2$, which satisfies conditions (2.2), (2.3), and (2.5). Now we show that (2.4) is satisfied:

$$\dot{V}(x) = \big\langle 2x, g(t,x)\big\rangle = -2\sum_{i=1}^{n} x_i \operatorname{sign}(x_i)|x_i|^{1/2} - \frac{2}{t+1}\sum_{i=1}^{n} x_i^2 \leqslant -2\big(\|x\|^2\big)^{3/4},$$

where $c(t) = 2$, $c \in \mathcal{P}$, and $\alpha = 3/4$. This means by Theorem 1 that the origin for (3.2) is globally finite-time stable.

Now we estimate the settling-time function using Theorem 2. Let us choose the antiderivative for c as $c_p(t) = 2t$ with inverse function ${c_p}^{-1}(\tau) = \tau/2$. Hence, by (3.1), for any initial conditions (t_0, x_0), $t_0 \geqslant 0$, and $x_0 \in \mathbb{R}^n$, we estimate the settling-time function as follows:

$$\bar{t} = t_0 + 2\|x_0\|^{1/4}. \tag{3.3}$$

Let us see that the settling-time described by (3.3) gives more accurate estimation than estimation commonly used in the literature (see Theorem 1, which uses the condition $\int_t^{\bar{t}} c(\tau)\,\mathrm{d}\tau = |w|^{1-\alpha}/(1-\alpha)$, where $w = V(x)$). As we see, this conditions requires a Lyapunov function. In this case the estimation is $\bar{t} = t_0 + 2\|x_0\|^{1/2}$ for any initial conditions (t_0, x_0), $t_0 \geqslant 0$, $x_0 \in \mathbb{R}^n$.

Example 2. Let us consider the system of differential equations

$$x' = g(t,x), \tag{3.4}$$

where

$$g(t,x) = \bigl(g_1(t,x_1),\dots,g_n(t,x_n)\bigr), \quad g_i(t,x_i) = \frac{-6\,\mathrm{sign}(x_i)|x_i|}{7(t+1)} - \mathrm{sign}(x_i)|x_i|^{1/6}$$

for $t \in [0,\infty)$, $x \in \mathbb{R}^n$, $n \in \mathbb{N}$. It is obvious that $g(t,x)$ is continuous in $\mathbb{R}^n$, $g(t,0) = 0$, and $\|g(t,x)\| \leqslant kn + (kn)^{1/6}$ in some ball $\bar{B}(0,k)$, $k \in \mathbb{N}$.

Take the time-varying Lyapunov candidate function $V(t,x) = (t+1)^2\|x\|^{7/3}$, which satisfies conditions (2.2), (2.3), and (2.5). Now we show that (2.4) is satisfied:

$$\begin{aligned}
\dot{V}(t,x) &= 2(t+1)\|x\|^{7/3} + \left\langle \frac{7}{3}(t+1)^2 x^{4/3},\ g(t,x)\right\rangle \\
&= 2(t+1)\|x\|^{7/3} - 2(t+1)\sum_{i=1}^{n} {x_i}^{4/3}\,\mathrm{sign}(x_i)|x_i| \\
&\quad - \frac{7}{3}(t+1)^2\sum_{i=1}^{n} {x_i}^{4/3}\,\mathrm{sign}(x_i)|x_i|^{1/6} \\
&\leqslant -\frac{7}{3}(t+1)^2\|x\|^{3/2} = -\frac{7}{3}(t+1)^{5/7}\bigl((t+1)^2\|x\|^{7/3}\bigr)^{9/14},
\end{aligned}$$

where $c(t) = (7/3)(t+1)^{5/7}$, $c \in \mathcal{P}$, and $\alpha = 9/14$. This means by Theorem 1 that the origin for (3.4) is globally finite-time stable.

Now we estimate the settling-time function using Theorem 2. Let us choose the antiderivative for c as $c_p(t) = (49/36)(t+1)^{12/7}$ with inverse function ${c_p}^{-1}(\tau) = ((36/49)\tau)^{7/12} - 1$. Hence, by (3.1), for any initial conditions (t_0,x_0), $t_0 \geqslant 0$, $x_0 \in \mathbb{R}^n$, we estimate the settling-time function as follows:

$$\bar{t} = \left((t_0+1)^{12/7} + \frac{72}{35}\|x_0\|^{5/14}\right)^{7/12} - 1. \tag{3.5}$$

This example shows also that the settling-time described by (3.5) gives more accurate estimation than methods using Lyapunov functions. This estimation via a Lyapunov function is

$$\bar{t} = \left((t_0+1)^{12/7} + \frac{72}{35}(t_0+1)^{5/7}\|x_0\|^{5/6}\right)^{7/12} - 1 \tag{3.6}$$

for any initial conditions (t_0,x_0), $t_0 \geqslant 0$, $x_0 \in \mathbb{R}^n$.

As we see in Fig. 1, the settling-time function given by (3.5) lies below the settling-time given by (3.6), which uses information about a Lyapunov function. This means that estimation using an inverse function gives more precise information.

Now we give an assumption sufficient to prove the fixed-time stability of the origin for (2.1). The conditions come from [13, Lemma 1, p. 4].

ASSUMPTION 1. Let $V : \mathbb{R}^n \to [0,\infty)$ be a continuous radially unbounded function such that for $x \in B(0,\varepsilon)\setminus\{0\}$ and $\alpha,\beta,p,q,k > 0$ such that $pk < 1$ and $qk > 1$, the following conditions hold:

$$V(0) = 0, \qquad \dot{V}(x) \leqslant -\bigl(\alpha\bigl(V(x)\bigr)^p + \beta\bigl(V(x)\bigr)^q\bigr)^k.$$

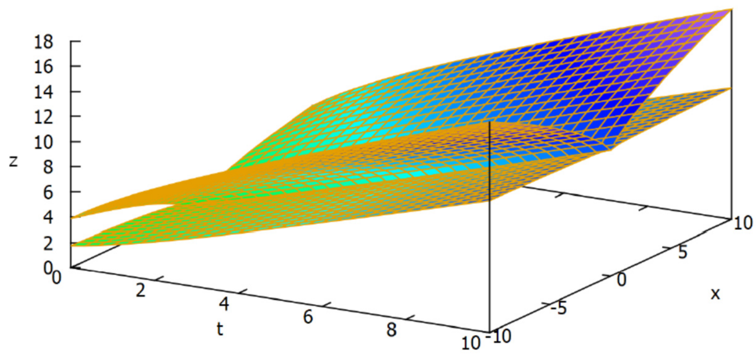

Figure 1. Comparison of the settling-time functions.

Lemma 1. *Let $V:\mathbb{R}^n\to[0,\infty)$ be a continuous radially unbounded function satisfying Assumption* 1 *with respect to the differential equation* (2.1)*. Then for any $x_0\in\mathbb{R}^n$, system* (2.1) *is globally fixed-time stable, and for any $x_0\in\mathbb{R}^n$, the settling-time function*

$$T(x_0)\leqslant\frac{1}{\alpha^k(1-pk)}+\frac{1}{\beta(qk-1)}.$$

Proof. See [13, Lemma 1, p. 4]. □

Let us consider the following system of differential equations for $\alpha,\beta,p,q,k>0$:

$$y'=-\alpha c(t)\operatorname{sign}(y)|y|^{pk},\quad pk<1,\tag{3.7}$$

$$y'=-\beta c(t)\operatorname{sign}(y)|y|^{qk},\quad qk>1,\tag{3.8}$$

$$y(t)=w.\tag{3.9}$$

A solution of the above system takes a form

$$u(t)=\begin{cases}\operatorname{sign}(w)(|w|^{1-pk}-\alpha(1-pk)\int_s^t c(\tau)\,\mathrm{d}\tau)^{1/(1-pk)} & \text{for (3.7) and (3.9)},\\ \operatorname{sign}(w)(|w|^{qk-1}-\beta(qk-1)\int_s^t c(\tau)\,\mathrm{d}\tau)^{1/(qk-1)} & \text{for (3.8) and (3.9)}.\end{cases}$$

Now we modify Assumption 1, which allows us to prove fixed-time stability for system of differential equations (3.7)–(3.8).

Assumption 2. Let $V:[0,\infty)\times\mathbb{R}^n\to[0,\infty)$ be a continuously differentiable function. There exist $\varepsilon>0$ and a function $c\in\bar{\mathcal{P}}$ such that for $t\in[0,\infty)$, $x\in B(0,\varepsilon)\setminus\{0\}$, and $\alpha,\beta,p,q,k>0$ such that $pk<1$ and $qk>1$, the following conditions hold:

$$V(t,0)=0,$$

$$\dot V(t,x)\leqslant -c(t)\big(\alpha\big(V(t,x)\big)^p+\beta\big(V(t,x)\big)^q\big)^k,\tag{3.10}$$

where $\dot V(t,x)=(\partial/\partial t)V(t,x)+\langle\nabla_x V(t,x),f(t,x)\rangle$, and $f(t,x)$ is the right-hand side of the investigated differential equation.

Lemma 2. *Let $V : [0,\infty)\times\mathbb{R}^n \to [0,\infty)$ be a function of class C^1 satisfying Assumption 2 with respect to the differential equation (3.7)–(3.8). Then for any initial conditions (t_0, x_0), $t_0 \geqslant 0$, $x_0 \in \mathbb{R}^n$, system (3.7)–(3.8) is globally fixed-time stable, and the settling-time function*

$$T(t_0,x_0) \leqslant {c_p}^{-1}\biggl(\frac{\|x_0\|^{1-pk}}{\alpha(1-pk)} + c_p(t_0)\biggr) + {c_p}^{-1}\biggl(\frac{\|x_0\|^{qk-1}}{\beta(qk-1)} + c_p(t_0)\biggr),$$

where c_p is an antiderivative for c.

Proof. From (3.10) we have that

$$\dot V(t,x) \leqslant -\alpha^k c(t)\bigl(V(t,x)\bigr)^{pk} \quad \text{for } V(t,x) \leqslant 1 \tag{3.11}$$

and

$$\dot V(t,x) \leqslant -\beta^k c(t)\bigl(V(t,x)\bigr)^{qk} \quad \text{for } V(t,x) > 1. \tag{3.12}$$

Then for any solution $y(t)$ of the differential equation (3.7) and $w = \|x_0\|$, condition (3.11) guarantees that (using a comparison lemma) $V(t,y(t)) \leqslant 1$ for

$$t \geqslant {c_p}^{-1}\biggl(\|\frac{x_0\|^{qk-1}}{\beta(qk-1)} + c_p(t_0)\biggr).$$

For any solution $y(t)$ of (3.8) such that $V(t_0,y(t_0)) \leqslant 1$, condition (3.12) guarantees that $V(t,y(t)) \equiv 0$ for

$$t \geqslant t_0 + {c_p}^{-1}\biggl(\frac{\|x_0\|^{1-pk}}{\alpha(1-pk)} + c_p(t_0)\biggr).$$

Hence $V(s,x(s)) \equiv 0$ for all

$$s \geqslant {c_p}^{-1}\biggl(\frac{\|x_0\|^{1-pk}}{\alpha(1-pk)} + c_p(t_0)\biggr) + {c_p}^{-1}\biggl(\frac{\|x_0\|^{qk-1}}{\beta(qk-1)} + c_p(t_0)\biggr)$$

and for any solution of (2.1). □

4 Main results

In this section, we use our theorems to show the fixed-time stability of the following system:

$$\begin{aligned} x(t) &= z,\\ x'(t) &= g\bigl(t,x(t)\bigr) + A(t)u\bigl(t,x(t)\bigr), \end{aligned} \tag{4.1}$$

where $t \geqslant 0$, $z \in \mathbb{R}^n$, $x(t) \in \mathbb{R}^n$ is a state vector, $g : [0,\infty)\times\mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory function describing the system uncertainties, $A(t) = (a_{ij}(t))_{n\times n}$ is a measurable matrix, and $u : [0,\infty)\times\mathbb{R}^n \to \mathbb{R}^n$ is a coupling function.

Theorem 3. *Let $V(t,x)$ be a Lyapunov function and consider two cases*:

(i) $\sum_{i=1}^n \sum_{j=1}^n a_{ij}(t)|x_j|^{pk} \geqslant \tilde L\|x\|^{pk}$ *for* $\|x\|^{pk} \leqslant 1$,
(ii) $\sum_{i=1}^n \sum_{j=1}^n a_{ij}(t)|x_j|^{qk} \geqslant \tilde L\|x\|^{qk}$ *for* $\|x\|^{qk} > 1$,

$$g(t,x) = \big(g_1(t,x_1),\dots,g_n(t,x_n)\big) = \left(-\operatorname{sign}(x_1)\left|\frac{\frac{\partial}{\partial t}V_1(t,x_1)}{\frac{\partial}{\partial x_1}V_1(t,x_1)}\right|,\dots,-\operatorname{sign}(x_n)\left|\frac{\frac{\partial}{\partial t}V_n(t,x_n)}{\frac{\partial}{\partial x_n}V_n(t,x_n)}\right|\right),$$

$L > 0$, $pk < 1$, $qk > 1$, $k > 0$, and $u(t,x) = (u_1(t,x_1),\dots,u_n(t,x_n))$, where

$$u_i(t,x_i) = -\frac{3}{4}\tilde{c}(t)\alpha^k \operatorname{sign}(x_i)|x_i|^{pk-1/3} \quad \textit{for } V(t,x) \leqslant 1, \tag{4.2}$$

and

$$u_i(t,x_i) = -\frac{3}{4}\tilde{c}(t)\beta^k \operatorname{sign}(x_i)|x_i|^{qk-1/3} \quad \textit{for } V(t,x) > 1, \tag{4.3}$$

$\alpha > 0$, $\beta > 0$, $i = 1,\dots,n$, $n \in \mathbb{N}$, and $\tilde{c} \in \bar{\mathcal{P}}$. Then the origin for system (4.1) *is globally fixed-time stable, and for any initial conditions (t_0,x_0), $t_0 \geqslant 0$, $x_0 \in \mathbb{R}^n$, and any solution to* (4.1)*, the settling-time function is bounded by*

$$T_{\max} = {c_p}^{-1}\left(\frac{\|x_0\|^{1-pk}}{\alpha(1-pk)} + c_p(t_0)\right) + {c_p}^{-1}\left(\frac{\|x_0\|^{qk-1}}{\beta(qk-1)} + c_p(t_0)\right),$$

where c_p is an antiderivative for c.

Proof. Let us consider the Lyapunov candidate function $V(t,x) = (t+1)^2\|x\|^{4/3}$. Then

$$g(t,x) = \left(-\operatorname{sign}(x_1)\frac{3|x_1|}{2(t+1)},\dots,-\operatorname{sign}(x_n)\frac{3|x_n|}{2(t+1)}\right).$$

Calculate the derivative along the trajectories of system (4.1). Then for $\|x\|^{pk} \leqslant 1$, using (4.2), we get

$$\begin{aligned}\dot{V}(t,x) &= 2(t+1)\|x\|^{4/3} + \left\langle \frac{4}{3}(t+1)^2x^{1/3},\ g(t,x) + A(t)u(t,x)\right\rangle \\ &= 2(t+1)\|x\|^{4/3} - 2(t+1)\sum_{i=1}^{n} x_i^{1/3}\operatorname{sign}(x_i)|x_i| \\ &\quad - (t+1)^2\tilde{c}(t)\alpha^k\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(t)x_j^{1/3}\operatorname{sign}(x_j)|x_j|^{pk-1/3} \\ &\leqslant -(t+1)^2\tilde{c}(t)\alpha^k\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(t)|x_j|^{pk} \leqslant -(t+1)^2\tilde{c}(t)\alpha^k\tilde{L}\|x\|^{pk} \\ &= -\tilde{c}(t)L\alpha^k\|x\|^{pk}, \quad L = \tilde{L}(t+1)^2.\end{aligned} \tag{4.4}$$

Analogously, for $\|x\|^{qk} > 1$, using (4.3), we obtain

$$\dot{V}(t,x) \leqslant -\tilde{c}(t)L\beta^k\|x\|^{qk}, \quad L = \tilde{L}(t+1)^2. \tag{4.5}$$

From (4.4)–(4.5) and from Lemma 2 it follows that $\dot{V}(t,x) \leqslant -c(t)(\alpha(V(t,x))^p + \beta(V(t,x))^q)^k$, where $c(t) = L\tilde{c}(t) = \tilde{L}(t+1)^2\tilde{c}(t)$. Hence, according to Lemma 2, we conclude that the origin for system (4.1) is globally fixed-time stable and for any initial conditions (t_0,x_0), $t_0 \geqslant 0$, $x_0 \in \mathbb{R}^n$, and for any solution to (4.1), the settling-time function is bounded by $T_{\max} = {c_p}^{-1}(\|x_0\|^{1-pk}/(\alpha(1-pk)) + c_p(t_0)) + {c_p}^{-1}(\|x_0\|^{qk-1}/(\beta(qk-1)) + c_p(t_0))$. □

Example 3. Let us consider the neural network

$$x'(t) = g\big(t, x(t)\big) + A(t)u\big(t, x(t)\big), \tag{4.6}$$

where $t \geqslant 0$, $A(t) = I$, $x \in \mathbb{R}^n$,

$$g(t,x) = \big(g_1(t,x_1), \dots, g_n(t,x_n)\big) = \left(-\frac{3\,\mathrm{sign}(x_1)|x_1|}{2(t+1)}, \dots, -\frac{3\,\mathrm{sign}(x_n)|x_n|}{2(t+1)}\right), \tag{4.7}$$

$$\begin{aligned} u(t,x) &= \big(u_1(t,x_1), \dots, u_n(t,x_n)\big) \\ &= \left(-\frac{3\,\mathrm{sign}(x_1)|x_1|^{1/3}}{(t+1)^{1/2}}, \dots, -\frac{3\,\mathrm{sign}(x_n)|x_n|^{1/3}}{(t+1)^{1/2}}\right) \quad \text{for } \|x\| \leqslant 1, \end{aligned} \tag{4.8}$$

and

$$\begin{aligned} u(t,x) &= \big(u_1(t,x_1), \dots, u_n(t,x_n)\big) \\ &= \left(-\frac{27\,\mathrm{sign}(x_1)|x_1|}{4(t+1)^{1/2}}, \dots, -\frac{27\,\mathrm{sign}(x_n)|x_n|}{4(t+1)^{1/2}}\right) \quad \text{for } \|x\| > 1. \end{aligned} \tag{4.9}$$

Let us consider the Lyapunov candidate function $V(t,x) = (t+1)^2\|x\|^{4/3}$. Then, using (4.8), we get

$$\begin{aligned} \dot V(t,x) &= 2(t+1)\|x\|^{4/3} - 2(t+1)\sum_{i=1}^{n} x_i^{1/3}\,\mathrm{sign}(x_i)|x_i| \\ &\quad - 4(t+1)^{3/2}\sum_{i=1}^{n} a_{ii}x_i^{1/3}\,\mathrm{sign}(x_i)|x_i|^{1/3} \\ &\leqslant 2(t+1)\|x\|^{4/3} - 2(t+1)\|x\|^{4/3} - 4(t+1)^{3/2}\|x\|^{2/3} \\ &= -4(t+1)^{3/2}\|x\|^{2/3} = -\tilde c(t)L\alpha^k\|x\|^{pk}, \end{aligned} \tag{4.10}$$

where $\tilde c(t) = (t+1)^{3/2}$, $L = 1$, $\alpha = 2$, $k = 2$, and $p = 1/3$.

Analogously, for (4.9), we obtain

$$\begin{aligned} \dot V(t,x) &= 2(t+1)\|x\|^{4/3} - 2(t+1)\sum_{i=1}^{n} x_i^{1/3}\,\mathrm{sign}(x_i)|x_i| \\ &\quad - 9(t+1)^{3/2}\sum_{i=1}^{n} a_{ii}x_i^{1/3}\,\mathrm{sign}(x_i)|x_i| \\ &\leqslant -9(t+1)^{3/2}\|x\|^{4/3} = -\tilde c(t)L\beta^k\|x\|^{qk}, \end{aligned} \tag{4.11}$$

where $\tilde c(t) = (t+1)^{3/2}$, $L = 1$, $\beta = 3$, $k = 2$, and $q = 2/3$.

All conditions of Theorem 3 are satisfied, because $pk < 1$, $qk > 1$,

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(t)|x_j|^{pk} \geqslant L\|x\|^{pk} \quad \text{for } \|x\|^{pk} \leqslant 1,$$
$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(t)|x_j|^{qk} \geqslant L\|x\|^{qk} \quad \text{for } \|x\|^{qk} > 1,$$

and

$$g(t,x) = \big(g_1(t,x_1),\dots,g_n(t,x_n)\big) = \left(-\operatorname{sign}(x_1)\left|\frac{\frac{\partial}{\partial t}V_1(t,x_1)}{\frac{\partial}{\partial x_1}V_1(t,x_1)}\right|,\dots,-\operatorname{sign}(x_n)\left|\frac{\frac{\partial}{\partial t}V_n(t,x_n)}{\frac{\partial}{\partial x_n}V_n(t,x_n)}\right|\right).$$

From (4.10) and (4.11) we have

$$\dot V(t,x) \leqslant -(t+1)^{3/2}\big(2(t+1)^{2/3}\|x\|^{1/3} + 3(t+1)^{4/3}\|x\|^{2/3}\big)^2,$$
$$c(t) = (t+1)^{3/2}, \quad c \in \bar{\mathcal{P}},$$

which shows that the solution of system (4.6) is fixed-time stable and for any initial conditions (t_0, x_0), $t_0 \geqslant 0$, $x_0 \in \mathbb{R}^n$, the settling-time function is estimated by

$$T_{\max} = \frac{5}{2}\left(\frac{3\|x_0\|^{1/3}}{2} + \frac{2}{5}(t_0+1)^{5/2}\right)^{2/5} + \frac{5}{2}\left(\|x_0\|^{1/3} + \frac{2}{5}(t_0+1)^{5/2}\right)^{2/5} - 2.$$

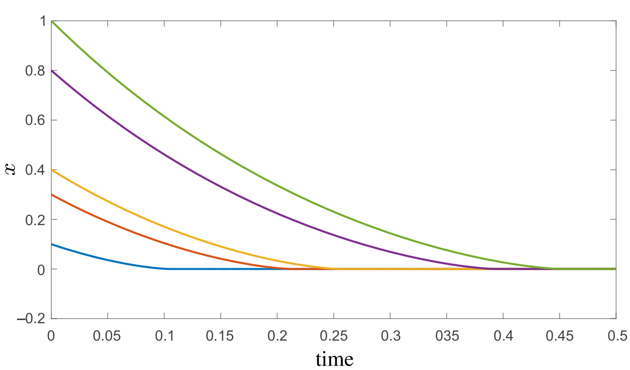

Figure 2. Trajectories for system (4.6) given by (4.7) and (4.8) with initial conditions 0.1, 0.3, 0.4, 0.8, and 1.

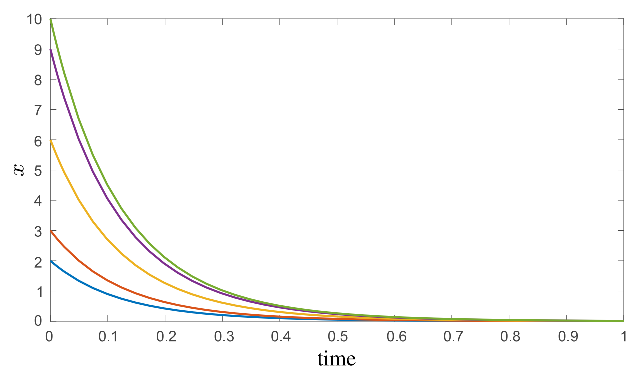

Figure 3. Trajectories for system (4.6) given by (4.7) and (4.9) with initial conditions 2, 3, 6, 9, and 10.

5 Conclusions

In this paper, we prove the fixed-time stability for a wide class of neural networks described by the system of differential equations satisfying Carathéodory conditions. We apply a different approach to fixed-time stability compared to the known results in the literature, because the settling-time function is estimated by a known function from a special class and not by a Lyapunov function, which is not known a priori. This class of functions requires only the measurability of functions, which can take the zero value even on sets of positive measure. In addition, the settling-time function does not depend on initial values.

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