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Orthonormal systems in spaces of number theoretical functions

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Abstract. In this paper, we consider some examples of set algebras $\mathcal{A}$ on $\mathbb{N}$. If $\mathcal{E}(\mathcal{A})$ is the set of simple functions on $\mathcal{A}$, then $\mathcal{L}^{*\alpha}(\mathcal{A})$ denotes the $\|\cdot\|_\alpha$-closure of $\mathcal{E}(\mathcal{A})$. Our aim is to determine a complete orthonormal system for the Hilbert space $L^{*2}(\mathcal{A})$ in each regarded case. Here $L^{*2}(\mathcal{A})$ denotes the quotient space $\mathcal{L}^{*2}(\mathcal{A})$ modulo null-functions.

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1 Introduction

For a function $f:\mathbb{N}\to\mathbb{C}$, we define $\|\cdot\|_\alpha$ by

$$\|f\|_\alpha := \left\{\limsup_{x\to\infty}\frac{1}{x}\sum_{n\leqslant x}\left|f(n)\right|^\alpha\right\}^{1/\alpha}, \quad 1\leqslant\alpha<\infty.$$

Let $\mathcal{L}^\alpha := \{f:\mathbb{N}\to\mathbb{C}\colon \|f\|_\alpha<\infty\}$ be the linear space of functions on $\mathbb{N}$ with bounded seminorm $\|f\|_\alpha$. By L^α we denote the quotient space $\mathcal{L}^\alpha$ modulo null-functions (i.e., functions f with $\|f\|_\alpha=0$). For $\alpha\geqslant 1$, the norm space L^α is complete [7].

Let $\mathcal{A}$ be an algebra of subsets of $\mathbb{N}$. Then

$$\mathcal{E}(\mathcal{A}) := \left\{s\in\mathcal{E}\colon\ s=\sum_{j=1}^m \alpha_j\mathbf{1}_{A_j},\ \alpha_j\in\mathbb{C},\ A_j\in\mathcal{A},\ j=1,\dots,m,\ m\in\mathbb{N}\right\}$$

denotes the space of simple functions on $\mathcal{A}$.

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DEFINITION 1. For a given algebra $\mathcal{A}$ and $1 \leqslant \alpha < \infty$, the space $\mathcal{L}^{*\alpha}(\mathcal{A})$ is defined as the $\|\cdot\|_\alpha$-closure of $\mathcal{E}(\mathcal{A})$. A function $f \in \mathcal{L}^{*\alpha}(\mathcal{A})$ is called *uniformly* $(\mathcal{A}) - \alpha$ *summable*. By $L^\alpha(\mathcal{A})$ we denote the quotient space $\mathcal{L}^{*\alpha}(\mathcal{A})$ modulo null functions.

Remark 1. If $\mathcal{A} = \mathcal{P}(\mathbb{N})$ is the algebra of *all* subsets of $\mathbb{N}$, then $\mathcal{L}^{*1}(\mathcal{A})$ is the $\|\cdot\|_1$-closure of l^∞ is the space $\mathcal{L}^*$ of *uniformly summable functions* introduced by Indlekofer [2].

Here we consider algebras $\mathcal{A}$ where every $A \in \mathcal{A}$ possesses an *asymptotic density* $\delta(A)$ defined by

$$\delta(A) := \lim_{n\to\infty} \frac{1}{n} \sum_{\substack{m\leqslant n\\ m\in A}} 1$$

if the limit exists. Then δ is *finitely additive* on $\mathcal{A}$, that is, δ is a *content* on $\mathcal{A}$.

We say that an arithmetical function f possesses an (*arithmetical*) *mean value* $M(f)$ if

$$M(f) := \lim_{n\to\infty} \frac{1}{n} \sum_{m\leqslant n} f(m)$$

exists. If every $A \in \mathcal{A}$ possesses an asymptotic density, then every $f \in \mathcal{L}^{*1}(\mathcal{A})$ possesses a mean value. Further, we define an inner product on $L^{*2}(\mathcal{A})$ by

$$\langle f, g\rangle := M(f\bar{g}), \quad f, g \in L^{*2}(\mathcal{A}).$$

This product is well-defined. For this, let $f, g \in L^{*2}(\mathcal{A})$. If $\varepsilon > 0$, then there exist $s_1, s_2 \in \mathcal{E}(\mathcal{A})$ such that $\|f - s_1\|_2 < \varepsilon$ and $\|g - s_2\|_2 < \varepsilon$. Put $\varepsilon := \varepsilon^*/(\|f\|_2 + 2\|g\|_2)$. Then

$$\begin{aligned}\|f\bar{g} - s_1\bar{s_2}\|_1 &\leqslant \big\|f(\bar{g} - \bar{s_2})\big\|_1 + \big\|(f - s_1)\bar{s_2}\big\|_1\\ &\leqslant \|f\|_2\|\bar{g} - \bar{s_2}\|_2 + \|f - \bar{s_1}\|_2\|\bar{s_2}\|_2\\ &\leqslant \varepsilon\big(\|f\|_2 + 2\|g\|_2\big) \leqslant \varepsilon^*,\end{aligned}$$

and $f\bar{g} \in L^{*1}(\mathcal{A})$. Since $L^{*2}(\mathcal{A})$ is complete, the space $L^{*2}(\mathcal{A})$ is a Banach space. Therefore the space $L^{*2}(\mathcal{A})$ is a Hilbert space with the inner product defined above.

In this paper, we investigate examples of Hilbert spaces $L^{*2}(\mathcal{A})$ together with associated (complete) orthonormal systems.

Remark 2. The described construction of $\mathcal{L}^{*\alpha}(\mathcal{A})$ was the starting point of an integration theory by Indlekofer (see [4, 5]).

Embedding $\mathbb{N}$, endowed with the discrete topology, in the compact space $\beta\mathbb{N}$, the Stone–Čech compactification of $\mathbb{N}$, we get:

$$\bar{\mathcal{A}} := \{\bar{A}\colon\ A \in \mathcal{A}\}, \quad \text{where } \bar{A} := \operatorname{clos}_{\beta\mathbb{N}} A,$$

is an algebra in $\beta\mathbb{N}$ (for details, see [4, 5]).

Let δ be a content on $\mathcal{A}$, that is, $\delta : \mathcal{A} \to \mathbb{R}_{\geqslant 0}$ is finitely additive, and define $\bar{\delta}$ on $\bar{\mathcal{A}}$ by

$$\bar{\delta}(\bar{A}) = \delta(A), \quad \bar{A} \in \bar{\mathcal{A}}.$$

Then $\bar{\delta}$ is a pseudo-measure on $\bar{\mathcal{A}}$ and can be extended to a measure on $\sigma(\bar{\mathcal{A}})$, which we also denote by $\bar{\delta}$. This leads to the measure space $(\beta\mathbb{N}, \sigma(\bar{\mathcal{A}}), \bar{\delta})$.

2 Some Hilbert spaces and corresponding orthonormal systems

2.1 A simple case

Let $\mathcal{A}_0$ be the algebra generated by the sets $A_p := \{n \in \mathbb{N}\colon\ p \mid n\}$, p prime, and put

$$\delta(A_p) := M(\mathbf{1}_{A_p}) = \lim_{n\to\infty} \frac{1}{n} \sum_{\substack{m\leqslant n\\ p\mid m}} 1 = \frac{1}{p}.$$

Note that the following relations of the characteristic functions

$$\mathbf{1}_{A\cap B} = \mathbf{1}_A \cdot \mathbf{1}_B, \qquad \mathbf{1}_{A\setminus B} = \mathbf{1}_A - \mathbf{1}_A \cdot \mathbf{1}_B, \qquad \mathbf{1}_{A\cup B} = \mathbf{1}_A + \mathbf{1}_B - \mathbf{1}_A \cdot \mathbf{1}_B$$

imply that the characteristic function of a set $A \in \mathcal{A}$ is a finite linear combination of products $\mathbf{1}_{A_{p_1}} \cdots \mathbf{1}_{A_{p_r}}$. Thus the asymptotic density $\delta(A)$ exists for all $A \in \mathcal{A}_0$.

For every prime p, put

$$h_p := p\mathbf{1}_{A_p} - 1$$

and define $h_n : \mathbb{N} \to \mathbb{Z}$ by $h_n = 1$ for $n = 1$ and

$$h_n := \prod_{p\mid n} h_p \quad \text{for every square-free } n \in \mathbb{N}.$$

Obviously, for every prime p,

$$M(h_p) = 0, \qquad M\bigl(h_p^2\bigr) = p - 1.$$

Now, if $f : \mathbb{N} \to \mathbb{C}$ is such that $M(f)$ exists and $f(pm) = f(m)$ for all $m \in \mathbb{N}$, then we conclude that

$$\sum_{m\leqslant x} h_p(m) f(m) = p \sum_{pm\leqslant x} f(pm) - \sum_{m\leqslant x} f(m) = p \sum_{m\leqslant x/p} f(m) - \sum_{m\leqslant x} f(m)$$

and $M(h_p f) = 0$, that is,

$$M(h_n) = 0 \quad \text{if } \mu^2(n) = 1,\ n > 1,$$

and

$$M(h_n h_{n'}) = 0 \quad \text{if } \mu^2(n) = \mu^2(n') = 1 \text{ and } n \neq n'.$$

In the same way, we obtain

$$M\bigl(h_p^2 f\bigr) = (p-1) M(f).$$

By induction this leads to

$$M\bigl(h_n^2\bigr) = \varphi(n) \quad \text{if } \mu^2(n) = 1.$$

Putting $h_n^* := (\varphi(n))^{-1/2} h_n$ ($\mu^2(n) = 1$), we have shown the following:

Theorem 1. *The set $\{h_n^*\colon\ n$ square-free$\}$ is a complete orthonormal system for $L^{*2}(\mathcal{A}_0)$.*

Remark 3. We easily to see that the function $h_n : \mathbb{N} \to \mathbb{Z}$ satisfies $h_{n_1 n_2} = h_{n_1} \cdot h_{n_2}$ if $(n_1, n_2) = 1$ and $\mu(n_1) = \mu(n_2) = 1$, that is, $n = 1$, or n is a product of an even number of different primes.

Remark 4. Every $f \in \mathcal{E}(\mathcal{A}_0)$ can be written as a linear combination of multiplicative g_j such that $g_j(p^l) = 1$ for all $p \geqslant k_j$ and $l \in \mathbb{N}$, since $g = 1 - \mathbf{1}_{A_p}$ is multiplicative.

2.2 Almost even functions

For primes p and $k = 0, 1, 2, \ldots$, let $A_{p^k} := \{n \in \mathbb{N}\colon\ p^k \mid n\}$ be the set of natural numbers divisible by p^k. Let $\mathcal{A}_1$ be the algebra generated by the sets $\{A_{p^k}\}$. Then, for all A_{p^k}, the asymptotic density $\delta(A_{p^k})$ exists and equals $1/p^k$, and, as before, the asymptotic density $\delta(A)$ exists for all $A \in \mathcal{A}_1$.

Schwarz and Spilker [8, Chap. VI] considered the space $\mathcal{B}$ of *even functions* and characterized the sets of *α-almost even functions* (see also [3]).

It is well known that $\mathcal{E}(\mathcal{A}_1)$ equals $\mathcal{B}$ and $\mathcal{L}^\alpha(\mathcal{A}_1)$ is exactly the space of α-almost even functions (see [5]).

Remark 5. Every $f \in \mathcal{E}(\mathcal{A}_1)$ can be written as a linear combination of multiplicative functions g_j such that $g_j(p^l) = 1$ for all $p \geqslant k_j$ and $l \in \mathbb{N}$.

Remark 6. Let $f : \mathbb{N} \to \mathbb{C}$ be a multiplicative function such that $|f| \leqslant 1$. Then the following statements hold.

(i) If $M(f) \neq 0$, then $f \in \mathcal{L}^{*\alpha}(\mathcal{A}_1)$ for every $\alpha \geqslant 1$.
(ii) $M(|f|) = 0$ if and only if $\sum_{p \text{ prime}}(1 - |f(p)|)/p = \infty$; especially, if $\sum_{p \text{ prime},\, f(p)=0} 1/p = \infty$, then $M(|f|) = 0$.
(iii) $M(|f|) = 0$ if and only if $M(|f|^2) = 0$.

Put

$$h_p := p\mathbf{1}_{A_p} - 1 \quad \text{for prime } p$$

and

$$h_{p^k} := p^k\mathbf{1}_{A_{p^k}} - p^{k-1}\mathbf{1}_{A_{p^{k-1}}} \quad \text{for } k > 2.$$

Define $h_n = 1$ for $n = 1$ and

$$h_n := \prod_{p^k \| n} h_{p^k} \quad \text{for } n > 1.$$

Putting

$$h_n^* = \frac{1}{(\varphi(n))^{1/2}} h_n, \tag{2.1}$$

where φ is Euler's function, it is easy to show (see above) that $\{h_n^*\}$ is an orthonormal system. We conclude by the following:

Theorem 2. *The set $\{h_n^*\colon\ n \in \mathbb{N}\}$ is a complete orthonormal system for $L^{*2}(\mathcal{A}_1)$.*

Remark 7. The functions h_n appear in a very natural way. It is not difficult to show (see [8, pp. 16–17]) that h_n is just the Ramanujan sum c_n for every n.

2.3 Limit periodic functions

Let $\mathcal{A}_2$ be the algebra generated by all residue classes $A_{a,r} := \{n \in \mathbb{N}\colon\ n \equiv a \bmod r\}$, $1 \leqslant a \leqslant r$, $r \in \mathbb{N}$.

Here again the asymptotic density δ is a finite additive function on $\mathcal{A}_2$. Then we have the following lemma.

Lemma 1. *$\mathcal{E}(\mathcal{A}_2)$ is the space of all periodic functions on $\mathbb{N}$.*

The space $L^{*\alpha}(\mathcal{A}_2)$ is the space of *α-limit-periodic functions*.

Defining $e_{a/r} : \mathbb{N} \to \mathbb{C}$ by

$$e_{a/r}(n) := \exp\left(2\pi \mathrm{i}\frac{a}{r}n\right),$$

we have the following result (see [8, p. 207]).

Theorem 3. *The set $\{e_{a/r}\colon\ 1 \leqslant a \leqslant r,\ \gcd(a, r) = 1,\ r = 1, 2, \ldots\}$ is a complete orthonormal system in $L^{*2}(\mathcal{A}_2)$.*

2.4 Almost periodic functions

For $\beta \in \mathbb{R}$, the function $e_\beta : \mathbb{N} \to \mathbb{C}$ defined by

$$e_\beta(n) = \exp(2\pi \mathrm{i} \beta n), \quad n \in \mathbb{N},$$

possesses a mean value $M(e_\beta)$.

Let $\mathcal{C}$ be the family of all half-open subsets of $[0,1]$ and denote by $\mathcal{A}_3$ the algebra generated by the sets $A(\beta, E) := \{n \in \mathbb{N}\colon \{\beta n\} \in E\}$, where $\beta \in [0,1)$, $E \in \mathcal{C}$, and $\beta n = [\beta n] + \{\beta n\}$ ($0 \leqslant \beta n < 1$). Then (see [8, p. 207]) we have the following:

Theorem 4. *The set* $\{e_\beta : \beta \in [0,1]\}$ *is a complete orthonormal system in* $\mathcal{L}^{*2}(\mathcal{A}_3)$.

2.5 Almost multiplicative functions

Let f be a multiplicative function taking only the values $\{-1, 0, 1\}$ and define the sets

$$A_f^+ := \{n\colon f(n) = 1\}, \quad A_f^0 := \{n\colon f(n) = 0\}, \quad \text{and} \quad A_f^- := \{n\colon f(n) = -1\}$$

with characteristic functions f^+, f^0, and f^-, respectively. Obviously,

$$f^+ = \frac{1}{2}\big(|f| + f\big), \qquad f^0 = 1 - f^+ - f^-, \qquad f^- = \frac{1}{2}\big(|f| - f\big).$$

We define the algebra $\mathcal{A}_4$ to be the algebra generated by the sets A_f^+, A_f^0, A_f^- for all multiplicative functions f with $f(\mathbb{N}) \subset \{-1, 0, 1\}$. Every $A \in \mathcal{A}_4$ possesses an asymptotic density by Wirsing's theorem. An arbitrary element A of $\mathcal{A}_4$ has a characteristic function that is a linear combination of such multiplicative functions. Thus the asymptotic density $\delta(A)$ exists. Let $\mathcal{E}(\mathcal{A}_4)$ be the vector space of simple functions on $\mathcal{A}_4$. Let $\mathcal{L}^{*\alpha}(\mathcal{A}_4)$ be the $\|\cdot\|_\alpha$-closure of $\mathcal{E}(\mathcal{A}_4)$.

DEFINITION 2. A function $f \in \mathcal{L}^{*\alpha}(\mathcal{A}_4)$ is called an α-*almost multiplicative* function.

First, we show $\mathcal{A}_1 \subset \mathcal{A}_4$. For the proof, consider

$$f^*(n) := (1 - \mathbf{1}_{A_{p^k}})(n) = \begin{cases} 0, & p^k \mid n, \\ 1 & \text{otherwise.} \end{cases}$$

Then f^* is multiplicative. Since $\mathbf{1}_{\mathbb{N}\setminus A_{p^k}} = \mathbf{1}_{\mathbb{N}} - \mathbf{1}_{A_{p^k}} = 1 - \mathbf{1}_{A_{p^k}} \in \mathcal{E}(\mathcal{A}_4)$, we have $\mathbb{N} \setminus A_{p^k} \in \mathcal{A}_4$. This implies that $A_{p^k} \in \mathcal{A}_4$.

Since $h_n \in \mathcal{E}(\mathcal{A}_1)$, we have $h_n \in \mathcal{E}(\mathcal{A}_4)$. Every h_n can be written as a finite linear combination of $\mathbf{1}_{A_{p_1^{\alpha_1}}} \cdot \dots \cdot \mathbf{1}_{A_{p_m^{\alpha_m}}}$, where $m \in \mathbb{N}$.

Theorem 5. *Let* $f : \mathbb{N} \to \mathbb{R}$ *be multiplicative with* $|f| \leqslant 1$. *Then* $f \in \mathcal{L}^{*\alpha}(\mathcal{A}_4)$ *for all* $\alpha \geqslant 1$.

Proof. Put $f = |f| \operatorname{sign}_f$, where sign_f is multiplicative with

$$(\operatorname{sign}_f)\big(p^l\big) = \begin{cases} 1 & \text{if } f(p^l) > 0, \\ 0 & \text{if } f(p^l) = 0, \\ -1 & \text{if } f(p^l) < 0. \end{cases}$$

Since $|f| \in \mathcal{L}^{*\alpha}(\mathcal{A}_1)$ and $\operatorname{sign} f \in \mathcal{E}(\mathcal{A}_4)$, we find $s_1, s_s \in \mathcal{E}(\mathcal{A}_4)$ such that $\||f| - s_1|\|_\alpha^\alpha \leqslant \varepsilon^\alpha$ and $\|\operatorname{sign} f - s_2\|_\alpha^\alpha \leqslant \varepsilon^\alpha$. Note that there exist $c_1(\alpha), c_2(\alpha) > 0$ such that

$$\|s_2\|_\alpha^\alpha \leqslant \big\|1 + \big(s_2 - |f|\big)\big\|_\alpha^\alpha \leqslant c_1(\alpha)$$

and

$$\big(|a| + |b|\big)^\alpha \leqslant c_2(\alpha)\big(|a|^\alpha + |b|^\alpha\big).$$

Put $\varepsilon^\alpha := \varepsilon^{*\alpha}/(1 + c_1(\alpha)c_2(\alpha))$. Then

$$\begin{aligned}\|f - s_1 s_2\|_\alpha^\alpha &= c_2(\alpha)\big\{\big\||f|(\operatorname{sign}_f - s_2)\big\|_\alpha^\alpha + \big\|\big(|f| - s_1\big)s_2\big\|_\alpha^\alpha\big\} \\ &\leqslant c_2(\alpha)\big\{\big\|(\operatorname{sign}_f - s_2)\big\|_\alpha^\alpha + \big\|\big(|f| - s_1\big)\big\|_\alpha^\alpha \|s_2\|_\alpha^\alpha\big\} \\ &\leqslant c_2(\alpha)\varepsilon^\alpha + \varepsilon^\alpha c_2(\alpha)c_1(\alpha) < \varepsilon^{*q}.\end{aligned}$$

This proves Theorem 5. □

Next, we construct an orthonormal system for the space $L^{*2}(\mathcal{A}_4)$.

Let $\mathcal{R}_0$ be the set of all multiplicative functions with $f(\mathbb{N}) \subset \{-1, 0, 1\}$ and $M(|f|) \neq 0$. Define the relation $\sim$ on $\mathcal{R}_0$ by

$$f \sim g \quad \text{if and only if} \quad \sum_{\substack{p \\ f(p) \neq g(p)}} \frac{1}{p} < \infty.$$

Observe, that in this case, by (ii) of Remark 6, $\sum_{p,\ f(p)=0} 1/p < \infty$. Obviously, $\sim$ is an equivalence relation on $\mathcal{R}_0$.

Now choose a representative from each residue class that takes only the values ± 1 and denote this set by $\mathcal{F}_1$. Then $\mathcal{F}_1$ forms an orthonormal system. For this, let $f, g \in \mathcal{F}_1$ and observe that $\sum_{p,\ f(p)\neq g(p)} 1/p = \infty$. Then, by (ii) of Remark 6, $M(f\bar{g}) = \langle f, g\rangle = 0$. Furthermore, for $f \in \mathcal{F}_1$, we have $f^2 = 1$ and $\langle f, f\rangle = M(f^2) = 1$.

This shows that $\mathcal{F}_1$ is an orthonormal system in $L^{*2}(\mathcal{A}_4)$. Consider, for $f \in \mathcal{F}_1$, the system

$$\mathcal{F}_2 := \big\{f h_n^*\colon\ f \in \mathcal{F}_1,\ n \in \mathbb{N}\big\},$$

where h_n^* is the normalized function (2.1).

Theorem 6. *$\mathcal{F}_2$ is a complete orthonormal system for $L^{*2}(\mathcal{A}_4)$.*

Proof. First, we show that $\mathcal{F}_2$ is an orthonormal system. For this, let $h_n^* f \neq h_{\tilde{n}}^* g$. This holds if and only if $f \neq g$ and $n, \tilde{n}$ are arbitrary or $f = g$ and $n \neq \tilde{n}$. Assume that $f \neq g$ and $n, \tilde{n}$ are arbitrary. Then

$$\langle h_n^* f, h_{\tilde{n}}^* g\rangle = M(h_n^* f h_{\bar{n}} g) = M(fgh^*),$$

where h^* is (see Remark 6) a finite linear combination of multiplicative functions g_j with $|g_j| = 1$ and $g_j(p) = 1$ for $p > k_j$. Therefore

$$\sum_p \frac{1 - f(p)g(p)g_j(p)}{p} = \infty \quad \text{and} \quad M(fgg_j) = 0.$$

So we obtain $\langle h_n^* f, h_{\tilde{n}}^* g\rangle = 0$ if $f \neq g$. In the case $f = g$ and $n \neq \bar{n}$, obviously, $\langle h_n^* f, h_{\tilde{n}}^* f\rangle = M(h_n^* h_{\tilde{n}}^*) = 0$. Since $\langle h_n^* f, h_{\tilde{n}}^* f\rangle = M(|h_n^* f|^2) = 1$, $\mathcal{F}_2$ is an orthonormal system.

For the proof of the completeness of $\mathcal{F}_2$, let $g \in \mathcal{L}^*(\mathcal{A}_4)$. Then g can be approximated in the $\|\cdot\|_2$ norm by some $g^* \in \mathcal{E}(\mathcal{A}_4)$,

$$g^* = \sum_{j=1}^{m} \alpha_j \mathbf{1}_{A_j}, \quad \alpha_j \in \mathbb{C},\ A_j \in \mathcal{A}_4.$$

Note that $\mathbf{1}_A$ for $A \in \mathcal{A}_4$ is a finite linear combination of products of multiplicative functions f taking only the values $\{-1, 0, 1\}$.

Therefore it suffices to prove that each real-valued multiplicative function f with values $f(\mathbb{N}) \subset \{-1, 0, 1\}$ can be approximated by a linear combination of functions from $\mathcal{F}_2$. Choose $g \in \mathcal{F}_1$ that is equivalent to f. Then $f = hg$ where $h = fg$, since $g^2 = 1$. Then

$$\sum_p \frac{1 - h(p)}{p} = \sum_p \frac{2}{p} < \infty$$

and $h \in \mathcal{L}^{*2}(\mathcal{A}_1)$. Thus h can be approximated by a linear combination of functions $h_1, \dots, h_m$, that is, for $\varepsilon > 0$, there exist $\alpha_j \in \mathbb{C}$ such that

$$\left\| h - \sum_{j=1}^{m} \alpha_j h_j \right\|_2 < \varepsilon, \quad \text{which implies} \quad \left\| f - \sum_{j=1}^{m} \alpha_j h_j g \right\|_2 < \varepsilon.$$

This ends the proof of the completeness of $\mathcal{F}_2$. □

2.6 q-ary almost even functions

First, we introduce *q-multiplicative* functions. Let $q \geqslant 2$ be an integer, and let $\mathbb{A} = \{0, 1, \dots, q-1\}$. The *$q$-ary expansion* of some $n \in \mathbb{N}_0$ is defined as the unique sequence $\varepsilon_0(n), \varepsilon_1(n), \dots$ for which

$$n = \sum_{j=0}^{\infty} \varepsilon_j(n) q^j, \quad \varepsilon_j(n) \in \mathbb{A}.$$

The numbers $\varepsilon_0(n), \varepsilon_1(n), \dots$ are called the *digits* in the q-ary expansion of n. In fact, $\varepsilon_r(n) = 0$ if $r > \log n / \log q$. A function $f : \mathbb{N}_0 \to \mathbb{C}$ is called *q-multiplicative* if $f(0) = 1$ and for every $n \in \mathbb{N}_0$,

$$f(n) = \prod_{j=0}^{\infty} f\bigl(\varepsilon_j(n) q^j\bigr).$$

Let the algebra $\mathcal{A}_5$ be generated by the sets $A_{j,a} := \{n \in \mathbb{N}\colon \varepsilon_j(n) = a\}$, $j \in \mathbb{N}_0, a \in \mathbb{A}$. Every $A \in \mathcal{A}_5$ possesses an asymptotic density $\delta(A)$.

Let $\mathcal{L}^{*1}(\mathcal{A}_5)$ be the $\|\cdot\|_1$-closure of $\mathcal{E}(\mathcal{A}_5)$. Here $\mathcal{E}(\mathcal{A}_5)$ is called the space of *q-ary even functions*. Then $\mathcal{L}^{*1}(\mathcal{A}_5)$ is called the space of *q-ary almost even* functions.

Remark 8. Let f be a real-valued q-multiplicative function of modulus $\leqslant 1$. Then the mean values $M(|f|)$ and $M(f)$ always exist (see [6]). Especially, we have:

(i) If $\|f\|_1 = M(|f|) > 0$, then

$$\sum_{j=0}^{\infty} \sum_{a \in \mathbb{A}} \bigl(1 - \bigl|f\bigl(aq^j\bigr)\bigr|\bigr) < \infty.$$

(ii) If

$$\sum_{a\in\mathbb{A}} f\big(aq^j\big) \neq 0 \quad \text{for all } j \in \mathbb{N}_0 \quad \text{and} \quad \sum_{j=0}^{\infty}\sum_{a\in\mathbb{A}} \big(1 - f\big(aq^j\big)\big) < \infty,$$

then $M(f) \neq 0$.

As an immediate consequence, we have the following:

Corollary 1. *Let f be a real-valued q-multiplicative function of modulus $\leqslant 1$. If*

$$\sum_{j=0}^{\infty}\sum_{a\in\mathbb{A}} \big(1 - f\big(aq^j\big)\big) < \infty,$$

then $f \in \mathcal{L}^{*1}(\mathcal{A}_5)$.

This ends Remark 8.

Let $\mathcal{L}^{*2}(\mathcal{A}_5)$ be the $\|\cdot\|_2$-closure of $\mathcal{E}(\mathcal{A}_5)$. Then we define a complete orthonormal system for the space $L^{*2}(\mathcal{A}_5)$.

Theorem 7. *The set $\{h_{a_0,\dots,a_r}\}$ of q- multiplicative functions with*

$$h_{a_0,\dots,a_r}(n) := \prod_{j=0}^{r} \exp\bigg(\frac{2\pi \mathrm{i} a_j}{q}\varepsilon_j(n)\bigg),$$

$a_j \in \mathbb{A}, j = 0,\dots,r$, $r \in \mathbb{N}_0$, is a complete orthonormal system for $L^{*2}(\mathcal{A}_5)$.

The proof is easy and is left to the reader.

2.7 Almost q-multiplicative functions

Let f be a q- multiplicative function taking only the values $\{-1, 0, 1\}$ and define the sets

$$A_f^+ := \{n\colon f(n) = 1\}, \quad A_f^0 := \{n\colon f(n) = 0\}, \quad \text{and} \quad A_f^- := \{n\colon f(n) = -1\}$$

with characteristic functions f^+, f^0, and f^-, respectively. We denote by $\mathcal{A}_6$ the algebra generated by the sets A_f^+, A_f^0, A_f^- for all q-multiplicative f with $f(\mathbb{N}) \subset \{-1, 0, 1\}$.

An arbitrary element A of $\mathcal{A}_6$ has a characteristic function that is a linear combination of q-multiplicative functions. From this and by the theorem of Delange [1] the asymptotic density $\delta(A)$ exists. Let $\mathcal{E}(\mathcal{A}_6)$ be the space of simple functions on $\mathcal{A}_6$. Let $\mathcal{L}^*(\mathcal{A}_6)$ be the $\|\cdot\|_1$-closure of $\mathcal{E}(\mathcal{A}_6)$.

DEFINITION 3. Functions $f \in \mathcal{L}^*(\mathcal{A}_6)$ are called *almost q-ary multiplicative* functions.

Next, we define a complete orthonormal system for $L^{*2}(\mathcal{A}_6)$.

Let $\mathcal{G} := \{f : \mathbb{N} \to \mathbb{R}\colon\ f\ q-\text{multiplicative},\ f(n) \in \{-1, 0, 1\}$ for all $n \in \mathbb{N}$ with $\|f\|_2 \neq 0\}$. Define the relation $\sim$ on $\mathcal{G}$ by

$$f \sim g \quad \text{if and only if } \sum_{l=0}^{\infty}\sum_{a=0}^{q-1} \big(1 - f\big(aq^l\big)g\big(aq^l\big)\big) < \infty.$$

Obviously, $\sim$ is an equivalence relation on $\mathcal{G}$.

Now from each equivalence class we choose a representative that is $\neq 0$ for all $n \in \mathbb{N}$. We denote this set of representatives by $\mathcal{F}_3$. We consider $\mathcal{F}_4 := \{h_{a_0,\dots,a_r} f\colon\ f \in \mathcal{F}_3,\ a_j \in A,\ j = 0,\dots,r,\ r \in \mathbb{N}\}$ and show the following:

Theorem 8. *$\mathcal{F}_4$ is a complete orthonormal system for $L^{*2}(\mathcal{A}_6)$.*

Proof. First, we show that $\mathcal{F}_4$ is an orthonormal system. In the case $g_1 = g_2 = g$, since $g^2 = 1$, we have

$$M(h_{a_0,\dots,a_r} g \bar{h}_{b_0,\dots,b_s} g) = M(h_{a_0,\dots,a_r} \bar{h}_{b_0,\dots,b_r}) = 0$$

if $h_{a_0,\dots,a_r} \neq h_{b_0,\dots,b_r}$ and 1 otherwise. If $g_1 \neq g_2$, then

$$(h_{a_0,\dots,a_r} g_1 \bar{h}_{b_1,\dots,b_s} g_2)\big(aq^j\big) = (g_1 g_2)\big(aq^j\big)$$

if j is large enough. Obviously, $M(g_1 g_2) = 0$, and $\mathcal{F}_4$ is an orthonormal system.

To prove the completeness of $\mathcal{F}_4$, it suffices to show that every q-multiplicative f with $M(|f|) \neq 0$ and $f(n) \in \{-1, 0, 1\}$ for all $n \in \mathbb{N}_0$ can be approximated by linear combinations of elements of $\mathcal{F}_4$.

Let f be such a function. Then $f = |f| \operatorname{sign}_f$ and $\operatorname{sign}_f \sim g$, $g \in \mathcal{F}_3$, and

$$f = |f| \operatorname{sign}_f g^2 = \big(|f| \operatorname{sign}_f g\big) g.$$

Now $|f| \operatorname{sign}_f g$ is a q-ary even function and can therefore be approximated by a linear combination of some $h_{a_0,\dots,a_r}$. This proves Theorem 8. □

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