

# Simultaneous ruin probability for two-dimensional fractional Brownian motion risk process over discrete grid

Grigori Jasnovidov

Department of Actuarial Science, University of Lausanne, UNIL-Dorigny, 1015 Lausanne, Switzerland  
(e-mail: grigori.jasnovidov@unil.ch)

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**Abstract.** We derive the asymptotic behavior of the following ruin probability:

$$\mathbf{P}\{\exists t \in G(\delta): B_H(t) - c_1 t > q_1 u, B_H(t) - c_2 t > q_2 u\}, \quad H \in (0, 1), u \rightarrow \infty,$$

where  $B_H$  is a standard fractional Brownian motion,  $c_1, q_1, c_2, q_2 > 0$ , and  $G(\delta)$  denotes the regular grid  $\{0, \delta, 2\delta, \dots\}$  for some  $\delta > 0$ . The approximation depends on  $H$ ,  $\delta$  (only when  $H \leq 1/2$ ) and the relations between parameters  $c_1, q_1, c_2, q_2$ .

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## 1 Introduction

Let  $B_H(t)$ ,  $t \in \mathbb{R}$ , be a standard fractional Brownian motion (fBM) with zero mean and covariance function

$$\text{cov}(B_H(t), B_H(s)) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t - s|^{2H}), \quad H \in (0, 1), s, t \in \mathbb{R}.$$

Define two risk processes

$$R_{i,u}^{(H)}(t) = q_i u + c_i t - B_H(t), \quad i = 1, 2,$$

where  $c_i, q_i > 0$ . The discrete simultaneous ruin probability over the infinite time horizon is defined by

$$\bar{\psi}_{\delta,H}(u) = \mathbf{P}\{\exists t \in G(\delta): R_{1,u}^{(H)}(t) < 0, R_{2,u}^{(H)}(t) < 0\},$$

where  $G(\delta)$  denotes the grid  $\{0, \delta, 2\delta, \dots\}$  (for  $\delta = 0$ ,  $G(\delta) = [0, \infty)$ ). For positive  $\delta$ , the simultaneous ruin probability is of interest both for theory-oriented studies and for applications in reinsurance (see, e.g., [9] and references therein). In this paper, we investigate only the discrete setup; the continuous problem is already

solved in [9]. For any possible choices of positive  $\delta$  and  $H \in (0, 1)$ , it is not possible to calculate  $\bar{\psi}_{\delta,H}(u)$  explicitly. When there are no explicit formulas, a natural question is how can we approximate  $\bar{\psi}_{\delta,H}(u)$  for large  $u$ . Also of interest is to know what is the role of  $\delta$  and whether it affects the ruin probability in the considered risk model. Theorem 1 gives detailed answers to these questions. Our results show that the discrete-time ruin probabilities behave differently from continuous if  $H \leq 1/2$ . We refer to [8] for some alternative proofs of the results.

Also of certain interest is the finite time horizon setup of the problem. For fixed  $T > 0$ , the discrete simultaneous ruin probability over a finite time horizon is

$$\bar{\zeta}_{H,T}(u) = \mathbf{P}\{\exists t \in [0, T]: R_{1,u}^{(H)}(t) < 0, R_{2,u}^{(H)}(t) < 0\}.$$

The corresponding discrete ruin problem over a finite time horizon is trivial, since the set  $[0, T] \cap G(\delta)$  consists of finite number of elements, and hence the asymptotics of the large deviation is determined by the unique maximizer of the variance of the process (this, e.g., immediately follows from Lemma 2.3 in [11] or Proposition 2.4.2 in [12]). Thus we are concerned only with the continuous ruin problem over a finite horizon. The asymptotics of  $\bar{\zeta}_{H,T}(u)$  is discussed in Remark 2. We organize the paper in the following way. The next section gives notation, necessary assumptions, and the main results. All proofs are relegated to Section 3, whereas some technical calculations are presented in the Appendix.

## 2 Main results

First, we eliminate the trends via self-similarity of fBM. For any  $u > 0$ , we have

$$\begin{aligned} \bar{\psi}_{\delta,H}(u) &= \mathbf{P}\{\exists t \in G(\delta): B_H(t) > q_1u + c_1t, B_H(t) > q_2u + c_2t\} \\ &= \mathbf{P}\left\{\exists t \in G\left(\frac{\delta}{u}\right): \frac{B_H(t)}{\max(c_1t + q_1, c_2t + q_2)} > u^{1-H}\right\}. \end{aligned}$$

If two lines  $q_1 + c_1t$  and  $q_2 + c_2t$  do not intersect over  $(0, \infty)$ , then the problem degenerates to the one-dimensional case, which is discussed in Theorem 2. In consideration of that dealing with  $\bar{\psi}_{\delta,H}(u)$ , we always suppose that

$$c_1 > c_2, \quad q_2 > q_1. \tag{2.1}$$

It turns out that the variance of  $B_H(t)/\max(c_1t + q_1, c_2t + q_2)$  can achieve its unique maximum only at one of the following points:

$$t_1 = \frac{Hq_1}{c_1(1-H)}, \quad t_2 = \frac{Hq_2}{c_2(1-H)}, \quad t^* = \frac{q_2 - q_1}{c_1 - c_2}.$$

It follows from (2.1) that  $t_1 < t_2$ . As we show later, the order between  $t_1, t_2$ , and  $t^*$  determines the asymptotics of  $\bar{\psi}_{\delta,H}(u)$  as  $u \rightarrow \infty$ .

Denote by  $\Phi$  and  $\bar{\Phi}$  the distribution and survival functions of a standard normal random variable, respectively. For notational simplicity, we write  $B(t)$  instead of  $B_{1/2}(t)$  and  $\bar{\psi}_{\delta}(u)$  instead of  $\bar{\psi}_{\delta,1/2}(u)$ . Define the Pickands constants for  $H \in (0, 1]$  and  $\eta > 0$  by

$$\mathcal{H}_{2H} = \lim_{S \rightarrow \infty} \frac{1}{S} \mathbf{E}\left\{\sup_{t \in [0, S]} \exp(\sqrt{2}B_H(t) - |t|^{2H})\right\}, \quad \mathcal{H}_{\eta} = \mathbf{E}\left\{\frac{\sup_{t \in \eta\mathbb{Z}} \exp(\sqrt{2}B(t) - |t|)}{\eta \sum_{t \in \eta\mathbb{Z}} \exp(\sqrt{2}B(t) - |t|)}\right\}.$$

It is known that  $\mathcal{H}_{2H}$  and  $\mathcal{H}_{\eta}$  are positive and finite (see [1, 11, 12]). Define for some real function  $k(t)$  the constant

$$\mathcal{H}_{\alpha}^k = \lim_{T \rightarrow \infty} \mathbf{E}\left\{\sup_{t \in [-T, T] \cap \alpha\mathbb{Z}} \exp(\sqrt{2}B(t) - |t| + k(t))\right\}$$

when the expectation above is finite, and for  $\delta \geq 0$ , set

$$d_\delta(t) = \mathbf{I}(t < 0) \frac{(q_2 c_1 + c_2 q_1 - 2q_2 c_2)t}{c_1 q_2 - q_1 c_2} + \mathbf{I}(t \geq 0) \frac{(2c_1 q_1 - c_1 q_2 - q_1 c_2)t}{c_1 q_2 - q_1 c_2} - \delta \mathbf{I}(t \geq 0) \frac{(c_1 q_2 - q_1 c_2)(c_1 - c_2)}{q_2 - q_1},$$

where  $\mathbf{I}(\cdot)$  is the indicator function. Define the constants

$$C_H^{(i)} = \frac{c_i^H q_i^{1-H}}{H^H (1-H)^{1-H}}, \quad i = 1, 2.$$

The following theorem establishes the asymptotics of  $\bar{\psi}_{\delta,H}(u)$ .

**Theorem 1.** For  $\delta > 0$ , as  $u \rightarrow \infty$ ,

(i) if  $t^* \notin (t_1, t_2)$ , then

$$\bar{\psi}_{\delta,H}(u) \sim \left(\frac{1}{2}\right)^{\mathbf{I}(t^*=t_i)} \times \begin{cases} \mathcal{H}_{2H} \frac{2^{1/2-1/(2H)} \sqrt{\pi}}{H^{1/2}(1-H)^{1/2}} (C_H^{(i)} u^{1-H})^{1/H-1} \bar{\Phi}(C_H^{(i)} u^{1-H}), & H > \frac{1}{2}, \\ \mathcal{H}_{2c_i^2 \delta} \exp(-2c_i q_i u), & H = \frac{1}{2}, \\ \frac{\sqrt{2\pi} H^{H+1/2} q_i^H u^H}{\delta c_i^{H+1} (1-H)^{H+1/2}} \bar{\Phi}(C_H^{(i)} u^{1-H}), & H < \frac{1}{2}, \end{cases}$$

where  $i = 1$  if  $t^* \leq t_1$  and  $i = 2$  if  $t^* \geq t_2$ ,

(ii) if  $t^* \in (t_1, t_2)$ , then with  $\mathcal{D}_H = (c_1 t^* + q_1)/(t^*)^H$  when  $H > 1/2$ ,

$$\bar{\psi}_{\delta,H}(u) \sim \bar{\Phi}(\mathcal{D}_H u^{1-H});$$

when  $H = 1/2$ ,

$$\mathcal{H}_\gamma^{d_\delta} \bar{\Phi}(\mathcal{D}_{1/2} \sqrt{u})(1 + o(1)) \leq \bar{\psi}_\delta(u) \leq A \mathcal{H}_\gamma^{d_0} \bar{\Phi}(\mathcal{D}_{1/2} \sqrt{u})(1 + o(1)),$$

where  $\mathcal{H}_\gamma^d, \mathcal{H}_\gamma^{d_\delta} \in (0, \infty)$ , and

$$A = \exp\left(\delta \frac{(c_1 q_2 - c_2 q_1)(c_1 q_2 + q_1 c_2 - 2c_2 q_2)}{2(q_2 - q_1)^2}\right) > 1, \quad \gamma = \frac{\delta(c_1 q_2 - q_1 c_2)^2}{2(q_2 - q_1)^2};$$

when  $H < 1/2$ ,

$$2 \exp(-B u^{1-H}) \bar{\Phi}(\mathcal{D}_H u^{1-H})(1 + o(1)) \leq \bar{\psi}_{\delta,H}(u) \leq \bar{\Phi}(\mathcal{D}_H u^{1-H})(1 + o(1)),$$

where

$$B = -\frac{\delta w_1'(t^*) w_2'(t^*)}{2(w_1'(t^*) - w_2'(t^*))} > 0, \quad w_i(t) = \frac{(q_i + c_i t)^2}{t^{2H}}, \quad i = 1, 2. \tag{2.2}$$

To study the asymptotics of the two-dimensional ruin probability over the infinite time horizon, the asymptotic approximation of the one-dimensional one is crucial. The asymptotics of this ruin probability was already studied in [10]. Since there are some inaccuracies, we give the following corrected result.

**Theorem 2.** For any  $\delta > 0$  with  $C_H = c^H / (H^H(1 - H)^{1-H})$ , as  $u \rightarrow \infty$ ,

$$\mathbf{P}\{\exists t \in G(\delta): B_H(t) - ct > u\} \sim \begin{cases} \mathcal{H}_{2H} \frac{2^{1/2-1/(2H)} \sqrt{\pi}}{H^{1/2}(1-H)^{1/2}} (C_H u^{1-H})^{1/H-1} \bar{\Phi}(C_H u^{1-H}), & H > \frac{1}{2}, \\ \mathcal{H}_{2c^2\delta} \exp(-2cu), & H = \frac{1}{2}, \\ \frac{\sqrt{2\pi} H^{H+1/2} u^H}{\delta c^{H+1} (1-H)^{H+1/2}} \bar{\Phi}(C_H u^{1-H}), & H < \frac{1}{2}. \end{cases} \quad (2.3)$$

*Remark 1.* If  $H > 1/2$ , then the asymptotics of the discrete probabilities in Theorems 1 and 2 are the same as in the continuous case and do not depend on  $\delta$ . If  $H = 1/2$ , then the asymptotics differ only by constants. If  $H < 1/2$ , then the discrete asymptotics are infinitely smaller than the corresponding continuous asymptotics. All these statements directly follow from Theorems 1 and 2 and Corollary 2 in [6] and Theorem 3.1 in [9].

Next, we discuss the finite time horizon case. Here for large  $u$ , the two-dimensional ruin probability always reduces to the one-dimensional one, which was already studied in [4, 5]. More precisely, we have

*Remark 2.* For any  $T > 0$  with  $\lambda(u) = \max(q_1u + c_1T, q_2u + c_2T)/T^H$ , as  $u \rightarrow \infty$ ,

$$\bar{\zeta}_{H,T}(u) \sim \begin{cases} \mathcal{H}_{2H} (\lambda(u))^{(1-2H)/H} \frac{(1/2)^{(1/2H)}}{H} \bar{\Phi}(\lambda(u)), & H < \frac{1}{2}, \\ \bar{\Phi}(\lambda(u)), & H > \frac{1}{2}, \end{cases}$$

and

$$\bar{\zeta}_{1/2,T}(u) = \bar{\Phi}\left(\frac{uq_i}{\sqrt{T}} + c_i\sqrt{T}\right) + \exp(-2c_iq_iu) \bar{\Phi}\left(\frac{uq_i}{\sqrt{T}} - c_i\sqrt{T}\right), \quad i = 1, 2,$$

where  $i = 1$  if  $(q_1, c_1) \geq (q_2, c_2)$  in the alphabetical order and  $i = 2$  otherwise.

### 3 Proofs

For any real numbers  $a < b$  and  $\alpha > 0$ , denote  $[a, b]_\alpha = [a, b] \cap \alpha\mathbb{Z}$ . We reserve the letters  $C, C_1$  for some positive constants that do not depend on  $u$  and may be different in different places.

*Proof of Theorem 1.* Denote

$$V_i(t) = \frac{B_H(t)}{c_it + q_i}, \quad i = 1, 2. \quad (3.1)$$

*Case (i).* Assume that  $t^* < t_1$ . We have by the self-similarity of fBM

$$\bar{\psi}_{\delta,H}(u) \leq \mathbf{P}\left\{\exists t \in G\left(\frac{\delta}{u}\right): V_1(t) > u^{1-H}\right\} =: \psi_{\delta,H}^{(1)}(u). \quad (3.2)$$

Since  $t^* < t_1$  for any  $0 < \varepsilon < t_1 - t^*$ , we have

$$\begin{aligned} \bar{\psi}_{\delta,H}(u) &\geq \mathbf{P}\{\exists t \in [t_1 - \varepsilon, t_1 + \varepsilon]_{\delta/u}: V_1(t) > u^{1-H}, V_2(t) > u^{1-H}\} \\ &= \mathbf{P}\{\exists t \in [t_1 - \varepsilon, t_1 + \varepsilon]_{\delta/u}: V_1(t) > u^{1-H}\} \sim \psi_{\delta,H}^{(1)}(u), \quad u \rightarrow \infty. \end{aligned} \quad (3.3)$$

For a detailed proof of the last line, see the Appendix. Thus by (3.2)

$$\bar{\psi}_{\delta,H}(u) \sim \psi_{\delta,H}^{(1)}(u), \quad u \rightarrow \infty,$$

and by Theorem 2 the claim is established.

Let  $t^* = t_1$ . We have

$$\begin{aligned} & \mathbf{P}\left\{\sup_{t \in [t^*, \infty)_{\delta/u}} V_1(t) > u^{1-H}\right\} \\ & \leq \bar{\psi}_{\delta, H}(u) \leq \mathbf{P}\left\{\sup_{t \in [t^*, \infty)_{\delta/u}} V_1(t) > u^{1-H}\right\} + \mathbf{P}\left\{\sup_{t \in [0, t^*)_{\delta/u}} V_2(t) > u^{1-H}\right\}. \end{aligned} \quad (3.4)$$

Since  $t^*$  is the unique maximizer of  $\text{Var}\{V_1(t)\}$  (the details are given in the Appendix),

$$\mathbf{P}\left\{\sup_{t \in [t^*, \infty)_{\delta/u}} V_1(t) > u^{1-H}\right\} \sim \frac{1}{2}\psi_{\delta, H}^{(1)}(u), \quad H \in (0, 1), \quad u \rightarrow \infty. \quad (3.5)$$

Next, we prove that

$$\mathbf{P}\left\{\sup_{t \in [0, t^*)_{\delta/u}} V_2(t) > u^{1-H}\right\} = o(\psi_{\delta, H}^{(1)}(u)), \quad u \rightarrow \infty. \quad (3.6)$$

*Case  $H \geq 1/2$ .* As follows from Corollary 2 in [6] and Theorem 2 for  $H \geq 1/2$ , for all large  $u$  and some  $C > 0$  independent of  $u$ ,

$$C\psi_{0, H}^{(1)}(u) \leq \psi_{\delta, H}^{(1)}(u) \leq \psi_{0, H}^{(1)}(u).$$

Hence with the same constant  $C$ , we have that

$$\begin{aligned} & \mathbf{P}\left\{\sup_{t \in [0, t^*)_{\delta/u}} V_2(t) > u^{1-H}\right\} (\psi_{\delta, H}^{(1)}(u))^{-1} \\ & \leq C^{-1} \mathbf{P}\left\{\sup_{t \in [0, t^*)_{\delta/u}} V_2(t) > u^{1-H}\right\} (\psi_{0, H}^{(1)}(u))^{-1} \rightarrow 0, \quad u \rightarrow \infty, \end{aligned}$$

where the last convergence follows from the proof of Theorem 3.1, case (4),  $H \geq 1/2$ , in [9].

*Case  $H < 1/2$ .* Let  $\theta_u \in [0, \delta)$  be such that  $t^* + \theta_u/u \in G(\delta/u)$ . Denote

$$t_u = t^* + \frac{\theta_u}{u}. \quad (3.7)$$

Notice that with  $t_u^- = t_u - \delta/u$ , by the Mill ratio  $\bar{\Phi}(x) \sim (1/\sqrt{2\pi}x) \exp(-x^2/2)$ ,  $x \rightarrow \infty$  (see, e.g., [11, Lemma 2.1]) and Taylor's theorem

$$\begin{aligned} & \mathbf{P}\left\{\sup_{t \in [0, t^*)_{\delta/u}} V_2(t) > u^{1-H}\right\} \leq \mathbf{P}\{V_2(t_u^-) > u^{1-H}\} + Cu \sup_{t \in [0, t_u^- - \delta/u]_{\delta/u}} \mathbf{P}\{V_2(t) > u^{1-H}\} \\ & = \mathbf{P}\{V_2(t_u^-) > u^{1-H}\} + Cu \mathbf{P}\left\{V_2\left(t_u^- - \frac{\delta}{u}\right) > u^{1-H}\right\} \\ & \leq \left(1 + Cu \exp\left(u^{1-2H} \frac{\delta w_2'(t^*)}{2}\right)\right) \mathbf{P}\{V_2(t_u^-) > u^{1-H}\}, \end{aligned}$$

where  $w_2(t)$  is defined in (2.2). Since  $H < 1/2$  and  $w_2'(t^*) < 0$ , it follows from (2.3) that the last expression equals  $o(\psi_{\delta, H}^{(1)}(u))$  as  $u \rightarrow \infty$  and (3.6) holds. Thus from (3.4), (3.5), and (3.6) it follows that

$$\bar{\psi}_{\delta, H}(u) \sim \frac{1}{2}\psi_{\delta, H}^{(1)}(u), \quad u \rightarrow \infty,$$

establishing the claim by (2.3). Case  $t^* \geq t_2$  follows by the same arguments.

Case (ii). Denote

$$Z_H(t) = \frac{B_H(t)}{\max(c_1t + q_1, c_2t + q_2)} \quad \text{and} \quad \sigma_H^2(t) = \text{Var}\{Z_H(t)\}.$$

Notice that if  $t^* \in [t_1, t_2]$ , then  $t^*$  is the unique maximizer of  $\sigma_H(t)$ . Moreover,  $\sigma_H(t)$  increases over  $[0, t^*]$  and decreases over  $[t^*, \infty)$ .

Case  $H > 1/2$ . From Theorem 3.1, case (3),  $H > 1/2$ , in [9] it follows that

$$\bar{\psi}_{\delta,H}(u) \leq \bar{\Phi}(\mathcal{D}_H u^{1-H})(1 + o(1)), \quad u \rightarrow \infty.$$

We have (recall that  $t_u$  is defined in (3.7))

$$\begin{aligned} \bar{\psi}_{\delta,H}(u) &= \mathbf{P}\left\{ \sup_{t \in G(\delta/u)} Z_H(t) > u^{1-H} \right\} \geq \mathbf{P}\{Z_H(t_u) > u^{1-H}\} \\ &\sim \bar{\Phi}(\mathcal{D}_H u^{1-H}), \quad u \rightarrow \infty. \end{aligned}$$

Combining two statements above, we establish the claim.

Case  $H = 1/2$ . For notational simplicity, we write  $Z(t)$  instead of  $Z_{1/2}(t)$ . It follows from [9] and (3.10) that with  $\Delta = [t_u - S/u, t_u + S/u]_{\delta/u}$ , as  $u \rightarrow \infty$  and then as  $S \rightarrow \infty$ ,

$$\bar{\psi}_{\delta}(u) \sim \mathbf{P}\{\exists t \in \Delta: Z(t) > \sqrt{u}\}.$$

Let  $B^*(t)$  be an independent copy of BM, let  $\bar{B}^*(t) = B^*(t) - c_1t$ , let  $\phi_u(x)$  be the probability density function of  $B(ut_u)$ , and define

$$\eta = q_1 + c_1t^* = q_2 + c_2t^* = \frac{c_1q_2 - q_1c_2}{c_1 - c_2}.$$

By the self-similarity and independence of the increments of BM we have, as  $u \rightarrow \infty$ ,

$$\begin{aligned} &\mathbf{P}\left\{ \sup_{t \in \Delta} Z(t) > \sqrt{u} \right\} \\ &= \mathbf{P}\left\{ \exists \hat{t} \in [ut_u - S, ut_u]_{\delta}: B(\hat{t}) > q_2u + c_2\hat{t} \text{ or} \right. \\ &\quad \left. \exists t \in [ut_u, ut_u + S]_{\delta}: (B(t) - B(ut_u)) + B(ut_u) > q_1u + c_1t \right\} \\ &= \mathbf{P}\left\{ \exists \hat{t} \in [ut_u - S, ut_u]_{\delta}: B(\hat{t}) > q_2u + c_2ut_u + c_2(\hat{t} - ut_u) \text{ or} \right. \\ &\quad \left. \exists t \in [ut_u, ut_u + S]_{\delta}: B^*(t - ut_u) + B(ut_u) > q_1u + c_1ut_u + c_1(t - ut_u) \right\} \\ &= \int_{\mathbb{R}} \phi_u(\eta u - x) \mathbf{P}\left\{ \exists \hat{t} \in [ut_u - S, ut_u]_{\delta}: B(\hat{t}) > q_2u + c_2ut_u + c_2(\hat{t} - ut_u) \text{ or} \right. \\ &\quad \left. \exists t \in [ut_u, ut_u + S]_{\delta}: B^*(t - ut_u) + \eta u - x > q_1u + c_1ut_u + c_1(t - ut_u) \mid \right. \\ &\quad \left. B(ut_u) = \eta u - x \right\} dx \\ &= \int_{\mathbb{R}} \mathbf{P}\left\{ \exists \hat{t} \in [-S, 0]_{\delta}: Z_u(\hat{t}) > x + c_2\theta_u \text{ or } \exists t \in [0, S]_{\delta}: B^*(t) - c_1t > x + c_1\theta_u \right\} \phi_u(\eta u - x) dx \end{aligned}$$

$$\begin{aligned}
&= \frac{\exp(-\frac{\eta^2 u}{2t_u})}{\sqrt{2\pi ut_u}} \int_{\mathbb{R}} \mathbf{P}\{\exists \hat{t} \in [-S, 0)_\delta: Z_u(\hat{t}) > x + c_2\theta_u \text{ or } \exists t \in [0, S]_\delta: \overline{B}^*(t) > x + c_1\theta_u\} \\
&\quad \times \exp\left(\frac{\eta x}{t_u} - \frac{x^2}{2ut_u}\right) dx \\
&\sim \frac{\exp(-\frac{\eta^2 u}{2t^*})}{\sqrt{2\pi ut^*}} \exp\left(\frac{\eta^2 \theta_u}{2(t^*)^2} - \frac{\eta c_2 \theta_u}{t^*}\right) \\
&\quad \times \int_{\mathbb{R}} \mathbf{P}\{\exists \hat{t} \in [-S, 0)_\delta: Z_u(\hat{t}) > x \text{ or } \exists t \in [0, S]_\delta: \overline{B}^*(t) > x + (c_1 - c_2)\theta_u\} \\
&\quad \times \exp\left(\frac{\eta x}{t_u} - \frac{(x - c_2\theta_u)^2}{2ut_u}\right) dx,
\end{aligned}$$

where  $Z_u(\hat{t})$  is a Gaussian process independent of  $\overline{B}^*(t)$  with expectation and covariance

$$\mathbf{E}\{Z_u(\hat{t})\} = \frac{uq_2 - x - c_2\theta_u}{ut_u} \hat{t} \quad \text{and} \quad \text{cov}(Z_u(\hat{s}), Z_u(\hat{t})) = \frac{-\hat{s}\hat{t}}{ut_u} - \hat{t}, \quad -S \leq \hat{s} \leq \hat{t} \leq 0.$$

Since  $\eta - 2t^*c_2 > 0$ , we have

$$\begin{aligned}
&\int_{\mathbb{R}} \mathbf{P}\{\exists \hat{t} \in [-S, 0)_\delta: Z_u(\hat{t}) > x \text{ or } \exists t \in [0, S]_\delta: \overline{B}^*(t) > x + (c_1 - c_2)\delta\} \exp\left(\frac{\eta x}{t_u} - \frac{(x - c_2\theta_u)^2}{2ut_u}\right) dx \\
&\leq \exp\left(\frac{\theta_u \eta (\eta - 2t^*c_2)}{2(t^*)^2}\right) \int_{\mathbb{R}} \mathbf{P}\{\exists \hat{t} \in [-S, 0)_\delta: Z_u(\hat{t}) > x \text{ or } \exists t \in [0, S]_\delta: \overline{B}^*(t) > x + (c_1 - c_2)\theta_u\} \\
&\quad \times \exp\left(\frac{\eta x}{t_u} - \frac{(x - c_2\theta_u)^2}{2ut_u}\right) dx \\
&\leq \exp\left(\frac{\delta \eta (\eta - 2t^*c_2)}{2(t^*)^2}\right) \int_{\mathbb{R}} \mathbf{P}\{\exists \hat{t} \in [-S, 0)_\delta: Z_u(\hat{t}) > x \text{ or } \exists t \in [0, S]_\delta: \overline{B}^*(t) > x\} \\
&\quad \times \exp\left(\frac{\eta x}{t_u} - \frac{(x - c_2\theta_u)^2}{2ut_u}\right) dx. \tag{3.8}
\end{aligned}$$

We estimate the integral in the lower bound. Assume that BM is defined on  $\mathbb{R}$  (centered Gaussian process with  $\text{cov}(B(t), B(s)) = (|t| + |s| - |s - t|)/2$ ). As  $u \rightarrow \infty$ , the covariance and expectation of  $Z_u(t) - q_2t/t^*$  converge to those of BM, and hence  $Z_u(t) - q_2t/t^*$  converges to  $B(t)$  for  $t < 0$  in the sense of convergence of finite-dimensional distributions. Thus with  $\zeta = q_2/t^*$  as  $u \rightarrow \infty$  (proof is given in the Appendix),

$$\begin{aligned}
&\int_{\mathbb{R}} \mathbf{P}\{\exists \hat{t} \in [-S, 0)_\delta: Z_u(\hat{t}) > x \text{ or } \exists t \in [0, S]_\delta: \overline{B}^*(t) > x + (c_1 - c_2)\delta\} \exp\left(\frac{\eta x}{t_u} - \frac{(x - c_2\theta_u)^2}{2ut_u}\right) dx \\
&\sim \int_{\mathbb{R}} \mathbf{P}\{\exists \hat{t} \in [-S, 0)_\delta: B(\hat{t}) + \zeta \hat{t} > x \text{ or } \exists t \in [0, S]_\delta: B(t) - c_1t > x + (c_1 - c_2)\delta\} \exp\left(\frac{\eta x}{t^*}\right) dx \\
&=: I(S). \tag{3.9}
\end{aligned}$$

By the explicit formula  $\mathbf{P}\{\sup_{t \geq 0}(B(t) - ct) > x\} = \exp(-2cx)$ ,  $c, x > 0$  (see [3]), we have

$$\begin{aligned} I(S) &\leq \int_{-\infty}^0 \exp\left(\frac{\eta x}{t^*}\right) dx \\ &\quad + \int_0^{\infty} (\mathbf{P}\{\exists t \geq 0: B(t) - \zeta t > x\} + \mathbf{P}\{\exists t \geq 0: B(t) - c_1 t > x\}) \exp\left(\frac{\eta x}{t^*}\right) dx \\ &= \frac{t^*}{\eta} + \int_0^{\infty} \left( \exp\left(\left(-2\zeta + \frac{\eta}{t^*}\right)x\right) + \exp\left(\left(-2c_1 + \frac{\eta}{t^*}\right)x\right) \right) dx < \infty, \end{aligned}$$

provided that  $\min(2\zeta t^*, 2c_1 t^*) > \eta$ . Since  $I(S)$  is a nondecreasing function, this implies  $\lim_{S \rightarrow \infty} I(S) \in (0, \infty)$ . With  $\xi = \eta^2/(t^*)^2$  and  $\hat{d}_\delta(t) = \mathbf{I}(t < 0)\zeta t^* t/\eta - \mathbf{I}(t \geq 0)(c_1 t^* t/\eta + \eta(c_1 - c_2)\delta/t^*)$ , we have

$$\begin{aligned} I(S) &= \frac{t^*}{\eta} \int_{\mathbb{R}} \mathbf{P}\left\{\exists t \in [-S, S]_\delta: \frac{\eta}{t^*} B(t) + t \left( \mathbf{I}(t \leq 0) \frac{\eta \zeta}{t^*} - \mathbf{I}(t \geq 0) \frac{\eta c_1}{t^*} \right) \right. \\ &\quad \left. - \mathbf{I}(t \geq 0) \frac{\eta(c_1 - c_2)\delta}{t^*} > \frac{\eta x}{t^*} \right\} \exp\left(\frac{\eta x}{t^*}\right) d\left(\frac{x\eta}{t^*}\right) \\ &= \frac{t^*}{\eta} \int_{\mathbb{R}} \mathbf{P}\left\{\exists \xi t \in [-\xi S, \xi S]_{\xi \delta}: B(\xi t) + \xi t \left( \mathbf{I}(t \leq 0) \frac{t^* \zeta}{\eta} - \mathbf{I}(t \geq 0) \frac{t^* c_1}{\eta} \right) \right. \\ &\quad \left. - \mathbf{I}(t \geq 0) \frac{\eta(c_1 - c_2)\delta}{t^*} > x \right\} e^x dx \\ &= \int_{\mathbb{R}} \mathbf{P}\{\exists t \in [-S\xi, S\xi]_{\xi \delta}: B(t) + \hat{d}_\delta(t) > x\} e^x dx \\ &= \mathbf{E}\left\{ \sup_{t \in [-S\xi/2, S\xi/2]_{\delta \xi/2}} \exp(\sqrt{2}B(t) - |t| + \hat{d}_\delta(2t) + |t|) \right\}. \end{aligned}$$

Since  $\lim_{S \rightarrow \infty} I(S) \in (0, \infty)$ ,  $\hat{d}_\delta(2t) + |t| = d_\delta(t)$ , and  $\delta \xi/2 = \gamma$ , we have that this expression tends to  $\mathcal{H}_\gamma^{d_\delta} \in (0, \infty)$  as  $S \rightarrow \infty$ . Thus, summarizing the calculations, we conclude that, as  $u \rightarrow \infty$  and then  $S \rightarrow \infty$ ,

$$\mathbf{P}\left\{\sup_{t \in \Delta} Z(t) > \sqrt{u}\right\} \geq \mathcal{H}_\gamma^{d_\delta} \bar{\Phi}(\mathcal{D}_{1/2} \sqrt{u})(1 + o(1)). \tag{3.10}$$

For the same reasons as estimating the upper bound in (3.8), we have that, as  $u \rightarrow \infty$  and then  $S \rightarrow \infty$ ,

$$\mathbf{P}\left\{\sup_{t \in \Delta} Z(t) > \sqrt{u}\right\} \leq \mathcal{H}_\gamma^{d_0} \exp\left(\frac{\delta \eta (\eta - 2t^* c_2)}{2(t^*)^2}\right) \bar{\Phi}(\mathcal{D}_{1/2} \sqrt{u})(1 + o(1)),$$

and the claim is established.

*Case  $H < 1/2$ .* As shown in the Appendix (recall that  $t_u^- = t_u - \delta/u$  and  $V_1(t), V_2(t)$  are defined in (3.1)),

$$\bar{\psi}_{\delta, H}(u) \sim \mathbf{P}\{V_2(t_u^-) > u^{1-H}\} + \mathbf{P}\{V_1(t_u) > u^{1-H}\}, \quad u \rightarrow \infty. \tag{3.11}$$



We have (recall that  $w_i(t) = (q_i + c_i t)^2/t^{2H}$ ,  $i = 1, 2$ )

$$\begin{aligned} \mathbf{P}\{V_1(t_u) > u^{1-H}\} &\sim \bar{\Phi}(\mathcal{D}_H u^{1-H}) \exp\left(-\frac{\theta_u w'_1(t^*) u^{1-2H}}{2}\right), \\ \mathbf{P}\{V_2(t_u^-) > u^{1-H}\} &\sim \bar{\Phi}(\mathcal{D}_H u^{1-H}) \exp\left(-\frac{-(\delta - \theta_u) w'_2(t^*) u^{1-2H}}{2}\right), \quad u \rightarrow \infty. \end{aligned}$$

Thus

$$\bar{\psi}_{\delta,H}(u) \sim \bar{\Phi}(\mathcal{D}_H u^{1-H}) \left( \exp\left(-\frac{\theta_u w'_1(t^*) u^{1-2H}}{2}\right) + \exp\left(-\frac{-(\delta - \theta_u) w'_2(t^*) u^{1-2H}}{2}\right) \right), \quad u \rightarrow \infty,$$

and hence the claim follows from the inequality (recall that  $B = -\delta w'_1(t^*) w'_2(t^*) / (2(w'_1(t^*) - w'_2(t^*))) > 0$ )

$$\begin{aligned} &2 \exp(-B u^{1-2H}) (1 + o(1)) \\ &\leq \exp\left(-\frac{\theta_u w'_1(t^*) u^{1-2H}}{2}\right) + \exp\left(-\frac{-(\delta - \theta_u) w'_2(t^*) u^{1-2H}}{2}\right) \leq 1 + o(1), \quad u \rightarrow \infty. \quad \square \end{aligned}$$

*Proof of Theorem 2.* When  $H = 1/2$ , the statement of the theorem follows from [7] and [10].

*Case  $H > 1/2$ .* For large  $u$ , we have

$$\begin{aligned} &\mathbf{P}\left\{\exists t \geq 0: \inf_{s \in [t, t+u^{(2H-1)/(2H)}]} (B_H(s) - cs) > u\right\} \\ &\leq \mathbf{P}\left\{\sup_{t \in G(\delta)} (B_H(t) - ct) > u\right\} \leq \mathbf{P}\left\{\sup_{t \geq 0} (B_H(t) - ct) > u\right\}. \end{aligned}$$

In view of Remark 3.2 in [2], the lower and upper bounds above are asymptotically equivalent, and hence

$$\mathbf{P}\left\{\sup_{t \in G(\delta)} (B_H(t) - ct) > u\right\} \sim \mathbf{P}\left\{\sup_{t \geq 0} (B_H(t) - ct) > u\right\}, \quad u \rightarrow \infty.$$

The asymptotics of the last probability is given, for example, in Corollary 3.1 in [2], and thus the claim follows.

*Case  $H < 1/2$ .* By the self-similarity of fBM we have

$$\begin{aligned} \psi_{\delta,H}(u) &:= \mathbf{P}\{\exists t \in G(\delta): B_H(t) > u + ct\} \\ &= \mathbf{P}\left\{\exists t \in G\left(\frac{\delta}{u}\right): \frac{B_H(t)}{1+ct} > u^{1-H}\right\} =: \mathbf{P}\left\{\exists t \in G\left(\frac{\delta}{u}\right): V(t) > u^{1-H}\right\}. \end{aligned}$$

Note that the variance of  $V(t)$  achieves its unique maxima at  $t_0 = H/(c(1-H))$ . As shown in the Appendix,

$$\psi_{\delta,H}(u) \sim \sum_{t \in I(t_0)} \mathbf{P}\{V(t) > u^{1-H}\}, \quad u \rightarrow \infty, \tag{3.12}$$

where  $I(t_0) = (-1/\sqrt{u} + t_0, 1/\sqrt{u} + t_0)_{\delta/u}$ . We have that with  $\hat{u} = u^{1-H} c^H / (H^H (1-H)^{1-H})$ , as  $u \rightarrow \infty$ ,

$$\sum_{t \in I(t_0)} \mathbf{P}\{V(t) > u^{1-H}\} = \sum_{t \in I(t_0)} \bar{\Phi}\left(u^{1-H} \frac{1+ct}{t^H}\right) \sim \sum_{t \in I(t_0)} \frac{1}{\sqrt{2\pi\hat{u}}} \exp\left(-\frac{1}{2} \left(u^{1-H} \frac{1+ct}{t^H}\right)^2\right).$$

Setting  $f_H(t) = (1 + ct)^2/t^{2H}$ , we have  $f'_H(t_0) = 0$  and  $f''_H(t_0) = 2c^{2+2H}(1 - H)^{2H+1}/H^{2H+1} > 0$ . Since  $f_H(t) \approx f_H(t_0) + (t - t_0)^2 f''_H(t_0)/2$ ,  $t \in I(t_0)$ , we write (a strict proof is given in the Appendix in [8])

$$\begin{aligned} & \sum_{t \in I(t_0)} \frac{1}{\sqrt{2\pi\hat{u}}} \exp\left(-\frac{1}{2}\left(u^{1-H} \frac{1+ct}{t^H}\right)^2\right) \\ &= \frac{1}{\sqrt{2\pi\hat{u}}} \exp\left(-\frac{\hat{u}^2}{2}\right) \sum_{t \in I(t_0)} \exp\left(-\frac{1}{2}u^{2-2H}\left(\frac{(1+ct)^2}{t^{2H}} - \frac{(1+ct_0)^2}{t_0^{2H}}\right)\right) \\ &\sim \bar{\Phi}(\hat{u}) \sum_{t \in I(t_0)} \exp\left(-\frac{1}{2}u^{2-2H} \frac{f''_H(t_0)}{2}(t - t_0)^2\right), \quad u \rightarrow \infty. \end{aligned} \tag{3.13}$$

Next, setting  $F = f''_H(t_0)/4 = c^{2+2H}(1 - H)^{2H+1}/(2H^{2H+1})$ , we have

$$\begin{aligned} & \sum_{t \in I(t_0)} \exp\left(-\frac{1}{2}u^{2-2H} \frac{f''_H(t_0)}{2}(t - t_0)^2\right) \\ &\sim 2 \sum_{t \in (0, u^{-1/2})_{\delta u}} \exp(-Fu^{2-2H}t^2) = 2 \sum_{tu^{1-H} \in (0, u^{1/2-H})_{\delta u-H}} \exp(-F(tu^{1-H})^2) \\ &= \frac{2u^H}{\delta} \left( \delta u^{-H} \sum_{t \in (0, u^{1/2-H})_{\delta u-H}} \exp(-Ft^2) \right) \sim \frac{2u^H}{\delta\sqrt{F}} \int_0^\infty \exp(-Ft^2) d(\sqrt{F}t) \\ &= \frac{\sqrt{\pi}u^H}{\delta\sqrt{F}}, \quad u \rightarrow \infty. \end{aligned}$$

Combining this with (3.13) and (3.12), we have

$$\psi_{\delta,H}(u) \sim \bar{\Phi}\left(\frac{u^{1-H}c^H}{HH(1-H)^{1-H}}\right) \frac{\sqrt{2\pi}H^{H+1/2}u^H}{\delta c^{H+1}(1-H)^{H+1/2}}, \quad u \rightarrow \infty. \quad \square \tag{3.14}$$

*Proof of Remark 2.* Assume that  $(q_1, c_1) \geq (q_2, c_2)$  in the alphabetical order; the other case follows by the same arguments. For large  $u$ , we have that  $q_1u + c_1t \geq q_2u + c_2t$  for all  $t \in [0, T]$ , implying

$$\bar{\zeta}_H(u) = \mathbf{P}\{\exists t \in [0, T]: B_H(t) > c_1t + q_1u\}.$$

Thus for  $H = 1/2$ , the claim follows by [3]. For  $H \neq 1/2$ , Theorem 2.1 in [5] completes the proof.  $\square$

### Appendix

*Proof of (3.3).* To establish the claim, it suffices to show that

$$\mathbf{P}\{\exists t \notin [t_1 - \varepsilon, t_1 + \varepsilon]: V_1(t) > u^{1-H}\} = o(\psi_{\delta,H}^{(1)}(u)), \quad u \rightarrow \infty.$$

We will prove that  $V_1(t)$  is a.s. bounded on  $[0, \infty)$ . By [12, Chap. 4, p. 31] it is equivalent to  $\mathbf{P}\{V_1(t)$  is bounded for  $t \geq 0\} > 0$ . By Corollary 2 in [6] we have

$$\mathbf{P}\left\{\sup_{t \geq 0} V_1(t) \leq u\right\} = 1 - \mathbf{P}\left\{\sup_{t \geq 0} V_1(t) > u\right\} \rightarrow 1, \quad u \rightarrow \infty.$$

Thus  $V_1(t)$  is bounded a.s. Note that the variance  $v(t)$  of  $V_1(t)$  achieves its unique maximum at  $t_1$ . Denote

$$m = \max_{t \in [0, t_1 - \varepsilon] \cup [t_1 + \varepsilon, \infty)} v(t), \quad M = \mathbf{E} \left\{ \sup_{t \in [0, t_1 - \varepsilon] \cup [t_1 + \varepsilon, \infty)} V_1(t) \right\}.$$

By the Borell–TIS inequality (see [9, Lemma 5.3]) we have that  $M < \infty$  and for all  $u$  large enough,

$$\mathbf{P} \{ \exists t \notin [t_1 - \varepsilon, t_1 + \varepsilon]: V_1(t) > u^{1-H} \} \leq \exp \left( - \frac{(u^{1-H} - M)^2}{2m} \right).$$

From Theorem 2 and the inequality  $m < v(t_1)$  it follows that

$$\exp \left( - \frac{(u^{1-H} - M)^2}{2m} \right) = o(\psi_{\delta, H}^{(1)}(u)), \quad u \rightarrow \infty,$$

and thus (3.3) holds.  $\square$

*Proof of (3.5).* Assume that  $H < 1/2$ . Since  $t^* = t_1$  is the unique maximizer of  $\text{Var}\{V_1(t)\}$ , repeating the proof of Theorem 2, we obtain

$$\mathbf{P} \left\{ \sup_{t \in [t^*, \infty)_{\delta/u}} V_1(t) > u^{1-H} \right\} \sim \sum_{t \in [t_1, t_1 + 1/\sqrt{u}]_{\delta/u}} \mathbf{P} \{ V_1(t) > u^{1-H} \}, \quad u \rightarrow \infty.$$

The method of computation of the asymptotics of the last sum is the same as in the proof of Theorem 2; see the calculation of the analogous sum in (3.12). The difference is only in the intervals of summation: in Theorem 2,  $I(t_0)$  is symmetric about  $t_0$ , whereas  $[t_1, t_1 + 1/\sqrt{u}]_{\delta/u}$  has only the right part. Thus the factor 1/2 appears before the final asymptotics.

Assume that  $H = 1/2$ . Then the claim follows from the proof of Theorem 1.1 in [7]. The index of summation in (19) in [7] in our case will be  $1 \leq j \leq N_u$ , and thus the factor 1/2 appears before the final asymptotics. The claim can also be established by Theorem 1(ii) in [10].

Assume that  $H > 1/2$ . As in the proof of Theorem 2, case  $H > 1/2$ , we have

$$\begin{aligned} & \mathbf{P} \left\{ \exists t \geq t^*: \inf_{s \in [t, t + u^{-1/(2H)}]} V_1(t) > u^{1-H} \right\} \\ & \leq \mathbf{P} \left\{ \sup_{t \in [t^*, \infty)_{\delta/u}} V_1(t) > u^{1-H} \right\} \leq \mathbf{P} \left\{ \sup_{t \in [t^*, \infty)} V_1(t) > u^{1-H} \right\}. \end{aligned}$$

As follows from [12], the upper bound in the inequality above is equivalent with  $\psi_{0, H}^{(1)}(u)/2$ ,  $u \rightarrow \infty$ , and by Theorem 2.1 in [2] the lower bound has the same asymptotics. Since for  $H > 1/2$  (see Theorem 2),  $\psi_{0, H}^{(1)}(u) \sim \psi_{\delta, H}^{(1)}(u)$ ,  $u \rightarrow \infty$ , we obtain the claim.  $\square$

*Proof of (3.9).* First, we show that with  $\bar{\delta} = (c_1 - c_2)\delta$ ,

$$\begin{aligned} & \int_{\mathbb{R}} \mathbf{P} \{ \exists \hat{t} \in [-S, 0]_{\delta}: Z_u(\hat{t}) > x \text{ or } \exists t \in [0, S]_{\delta}: \bar{B}^*(t) > x + \bar{\delta} \} \exp \left( \frac{\eta x}{t_u} - \frac{(x - c_2 \theta_u)^2}{2ut_u} \right) dx \\ & = \int_{-M}^M \mathbf{P} \{ \exists \hat{t} \in [-S, 0]_{\delta}: Z_u(\hat{t}) > x \text{ or } \exists t \in [0, S]_{\delta}: \bar{B}^*(t) > x + \bar{\delta} \} \exp \left( \frac{\eta x}{t^*} \right) dx + B_{M, v}, \quad (\text{A.1}) \end{aligned}$$

where  $B_{M,v} \rightarrow 0$  as  $u \rightarrow \infty$  and then  $M \rightarrow \infty$ . We have

$$\begin{aligned} |B_{M,v}| &\leq \left| \int_{-M}^M \mathbf{P}\{\exists \hat{t} \in [-S, 0]_\delta: Z_u(\hat{t}) > x \text{ or } \exists t \in [0, S]_\delta: \bar{B}^*(t) > x + \bar{\delta}\} \right. \\ &\quad \left. \times \left( \exp\left(\frac{\eta x}{t_u} - \frac{(x - c_2\theta_u)^2}{2ut_u}\right) - \exp\left(\frac{\eta x}{t^*}\right) \right) dx \right| \\ &\quad + \int_{|x|>M} \mathbf{P}\{\exists \hat{t} \in [-S, 0]_\delta: Z_u(\hat{t}) > x \text{ or } \exists t \in [0, S]_\delta: \bar{B}^*(t) > x + \bar{\delta}\} \\ &\quad \times \exp\left(\frac{\eta x}{t_u} - \frac{(x - c_2\theta_u)^2}{2ut_u}\right) dx \\ &=: |I_1| + I_2. \end{aligned}$$

Since  $\text{Var}\{Z_u(t)\}$  is bounded and  $\mathbf{E}\{Z_u(t)\} < 0$  for large  $u$  and all  $t \in [-S, 0]$ , by the Borell–TIS inequality, for  $x > 0$  and some  $C > 0$ , we have

$$\begin{aligned} &\mathbf{P}\left\{ \sup_{\hat{t} \in [-S, 0]_\delta} Z_u(\hat{t}) > x \text{ or } \sup_{t \in [0, S]_\delta} \bar{B}^*(t) > x + \bar{\delta} \right\} \\ &\leq \mathbf{P}\left\{ \sup_{t \in [-S, 0]} (Z_u(t) - \mathbf{E}\{Z_u(t)\}) > x \right\} + \mathbf{P}\left\{ \sup_{t \in [0, S]} B(t) > x \right\} \leq \exp\left(-\frac{x^2}{C}\right). \end{aligned}$$

Thus, as  $u \rightarrow \infty$ ,

$$I_2 \leq \int_{x>M} \exp\left(-\frac{x^2}{C} + \frac{\eta x}{t_u}\right) dx + \exp\left(\frac{\eta x}{2t^*}\right) dx \rightarrow 0, \quad M \rightarrow \infty.$$

For  $u \geq M^3$ , we have

$$\begin{aligned} |I_1| &\leq \int_{-M}^M \exp\left(-\frac{x^2}{C} + \frac{\eta x}{t^*}\right) \left| \exp\left(-\frac{x\eta\theta_u}{ut^*t_u} - \frac{(x - c_2\theta_u)^2}{2ut_u}\right) - 1 \right| dx \\ &\leq \int_{\mathbb{R}} \exp\left(-\frac{x^2}{C} + \frac{\eta x}{t^*}\right) dx \sup_{x \in [-M, M]} \left| \exp\left(-\frac{x\eta\theta_u}{ut^*t_u} - \frac{(x - c_2\theta_u)^2}{2ut_u}\right) - 1 \right| \leq \frac{C}{M}. \end{aligned}$$

Thus  $\lim_{M \rightarrow \infty} \lim_{u \rightarrow \infty} (|I_1| + I_2) = 0$ , and (A.1) holds. Since for  $t \in [-S, 0]$ ,  $Z_u(t)$  converges to  $B(t) + \zeta t$  as  $u \rightarrow \infty$  in the sense of convergence of finite-dimensional distributions, we have

$$\begin{aligned} &\int_{-M}^M \mathbf{P}\{\exists \hat{t} \in [-S, 0]_\delta: Z_u(\hat{t}) > x \text{ or } \exists t \in [0, S]_\delta: \bar{B}^*(t) > x + \bar{\delta}\} \exp\left(\frac{\eta x}{t^*}\right) dx \\ &\rightarrow \int_{-M}^M \mathbf{P}\{\exists \hat{t} \in [-S, 0]_\delta: B(t) + \zeta \hat{t} > x \text{ or } \exists t \in [0, S]_\delta: \bar{B}^*(t) > x + \bar{\delta}\} \exp\left(\frac{\eta x}{t^*}\right) dx, \quad u \rightarrow \infty. \end{aligned}$$

By the monotone convergence theorem the expression above tends to

$$\int_{\mathbb{R}} \mathbf{P}\{\exists \hat{t} \in [-S, 0)_{\delta}: B(\hat{t}) + \zeta \hat{t} > x \text{ or } \exists t \in [0, S]_{\delta}: B(t) - c_1 t > x + \bar{\delta}\} \exp\left(\frac{\eta x}{t^*}\right) dx, \quad M \rightarrow \infty,$$

and the claim is established.  $\square$

*Proof of (3.11).* By Lemma 2.3 in [11], for all large  $u$  with  $w = u^{1-H}$  (recall that  $t_u^- = t_u - \delta/u$ ), we have

$$\begin{aligned} \bar{\psi}_{\delta, H}(u) &\geq \mathbf{P}\left\{\sup_{t \in \{t_u^-, t_u\}} Z_H(t) > w\right\} \\ &= \mathbf{P}\{V_1(t_u) > w\} + \mathbf{P}\{V_2(t_u^-) > w\} - \mathbf{P}\{V_1(t_u) > w, V_2(t_u^-) > w\} \\ &\sim \mathbf{P}\{V_1(t_u) > w\} + \mathbf{P}\{V_2(t_u^-) > w\}, \quad u \rightarrow \infty. \end{aligned} \quad (\text{A.2})$$

Next, we prove that

$$\mathbf{P}\left\{\exists t \in G\left(\frac{\delta}{u}\right), t \geq t^*: V_1(t) > w\right\} \sim \mathbf{P}\{V_1(t_u) > w\}, \quad u \rightarrow \infty. \quad (\text{A.3})$$

Fix some  $\varepsilon > 0$ . Since  $\sigma_H^2(t)$  is decreasing over  $[t^*, \infty)$ , by the Borell–TIS inequality we have, as  $u \rightarrow \infty$ ,

$$\mathbf{P}\left\{\exists t \in G\left(\frac{\delta}{u}\right), t \geq t^* + \varepsilon: V_1(t) > w\right\} = o(\mathbf{P}\{V_1(t_u) > w\}). \quad (\text{A.4})$$

With  $t_u^+ = t_u + \delta/u$  and  $w_1(t)$  defined in (2.2), we have, as  $u \rightarrow \infty$ ,

$$\begin{aligned} &\mathbf{P}\left\{\exists t \in G\left(\frac{\delta}{u}\right), t_u^+ \leq t \leq t^* + \varepsilon: V_1(t) > w\right\} \\ &\leq Cu \sup_{t \in G(\delta/u), t_u^+ \leq t \leq t^* + \varepsilon} \mathbf{P}\{V_1(t) > w\} \leq Cu \mathbf{P}\{V_1(t_u^+) > w\} \\ &\sim Cu \mathbf{P}\{V_1(t_u) > w\} \exp\left(-\frac{w_1'(t^*)\delta}{2} w\right) = o(\mathbf{P}\{V_1(t_u) > w\}). \end{aligned}$$

Combining this with (A.4), we establish (A.3). By the same arguments we have

$$\mathbf{P}\left\{\exists t \in G\left(\frac{\delta}{u}\right), t < t^*: V_2(t) > w\right\} \sim \mathbf{P}\{V_2(t_u^-) > w\}, \quad u \rightarrow \infty,$$

implying, together with (A.3),

$$\begin{aligned} \bar{\psi}_{\delta, H}(u) &\leq \mathbf{P}\left\{\exists t \in G\left(\frac{\delta}{u}\right), t < t^*: V_2(t) > w\right\} + \mathbf{P}\left\{\exists t \in G\left(\frac{\delta}{u}\right), t \geq t^*: V_1(t) > w\right\} \\ &= (\mathbf{P}\{V_1(t_u) > w\} + \mathbf{P}\{V_2(t_u^-) > w\})(1 + o(1)), \quad u \rightarrow \infty. \end{aligned}$$

By this and (A.2) we obtain the claim.  $\square$

*Proof of (3.12).* First, we prove that with  $\bar{I}(t_0) = (-1/\sqrt{u} + t_0, t_0 + 1/\sqrt{u})$ ,

$$\mathbf{P}\left\{\sup_{t \in G(\delta/u) \setminus \bar{I}(t_0)} V(t) > u^{1-H}\right\} = o(\psi_{\delta, H}(u)), \quad u \rightarrow \infty. \quad (\text{A.5})$$

Denote  $\varepsilon(t_0) = (-\varepsilon + t_0, \varepsilon + t_0)_{\delta/u}$  and  $\bar{\varepsilon}(t_0) = (-\varepsilon + t_0, \varepsilon + t_0)$  for some  $\varepsilon > 0$ . We have

$$\mathbf{P}\left\{\sup_{t \in G(\delta/u) \setminus \bar{I}(t_0)} V(t) > u^{1-H}\right\} \leq \mathbf{P}\left\{\sup_{t \in \varepsilon(t_0) \setminus \bar{I}(t_0)} V(t) > u^{1-H}\right\} + \mathbf{P}\left\{\sup_{t \in [0, \infty) \setminus \bar{\varepsilon}(t_0)} V(t) > u^{1-H}\right\}.$$

The second summand is negligible by the Borell–TIS inequality. Notice that

$$\begin{aligned} \mathbf{P}\left\{\sup_{t \in \varepsilon(t_0) \setminus \bar{I}(t_0)} V(t) > u^{1-H}\right\} &\leq Cu \sup_{t \in \varepsilon(t_0) \setminus \bar{I}(t_0)} \mathbf{P}\{V(t) > u^{1-H}\} \\ &\leq Cu \left( \mathbf{P}\left\{V\left(t_0 - \frac{1}{\sqrt{u}}\right) > u^{1-H}\right\} + \mathbf{P}\left\{V\left(t_0 + \frac{1}{\sqrt{u}}\right) > u^{1-H}\right\} \right) \\ &\leq 3Cu \bar{\Phi}\left(u^{1-H} \frac{1 + ct_0}{t_0^H}\right) \exp\left(-\frac{1}{4} f_H''(t_0) u^{1-2H}\right), \quad u \rightarrow \infty, \end{aligned}$$

and recall that  $f_H(t) = (1 + ct)^2/t^{2H}$  and  $f_H''(t_0) > 0$ . Hence we have

$$\mathbf{P}\left\{\sup_{t \in \varepsilon(t_0) \setminus \bar{I}(t_0)} V(t) > u^{1-H}\right\} = o\left(\bar{\Phi}\left(u^{1-H} \frac{1 + ct_0}{t_0^H}\right)\right), \quad u \rightarrow \infty,$$

and thus (A.5) follows from (3.14). Next, by the Bonferroni inequality

$$\sum_{t \in I(t_0)} \mathbf{P}\{V(t) > u^{1-H}\} - \Pi(u) \leq \mathbf{P}\left\{\sup_{t \in I(t_0)} V(t) > u^{1-H}\right\} \leq \sum_{t \in I(t_0)} \mathbf{P}\{V(t) > u^{1-H}\}, \quad (\text{A.6})$$

where

$$\Pi(u) = \sum_{t_1 < t_2 \in I(t_0)} \mathbf{P}\{V(t_1) > u^{1-H}, V(t_2) > u^{1-H}\}.$$

Fix some numbers  $t_1, t_2 \in I(t_0)$ . We have (recall that  $\hat{u} = u^{1-H} c^H / (H^H (1 - H)^{1-H})$ )

$$\begin{aligned} \mathbf{P}\{V(t_1) > u^{1-H}, V(t_2) > u^{1-H}\} &\leq \mathbf{P}\left\{\frac{\varepsilon^{(1)} V(t_1)}{\sigma_H(t_0)} > \frac{u^{1-H}}{\sigma_H(t_0)}, \frac{\varepsilon^{(2)} V(t_2)}{\sigma_H(t_0)} > \frac{u^{1-H}}{\sigma_H(t_0)}\right\} \\ &=: \mathbf{P}\{W_1 > \hat{u}, W_2 > \hat{u}\}, \end{aligned}$$

where the numbers  $\varepsilon^{(1)}, \varepsilon^{(2)} \geq 1$  are chosen such that

$$\text{Var}\left\{\frac{\varepsilon^{(1)} V(t_1)}{\sigma_H(t_0)}\right\} = \text{Var}\left\{\frac{\varepsilon^{(2)} V(t_2)}{\sigma_H(t_0)}\right\} = 1.$$

We have that the correlation  $r_w$  of  $(W_1, W_2)$  has the expansion

$$r_w(t_1, t_2) = 1 - C|t_1 - t_2|^{2H} + o(|t_1 - t_2|^{2H}), \quad t_1, t_2 \rightarrow t_0,$$

and hence  $\sqrt{|t_1 - t_2|^{2H}} \geq \delta^H u^{-H}$  for all  $t_1, t_2 \in I(t_0)$ . Thus by Lemma 2.3 in [11] we have

$$\mathbf{P}\{W_1 > \hat{u}, W_2 > \hat{u}\} \leq \bar{\Phi}(\hat{u}) \bar{\Phi}(\hat{C} u^{1-2H}), \quad u \rightarrow \infty,$$

implying that for all  $t_1, t_2 \in I(t_0)$ ,

$$\mathbf{P}\{V(t_1) > u^{1-H}, V(t_2) > u^{1-H}\} \leq \bar{\Phi}(\hat{u})\bar{\Phi}(\hat{C}u^{1-2H}), \quad u \rightarrow \infty.$$

There are less than  $Cu^2$  summands in  $\Pi(u)$ , and hence from the last inequality, (A.6), and (3.14) it follows that

$$\mathbf{P}\left\{\sup_{t \in I(t_0)} V(t) > u^{1-H}\right\} \sim \sum_{t \in I(t_0)} \mathbf{P}\{V(t) > u^{1-H}\}, \quad u \rightarrow \infty,$$

which, combined with (A.5), yields the claim.  $\square$

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