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$\mathcal{A}$-continuity and measure

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Abstract. We introduce the notions of λ-Baire property and λ-semiopen set using sets of Lebesgue measure zero. For a family $\mathcal{A}$ of subsets of the real line, we define the (λ^*)-property analogously as it was done in the category case for the $(*)$-property. The main result is that the family $\mathcal{A}$ of all subsets of the real line having the λ-Baire property has the (λ^*)-property iff $\mathcal{A}$ is situated between the Euclidean topology and the family of λ-semiopen sets.

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1 Introduction

The considerations concerning some generalization of the continuity are interesting for many mathematicians. Such studies have a long tradition and have been expanded in two directions, changing the topology in the domain or replacing this topology by some family $\mathcal{A}$ (not necessary topology) of subsets of the domain. For example, the notion of a quasicontinuous function in the sense of Kempisty [6] can be introduced using the second method. In this case, for the family $\mathcal{A}$, the family of semiopen sets on the real line can be used (see [10]).

Such studies concerning $\mathcal{A}$-continuity are carried out for the category case in [4]. There it is proved, among others, that if $\mathcal{A}$ has the $(*)$-property, then each $\mathcal{A}$-continuous function is quasicontinuous, and a family $\mathcal{A}$ has the $(*)$-property iff $\mathcal{A}$ is situated between the Euclidean topology and the family of $\mathcal{I}$-semiopen sets. In [4], it is proved that each family of all $\mathcal{A}$-continuous functions is a strongly porous set in the space of quasicontinuous functions if $\mathcal{A}$ is a translation-invariant topology having the $(*)$-property.

Another example of such studies is given in [2]. Here the Baire property and the $(*)$-property are replaced by measurability in the Lebesgue sense and the (d^*)-property, respectively. The authors consider families of sets situated between the Euclidean topology and the family of measurable sets such that each set from this family is not of measure zero at each of its point. They prove, among others, that if $\mathcal{A}$ has the (d^*)-property, then the set of all functions having the $\mathcal{A}$-Darboux property is strongly porous in the space of Darboux functions.

In our paper, we introduce the notions of the λ-Baire property and the λ-semiopen sets, and replace the topology in the domain by some family $\mathcal{A}$ of subsets of the real line having the (λ^*)-property. We prove that the family $\mathcal{A}$ of sets having the λ-Baire property has the (λ^*)-property iff $\mathcal{A}$ is situated between the Euclidean topology and the family of λ-semiopen sets.

The paper ends with a presentation of the relationships between measurable functions and functions having the λ-Baire property.

2 Terminology, notations, and previous results

Let $A \in \mathcal{P}(\mathbb{R})$, where $\mathcal{P}(\mathbb{R})$ is a family of all subsets of $\mathbb{R}$. We say that A is open (closed) if it is open (closed) in the Euclidean topology τ_{e}. We denote by $\overline{A}$, $\operatorname{Int} A$, and $\operatorname{Fr} A$ the closure, interior, and boundary of a set A in the Euclidean topology. By $\lambda(A)$ we denote the Lebesgue measure of a measurable set A. By "functions" we mean real-valued functions on the real line.

First, let us recall some necessary definitions. Let $\mathcal{A} \subset \mathcal{P}(\mathbb{R})$.

DEFINITION 1. A function f is $\mathcal{A}$-continuous at a point $x \in \mathbb{R}$ if for each open set $V \subset \mathbb{R}$ with $f(x) \in V$, there exists a set $A \in \mathcal{A}$ such that $x \in A$ and $f(A) \subset V$. A function f is *$\mathcal{A}$-continuous* (briefly, $f \in C_{\mathcal{A}}$) if f is $\mathcal{A}$-continuous at each point $x \in \mathbb{R}$.

It is clear that if $\mathcal{A}$ is an arbitrary topology τ on $\mathbb{R}$, then the $\mathcal{A}$-continuity coincides with the continuity between topological spaces $(\mathbb{R}, \tau)$ and $(\mathbb{R}, \tau_{\mathrm{e}})$. In particular, if $\mathcal{A}$ is the Euclidean topology τ_{e} or the density topology τ_{d}, then the notion of $\mathcal{A}$-continuity is equivalent to the notions of the continuity in the classical sense and the approximate continuity, respectively.

A set $A \subset \mathbb{R}$ is said to be *semiopen* if there exists an open set U such that $U \subset A \subset \overline{U}$ (see [8]). Let $\mathcal{S}$ denote the family of all semiopen sets. It is not difficult to see that $A \in \mathcal{S}$ iff $A \subset \overline{\operatorname{Int} A}$.

DEFINITION 2. (See [6].) A function f is quasicontinuous at a point $x \in \mathbb{R}$ if for every neighborhood U of x and for every neighborhood V of $f(x)$, there exists a nonempty open set $G \subset U$ such that $f(G) \subset V$. A function f is *quasicontinuous* (briefly, $f \in \mathcal{Q}$) if it is quasicontinuous at each point $x \in \mathbb{R}$.

Neubrunnová [10] showed that $\mathcal{C}_{\mathcal{S}} = \mathcal{Q}$.

A set A is *of the first category at a point* x (see [7]) if there exists an open neighborhood G of x such that $A \cap G$ is of the first category. We denote by $D(A)$ the set of all points at which A is not of the first category.

Let $\mathcal{B}a$ be a family of all sets having the Baire property.

DEFINITION 3. (See [3].) A family $\mathcal{A} \subset \mathcal{P}(\mathbb{R})$ has the $(*)$-property if

(i) $\tau_{\mathrm{e}} \subset \mathcal{A} \subset \mathcal{B}a$;
(ii) $A \subset D(A)$ for each $A \in \mathcal{A}$.

The following families have the $(*)$-property: the Euclidean topology, the Hashimoto-type topology generated by the σ-ideal of sets of the first category [1], the $\mathcal{I}$-density topology [13, 14, 19, 20], the topologies introduced by Łazarow, Johnson, and Wilczyński [5], the topology considered by Wiertelak [18], and also some families of sets that are not topologies, for example, the family of semiopen sets, but the density topology does not have it.

We observed in [4] that not always $\mathcal{A} \subset \mathcal{S}$, even if $\mathcal{C}_{\mathcal{A}} \subset \mathcal{C}_{\mathcal{S}}$, and that there exists a family $\mathcal{A}$ having the $(*)$-property such that the families $\mathcal{A}$ and $\mathcal{S}$ are incomparable and $\mathcal{C}_{\mathcal{A}} = \mathcal{C}_{\tau_{\mathrm{e}}} \subset \mathcal{C}_{\mathcal{S}} = \mathcal{Q}$.

In [2] the authors introduced the notion analogous to the $(*)$-property for measures. Let $\mathcal{L}$ be a family of all measurable sets. We say that a set A is *of measure zero at a point* x if there exists an open neighborhood G of x such that $A \cap G$ is a nullset. We denote by $D_\lambda(A)$ the set of all points at which A is not of measure zero.

DEFINITION 4. (See [2].) A family $\mathcal{A} \subset \mathcal{P}(\mathbb{R})$ has the (d^*)-property if

(i) $\tau_{\mathrm{e}} \subset \mathcal{A} \subset \mathcal{L}$;
(ii) $A \subset D_\lambda(A)$ for each $A \in \mathcal{A}$.

For example, the following topologies have the (d^*)-property: Euclidean topology, the Hashimoto-type topology generated by the σ-ideal of nullsets, the density topology, an arbitrary ψ-density topology (see [16, 17]), and the topologies constructed by Strobin and Wiertelak [15].

Remark 1. Observe that the family $\mathcal{S}$ of semiopen sets does not have the (d^*)-property since $\mathcal{S} \not\subseteq \mathcal{L}$. For this purpose, let $C_1 \subset [0, 1]$ be a Cantor-type set of positive measure, and let H be a nonmeasurable subset of C_1. Then $A = [0, 1] \setminus H$ is nonmeasurable and semiopen since $\overline{\operatorname{Int} A} = [0, 1]$, so that $A \in \mathcal{S} \setminus \mathcal{L}$.

3 Main results

First we introduce a family of subsets of the real line, which is analogous in the measure sense to the family $\mathcal{B}a$ of all sets having the Baire property.

DEFINITION 5. A set $A \in \mathcal{P}(\mathbb{R})$ has the λ-Baire property if it can be represented as a symmetric difference of G and N, where G is open, and N is of measure zero.

We denote the family of all sets having the λ-Baire property by $\mathcal{B}a_\lambda$, that is,

$$\mathcal{B}a_\lambda = \{A \subset \mathbb{R}\colon\ A = G \triangle N,\ G \in \tau_{\mathrm{e}},\ \lambda(N) = 0\}.$$

Clearly, each open set and each nullset have the λ-Baire property, and A has the λ-Baire property iff $A = (G \setminus N_1) \cup N_2$, where G is open in τ_{e}, and N_1 and N_2 are nullsets such that $N_1 \subset G$ and $G \cap N_2 = \emptyset$.

It is easy to see that $\mathcal{B}a_\lambda \subset \mathcal{L}$. Note that $\mathcal{B}a_\lambda \neq \mathcal{L}$, since $\mathcal{B}a_\lambda$ is not a σ-algebra.

Theorem 1. *The family $\mathcal{B}a_\lambda$ is not closed under taking the complement.*

Proof. Let $C \subset [0,1]$ be a Cantor-type set, that is, closed, nowhere dense, and such that $0, 1 \in C$ and $\lambda(C) = 1/2$. Put $G_0 = \mathbb{R} \setminus C$. Clearly, $G_0 \in \mathcal{B}a_\lambda$, $\lambda(G_0 \cap [0,1]) = 1/2$, and $C = \mathbb{R} \setminus G_0$.

We will prove that $C \notin \mathcal{B}a_\lambda$. Suppose on the contrary that $C = (G_1 \setminus N_1) \cup N_2$, where $G_1 \in \tau_{\mathrm{e}}$, $\lambda(N_1) = \lambda(N_2) = 0$, $N_1 \subset G_1$, and $N_2 \cap G_1 = \emptyset$. Then $\lambda(G_1) = 1/2$ and $G_1 \subset [0,1]$. Since $G_1 = (G_1 \cap C) \cup (G_1 \cap G_0)$ and $\lambda(G_1 \cap C) = 1/2$, we obtain $G_1 \cap G_0 = \emptyset$. Hence $G_1 \subset C$, a contradiction. Consequently, $C \notin \mathcal{B}a_\lambda$. $\square$

Corollary 1. $\mathcal{B}a_\lambda \subsetneq \mathcal{L}$.

Theorem 2. *The families $\mathcal{B}a$ and $\mathcal{B}a_\lambda$ are incomparable.*

Proof. It is well known that the real line can be represented as a union of some nullset A and a set B of the first category (see, e.g., [12]). Hence there exists a nullset $A_1 \subset A$ thats does not have the Baire property, so that $A_1 \in \mathcal{B}a_\lambda \setminus \mathcal{B}a$. Analogously, we can find a meager set $B_1 \subset B$ that is not measurable, so $B_1 \in \mathcal{B}a \setminus \mathcal{B}a_\lambda$. $\square$

We can define the property for measurable sets, analogously to the $(*)$-property and the (d*)-property, in the following way.

DEFINITION 6. A family $\mathcal{A} \subset \mathcal{P}(\mathbb{R})$ has the *(λ^*)*-property if

(i) $\tau_{\mathrm{e}} \subset \mathcal{A} \subset \mathcal{B}a_\lambda$;
(ii) $A \subset D_\lambda(A)$ for each $A \in \mathcal{A}$.

For example, the Hashimoto-type topology generated by the σ-ideal of nullsets has this property, but the family of semiopen sets has not since $\mathcal{S} \nsubseteq \mathcal{L}$ (see Remark 1). Moreover, we can show the following:

Lemma 1. *There exist two families $\mathcal{A}_1$ and $\mathcal{A}_2$ such that*:

(i) $\mathcal{A}_1$ *and* $\mathcal{A}_2$ *are incomparable*;
(ii) $\mathcal{A}_1$ *has the* $(*)$*-property, and* $\mathcal{A}_2$ *has the* (λ^*)*-property*;
(iii) $\mathcal{A}_1$ *does not have the* (λ^*)*-property, and* $\mathcal{A}_2$ *does not have the* $(*)$*-property*;
(iv) $\mathcal{C}_{\mathcal{A}_1} = \mathcal{C}_{\mathcal{A}_2} = \mathcal{C}_{\tau_{\mathrm{e}}}$.

Proof. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be the Hashimoto-type topologies considered for the σ-ideals of meager sets and nullsets, respectively, that is,

$$\mathcal{A}_1 = \{V \setminus P\colon\ V \in \tau_{\mathrm{e}},\ P \text{ is of the first category}\},$$
$$\mathcal{A}_2 = \{V \setminus N\colon\ V \in \tau_{\mathrm{e}},\ N \text{ is of measure zero}\}.$$

Let us represent the real line as a union of two disjoint sets A and B, where A is a nullset, and B is of the first category. Clearly, $A \in \mathcal{A}_1 \setminus \mathcal{A}_2$ and $B \in \mathcal{A}_2 \setminus \mathcal{A}_1$, so that $\mathcal{A}_1$ and $\mathcal{A}_2$ are incomparable. Conditions (ii) and (iii) are fulfilled since $A \in \mathcal{A}_1$ and $B \in \mathcal{A}_2$. As is well known, $\mathcal{C}_{\mathcal{A}_1} = \mathcal{C}_{\mathcal{A}_2} = \mathcal{C}_{\tau_e}$ (see [9]). □

It is easy to see that if $\mathcal{A}$ has the (λ^*)-property, then it has the (d*)-property since $\mathcal{B}a_\lambda \subset \mathcal{L}$. Note that the opposite condition does not hold.

Theorem 3. *There exists a family $\mathcal{A}$ that has the* (d*)*-property and does not have the* (λ^*)*-property.*

Proof. Let $\mathcal{A}$ be the density topology τ_d. Then $\mathcal{A}$ has the (d*)-property since $\tau_e \subset \mathcal{A} \subset \mathcal{L}$, and if $A \in \tau_d$, then each point of A is its density point, so $A \subset D_\lambda(A)$.

On the other hand, note that τ_d does not have the (λ^*)-property since τ_d is not contained in $\mathcal{B}a_\lambda$. For this purpose, consider the set C from Theorem 1, and let A be the interior of C in the density topology.

Suppose that $A \in \mathcal{B}a_\lambda$, that is, $A = G_1 \triangle N_1$, where $G_1 \in \tau_e$ and $\lambda(N_1) = 0$. On the other hand, $A \subset C$ and $\lambda(C \setminus A) = 0$ from the Lebesgue density theorem.

Put $N_2 = C \setminus A$. Then $N_2 \cap A = \emptyset$, and

$$C = A \cup N_2 = A \triangle N_2 = (G_1 \triangle N_1) \triangle N_2 = G_1 \triangle (N_1 \triangle N_2),$$

where $G_1 \in \tau_e$ and $\lambda(N_1 \triangle N_2) = 0$. Consequently, $C \in Ba_\lambda$, a contradiction with the proof of Theorem 1. Finally, $A \in \tau_d \setminus \mathcal{B}a_\lambda$, so τ_d is not contained in $\mathcal{B}a_\lambda$, and the family $\mathcal{A} = \tau_d$ has the (d*)-property but does not have the (λ^*)-property. □

In [4], we showed that if $\mathcal{A}$ has the ($*$)-property, then each $\mathcal{A}$-continuous function is quasicontinuous, that is, for each family $\mathcal{A}$ having the ($*$)-property, we have $\mathcal{C}_{\tau_e} \subset \mathcal{C}_{\mathcal{A}} \subset \mathcal{Q}$. It is not difficult to see that these inclusions can be proper for some families. For example, if $\mathcal{A} = \tau_e$, then $\mathcal{C}_{\tau_e} = \mathcal{C}_{\mathcal{A}} \subsetneq \mathcal{Q}$. If $\mathcal{A} = \mathcal{S}$, then $\mathcal{C}_{\tau_e} \subsetneq \mathcal{C}_{\mathcal{A}} = \mathcal{Q}$. In [4] the proof that for the $\mathcal{I}$-density topology, both inclusions are proper, is given.

Let us show that these properties hold also in the measure case.

Theorem 4. *If $\mathcal{A}$ has the* (λ^*)*-property, then $\mathcal{C}_{\tau_e} \subset \mathcal{C}_{\mathcal{A}} \subset \mathcal{Q}$.*

Proof. Let $\mathcal{A}$ have the (λ^*)-property. Then $\tau_e \subset \mathcal{A}$ and $\mathcal{C}_{\tau_e} \subset \mathcal{C}_{\mathcal{A}}$.

Let us show that $\mathcal{C}_{\mathcal{A}} \subset \mathcal{Q}$. For this purpose, fix $f \in \mathcal{C}_{\mathcal{A}}$, $x \in \mathbb{R}$, and two positive numbers ϵ, ϵ_0 such that $\epsilon < \epsilon_0$. Put $I = (f(x) - \epsilon, f(x) + \epsilon)$. Then there exists a set $A_x \in \mathcal{A}$ with $x \in A_x$ and $f(A_x) \subset I$. As A_x has the λ-Baire property, $A_x = G \triangle N$, where G is open, and N is a nullset.

Clearly, $f(G \triangle N) \subset I$. Note that $f(G) \subset \overline{I}$. Indeed, suppose opposite that there exists a point $x' \in G \cap N$ such that $f(x') \notin \overline{I}$. Let $\epsilon' \in (0, |f(x) - f(x')| - \epsilon)$. The function f is $\mathcal{A}$-continuous at x', so there exists a set $A_{x'} \in \mathcal{A}$ such that $x' \in A_{x'}$ and $f(A_{x'}) \subset (f(x') - \epsilon', f(x') + \epsilon')$. Obviously, $f(A_{x'}) \cap \overline{I} = \emptyset$ and $f(G \setminus N) \subset I$, and hence

$$x' \in A_{x'} \cap G \subset f^{-1}\big((f(x') - \epsilon', f(x') + \epsilon')\big) \cap N \subset N,$$

so $x' \in A_{x'} \setminus D_\lambda(A_{x'})$, a contradiction with the (λ^*)-property.

It is easy to see that $x \in \overline{G}$ since $A_x \subset D_\lambda(A_x)$, so $\{x\} \cup G$ is semiopen. Hence we find a semiopen set $\{x\} \cup G$ containing x and such that $f(\{x\} \cup G) \subset (f(x) - \epsilon_0, f(x) + \epsilon_0)$. Consequently, f is quasicontinuous at x. □

Let us show that

(i) both inclusions in the previous theorem can be proper,
(ii) there exists a family $\mathcal{A}$ having the (λ^*)-property, incomparable with $\mathcal{S}$ such that $\mathcal{C}_{\mathcal{A}} \subsetneq \mathcal{Q}$, and
(iii) there exists a family $\mathcal{A} \subsetneq \mathcal{S}$ having the ($*$)- and (λ^*)-properties such that $\mathcal{C}_{\mathcal{A}} = \mathcal{Q}$.

Let us prove the first observation.

Theorem 5. *There exists a family $\mathcal{A} \subset \mathcal{S}$ having the (λ^*)-property such that $\mathcal{C}_{\tau_e} \subsetneq \mathcal{C}_{\mathcal{A}} \subsetneq \mathcal{Q}$.*

Proof. Put

$$\mathcal{A} = \bigl\{A \in \tau_d\colon\ A = G \cup M,\ G \in \tau_e,\ \lambda(M) = 0\bigr\}.$$

Then $\mathcal{A}$ forms the so-called a.e.-topology considered by O'Malley [11]. We easily see that $\mathcal{A} \subset \mathcal{S}$ and $\mathcal{A}$ has the (λ^*)-property. Let us show that there exists a function f such that $f \in \mathcal{Q} \setminus \mathcal{C}_{\mathcal{A}}$. Let A be a right-hand interval set at 0 (i.e., $A = \bigcup_{n=1}^{\infty}(a_n, b_n)$, where $0 < \cdots < a_n < b_n < a_{n-1} < b_{n-1} < \cdots < a_1 < b_1 < 1$ with $\lim_{n\to\infty} a_n = 0$) such that 0 is a density point of $(-\infty, 0) \cup A$.

Put

$$f(x) = \begin{cases} 1 - x & \text{for } x \leqslant 0,\\ 0 & \text{for } x \in [a_n, b_n],\ n \in \mathbb{N},\\ 1 & \text{for } x = \frac{a_n + b_{n+1}}{2},\ n \in \mathbb{N}, \text{ and for } x \in [b_1, \infty),\\ \text{linear} & \text{on the intervals } [b_{n+1}, \frac{a_n+b_{n+1}}{2}],\ [\frac{a_n+b_{n+1}}{2}, a_n],\ n \in \mathbb{N}. \end{cases}$$

Clearly, $f \in \mathcal{Q}$. Simultaneously, for each $B \in \mathcal{A}$, if $B \subset f^{-1}((1/2, 3/2))$, then 0 is not a density point of B, so $0 \notin B$, and f is not $\mathcal{A}$-continuous at 0. Therefore $f \in \mathcal{Q} \setminus \mathcal{C}_{\mathcal{A}}$.

On the other hand, we easily to see that the function

$$g(x) = \begin{cases} 1 - x & \text{for } x \leqslant 0,\\ 1 & \text{for } x \in [a_n, b_n],\ n \in \mathbb{N},\\ 0 & \text{for } x = \frac{a_n + b_{n+1}}{2},\ n \in \mathbb{N} \text{ and for } x \in [b_1, \infty),\\ \text{linear} & \text{on the intervals } [b_{n+1}, \frac{a_n+b_{n+1}}{2}],\ [\frac{a_n+b_{n+1}}{2}, a_n],\ n \in \mathbb{N}, \end{cases}$$

is $\mathcal{A}$-continuous but not continuous, so $g \in C_{\mathcal{A}} \setminus \mathcal{C}_{\tau_e}$. □

It is worth noting that the family $\mathcal{A}$ from the previous theorem has also the $(*)$-property and, clearly, the (d^*)-property.

Theorem 6. *There exists a family $\mathcal{A}$ having the (λ^*)-property incomparable with $\mathcal{S}$ such that $\mathcal{C}_{\mathcal{A}} = \mathcal{C}_{\tau_e}$.*

Proof. Let us consider the family $\mathcal{A}_2$ from Lemma 1. Then $\mathcal{C}_{\mathcal{A}_2} = \mathcal{C}_{\tau_e}$. Simultaneously, $\mathcal{A}_2$ and $\mathcal{S}$ are incomparable, since $\mathbb{R} \setminus \mathbb{Q} \in \mathcal{A}_2 \setminus \mathcal{S}$ and each closed nondegenerate interval belongs to $\mathcal{S} \setminus \mathcal{A}_2$. □

The family $\mathcal{S}$ of semiopen sets has the $(*)$-property and does not have the (λ^*)- and (d^*)-properties. But we can find a family $\mathcal{A} \subsetneq \mathcal{S}$ with the $(*)$- and (λ^*)-properties such that $\mathcal{C}_{\mathcal{A}} = \mathcal{C}_{\mathcal{S}} = \mathcal{Q}$.

Theorem 7. *There exists a family $\mathcal{A} \subsetneq \mathcal{S}$ having the $(*)$- and (λ^*)-properties such that $\mathcal{C}_{\mathcal{A}} = \mathcal{Q}$.*

Proof. Put

$$\mathcal{A} = \{A \subset \mathbb{R}\colon\ A = G \cup \{x\},\ G \in \tau_e,\ x \in \operatorname{Fr} G\}.$$

We easily see that $\mathcal{A}$ has the $(*)$- and (λ^*)-properties, so $\mathcal{C}_{\mathcal{A}} \subset \mathcal{Q}$ by Theorem 4.

Let $f \in \mathcal{Q}$ and $x \in \mathbb{R}$. Fix $\epsilon > 0$. As f is quasicontinuous, there exists a set $S \in \mathcal{S}$ such that $x \in S$ and $f(S) \subset (f(x) - \epsilon, f(x) + \epsilon)$.

We easily see that $\{x\} \cup \operatorname{Int} S \in \mathcal{A}$ and

$$f\bigl(\{x\} \cup \operatorname{Int} S\bigr) \subset \bigl(f(x) - \epsilon,\ f(x) + \epsilon\bigr),$$

so f is $\mathcal{A}$-continuous at x, and $\mathcal{C}_{\mathcal{A}} = \mathcal{Q}$. □

DEFINITION 7. (See [4].) A set $A \in \mathcal{P}(\mathbb{R})$ is $\mathcal{I}$-semiopen if $A = S \setminus P$, where S is semiopen, and P is of the first category.

We denote the family of all $\mathcal{I}$-semiopen sets by $\mathcal{S}_{\mathcal{I}}$. Clearly, $\mathcal{S}_{\mathcal{I}}$ has the $(*)$-property. Analogously, we have the following:

DEFINITION 8. A set $A \in \mathcal{P}(\mathbb{R})$ is λ-semiopen if $A = S \setminus N$, where S is semiopen, and N is of measure zero.

We denote the family of all λ-semiopen sets by $\mathcal{S}_\lambda$.

It is well known that each semiopen set has the Baire property, that is, $\mathcal{S} \subset \mathcal{B}a$. However, we will show that the analogous inclusion for the families $\mathcal{S}_\lambda$ and $\mathcal{B}a_\lambda$ does not hold, so the family $\mathcal{S}_\lambda$ does not have the (λ^*)-property. Moreover, there exists a set A that is λ-semiopen and does not have the Baire property.

Theorem 8. *The family $\mathcal{S}_\lambda$ is incomparable with $\mathcal{B}a$, $\mathcal{L}$, and $\mathcal{B}a_\lambda$.*

Proof. First, observe that the classical Cantor set belongs to $\mathcal{B}a$, $\mathcal{L}$, and $\mathcal{B}a_\lambda$ but does not belong to $\mathcal{S}_\lambda$.

Now let D be a nullset without the Baire property. Then $\mathbb{R} \setminus D \in \mathcal{S}_\lambda \setminus \mathcal{B}a$.

Let A be a set from Remark 1. Clearly, A is λ-semiopen, so $A \in \mathcal{S}_\lambda \setminus \mathcal{L}$ and $A \in \mathcal{S}_\lambda \setminus \mathcal{B}a_\lambda$.

Next, the classical Cantor set C has the λ-Baire property and is not λ-semiopen, that is, $C \in \mathcal{B}a_\lambda \setminus \mathcal{S}_\lambda$ and $C \in \mathcal{L} \setminus \mathcal{S}_\lambda$. □

In [4], we showed that a family $\mathcal{A}$ has the $(*)$-property iff $\tau_{\mathrm{e}} \subset \mathcal{A} \subset \mathcal{S}_{\mathcal{I}}$. We easily see that there also exist families with the $(*)$-property such that both inclusions are proper, for example, the $\mathcal{I}$-density topology. We will show that the analogous result holds also for the measure case.

Theorem 9. *The family $\mathcal{A} \subset \mathcal{B}a_\lambda$ has the (λ^*)-property iff $\tau_{\mathrm{e}} \subset \mathcal{A} \subset \mathcal{S}_\lambda$.*

Proof. Assume that $\mathcal{A}$ has the (λ^*)-property. Let us show that $\mathcal{A} \subset \mathcal{S}_\lambda$. For this purpose, fix $A \in \mathcal{A}$. Then $A \subset D_\lambda(A)$ and $A = (G \setminus N_1) \cup N_2$, where $G \in \tau_{\mathrm{e}}$, and N_1, N_2 are nullsets. We can assume that $N_1 \subset G$ and $N_2 \cap G = \emptyset$, so $A = (G \cup N_2) \setminus N_1$. Obviously, $N_2 \subset D_\lambda(A) = D_\lambda(G) \subset \overline{G}$, so $G \subset G \cup N_2 \subset \overline{G}$, and therefore $G \cup N_2$ is semiopen. Consequently, $(G \cup N_2) \setminus N_1 \in \mathcal{S}_\lambda$, i.e., $\mathcal{A} \subset \mathcal{S}_\lambda$.

Assume now that $\mathcal{A} \subset \mathcal{B}a_\lambda$ and $\tau_{\mathrm{e}} \subset \mathcal{A} \subset S_\lambda$. It suffices to show that $A \subset D_\lambda(A)$ for each $A \in \mathcal{A}$. Fix $A \in \mathcal{A}$. As $A \in \mathcal{S}_\lambda$, $A = S \setminus N$, where S is semiopen, and N is a nullset. By Corollary 1 both A and S are measurable. On the other hand, $A \in \mathcal{B}a_\lambda$, so $A = (G \setminus N_1) \cup N_2$, where $G \in \tau_{\mathrm{e}}$, and N_1, N_2 are nullsets. We can assume that $N_1 \subset G$ and $N_2 \cap G = \emptyset$. Clearly, $G \setminus N_1 \subset D_\lambda(A)$. Let us show that N_2 is a subset of $D_\lambda(A)$, too. Indeed, let $x \in N_2$. Clearly, $N_2 \subset S \setminus N$, so $x \in S$ and $S \neq \emptyset$. Therefore, as S is semiopen, also $\operatorname{Int} S \neq \emptyset$, and hence $\lambda(S) > 0$. Consequently, $\lambda(A) > 0$ and $G \neq \emptyset$.

As $x \in S$ and $S \subset \overline{\operatorname{Int} S}$, $x \in D_\lambda(\operatorname{Int} S) \subset D_\lambda(S) = D_\lambda(A)$. Consequently, $N_2 \subset D_\lambda(A)$, and also $A \subset D_\lambda(A)$. □

Let us observe that for the Hashimoto-type topology generated by the σ-ideal of nullsets, both inclusions in the previous theorem are proper.

A function f has the Baire property (λ-Baire property) if $f^{-1}(U) \in \mathcal{B}a$ ($f^{-1}(U) \in \mathcal{B}a_\lambda$) for each open set $U \in \tau_{\mathrm{e}}$ (c.f. [12, p. 36]).

It is well known (see [12, Thm. 8.1]) that a function f has the Baire property iff there exists a set P of the first category such that the restriction of f to $\mathbb{R} \setminus P$ is continuous. Analogously, we can prove the following:

Theorem 10. *A function f has the λ-Baire property iff there exists a nullset N such that the restriction of f to $\mathbb{R} \setminus N$ is continuous.*

Proof. (C.f. [12, p. 36]) Fix a function f having the λ-Baire property. Let $\{V_i\}_{i \in \mathbb{N}}$ be a base of τ_{e}. As f has the λ-Baire property, for each $i \in \mathbb{N}$, we can find a set $G_i \in \tau_{\mathrm{e}}$ and a nullset N_i such that $f^{-1}(V_i) = G_i \triangle N_i$. Put $N = \bigcup_{n=1}^{\infty} N_i$. Obviously, N is a nullset. Fix $n \in \mathbb{N}$. Then $(f|_{\mathbb{R} \setminus N})^{-1}(V_i) = f^{-1}(V_i) \setminus N = G_i \setminus N$, so the set $(f|_{\mathbb{R} \setminus N})^{-1}(V_i)$ is open relative to $\mathbb{R} \setminus N$, and $f|_{\mathbb{R} \setminus N}$ is continuous.

Assume now that there exists a nullset N such that $f|_{\mathbb{R} \setminus N}$ is continuous. Fix $V \in \tau_{\mathrm{e}}$. Then $(f|_{\mathbb{R} \setminus N})^{-1}(V) = U \setminus N$ for some $U \in \tau_{\mathrm{e}}$. Therefore $(f|_{\mathbb{R} \setminus N})^{-1}(V) \subset f^{-1}(V) \subset (f|_{\mathbb{R} \setminus N})^{-1}(V) \cup N$.

Hence $U \setminus N \subset f^{-1}(V) \subset U \cup N$ and $f^{-1}(V) = U \triangle M$ for some $M \subset N$, and therefore f has the λ-Baire property. □

In [12], it is also proved that a measurable function need not to be continuous on the complement of a nullset, so not each measurable function has the λ-Baire property.

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