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On the Probabilistic Representation of the Free Effective Resistance of Infinite Graphs

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Abstract
We completely characterize when the free effective resistance of an infinite graph whose vertices have finite degrees can be expressed in terms of simple hitting probabilities of the random walk on the graph.

Keywords Weighted graph · Electrical networks · Effective resistance · Random walk · Transience

Mathematics Subject Classification (2020) Primary 05C63 · 05C81; Secondary 60J10 · 05C12

1 Introduction

We consider undirected, connected graphs with no multiple edges and no self-loops. Each edge (x, y) is given a positive weight $c(x, y)$. A possible interpretation is that (x, y) is a resistor with resistance $1/c(x, y)$. The graph then becomes an electrical network.

More precisely, a *(weighted) graph* $G = (V, c)$ consists of an at most countable *set of vertices* V and a *weight function* $c : V \times V \to \mathbb{R}_{\geq 0}$ such that c is symmetric and for all $x \in V$, we have $c(x, x) = 0$ and

$$c_x := \sum_{y \in V} c(x, y) < \infty.$$

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We think of two vertices $x, y \in V$ as being *adjacent* if $c(x, y) > 0$. For $x \in V$, let $N(x) := \{y \in V \mid c(x, y) > 0\}$ be the *set of neighbors of* x. Throughout this work, we assume that every vertex has finite degree in G, i.e., $|N(x)| < \infty$ for every $x \in V$.

For $x \in V$, let $\mathbb{P}_x$ be the random walk on G starting at x. It is the Markov chain defined by the transition matrix

$$p(x, y) = \frac{c(x, y)}{c_x}, x, y \in V$$

and initial distribution δ_x. We will think of $\mathbb{P}_x$ as a probability measure on $\Omega = V^{\mathbb{N}_0}$ equipped with the σ-algebra $(2^V)^{\otimes \mathbb{N}_0}$. If not explicitly stated otherwise, we will from now on assume that every occurring graph is connected. In that case, $\mathbb{P}_x$ is irreducible.

For a set of vertices $A \subseteq V$ and $\omega = (\omega_k)_{k \in \mathbb{N}_0} \in \Omega$, let

$$\begin{aligned} \tau_A(\omega) &:= \inf\{k \geq 0 \mid \omega_k \in A\} \text{ and} \\ \tau_A^+(\omega) &:= \inf\{k \geq 1 \mid \omega_k \in A\} \ (\inf \emptyset := \infty), \end{aligned}$$

be *hitting times of* A. For $x \in V$, we use the shorthand notation $\tau_{\{x\}} =: \tau_x$.

Suppose that G is finite. Ohm's Law states that the *effective resistance* $R(x, y)$ between two vertices x, y is the voltage drop needed to induce an electrical current of exactly 1 ampere from x to y.

The relationship between electrical currents and the random walk of G has been studied intensively [3, 5–7]. For finite graphs, $x \neq y$, one has the following probabilistic representations

$$\begin{aligned} R(x, y) &= \frac{1}{c_x} \mathbb{E}_x \left[\sum_{k=0}^{\tau_y - 1} \mathbb{1}_x(\omega_k) \right] && (1.1) \\ &= \frac{1}{c_x \cdot \mathbb{P}_x[\tau_y \leq \tau_x^+]} && (1.2) \\ &= \frac{1}{c_x \cdot \mathbb{P}_x[\tau_y < \tau_x^+]}. && (1.3) \end{aligned}$$

Note that $(c_z)_{z \in V}$ is an invariant measure of p. A proof of the first equality in the unweighted case can be found in [7] and can be extended to fit our more general context. To see that (1.1) equals (1.2), realize that $\sum_{k=0}^{\tau_y - 1} \mathbb{1}_x(\omega_k)$ is geometrically distributed with parameter $\mathbb{P}_x[\tau_x^+ < \tau_y]$. For the last equality, use that any finite graph is recurrent and thus $\mathbb{P}_x[\tau_x^+ = \tau_y = \infty] = 0$.

The subject of effective resistances gets much more complicated on infinite graphs since those may admit multiple different notions of effective resistances. Recurrent graphs, however, have a property which is often referred to as *unique currents* [6] and consequently also have one unique effective resistance. In this case, the above representation holds [1, 8]. Indeed, [1, Lemma 2.61] states the more general inequalities

$$\frac{1}{c_x \cdot \mathbb{P}_x[\tau_y \leq \tau_x^+]} \leq R^F(x, y) \leq \frac{1}{c_x \cdot \mathbb{P}_x[\tau_y < \tau_x^+]} \tag{1.4}$$

for the free effective resistance R^F (see Sect. 2) of any infinite graph whose vertices have only finitely many neighbors. For the convenience of the reader, we include a proof of the result, see Lemma 2.4.

In [5, Corollaries 3.13 and 3.15], it is suggested that one seems to have

$$R^F(x, y) = \frac{1}{c_x \cdot \mathbb{P}_x[\tau_y < \tau_x^+]} \tag{1.5}$$

on all transient graphs. However, this is false as our example in Sect. 3 shows.

The main result of this work (Corollary 6.3) states that the free effective resistance of a transient graph $G = (V, c)$ admits the representation (1.5) for all $x, y \in V$ if and only if G is a subgraph of an infinite line. Corollary 6.5 states that the lower bound in (1.4) is attained if and only if G is recurrent.

2 Free Effective Resistance

Let $G = (V, c)$ be an infinite connected graph. For any $W \subseteq V$, let $G\restriction_W := (W, c\restriction_{W\times W})$ be the *subgraph of G induced by W*. We say a sequence $(V_n)_{n\in\mathbb{N}}$ of subsets of V is a *finite exhaustion of V* if $|V_n| < \infty$, $V_n \subseteq V_{n+1}$ and $V = \cup_{n\in\mathbb{N}} V_n$. Define $G_n = (V_n, c_n) := G\restriction_{V_n}$.

Definition 2.1 Let $(V_n)_{n\in\mathbb{N}}$ be any finite exhaustion of V such that G_n is connected. For $x, y \in V$, the *free effective resistance $R^F(x, y)$* of G is defined by

$$R^F(x, y) = \lim_{n\to\infty} R_{G_n}(x, y).$$

Remark 2.2 The fact that $R_{G_n}(x, y)$ converges with the limit independent of a choice of $(V_n)_{n\in\mathbb{N}}$ is due to Rayleigh's monotonicity principle (see, e.g., [2, 4]).

We denote by $\mathbb{P}_x^n$ the random walk on G_n starting at x with transition matrix p_n associated with c_n. Since we can extend it to a function on V by defining $p_n(x, y) = 0$ whenever $x \notin V_n$ or $y \notin V_n$, $\mathbb{P}_x^n$ is a probability measure on $\Omega = V^{\mathbb{N}_0}$ for each $x \in V_n$ and we have

$$p_n(x, y) = \frac{c_n(x, y)}{(c_n)_x} = \frac{c(x, y)}{\sum_{w\in V_n} c(x, w)}$$

for all $x, y \in V_n$.

Remark 2.3 Note that for any $x, y \in V_n$,

$$p_n(x, y) = p(x, y) \cdot \frac{c_x}{(c_n)_x} = p(x, y) \cdot \left(1 + \frac{\sum_{v \notin V_n} c(x, v)}{\sum_{v \in V_n} c(x, v)}\right) \geq p(x, y).$$

Lemma 2.4 ([1, Lemma 2.61]). Let $G = (V, c)$ be an infinite, connected graph such that $|N(x)| < \infty$ for all $x \in V$ and let R^F be the free effective resistance of G. Then,

$$\frac{1}{c_x \cdot \mathbb{P}_x[\tau_y \leq \tau_x^+]} \leq R^F(x, y) \leq \frac{1}{c_x \cdot \mathbb{P}_x[\tau_y < \tau_x^+]}$$

holds for all $x, y \in V$ with $x \neq y$.

Proof For any $v \in V$, we have $(c_n)_v \to c_v$ as $n \to \infty$ and thus $p_n(v, w) \to p(v, w)$ for all $v, w \in V$. By the definition of R^F and (1.3), we know that

$$\begin{aligned} R^F(x, y) &= \lim_{n\to\infty} R_{G_n}(x, y) = \lim_{n\to\infty} \frac{1}{(c_n)_x \cdot \mathbb{P}_x^n[\tau_y < \tau_x^+]} \\ &= \frac{1}{c_x \cdot (\lim_{n\to\infty} \mathbb{P}_x^n[\tau_y < \tau_x^+])} . \end{aligned} \tag{2.1}$$

In particular, $\lim_{n\to\infty} \mathbb{P}_x^n[\tau_y < \tau_x^+]$ exists. Hence, the claim is equivalent to

$$\mathbb{P}_x[\tau_y < \tau_x^+] \leq \lim_{n\to\infty} \mathbb{P}_x^n[\tau_y < \tau_x^+] \leq \mathbb{P}_x[\tau_y \leq \tau_x^+].$$

Consider the discrete topology on V and its product topology on $\Omega = V^{\mathbb{N}_0}$. Since $p_n \to p$ point-wise and $\{y \in V_n \mid c_n(x, y) > 0\} \subset N(x)$ and $|N(x)| < \infty$ for all $n \in \mathbb{N}$ and all $x \in V_n$, it follows that $(\mathbb{P}_x^n)_{n\in\mathbb{N}}$ converges weakly to $\mathbb{P}_x$. In the product topology, the sets $\{\omega \in \Omega \mid \tau_y(\omega) < \tau_x^+(\omega)\}$ and $\{\omega \in \Omega \mid \tau_y(\omega) \leq \tau_x^+(\omega)\}$ are open and closed, respectively. By the Portmanteau theorem, it follows that

$$\mathbb{P}_x[\tau_y < \tau_x^+] \leq \liminf_{n\to\infty} \mathbb{P}_x^n[\tau_y < \tau_x^+] = \lim_{n\to\infty} \mathbb{P}_x^n[\tau_y < \tau_x^+]$$

and

$$\lim_{n\to\infty} \mathbb{P}_x^n[\tau_y < \tau_x^+] \leq \limsup_{n\to\infty} \mathbb{P}_x^n[\tau_y \leq \tau_x^+] \leq \mathbb{P}_x[\tau_y \leq \tau_x^+].$$

□

In view of (2.1), equation (1.5) holds if and only if

$$\lim_{n\to\infty} \mathbb{P}_x^n[\tau_y < \tau_x^+] = \mathbb{P}_x[\tau_y < \tau_x^+]. \tag{2.2}$$

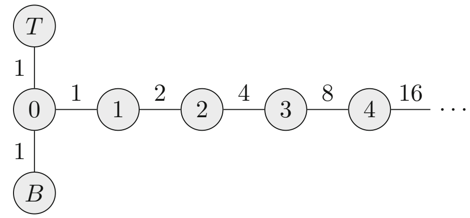


Fig. 1 The transient graph $\mathcal{T}$

Analogously, the lower bound of (1.4) is attained if and only if

$$\lim_{n\to\infty} \mathbb{P}_x^n[\tau_y < \tau_x^+] = \mathbb{P}_x[\tau_y \le \tau_x^+]. \tag{2.3}$$

3 The Transient $\mathcal{T}$

We will now show that (1.5) does not hold in general. Consider the graph $\mathcal{T}$ shown in Fig. 1. It is transient and we have $R^F(B, T) = 2$. However,

$$\begin{aligned} \mathbb{P}_B[\tau_T < \tau_B^+] &= \mathbb{P}_0[\tau_T < \tau_B] \\ &= 1 - \mathbb{P}_0[\tau_B \le \tau_T] \\ &= 1 - \mathbb{P}_0[\tau_B < \tau_T] - \mathbb{P}_0[\tau_B = \tau_T = \infty]. \end{aligned}$$

Due to the symmetry of $\mathcal{T}$ we have $\mathbb{P}_0[\tau_B < \tau_T] = \mathbb{P}_0[\tau_T < \tau_B]$. Together with the transience of $\mathcal{T}$, this implies

$$\mathbb{P}_B[\tau_T < \tau_B^+] = \mathbb{P}_0[\tau_T < \tau_B] = \frac{1 - \mathbb{P}_0[\tau_B = \tau_T = \infty]}{2} < \frac{1}{2}$$

and

$$\mathbb{P}_B[\tau_T \le \tau_B^+] = \mathbb{P}_0[\tau_T \le \tau_B] = \frac{1 + \mathbb{P}_0[\tau_B = \tau_T = \infty]}{2} > \frac{1}{2}.$$

More precisely, one can compute

$$\mathbb{P}_B[\tau_T < \tau_B^+] = \frac{2}{5} \text{ and } \mathbb{P}_B[\tau_T \le \tau_B^+] = \frac{3}{5}\,.$$

Hence,

$$\frac{1}{c_B \mathbb{P}_B[\tau_T < \tau_B^+]} \ne R^F(B, T)$$

and

$$\frac{1}{c_B}\mathbb{E}_B\left[\sum_{k=0}^{\tau_T-1}\mathbb{1}_B(\omega_k)\right]=\frac{1}{c_B\mathbb{P}_B[\tau_T\le\tau_B^+]}\neq R^F(B,T).$$

4 Probability of Paths

To check whether (2.2) holds, it is useful to write both sides as sums of probabilities of paths.

A sequence $\gamma=(\gamma_0,\dots,\gamma_n)\in V^{n+1}$ is called a *path (of length n)* in G if $c(\gamma_k,\gamma_{k+1})>0$ for all $k=0,\dots,n-1$. We denote by $L(\gamma)$ the length of γ and by Γ_G the set of all paths in G. A path γ is called *simple* if it does not contain any vertex twice. The probability of γ with respect to $\mathbb{P}_x$ is defined by

$$\mathbb{P}_x(\gamma):=\mathbb{P}_x(\{\gamma\}\times V^{\mathbb{N}})=\mathbb{1}_x(\gamma_0)\cdot\prod_{k=0}^{L(\gamma)-1}p(\gamma_k,\gamma_{k+1})\,.$$

We say γ is $x\to y$ if $\gamma_0=x$, $\gamma_{L(\gamma)}=y$ and $\gamma_k\notin\{x,y\}$ for all $k=1,\dots,L(\gamma)-1$. We denote by $\Gamma_G(x,y)$ the set of all paths $x\to y$ in G.

For $A\subseteq V$, let

$$\Gamma_G(x,y;A):=\{\gamma\in\Gamma_G(x,y)\mid\gamma_k\in A\text{ for all }k=0,\dots,L(\gamma)\}$$

be the set of all paths $x\to y$ in G that use only vertices in A.

Using this notion and $\Gamma_{G_n}(x,y)=\Gamma_G(x,y;V_n)$, we see that (2.2) becomes

$$\lim_{n\to\infty}\sum_{\gamma\in\Gamma_G(x,y;V_n)}\mathbb{P}_x^n(\gamma)=\sum_{\gamma\in\Gamma_G(x,y)}\mathbb{P}_x(\gamma).\tag{4.1}$$

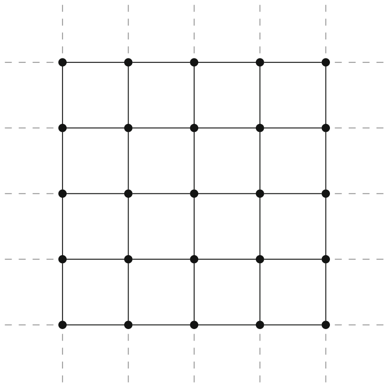
Since $\Gamma_G(x,y;V_n)$ increases to $\Gamma_G(x,y)$, this might look like an easy application of either the Monotone Convergence Theorem or the Dominated Convergence Theorem. However, neither is applicable since $\mathbb{P}_x^n(\gamma)$ may be strictly greater than $\mathbb{P}_x(\gamma)$.

To investigate when exactly (4.1) holds, we will introduce another random walk on V which can be considered an intermediary between $\mathbb{P}_x^n$ and $\mathbb{P}_x$.

5 Extended Finite Random Walk

The difference in the behavior of $\mathbb{P}_x$ and $\mathbb{P}_x^n$ occurs only when $\mathbb{P}_x$ leaves V_n. Instead, $\mathbb{P}_x^n$ is basically reflected back to a vertex in V_n. We will now construct an intermediary random walk which still has a finite state space, models the behavior of stepping out of V_n and has the same transition probabilities as $\mathbb{P}_x$ in V_n. This is done by adding *boundary vertices* to G_n wherever there is an edge from V_n to $V\setminus V_n$.

Fig. 2 The lattice $\mathbb{Z}^2$



For any set $A \subseteq V$, let

$$\partial_i A := \{v \in A \mid \exists\, w \in V \setminus A : c(v, w) > 0\}$$

be the *inner boundary* and $\partial_o A := \partial_i (V \setminus A)$ be the *outer boundary* of A in G.

For any $v \in \partial_i A$, let $\overline{v}$ be a copy of v. Define $\overline{G_n} = (\overline{V_n}, \overline{c_n})$ where

$$\overline{V_n} = V_n \cup \{\overline{v} \mid v \in \partial_i V_n\},$$

and $\overline{c_n}$ is defined as follows. For $x, y \in \overline{V_n}$, let

$$\overline{c_n}(x, y) = \overline{c_n}(y, x) = \begin{cases} c(x, y) & , \; x, y \in V_n \\ \sum_{z \notin V_n} c(x, z) & , \; y = \overline{x} \\ 0 & , \; \text{otherwise} \end{cases}.$$

In particular, we have $(\overline{c_n})_x = c_x$ for all $x \in V_n$. We denote by $\overline{\mathbb{P}^n_x}$ the random walk on $\overline{G_n}$ starting at x with transition matrix $\overline{p_n}$ given by

$$\overline{p_n}(x, y) = \frac{\overline{c_n}(x, y)}{(\overline{c_n})_x}.$$

Furthermore, let $V_n^* := \overline{V_n} \setminus V_n$.

Example 5.1 Let G be the lattice $\mathbb{Z}^2$ with unit weights, see Fig. 2. Furthermore, let $V_n := \{-n \ldots, 0, \ldots, n\}^2$. G_1 and $\overline{G_1}$ are illustrated in Fig. 3. Note that $c((1, 1), \overline{(1, 1)}) = 2$ since $(1, 1)$ has two edges leaving V_1 in G.

Lemma 5.2 *(*Relation of p_n, $\overline{p_n}$ and p*). For $x, y \in V_n$ we have*

$$p_n(x, y) \geq p(x, y) = \overline{p_n}(x, y).$$

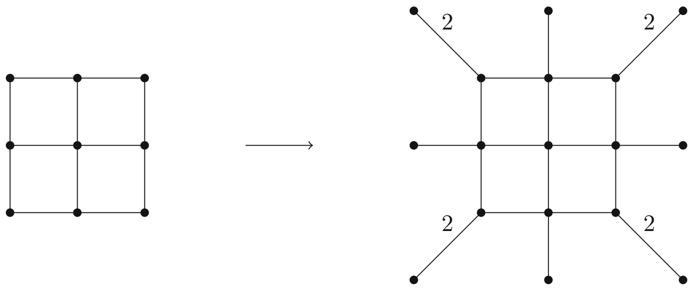


Fig. 3 G_1 (left) and $\overline{G_1}$ (right) for $G = \mathbb{Z}^2$

For $x, y \in V$ and $m \in \mathbb{N}$ such that $x, y \in V_m$, we have

$$\lim_{n\to\infty} p_n(x, y) = p(x, y) = \overline{p_m}(x, y).$$

Note that for $n \in \mathbb{N}$ and $x, y \in V_n$, we have

$$\Gamma_{G_n}(x, y) = \Gamma_{\overline{G_n}}(x, y; V_n) = \Gamma_G(x, y; V_n).$$

By Lemma 5.2, the following holds for all $x, y \in V_n$.

$$\forall\, m \geq n\ \forall\, \gamma \in \Gamma_G(x, y; V_m) : \mathbb{P}_x(\gamma) = \overline{\mathbb{P}^m_x}(\gamma).$$

The connection between $\mathbb{P}^n_x(\gamma)$ and $\overline{\mathbb{P}^n_x}(\gamma)$ is a bit more intricate. In order to investigate this connection, first consider what kind of paths exist in $\overline{G_n}$. Let $x, y \in V_n$, $x \neq y$ and $\overline{\gamma} \in \Gamma_{\overline{G_n}}(x, y)$ with $L(\overline{\gamma}) \geq 2$. Then, by the definition of $\overline{G_n}$, there exist $l \in \mathbb{N}$ with $l \geq 2$, $v_1, \ldots, v_{l-1} \in V_n \setminus \{x, y\}$ and $k_1, \ldots, k_{l-1} \in \mathbb{N}_0$ such that $k_j = 0$ for any $j \in \{1, \ldots, l-1\}$ with $v_j \notin \partial_i V_n$ and

$$\overline{\gamma} = (x, (v_1)_{k_1}, \ldots, (v_{n-1})_{k_{l-1}}, y) \tag{5.1}$$

where $(v)_k := (v, \underbrace{\overline{v}, v, \ldots, \overline{v}, v}_{k \text{ times}})$ for $(v, k) \in ((\partial_i V_n) \times \mathbb{N}_0) \cup ((V_n \setminus \partial_i V_n) \times \{0\})$. Note that the representation (5.1) is unique for $\overline{\gamma}$ since G_n does not contain any self-loops.

Definition 5.3 For $x, y \in V_n, x \neq y$, let $\pi : \Gamma_{\overline{G_n}}(x, y) \to \Gamma_{G_n}(x, y)$ be the projection of $\Gamma_{\overline{G_n}}(x, y)$ onto $\Gamma_{G_n}(x, y)$ which replaces all occurrences of $(v, \overline{v}, v)$ for any $v \in \partial_i V_n$ by (v).

More precisely, let $\overline{\gamma} \in \Gamma_{\overline{G_n}}(x, y)$. If $L(\overline{\gamma}) = 1$, then $\overline{\gamma} = (x, y)$ and we define $\pi(\overline{\gamma}) := (x, y)$. If $L(\overline{\gamma}) \geq 2$, it is of the form (5.1) and we define

$$\pi(\overline{\gamma}) := (x, v_1, \ldots, v_{l-1}, y). \tag{5.2}$$

Lemma 5.4 *For all $x, y \in V_n$, $x \neq y$ and $\gamma \in \Gamma_{G_n}(x, y)$, we have*

$$\mathbb{P}_x^n(\gamma) = \frac{c_x}{(c_n)_x} \cdot \sum_{\overline{\gamma} \in \pi^{-1}(\gamma)} \overline{\mathbb{P}_x^n}(\overline{\gamma}).$$

Proof For any $v, w \in V_n$, we have

$$\frac{c_v}{(c_n)_v} \cdot \overline{p_n}(v, w) = \frac{c_v}{(c_n)_v} \cdot \frac{c(v, w)}{c_v} = p_n(v, w) \tag{5.3}$$

and, if $v \in \partial_i V_n$,

$$\sum_{k=0}^{\infty} (\overline{p_n}(v, \overline{v}) \cdot \underbrace{\overline{p_n}(\overline{v}, v))^k}_{=1} = \frac{1}{1 - \overline{p_n}(v, \overline{v})} = \frac{c_v}{c_v - \sum_{w \notin V_n} c(v, w)} = \frac{c_v}{(c_n)_v} . \tag{5.4}$$

For $\gamma \in \Gamma_{G_n}(x, y)$ with $L(\gamma) = 1$, we have $\gamma = (x, y)$. Since any $\overline{\gamma} \in \pi^{-1}(\gamma)$ visits x and y only once, $\pi^{-1}(\gamma) = \{\gamma\}$ holds and

$$\frac{c_x}{(c_n)_x} \cdot \sum_{\overline{\gamma} \in \pi^{-1}(\gamma)} \overline{\mathbb{P}_x^n}(\overline{\gamma}) = \frac{c_x}{(c_n)_x} \cdot \overline{\mathbb{P}_x^n}(\gamma) = \frac{c_x}{(c_n)_x} \cdot \overline{p_n}(x, y) = p_n(x, y) = \mathbb{P}_x^n(\gamma).$$

Now let $\gamma = (x, v_1, \ldots, v_{l-1}, y) \in \Gamma_{G_n}(x, y)$ with $l = L(\gamma) \geq 2$. We define

$$A(\gamma) := \left\{(k_1, \ldots, k_{l-1}) \in (\mathbb{N}_0)^{l-1} \mid \text{for each } j \in \{1, \ldots, l-1\} : k_j = 0 \text{ if } v_j \notin \partial_i V_n\right\}.$$

It follows that

$$\begin{aligned}\pi^{-1}(\gamma) &= \left\{\overline{\gamma} \in \Gamma_{\overline{G_n}}(x, y) \mid \pi(\overline{\gamma}) = \gamma\right\} \\ &= \left\{(x, (v_1)_{k_1}, \ldots, (v_{l-1})_{k_{l-1}}, y) \mid (k_1, \ldots, k_{l-1}) \in A(\gamma)\right\}\end{aligned}$$

and we compute

$$\begin{aligned}\sum_{\overline{\gamma} \in \pi^{-1}(\gamma)} \overline{\mathbb{P}_x^n}(\overline{\gamma}) &= \sum_{(k_1, \ldots, k_{l-1}) \in A(\gamma)} \overline{\mathbb{P}_x^n}((x, (v_1)_{k_1}, \ldots, (v_{l-1})_{k_{l-1}}, y)) \\ &= \sum_{(k_1, \ldots, k_{l-1}) \in A(\gamma)} \left[\overline{\mathbb{P}_x^n}((x, v_1, \ldots, v_{l-1}, y)) \cdot \prod_{\substack{j=1,\ldots,l-1 \\ v_j \in \partial_i V_n}} \overline{p_n}(v_j, \overline{v_j})^{k_j} \right] \\ &\overset{(5.4)}{=} \overline{\mathbb{P}_x^n}((x, v_1, \ldots, v_{l-1}, y)) \cdot \prod_{j=1}^{l-1} \frac{c_{v_j}}{(c_n)_{v_j}}\end{aligned}$$

$$\overset{(5.3)}{=} \frac{(c_n)_x}{c_x} \cdot \mathbb{P}^n_x((x, v_1, \dots, v_{l-1}, y)) = \frac{(c_n)_x}{c_x} \cdot \mathbb{P}^n_x(\gamma).$$

□

Proposition 5.5 *For $x, y \in V_n$, $x \neq y$, we have*

$$\overline{\mathbb{P}^n_x}[\tau_y < \tau_x^+] = \frac{(c_n)_x}{c_x} \cdot \mathbb{P}^n_x[\tau_y < \tau_x^+].$$

Proof Using

$$\Gamma_{\overline{G_n}}(x, y) = \bigsqcup_{\gamma \in \Gamma_{G_n}(x,y)} \pi^{-1}(\gamma),$$

we compute

$$\begin{aligned}\mathbb{P}^n_x[\tau_y < \tau_x^+] &= \sum_{\gamma \in \Gamma_{G_n}(x,y)} \mathbb{P}^n_x(\gamma) = \sum_{\gamma \in \Gamma_{G_n}(x,y)} \left(\frac{c_x}{(c_n)_x} \cdot \sum_{\overline{\gamma} \in \pi^{-1}(\gamma)} \overline{\mathbb{P}^n_x}(\overline{\gamma}) \right) \\ &= \frac{c_x}{(c_n)_x} \cdot \sum_{\overline{\gamma} \in \Gamma_{\overline{G_n}}(x,y)} \overline{\mathbb{P}^n_x}(\overline{\gamma}) = \frac{c_x}{(c_n)_x} \cdot \overline{\mathbb{P}^n_x}[\tau_y < \tau_x^+].\end{aligned}$$

□

Since we now have clarified the relation between $\mathbb{P}^n_x$, $\overline{\mathbb{P}^n_x}$ and $\mathbb{P}_x$, we can return our attention to (2.2).

Proposition 5.6 *For $x, y \in V$, $x \neq y$, we have*

$$\lim_{n \to \infty} \mathbb{P}^n_x[\tau_y < \tau_x^+] = \mathbb{P}_x[\tau_y < \tau_x^+]$$

if and only if

$$\lim_{n \to \infty} \overline{\mathbb{P}^n_x}[\tau_{V_n^*} < \tau_y < \tau_x^+] = 0. \tag{5.5}$$

Proof We have

$$\begin{aligned}\mathbb{P}_x[\tau_y < \tau_x^+] &= \sum_{\gamma \in \Gamma_G(x,y)} \mathbb{P}_x(\gamma) = \lim_{n \to \infty} \sum_{\gamma \in \Gamma_G(x,y;V_n)} \mathbb{P}_x(\gamma) \\ &= \lim_{n \to \infty} \sum_{\gamma \in \Gamma_G(x,y;V_n)} \overline{\mathbb{P}^n_x}(\gamma)\end{aligned} \tag{5.6}$$

and

$$\mathbb{P}^n_x[\tau_y < \tau_x^+] = \frac{c_x}{(c_n)_x} \cdot \overline{\mathbb{P}^n_x}[\tau_y < \tau_x^+] = \frac{c_x}{(c_n)_x} \cdot \sum_{\gamma \in \Gamma_{\overline{G_n}}(x,y)} \overline{\mathbb{P}^n_x}(\gamma). \tag{5.7}$$

Since $\Gamma_G(x, y; V_n) = \Gamma_{\overline{G_n}}(x, y; V_n)$ and $(c_n)_x \to c_x$, it follows that $\mathbb{P}^n_x[\tau_y < \tau^+_x] \to \mathbb{P}_x[\tau_y < \tau^+_x]$ holds if and only if

$$\lim_{n\to\infty} \sum_{\substack{\gamma\in\Gamma_{\overline{G_n}}(x,y)\\ \gamma\notin\Gamma_{\overline{G_n}}(x,y;V_n)}} \overline{\mathbb{P}^n_x}(\gamma) = 0.$$

This is the same as

$$\lim_{n\to\infty} \overline{\mathbb{P}^n_x}[\tau_{V^*_n} < \tau_y < \tau^+_x] = 0.$$

□

Using the same approach, we can also characterize when (2.3) holds.

Proposition 5.7 *For $x, y \in V$, $x \neq y$, we have*

$$\lim_{n\to\infty} \mathbb{P}^n_x[\tau_y < \tau^+_x] = \mathbb{P}_x[\tau_y \leq \tau^+_x]$$

if and only if

$$\lim_{n\to\infty} \overline{\mathbb{P}^n_x}[\tau_{V^*_n} < \tau_y < \tau^+_x] = \mathbb{P}_x[\tau^+_x = \tau_y = \infty] \tag{5.8}$$

which in turn is equivalent to

$$\lim_{n\to\infty} \overline{\mathbb{P}^n_x}[\tau_{V^*_n} < \tau^+_x < \tau_y] = 0. \tag{5.9}$$

Proof Using (5.6) and (5.7) from the proof of Proposition 5.6, we have

$$\mathbb{P}_x[\tau_y \leq \tau^+_x] = \mathbb{P}_x[\tau^+_x = \tau_y = \infty] + \lim_{n\to\infty} \sum_{\gamma\in\Gamma_{\overline{G_n}}(x,y;V_n)} \overline{\mathbb{P}^n_x}(\gamma)$$

and

$$\lim_{n\to\infty} \mathbb{P}^n_x[\tau_y < \tau^+_x] = \lim_{n\to\infty} \sum_{\gamma\in\Gamma_{\overline{G_n}}(x,y)} \overline{\mathbb{P}^n_x}(\gamma)$$

provided either one of these two limits exists.

Hence, we have convergence as desired if and only if

$$\mathbb{P}_x[\tau^+_x = \tau_y = \infty] = \lim_{n\to\infty} \sum_{\substack{\gamma\in\Gamma_{\overline{G_n}}(x,y)\\ \gamma\notin\Gamma_{\overline{G_n}}(x,y;V_n)}} \overline{\mathbb{P}^n_x}(\gamma) = \lim_{n\to\infty} \overline{\mathbb{P}^n_x}[\tau_{V^*_n} < \tau_y < \tau^+_x].$$

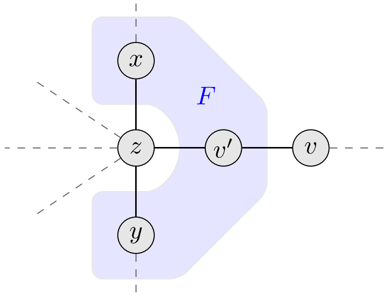


Fig. 4 Embedding $\mathcal{T}$ into a transient graph

On the other hand, we have

$$\begin{aligned}\mathbb{P}_x[\tau_x^+ = \tau_y = \infty] &= \lim_{n\to\infty} \mathbb{P}_x[\tau_{V\setminus V_n} < \min(\tau_x^+, \tau_y)] \\ &= \lim_{n\to\infty} \overline{\mathbb{P}_x^n}[\tau_{V_n^*} < \min(\tau_x^+, \tau_y)] \\ &= \lim_{n\to\infty} \left(\overline{\mathbb{P}_x^n}[\tau_{V_n^*} < \tau_x^+ < \tau_y] + \overline{\mathbb{P}_x^n}[\tau_{V_n^*} < \tau_y < \tau_x^+]\right) \qquad (5.10)\end{aligned}$$

which implies the second claim. □

Remark 5.8 An equivalent approach would be to consider a *lazy* random walk on G_n which has the same transition probabilities $p(v, w)$ as $\mathbb{P}_x$ for $v \neq w$ but stays at v with probability

$$\sum_{w\in V\setminus V_n} p(v, w) = \mathbb{P}_v[\omega_1 \in V \setminus V_n].$$

In that case the notion of "stepping out of V_n" would be modeled by staying at any vertex $v \in V_n$.

6 Embedding $\mathcal{T}$ into Transient Graphs

We will show that whenever a graph G is transient and not part of an infinite line, one can find a subgraph of G which is similar to $\mathcal{T}$ from Sect. 3. We will also show that this is sufficient for (5.5) not to hold.

Proposition 6.1 *Let G be a transient, connected graph which is not a subgraph of a line. Then, there exist $x, y, z \in V$ such that $x \neq y$, (x, z, y) is a path in G and*

$$\mathbb{P}_z[\tau_x = \tau_y = \infty] > 0.$$

Proof Since G is transient, it is infinite. If G is not a subgraph of a line, then there exists some $z \in V$ with at least three adjacent vertices. Let F be a set of exactly three neighbors of z. Since G is transient and F is finite, there exists $v \in \partial_o F$ such that

$$\mathbb{P}_v[\tau_F = \infty] > 0.$$

If $v = z$, we can choose $x, y \in F$, $x \neq y$, and get

$$\mathbb{P}_z[\tau_x = \tau_y = \infty] \geq \mathbb{P}_v[\tau_F = \infty] > 0.$$

If $v \neq z$, then there exists $v' \in F$ such that (z, v', v) is a path in G. Let $x, y \in V$ be such that $F = \{x, y, v'\}$, see Fig. 4. It follows that

$$\begin{aligned}
\mathbb{P}_z[\tau_x = \tau_y = \infty] &\geq \mathbb{P}_z[\omega_1 = v', \omega_2 = v, \tau_x = \tau_y = \infty] \\
&= p(z, v') \cdot p(v', v) \cdot \mathbb{P}_v[\tau_x = \tau_y = \infty] \\
&\geq p(z, v') \cdot p(v', v) \cdot \mathbb{P}_v[\tau_F = \infty] > 0.
\end{aligned}$$

□

Theorem 6.2 *Let G be a transient, connected graph. Then,*

$$\forall\, x, y \in V,\ x \neq y : \lim_{n\to\infty} \overline{\mathbb{P}^n_x}[\tau_{V_n^*} < \tau_y < \tau_x^+] = 0$$

holds if and only if G is a subgraph of an infinite line.

Proof First, assume that G is a subgraph of an infinite line and let $x, y \in V$, $x \neq y$. Then, for any $n \in \mathbb{N}$ sufficiently big, we have

$$\Gamma_{\overline{G_n}}(x, y) \setminus \Gamma_{\overline{G_n}}(x, y; V_n) = \emptyset,$$

i.e., there exists no path $x \to y$ which leaves V_n before reaching y. Hence,

$$\lim_{n\to\infty} \overline{\mathbb{P}^n_x}[\tau_{V_n^*} < \tau_y < \tau_x^+] = 0.$$

To prove the converse direction, suppose that G is not a subgraph of a line. By Proposition 6.1, we know that there exist distinct vertices $x, y, z \in V$ such that (x, z, y) is a path in G and $\mathbb{P}_z[\tau_x = \tau_y = \infty] > 0$. Hence,

$$\begin{aligned}
0 &< \mathbb{P}_z[\tau_x = \tau_y = \infty] \\
&= \lim_{n\to\infty} \mathbb{P}_z[\tau_{\partial_o V_n} < \min(\tau_x, \tau_y)] \\
&= \lim_{n\to\infty} \overline{\mathbb{P}^n_z}[\tau_{V_n^*} < \min(\tau_x, \tau_y)] \\
&= \lim_{n\to\infty} \left(\overline{\mathbb{P}^n_z}[\tau_{V_n^*} < \tau_x < \tau_y] + \overline{\mathbb{P}^n_z}[\tau_{V_n^*} < \tau_y < \tau_x]\right) \\
&\leq \limsup_n \overline{\mathbb{P}^n_z}[\tau_{V_n^*} < \tau_x < \tau_y] + \limsup_n \overline{\mathbb{P}^n_z}[\tau_{V_n^*} < \tau_y < \tau_x].
\end{aligned}$$

Without loss of generality assume that $\limsup_{n\to\infty} \overline{\mathbb{P}^n_z}[\tau_{V_n^*} < \tau_y < \tau_x] > 0$. It follows that $\limsup_{n\to\infty} \overline{\mathbb{P}^n_x}[\tau_{V_n^*} < \tau_y < \tau_x^+] > 0$ because for all $n \in \mathbb{N}$ with $\{x, y, z\} \subseteq V_n$, we have

$$\begin{aligned}\overline{\mathbb{P}^n_x}[\tau_{V_n^*} < \tau_y < \tau_x] &\geq \overline{\mathbb{P}^n_x}[\tau_z < \tau_{V_n^*} < \tau_y < \tau_x]\\ &= \overline{\mathbb{P}^n_x}[\tau_z < \min(\tau_{V_n^*}, \tau_x, \tau_y)] \cdot \overline{\mathbb{P}^n_z}[\tau_{V_n^*} < \tau_y < \tau_x]\\ &\geq p(x, z) \cdot \overline{\mathbb{P}^n_z}[\tau_{V_n^*} < \tau_y < \tau_x]. \qquad (6.1)\end{aligned}$$

□

Corollary 6.3 *Let G be a transient, connected graph with $|N(x)| < \infty$ for any $x \in V$. Then,*

$$R^F(x, y) = \frac{1}{c_x \cdot \mathbb{P}_x[\tau_y < \tau_x^+]}$$

holds for all $x, y \in V$ with $x \neq y$ if and only if G is a subgraph of an infinite line.

Proof As seen in (2.2), the desired probabilistic representation (1.5) holds if and only if

$$\lim_{n\to\infty} \mathbb{P}^n_x[\tau_y < \tau_x^+] = \mathbb{P}_x[\tau_y < \tau_x^+].$$

By Proposition 5.6, this is equivalent to

$$\lim_{n\to\infty} \overline{\mathbb{P}^n_x}[\tau_{V_n^*} < \tau_y < \tau_x^+] = 0$$

and the claim follows by Theorem 6.2. □

Theorem 6.4 *Let G be an infinite, connected graph. If*

$$\forall\, x, y \in V,\ x \neq y : \lim_{n\to\infty} \overline{\mathbb{P}^n_x}[\tau_{V_n^*} < \tau_y < \tau_x^+] = \mathbb{P}_x[\tau_x^+ = \tau_y = \infty]$$

holds, then G is recurrent.

Proof By Proposition 5.7, we have

$$\forall\, x, y \in V,\ x \neq y : \lim_{n\to\infty} \overline{\mathbb{P}^n_x}[\tau_{V_n^*} < \tau_x^+ < \tau_y] = 0. \qquad (6.2)$$

Suppose that G is transient and not a subgraph of a line. Using the same arguments as in the proof of Theorem 6.2, we see that there exist distinct vertices $x, y, z \in V$ such that $(x, z, y) \in \Gamma_G(x, y)$ and

$$\limsup_n \overline{\mathbb{P}^n_z}[\tau_{V_n^*} < \tau_x < \tau_y] + \limsup_n \overline{\mathbb{P}^n_z}[\tau_{V_n^*} < \tau_y < \tau_x] > 0.$$

Since the same argument as in (6.1) yields

$$\overline{\mathbb{P}^n_x}[\tau_{V_n^*} < \tau_x^+ < \tau_y] \geq p(x,z) \cdot \overline{\mathbb{P}^n_z}[\tau_{V_n^*} < \tau_x < \tau_y]$$

for all $n \in \mathbb{N}$ with $\{x, y, z\} \subseteq V_n$, it follows from (6.2) that

$$\limsup_n \overline{\mathbb{P}^n_z}[\tau_{V_n^*} < \tau_x < \tau_y] = 0$$

which implies

$$\limsup_{n\to\infty} \overline{\mathbb{P}^n_z}[\tau_{V_n^*} < \tau_y < \tau_x] > 0.$$

However, we also have

$$\overline{\mathbb{P}^n_y}[\tau_{V_n^*} < \tau_y^+ < \tau_x] \geq p(y,z) \cdot \overline{\mathbb{P}^n_z}[\tau_{V_n^*} < \tau_y < \tau_x]$$

for all $n \in \mathbb{N}$ with $\{x, y, z\} \subseteq V_n$ by the same argument as in (6.1), and it follows that

$$\limsup_n \overline{\mathbb{P}^n_y}[\tau_{V_n^*} < \tau_y^+ < \tau_x] > 0$$

which is a contradiction to (6.2).

Hence, if G is transient, then it must be a subgraph of a line. In this case,

$$\lim_{n\to\infty} \overline{\mathbb{P}^n_x}[\tau_{V_n^*} < \tau_y < \tau_x^+] = 0$$

follows for all $x, y \in V$ with $x \neq y$ by Theorem 6.2. Together with (6.2) and (5.10), this implies

$$\mathbb{P}_x[\tau_x^+ = \tau_y = \infty] = 0$$

for all $x, y \in V$ with $x \neq y$. However, this is a contradiction to the transience of G. □

Corollary 6.5 *Let G be an infinite, connected graph with $|N(x)| < \infty$ for any $x \in V$. Then,*

$$R^F(x,y) = \frac{1}{c_x \cdot \mathbb{P}_x[\tau_y \leq \tau_x^+]} \tag{6.3}$$

holds for all $x, y \in V$ with $x \neq y$ if and only if G is recurrent.

Proof If G is recurrent, we have $\mathbb{P}_x[\tau_x^+ = \tau_y = \infty] = 0$ for all $x, y \in V$. Hence,

$$\mathbb{P}_x[\tau_x^+ < \tau_y] = \mathbb{P}_x[\tau_x^+ \leq \tau_y]$$

and (1.4) implies the claim.

If (6.3) holds for all $x, y \in V$ with $x \neq y$, then by (2.3) we have

$$\lim_{n\to\infty} \mathbb{P}_x^n[\tau_y < \tau_x^+] = \mathbb{P}_x[\tau_y \leq \tau_x^+]$$

for all $x, y \in V$ with $x \neq y$, and Proposition 5.7 and Theorem 6.4 imply the recurrence of G. □

This shows that the lower bound in (1.4) is actually a strict inequality for some $x, y \in V$ with $x \neq y$ for any transient graph $G = (V, c)$.

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