Check for
updates

A Kolmogorov–Chentsov Type Theorem on General Metric Spaces with Applications to Limit Theorems for Banach-Valued Processes

Volker Krätschmer[1] · Mikhail Urusov[1]

Abstract

This paper deals with moduli of continuity for paths of random processes indexed by a general metric space Θ with values in a general metric space $\mathcal{X}$. Adapting the moment condition on the increments from the classical Kolmogorov–Chentsov theorem, the obtained result on the modulus of continuity allows for Hölder-continuous modifications if the metric space $\mathcal{X}$ is complete. This result is universal in the sense that its applicability depends only on the geometry of the space Θ. In particular, it is always applicable if Θ is a bounded subset of a Euclidean space or a relatively compact subset of a connected Riemannian manifold. The derivation is based on refined chaining techniques developed by Talagrand. As a consequence of the main result, a criterion is presented to guarantee uniform tightness of random processes with continuous paths. This is applied to find central limit theorems for Banach-valued random processes.

Keywords Kolmogorov–Chentsov type theorems · Covering numbers · Talagrand's chaining technique · Uniform tightness · Banach-valued central limit theorems

Mathematics Subject Classification Primary 60G17 · 60G60; Secondary 60B12 · 60F05

1 Introduction and Main Result

Let (Θ, d_Θ) be a totally bounded metric space. For subsets $\overline{\Theta} \subseteq \Theta$, the diameter of $\overline{\Theta}$ w.r.t. d_Θ will be denoted by $\Delta(\overline{\Theta})$, whereas $N(\overline{\Theta}, d_\Theta, \eta)$ stands for the minimal

✉ Volker Krätschmer
  volker.kraetschmer@uni-due.de

  Mikhail Urusov
  mikhail.urusov@uni-due.de

[1] Faculty of Mathematics, University Duisburg-Essen, Thea Leymann Str. 9, 45127 Essen, Germany

number to cover $\overline{\Theta}$ with closed d_Θ-metric balls of radius $\eta > 0$ with centers in $\overline{\Theta}$. We will often need the following assumption on the geometry of Θ:

$$\exists C, t > 0 \; \forall \eta \in]0, \Delta(\Theta)] : N(\Theta, d_\Theta, \eta) \leq C\eta^{-t}. \tag{1}$$

Furthermore, let $(\mathcal{X}, d_\mathcal{X})$ be a metric space. By $\mathcal{B}(\mathcal{X})$, we denote the Borel σ-algebra on $\mathcal{X}$. Let $(X_\theta)_{\theta \in \Theta}$ be an $\mathcal{X}$-valued random process on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, i.e., for all $\theta \in \Theta$, X_θ is a random element in $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$. Under a "Kolmogorov–Chentsov type theorem," we understand a theorem that, under an appropriate moment condition on the distance $d_\mathcal{X}(X_\theta, X_\vartheta)$ for $\theta, \vartheta \in \Theta$, yields existence of a continuous or Hölder-continuous modification (cf. [4]). We establish the following general result.

Theorem 1.1 *Assume* (1) *and*

$$(X_\theta, X_\vartheta) \text{ is } \mathcal{F} - \mathcal{B}(\mathcal{X}^2)\text{-measurable for all pairs } (\theta, \vartheta) \in \Theta^2 \text{ with} \theta \neq \vartheta. \tag{2}$$

Let $M, p > 0$ *and* $q > t$ *(with* t *from* (1)) *be such that*

$$\mathbb{E}\left[d_\mathcal{X}(X_\theta, X_\vartheta)^p \right] \leq M d_\Theta(\theta, \vartheta)^q \quad \text{for } \theta, \vartheta \in \Theta. \tag{3}$$

Then, for any $\beta \in]0, (q-t)/p[$, *there exists a finite constant* $\overline{L}(\Theta, C, t, p, q, \beta)$ *dependent on* $\Delta(\Theta), C, t, p, q$ *and* β *only such that, for every at most countable subset* $\overline{\Theta} \subseteq \Theta$ *with* $\Delta(\overline{\Theta}) > 0$,

$$\mathbb{E}\left[\sup_{\substack{\theta, \vartheta \in \overline{\Theta} \\ \theta \neq \vartheta}} \frac{d_\mathcal{X}(X_\theta, X_\vartheta)^p}{d_\Theta(\theta, \vartheta)^{\beta p}} \right] \leq L(\Theta, C, t, M, p, q, \beta) := M\overline{L}(\Theta, C, t, p, q, \beta).$$

$$\tag{4}$$

In particular, if $d_\mathcal{X}$ *is complete, then the random process* $(X_\theta)_{\theta \in \Theta}$ *has a modification which satisfies* (2) *such that all its paths are Hölder-continuous of all orders* $\beta \in]0, (q-t)/p[$.

Remark 1 Technical assumption (2) is always satisfied when $\mathcal{X}$ is a separable metric space because, in this case, $\mathcal{B}(\mathcal{X}^2) = \mathcal{B}(\mathcal{X}) \otimes \mathcal{B}(\mathcal{X})$. In general, we only have the inclusion $\mathcal{B}(\mathcal{X}^2) \supseteq \mathcal{B}(\mathcal{X}) \otimes \mathcal{B}(\mathcal{X})$, and the assumption is needed to ensure measurability of $d_\mathcal{X}(X_\theta, X_\vartheta)$.

We consider Theorem 1.1 as our main "building block." In the literature, Kolmogorov–Chentsov type theorems are sometimes formulated in a localized form. A localized version of Theorem 1.1 where Θ is not necessarily totally bounded is presented in Sect. 2.

Remark 2 The key assumption on the geometry of the parametric space Θ is (1), where the value of t is important, as we need to have $q > t$ in (3).[1] We remark that, if Θ is a bounded subset of $\mathbb{R}^m$ with the Euclidean metric $d_{m,2} = d_\Theta$, then (1) is always satisfied with $t = m$,[2] More generally, a relatively compact subset Θ of an m-dimensional connected Riemannian manifold always satisfies (1) with $t = m$ (we provide more detail in Sect. 3).

In the classical formulation of the Kolmogorov–Chentsov theorem, it is assumed that $\mathcal{X}$ is a Banach space and $\Theta = [0, 1]^m$ for some $m \in \mathbb{N}$ (see [22, Theorem I.2.1]), and the proof relies on the fact that the dyadic rationals are dense in $[0, 1]$. Since that time, there appeared many other versions of the Kolmogorov–Chentsov theorem that essentially allow to treat more general sets Θ. We mention [19, Theorem 2.1], [6, Theorem 3.9], [10, Lemma 2.19], [12, Proposition 3.9] for several recent formulations where Θ is a subset of $\mathbb{R}^m$. Some versions of the Kolmogorov–Chentsov theorem only guarantee that $\sup(d_\mathcal{X}(X_\theta, X_\vartheta)/d_\Theta(\theta, \vartheta)^\beta) < \infty$ a.s. (i.e., it is not claimed that the expectation of the p-th power of that quantity is finite). However, some applications such as the ones discussed in Sects. 4 and 5 require that the expectation is finite. As another example of this kind, we mention that the proof of Theorem 6.1 in [2] would not work without finiteness of such an expectation (see formula (106) in [2]).

In the aforementioned references, $\mathcal{X}$ is (a closed subset of) a Banach space and all X_θ are assumed to be in L^p (with p from (3)), and the proof involves a certain extension result for Banach-valued Hölder-continuous mappings. That extension result allows to pass from rectangular regions in $\mathbb{R}^m$ to general subsets $\Theta \subseteq \mathbb{R}^m$. In our situation, when $\mathcal{X}$ is only a metric space and we do not assume $\mathbb{E}[d_\mathcal{X}(a, X_\theta)^p] < \infty$ for all θ and some $a \in \mathcal{X}$ (or the like) such a method of the proof cannot work, so we use essentially different ideas to prove Theorem 1.1.

Another approach, used in [21, Theorem 2.9] (also see [17, Corollary 4.3]), is worth mentioning. In that reference, the existence of a locally Hölder-continuous modification is proved for $\mathcal{X} = \mathbb{R}$ under assumptions of a different kind. In particular, the assumption on Θ is that it is a dyadically separable metric space. The latter is a requirement of a different type than (1) on the geometry of Θ, which allows to pursue the arguments initially elaborated for rectangular regions in $\mathbb{R}^m$ in more general situations. The setup in [21] is quite different from ours, and the relation between the approaches still has to be worked out. Notice, however, that in the finite-dimensional situation $\Theta \subseteq \mathbb{R}^m$, the other approach imposes some restrictions on possible sets Θ (see[21, Theorem 4.1]), while our approach allows for arbitrary sets $\Theta \subseteq \mathbb{R}^m$ (see Proposition 2.1 and Remark 3).

We thus summarize the previous discussion by noting that we obtain inequality (4), essentially, only under requirement (1) on the geometry of the metric space Θ, which is satisfied for bounded subsets of $\mathbb{R}^m$ (with $t = m$) and allows to go beyond $\mathbb{R}^m$. It is also worth noting that the right-hand side of (4) is the same for all countable subsets $\overline{\Theta} \subseteq \Theta$ and that (4) is the right way to formulate the result in the case when $d_\mathcal{X}$ is incomplete (and thus a continuous modification may fail to exist).

---

[1] On the contrary, the value of C in (1) is not important.

[2] More precisely, we have $N(\overline{\Theta}d_\Theta, \eta) \leq \left[(8\Delta(\Theta) + \eta)/\eta\right]^m$ for all $\eta > 0$ (see [25, Lemma 2.5]). In particular, (1) holds with $t = m$ and $C = 9\Delta(\Theta)^m$.

In order to discuss applications of Theorem 1.1, we formulate the following immediate

Corollary 1.2 *Assume (1), (2) and (3). Let $\beta \in]0, (q - t)/p[$ (with p, q from (3) and t from (1)), and let $L(\Theta, C, t, M, p, q, \beta)$ be any constant satisfying (4). Then, for every at most countable subset $\overline{\Theta} \subseteq \Theta$ and arbitrary $\delta > 0$,*

$$\mathbb{E}\left[\sup_{\substack{\theta, \vartheta \in \overline{\Theta} \\ d(\theta, \vartheta) \leq \delta}} d_{\mathcal{X}}(X_\theta, X_\vartheta)^p\right] \leq L(\Theta, C, t, M, p, q, \beta)\delta^{\beta p}. \tag{5}$$

Notice that, like in Theorem 1.1, inequality (5) holds universally, i.e., independently of the random process satisfying (2) and (3). This will turn out to be useful when analyzing weak convergence of $\mathcal{X}$-valued random processes (see Sects. 4 and 5).

The crucial step for the proof of Theorem 1.1 is provided by the following auxiliary result. It is interesting in its own right.

Lemma 1.3 *Assume (1), (2) and (3). Let $\overline{\Theta}$ be some finite subset of Θ with diameter $\Delta(\overline{\Theta}) > 0$. Then, for any $\delta > 0$,*

$$\mathbb{E}\left[\sup_{\substack{\theta, \vartheta \in \overline{\Theta} \\ d_\Theta(\theta, \vartheta) \leq \delta}} d_{\mathcal{X}}(X_\theta, X_\vartheta)^p\right]$$

$$\leq 4^{t+2p+3q+2} M \left(N(\overline{\Theta}, d_\Theta, \delta/4) \left[\ln\left(N(\overline{\Theta}, d_\Theta, \delta/4)\right)\right]^q \delta^q + \frac{C \delta^{q-t}}{(2^{(q-t)/p} - 1)^p}\right).$$

In the case $\mathcal{X} = \mathbb{R}$, Theorem B.2.4 in [23] provides a result similar to Lemma 1.3. For the proof, a refined chaining technique is used there, which we shall adopt to derive Lemma 1.3.

The structure of the paper is as follows. In Sect. 2, we discuss a localized version of Theorem 1.1 where Θ is not necessarily totally bounded. As an example, in Sect. 3 we explicitly treat the case where Θ is a subset of a Riemannian manifold. In Sects. 4 and 5, we present some applications of Theorem 1.1 to weak convergence of Banach-valued processes. Lemma 1.3 and Theorem 1.1 are proved in Sect. 6.

2 Localized Version of Theorem 1.1

Since the literature in the case $\Theta \subseteq \mathbb{R}^m$ sometimes formulates Kolmogorov–Chentsov type theorems for unbounded Θ (by localizing the results of the type of Theorem 1.1), we now formulate and discuss the localized version of Theorem 1.1 for metric spaces (Θ, d_Θ) that are not necessarily totally bounded.

The setting is as follows. Let (Θ, d_Θ) be a metric space satisfying

Property (P) There exists an increasing sequence $\{\Theta_n\}_{n\in\mathbb{N}}$, $\Theta_n \subseteq \Theta_{n+1}$, $n \in \mathbb{N}$, of totally bounded open subsets of Θ such that $\Theta = \bigcup_{n\in\mathbb{N}} \Theta_n$ and

$$\forall n \in \mathbb{N} \; \exists C_n, t_n > 0 \; \forall \eta \in]0, \Delta(\Theta_n)] : N(\Theta_n, d_\Theta, \eta) \leq C_n \eta^{-t_n}. \tag{6}$$

Let $(\mathcal{X}, d_\mathcal{X})$ be a complete metric space and let $(X_\theta)_{\theta\in\Theta}$ be an $\mathcal{X}$-valued random process on some $(\Omega, \mathcal{F}, \mathbb{P})$.

Proposition 2.1 *Assume Property (P), that the process $(X_\theta)_{\theta\in\Theta}$ satisfies (2) and that, for all $n \in \mathbb{N}$, there exist $M_n, p_n, \rho_n > 0$ and $q_n > t_n$ (with t_n as in Property (P)) such that*

$$\mathbb{E}\left[d_\mathcal{X}(X_\theta, X_\vartheta)^{p_n} \right] \leq M_n \, d_\Theta(\theta, \vartheta)^{q_n} \quad \text{for } \theta, \vartheta \in \Theta_n, \; d_\Theta(\theta, \vartheta) < \rho_n, \; n \in \mathbb{N}. \tag{7}$$

Then, the random process $(X_\theta)_{\theta\in\Theta}$ has a modification $(\widetilde{X}_\theta)_{\theta\in\Theta}$ satisfying (2) such that all its paths are locally Hölder-continuous of all orders $\beta \in \bigcap_{n\in\mathbb{N}}[0, (q_n - t_n)/p_n[$, where the expression "Hölder-continuous of order 0" is understood as "uniformly continuous." Moreover, for $n \in \mathbb{N}$, $\overline{\theta} \in \Theta_n$, there is some open in Θ neighborhood $V(\overline{\theta})$ of $\overline{\theta}$ such that

$$\mathbb{E}\left[\sup_{\substack{\theta, \vartheta \in V(\overline{\theta}) \\ \theta \neq \vartheta}} \frac{d_\mathcal{X}(\widetilde{X}_\theta, \widetilde{X}_\vartheta)^{p_n}}{d_\Theta(\theta, \vartheta)^{\beta p_n}} \right] < \infty \quad \text{for all } \beta \in [0, (q_n - t_n)/p_n[. \tag{8}$$

Remark 3 Notice that any $\Theta \subseteq \mathbb{R}^m$ satisfies Property (P). We can take[3] $\Theta_n = \Theta \cap (]-n, n[^m)$ and all $t_n = m$, $n \in \mathbb{N}$, whereas the constants C_n indeed depend on n. Therefore, in the case of an arbitrary subset $\Theta \subseteq \mathbb{R}^m$ Proposition 2.1 includes, e.g., the following statement: There is a locally Hölder-continuous modification of all orders $\beta \in]0, (q-m)/p[$ whenever (7) holds with $\Theta_n = \Theta \cap (]-n, n[^m)$, $p_n = p > 0$ and $q_n = q > m$ not depending on n (on the contrary, M_n and ρ_n are allowed to depend on n). Moreover, in this case, for any $\overline{\theta} \in \Theta$, there exists an open in Θ neighborhood $V(\overline{\theta})$ of $\overline{\theta}$ such that (8) with $p_n \equiv p$ holds for all $\beta \in]0, (q-m)/p[$.

Although Proposition 2.1 follows from Theorem 1.1 via standard arguments, we present a proof to make the paper self-contained.

Proof of Proposition 2.1 Fix any $n \in \mathbb{N}$. The set Θ_n from Property (P) is totally bounded. Therefore, we can find open subsets $\Theta_{n,1}, \ldots, \Theta_{n,r_n}$ of Θ with diameters less than ρ_n such that

$$\Theta_n = \bigcup_{i=1}^{r_n} \overline{\Theta}_{n,i},$$

---

[3] We remark that although such sets Θ_n are in general not open in $\mathbb{R}^m$, they are always open in Θ, as required in Property (P).

where $\overline{\Theta}_{n,i} = \Theta_n \cap \Theta_{n,i}$. By (7), we can apply Theorem 1.1 on each $\overline{\Theta}_{n,i}$. Hence, each $(X_\theta)_{\theta \in \overline{\Theta}_{n,i}}$ has a modification $(\overline{X}_\theta^{n,i})_{\theta \in \overline{\Theta}_{n,i}}$ which satisfies (2) such that all its paths are Hölder-continuous on $\overline{\Theta}_{n,i}$ of all orders $\beta \in [0, (q_n - t_n)/p_n[$ with

$$
\mathbb{E}\left[\sup_{\substack{\theta, \vartheta \in \overline{\Theta}_{n,i} \\ \theta \neq \vartheta}} \frac{d_\mathcal{X}(\overline{X}_\theta^{n,i}, \overline{X}_\vartheta^{n,i})^{p_n}}{d_\Theta(\theta, \vartheta)^{\beta p_n}} \right] < \infty \quad \text{for all } \beta \in [0, (q_n - t_n)/p_n[.
$$

If $i, j \in \{1, \ldots, r_n\}$ are such that $\overline{\Theta}_{n,i} \cap \overline{\Theta}_{n,j} \neq \emptyset$, then the processes $(\overline{X}_\theta^{n,i})_{\theta \in \overline{\Theta}_{n,i} \cap \overline{\Theta}_{n,j}}$ and $(\overline{X}_\theta^{n,j})_{\theta \in \overline{\Theta}_{n,i} \cap \overline{\Theta}_{n,j}}$ are indistinguishable, as they are both continuous, modifications of each other and $\overline{\Theta}_{n,i} \cap \overline{\Theta}_{n,j}$ is separable (because totally bounded). Using this, it is straightforward to construct a modification $(\overline{X}_\theta^n)_{\theta \in \Theta_n}$ of $(X_\theta)_{\theta \in \Theta_n}$ which satisfies (2) such that all its paths are Hölder-continuous of all orders $\beta \in [0, (q_n - t_n)/p_n[$ on each $\overline{\Theta}_{n,i}$ with

$$
\mathbb{E}\left[\sup_{\substack{\theta, \vartheta \in \overline{\Theta}_{n,i} \\ \theta \neq \vartheta}} \frac{d_\mathcal{X}(\overline{X}_\theta^n, \overline{X}_\vartheta^n)^{p_n}}{d_\Theta(\theta, \vartheta)^{\beta p_n}} \right] < \infty \tag{9}
$$

for all $\beta \in [0, (q_n - t_n)/p_n[$ and $i \in \{1, \ldots, r_n\}$.

Now we vary $n \in \mathbb{N}$. Recall that $\Theta_n \subseteq \Theta_{n+1}$. Since $(\overline{X}_\theta^n)_{\theta \in \Theta_n}$ and $(\overline{X}_\theta^{n+1})_{\theta \in \Theta_n}$ are modifications of each other, both continuous and Θ_n is separable, then $(\overline{X}_\theta^n)_{\theta \in \Theta_n}$ and $(\overline{X}_\theta^{n+1})_{\theta \in \Theta_n}$ are indistinguishable. Therefore, there exists $\Omega_n \in \mathcal{F}$ with $\mathbb{P}(\Omega_n) = 1$ such that, for all $\omega \in \Omega_n$ and $\theta \in \Theta_n$, it holds $\overline{X}_\theta^n(\omega) = \overline{X}_\theta^{n+1}(\omega)$. We define $\Omega_\infty = \bigcap_{n \in \mathbb{N}} \Omega_n$ and the process $(\widetilde{X}_\theta)_{\theta \in \Theta}$ by the formula

$$
\widetilde{X}_\theta(\omega) = \begin{cases} \overline{X}_\theta^n(\omega), & \omega \in \Omega_\infty, \ \theta \in \Theta_n \setminus \Theta_{n-1}, \ n \in \mathbb{N}, \\ \bar{x}, & \omega \notin \Omega_\infty, \end{cases}
$$

where $\Theta_0 := \emptyset$ and $\bar{x} \in \mathcal{X}$ is arbitrary. This is a modification of $(X_\theta)_{\theta \in \Theta}$ which satisfies (2), and all its paths are Hölder-continuous of all orders β from the interval $[0, (q_n - t_n)/p_n[$ on each $\overline{\Theta}_{n,i}$, $n \in \mathbb{N}$, $i \in \{1, \ldots, r_n\}$. Recalling from Property (P) that each Θ_n is open in Θ, we get that each point $\theta \in \Theta$ belongs to some open subset $\overline{\Theta}_{n,i}$ of Θ (for some $n \in \mathbb{N}$ and $i \in \{1, \ldots, r_n\}$). In particular, all paths of $(\widetilde{X}_\theta)_{\theta \in \Theta}$ are locally Hölder-continuous of all orders $\beta \in \bigcap_{n \in \mathbb{N}}[0, (q_n - t_n)/p_n[$, while the last statement of Proposition 2.1 follows from (9). □

3 Example: Subsets of Riemannian Manifolds

In this section, we discuss applicability of Theorem 1.1 and Proposition 2.1 in the setting when Θ is a subset of an m-dimensional connected Riemannian manifold M. More precisely, we are going to understand restrictions (1) and Property (P) on Θ in this setting. Essentially, the results are:

- Every relatively compact $\Theta \subseteq M$ satisfies (1) with $t = m$ (Proposition 3.1);
- Every $\Theta \subseteq M$ satisfies Property (P) with $t_n = m$, $n \in \mathbb{N}$ (Corollary 3.2).

For basic concepts and results from differential geometry, we refer to standard text-books, e.g., [7], [8], [14] and [16].

Let (M, g) be any connected m-dimensional Riemannian manifold as defined in [8]. This means that M denotes an m-dimensional C^∞-manifold endowed with the Riemannian metric g. By definition, g is a mapping which associates with each point $p \in M$ an inner product g_p on the tangential space $T_p M$ at p such that for C^∞-vector fields $\mathcal{V}, \mathcal{W}$ on an open subset G of M the mapping

$$G \to \mathbb{R}, \ p \mapsto g_p(\mathcal{V}_p, \mathcal{W}_p)$$

is differentiable of class C^∞. Furthermore, let for p, q denote by $\mathcal{C}_{pq}$ the set of all C^∞-curves in M joining p to q. The length $L(c)$ of a curve $c \in \mathcal{C}_{pq}$ defined on the closed interval I_c of $\mathbb{R}$ is

$$L(c) := \int_{I_c} \sqrt{g_{c(t)}\big(c'(t), c'(t)\big)} \, dt,$$

where $c'(t)$ stands for the velocity of c at t. Since M is connected, the sets $\mathcal{C}_{pq}$ are always nonvoid (see [8, p. 146]), and the mapping

$$d_g : M \times M \to \mathbb{R}, \ (p, q) \mapsto \inf_{c \in \mathcal{C}_{pq}} L(c)$$

is a metric on M (see [8, Proposition 7.2.5]) sometimes called the *inner metric* (*induced by g*). Moreover, the topology induced by this metric coincides with the original topology on M (see [8, Proposition 7.2.6]).

Proposition 3.1 *(i) Let Θ be any relatively compact subset of M. Then there exist a compact subset K_m of $\mathbb{R}^m$ as well as $r \in \mathbb{N}$ and $\delta > 0$ such that*

$$N(\Theta, d_g, \eta) \leq r N(K_m, d_{m,2}, \eta/\delta) \ \text{ for all } \eta > 0,$$

where $d_{m,2}$ stands for the Euclidean metric on $\mathbb{R}^m$. As a consequence, Θ satisfies condition (1) with $t = m$ w.r.t. the metric d_g.

(ii) If d_g is complete, then every d_g-bounded subset Θ of M satisfies (1) with $t = m$ w.r.t. the metric d_g.

Corollary 3.2 *Every $\Theta \subseteq M$ satisfies Property (P) with $t_n = m$, $n \in \mathbb{N}$, w.r.t. the metric d_g.*

Proof Since M is a C^∞-manifold, we can find an open covering $\{\overline{\Theta}_n\}_{n \in \mathbb{N}}$ of M consisting of relatively compact subsets of M and satisfying $\overline{\Theta}_n \subseteq \overline{\Theta}_{n+1}$ for $n \in \mathbb{N}$ (see, e.g., [7, (16.1.4)]). By Proposition 3.1, this sequence of subsets satisfies (6) w.r.t. d_g with $t_n = m$ for $n \in \mathbb{N}$ (and the constants C_n indeed depend on n). Hence, every $\Theta \subseteq M$ satisfies Property (P) with $t_n = m$, $n \in \mathbb{N}$, w.r.t. d_g, as we can choose[4] $\Theta_n := \Theta \cap \overline{\Theta}_n, n \in \mathbb{N}$. □

In the rest of this section, we prove Proposition 3.1. The proof is based on a couple of auxiliary results.

Lemma 3.3 *Let $\overline{\Theta}$ be a nonvoid compact subset of M and assume $\overline{\Theta} \subseteq G$, where G is an open subset of M allowing a chart $u \colon G \to \mathbb{R}^m$ which satisfies that $u(\overline{\Theta})$ is convex. Then, there is some $\delta > 0$ such that*

$$N(\overline{\Theta}, d_g, \eta) \leq N\big(u(\overline{\Theta}), d_{m,2}, \eta/\delta\big) \ \ for \ \eta > 0.$$

Proof Let $\{e_1, \ldots, e_m\}$ stand for the standard basis on $\mathbb{R}^m$. For any C^∞-mapping $g : \mathcal{U} \to \mathbb{R}$ on some open subset $\mathcal{U}$ of $\mathbb{R}^m$, we shall use notation $d_x h$ to denote the differential of h at $x \in \mathcal{U}$.

Let us introduce for $p \in G$ the set $\mathcal{C}_M^\infty(p)$ of all real-valued C^∞-mappings on some open neighborhood of p. By definition, the tangential space $T_p M$ of M at p consists of real-valued mappings on $\mathcal{C}_M^\infty(p)$. The chart u provides the following basis of $T_p M$

$$\frac{\partial}{\partial u_i}\Big|_p : \mathcal{C}_M^\infty(p) \to \mathbb{R}, \ \varphi \mapsto d_p(\varphi \circ u^{-1})(e_i) \ \ (i \in \{1, \ldots, m\})$$

(see [8, p.8]). Moreover,

$$\Big(\frac{\partial}{\partial u_1}, \ldots, \frac{\partial}{\partial u_m}\Big) : G \to \bigcup_{p \in G} T_p M, \ p \mapsto \Big(\frac{\partial}{\partial u_1}\Big|_p, \ldots, \frac{\partial}{\partial u_m}\Big|_p\Big)$$

defines some C^∞-vector field (see [8, pp. 25f.]).

Next, let for $x \in u(G)$ denote by $d_x u^{-1}$ the differential of u^{-1} at x which is a linear mapping from $\mathbb{R}^m$ into $T_{u^{-1}(x)} M$ satisfying

$$d_x u^{-1}(e_i) = \frac{\partial}{\partial u_i}\Big|_{u^{-1}(x)} \ \ for \ i = 1, \ldots, m.$$

Since $g_{u^{-1}(x)}$ is an inner product on $T_{u^{-1}(x)} M$, we may observe for any $v = (v_1, \ldots, v_m) \in \mathbb{R}^m$

$$g_{u^{-1}(x)}\big(d_x u^{-1}(v), d_x u^{-1}(v)\big) = \sum_{i,j=1}^m v_i \, v_j \, g_{u^{-1}(x)}\Big(\frac{\partial}{\partial u_i}\Big|_{u^{-1}(x)}, \frac{\partial}{\partial u_j}\Big|_{u^{-1}(x)}\Big).$$

---

[4] Notice that, although such sets Θ_n are in general not open in M, they are always open in Θ, as required in Property (P).

Then, with S^{m-1} denoting the Euclidean sphere in $\mathbb{R}^m$, we may conclude from the defining properties of the Riemannian metric g that the mapping

$$f : u(\overline{\Theta}) \times S^{m-1} \to \mathbb{R}, \ (x, v) \mapsto \sqrt{g_{u^{-1}(x)}\big(d_x u^{-1}(v), d_x u^{-1}(v)\big)}$$

is continuous with strictly positive outcomes. Moreover, its domain is a compact subset of $\mathbb{R}^m \times \mathbb{R}^m$ so that it attains its maximum δ which is a positive number.

Now, let $p, q \in \overline{\Theta}$ with $p \neq q$. Since $u(\overline{\Theta})$ is assumed to be convex, the mapping

$$\overline{c} : [0, 1] \to \mathbb{R}^m, \ t \mapsto tu(q) + (1 - t)u(p)$$

is a C^∞-curve in $\mathbb{R}^m$ satisfying $\overline{c}(t) \in u(\overline{\Theta})$ for $t \in [0, 1]$. Then, $c := u^{-1} \circ \overline{c} \in \mathcal{C}_{pq}$, and by chain rule

$$c'(t) = d_{\overline{c}(t)}u^{-1}\big(u(q) - u(p)\big) \quad t \in [0, 1].$$

Since $g_{u^{-1}(\overline{c}(t))}$ is an inner product on $T_{u^{-1}(\overline{c}(t))}M$ and $d_{\overline{c}(t)}u^{-1}$ is linear for every $t \in [0, 1]$, we obtain

$$\sqrt{g_{u^{-1}(\overline{c}(t))}\big(c'(t), c'(t)\big)} = \|u(p) - u(q)\|_{m,2}\, f\big(\overline{c}(t), [u(q) - u(p)]/\|u(p) - u(q)\|_{m,2}\big)$$
$$\leq \delta\|u(p) - u(q)\|_{m,2} \quad \text{for } t \in [0, 1],$$

where $\|\cdot\|_{m,2}$ stands for the Euclidean norm on $\mathbb{R}^m$. Hence, by definition of the inner metric d_g we end up with

$$d_g(p, q) \leq L(c) \leq \delta\|u(p) - u(q)\|_{m,2}.$$

Since δ does not depend on p, q, we now easily derive the claim of Lemma 3.3. $\square$

In the next step, using Lemma 3.3, we prove the result of Proposition 3.1 first for compact subsets of M.

Lemma 3.4 *Let $\overline{\Theta} \subseteq M$ be nonvoid and compact. Then, there exists a nonvoid compact subset K_m of $\mathbb{R}^m$ as well as $r \in \mathbb{N}$ and $\delta > 0$ such that*

$$N(\overline{\Theta}, d_g, \eta) \leq r N\big(K_m, d_{m,2}, \eta/\delta\big) \ \text{ for } \eta > 0.$$

Proof For any $p \in \overline{\Theta}$, we may find a chart u_p, defined on an open subset G_{u_p} of M, and some $\varepsilon_p > 0$ such that $p \in G_{u_p}$ and

$$B_{\varepsilon_p}\big(u_p(p)\big) := \{x \in \mathbb{R}^m \mid d_{m,2}\big(x, u_p(p)\big) \leq \varepsilon_p\} \subseteq u_p(G_{u_p}).$$

Setting $U_{\varepsilon_p}\big(u_p(p)\big) := \{x \in \mathbb{R}^m \mid d_{m,2}\big(x, u_p(p)\big) < \varepsilon_p\}$ and $G^p := u_p^{-1}\big(U_{\varepsilon_p}(u_p(p))\big)$, we observe that $(G^p)_{p \in \overline{\Theta}}$ is an open covering of $\overline{\Theta}$ because

$U_{\varepsilon_p}\big(u_p(p)\big)$ is an open subset of $\mathbb{R}^m$. Hence, by compactness of $\overline{\Theta}$ there exist $p_1, \ldots, p_r \in M$ such that

$$\overline{\Theta} \subseteq \bigcup_{i=1}^{r} G^{p_i} \subseteq \bigcup_{i=1}^{r} \Theta^i,$$

where $\Theta^i := u_{p_i}^{-1}\left(B_{\varepsilon_{p_i}}\big(u_{p_i}(p_i)\big)\right)$ for $i = 1, \ldots, r$. For any $i \in \{1, \ldots, r\}$ the set Θ^i meets the requirements of Lemma 3.3. Hence, we may find $\delta_1, \ldots, \delta_r > 0$ such that

$$N(\Theta_i, d_g, \eta) \leq N\left(B_{p_i}\big(u_{p_i}(p_i)\big), d_{m,2}, \eta/\delta_i\right) \quad \text{for } i \in \{1, \ldots, r\}, \eta > 0.$$

The set

$$K_m := \bigcup_{i=1}^{r} B_{p_i}\big(u_{p_i}(p_i)\big)$$

is a compact subset of $\mathbb{R}^m$. Then, setting $\delta := 4 \max\{\delta_1, \ldots, \delta_r\}$, we end up with

$$N(\overline{\Theta}, d_g, \eta) \leq \sum_{i=1}^{r} N(\Theta^i, d_g, \eta/2) \leq \sum_{i=1}^{r} N\left(K_m, d_{m,2}, \eta/(4\delta_i)\right)$$
$$\leq r N\left(K_m, d_{m,2}, \eta/\delta\right) \quad \text{for } \eta > 0.$$

This completes the proof.   □

Finally, we are ready to prove Proposition 3.1.

Proof of Proposition 3.1 (i) Let Θ be a nonvoid relatively compact subset of M. The topological closure $\overline{\Theta}$ is compact, and $N(\Theta, d_g, \eta) \leq N(\overline{\Theta}, d_g, \eta/2)$ holds for every $\eta > 0$. Therefore, the first claim immediately follows from Lemma 3.4.
(ii) If d_g is complete, then by the Hopf–Rinow theorem (see, e.g., [8, Theorem 7.2.8]) every d_g-bounded subset of M is already relatively compact. Therefore, the second claim follows from the first one.   □

4 Tightness for Sequences of Random Processes

Let (Θ, d_Θ) be a compact metric space and $(\mathcal{X}, d_\mathcal{X})$ a complete metric space. We denote by $\mathcal{C}(\Theta, \mathcal{X})$ the space of all continuous mappings from Θ into $\mathcal{X}$ endowed with uniform metric d_∞ w.r.t. the metric $d_\mathcal{X}$ and the induced Borel σ-algebra $\mathcal{B}\big(\mathcal{C}(\Theta, \mathcal{X})\big)$.

Some of the results we are going to present simplify in the case when $\mathcal{C}(\Theta, \mathcal{X})$ is separable (hence Polish, as it is complete). For some discussions below we recall that, as Θ is compact, $\mathcal{C}(\Theta, \mathcal{X})$ is separable *if and only if* $\mathcal{X}$ is separable (see [1, Lemma 3.99]). We, however, stress at this point that we *never assume* $\mathcal{X}$ (equivalently, $\mathcal{C}(\Theta, \mathcal{X})$) to be separable.

Let us fix any sequence $(X_n)_{n\in\mathbb{N}}$ of Borel random elements $X_n \colon \Omega \to \mathcal{C}(\Theta, \mathcal{X})$ on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We show how Corollary 1.2 leads to a sufficient condition for uniform tightness in $\mathcal{C}(\Theta, \mathcal{X})$.

Proposition 4.1 *Let Θ fulfill property* (1) *with constants* $C, t > 0$. *Let* $\Theta' \subseteq \Theta$ *be dense in* Θ. *Assume that* $\big(X_n(\cdot, \theta)\big)_{n\in\mathbb{N}}$ *is a uniformly tight sequence of random elements in* $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$, *for all* $\theta \in \Theta'$ *and that there exist* $M, p > 0$ *and* $q > t$ *such that*

$$\sup_{n\in\mathbb{N}} \mathbb{E}\big[\, d_{\mathcal{X}}\big(X_n(\cdot, \theta), X_n(\cdot, \vartheta)\big)^p \,\big] \le M\, d_{\Theta}(\theta, \vartheta)^q \quad \text{for } \theta, \vartheta \in \Theta. \qquad (10)$$

Then, $(X_n)_{n\in\mathbb{N}}$ *is a uniformly tight sequence of Borel random elements in* $\mathcal{C}(\Theta, \mathcal{X})$.

We recall that (1) need not be assumed if Θ is a compact subset of $\mathbb{R}^m$ endowed with the Euclidean metric. In this case, it is enough only to require $q > m$ in (10) (see Remark 2).

Remark 4 Notice that (2) is satisfied for all processes X_n because they are assumed to be Borel random elements in $\mathcal{C}(\Theta, \mathcal{X})$ in this section and the projection map

$$\pi_{\theta,\vartheta} \colon \mathcal{C}(\Theta, \mathcal{X}) \to \mathcal{X}^2, \quad f \mapsto (f(\theta), f(\vartheta)),$$

is continuous for all $(\theta, \vartheta) \in \Theta^2$.

Remark 5 Observe that, if $\mathcal{X}$ is separable, then the statements
(A) $X_n \colon \Omega \to \mathcal{C}(\Theta, \mathcal{X})$ is a *Borel* random element, i.e., a random element in $\big(\mathcal{C}(\Theta, \mathcal{X}), \mathcal{B}(\mathcal{C}(\Theta, \mathcal{X}))\big)$; and
(B) $X_n = (X_n(\cdot, \theta))_{\theta\in\Theta}$ is an $\mathcal{X}$-valued process (i.e., for all $\theta \in \Theta$, $X_n(\cdot, \theta)$ is a random element in $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$) with continuous paths are equivalent (see [15, Lemma 14.1]). Thus, whenever $\mathcal{X}$ is a Polish space, in Proposition 4.1 (and in what follows) we essentially work with sequences of continuous $\mathcal{X}$-valued processes. In general, when (A) and (B) no longer coincide, the right choice is always (A), i.e., always to consider Borel random elements in $\mathcal{C}(\Theta, \mathcal{X})$, as the concept of tightness (in $\mathcal{C}(\Theta, \mathcal{X})$) discussed in Proposition 4.1 requires the Borel σ-algebra (in $\mathcal{C}(\Theta, \mathcal{X})$).

Proof of Proposition 4.1 We take an arbitrary $\beta \in\,]0, (q - t)/p[$. By compactness of Θ, there exists some at most countable dense subset $\overline{\Theta}$ of Θ. Corollary 1.2 together with the continuity of the processes X_n yields, for all $\delta > 0$ and $n \in \mathbb{N}$,

$$\mathbb{E}\left[\sup_{\substack{\theta,\vartheta\in\Theta \\ d_{\Theta}(\theta,\vartheta)\le\delta}} d_{\mathcal{X}}\big(X_n(\cdot,\theta), X_n(\cdot,\vartheta)\big)^p \right] = \mathbb{E}\left[\sup_{\substack{\theta,\vartheta\in\overline{\Theta} \\ d_{\Theta}(\theta,\vartheta)\le\delta}} d_{\mathcal{X}}\big(X_n(\cdot,\theta), X_n(\cdot,\vartheta)\big)^p \right]$$
$$\le L(\Theta, C, t, M, p, q, \beta)\delta^{\beta p}.$$

Using the Markov inequality, we conclude that, for every $\varepsilon > 0$,

$$\lim_{\delta\to 0+}\ \limsup_{n\to\infty}\ \mathbb{P}\left(\sup_{\substack{\theta,\vartheta\in\Theta \\ d_{\Theta}(\theta,\vartheta)\le\delta}} d_{\mathcal{X}}\big(X_n(\cdot,\theta), X_n(\cdot,\vartheta)\big) \ge \varepsilon \right) = 0.$$

Now the criterion for uniform tightness in $\mathcal{C}(\Theta, \mathcal{X})$ presented in Theorem A.1 applies and completes the proof. □

We observe that essentially the same condition achieves rather different aims in Theorem 1.1 and in Proposition 4.1. In Theorem 1.1, condition (3) ensures existence of a continuous modification for the process X (when $\mathcal{X}$ is complete, which is assumed in Sect. 4), while in Proposition 4.1, condition (10) implies the uniform tightness in $\mathcal{C}(\Theta, \mathcal{X})$ for the sequence (X_n). (Notice that (10) is nothing else but (3) required for all X_n uniformly in n.) It is, therefore, tempting to try to shift continuity of the processes into the conclusion of Proposition 4.1. And, indeed, this easily follows from the discussions above, although *at the cost of requiring $\mathcal{X}$ to be separable.*

Corollary 4.2 *Assume that $\mathcal{X}$ is separable. Let Θ fulfill property* (1) *with constants $C, t > 0$. We consider a sequence $(X_n)_{n \in \mathbb{N}}$ of $\mathcal{X}$-valued processes $X_n = (X_n(\cdot, \theta))_{\theta \in \Theta}$. Let $\Theta' \subseteq \Theta$ be dense in Θ. Assume that $(X_n(\cdot, \theta))_{n \in \mathbb{N}}$ is a uniformly tight sequence of random elements in $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$, for all $\theta \in \Theta'$, and that there exist $M, p > 0$ and $q > t$ such that*

$$\sup_{n \in \mathbb{N}} \mathbb{E} \left[d_{\mathcal{X}} \big(X_n(\cdot, \theta), X_n(\cdot, \vartheta) \big)^p \right] \le M \, d_\Theta(\theta, \vartheta)^q \quad \text{for } \theta, \vartheta \in \Theta. \tag{11}$$

Then, each process X_n admits a modification $\overline{X}_n = (\overline{X}_n(\cdot, \theta))_{\theta \in \Theta}$ that has continuous paths $\theta \mapsto \overline{X}_n(\omega, \theta)$ for all $\omega \in \Omega$, the processes $\overline{X}_n, n \in \mathbb{N}$, are Borel random elements in $\mathcal{C}(\Theta, \mathcal{X})$, and the sequence $(\overline{X}_n)_{n \in \mathbb{N}}$ is uniformly tight in $\mathcal{C}(\Theta, \mathcal{X})$.

Proof Theorem 1.1 ensures the existence of the continuous modifications $\overline{X}_n, n \in \mathbb{N}$. As $\mathcal{X}$ is separable, then, due to the equivalence between (A) and (B) in Remark 5, each $\overline{X}_n$ is a Borel random element in $\mathcal{C}(\Theta, \mathcal{X})$. The uniform tightness of the sequence $(\overline{X}_n)_{n \in \mathbb{N}}$ now follows from Proposition 4.1. □

Remark 6 If in Corollary 4.2 we additionally require that each process X_n is *separable* (the definition is recalled below), then we obtain that each process X_n is itself continuous almost surely, so that we obtain the uniform tightness for the sequence $(X_n)_{n \in \mathbb{N}}$ *itself.*[5] This immediately follows from Lemma 4.3. For when this remark can be useful, we observe that, in some situations, we are given processes that are *a priori* separable (e.g., càdlàg $\mathcal{X}$-valued processes in the case $\Theta = [0, 1]$).

It remains to justify the previous remark. Recall that an $\mathcal{X}$-valued process $(Y(\cdot, \theta))_{\theta \in \Theta}$ on some $(\Omega, \mathcal{F}, \mathbb{P})$ is called *separable*[6] if there exist an at most countable subset $\Theta_0 \subseteq \Theta$ dense in Θ and an event $\Omega_0 \in \mathcal{F}$ with $\mathbb{P}(\Omega_0) = 1$ such that for every open subset $\mathcal{G}$ of Θ, and any closed subset $\mathcal{D}$ of $\mathcal{X}$ the following equality holds true

$$\big\{ \omega \in \Omega_0 \mid Y(\omega, \theta) \in \mathcal{D} \text{ for all } \theta \in \mathcal{G} \cap \Theta_0 \big\} = \big\{ \omega \in \Omega_0 \mid Y(\omega, \theta) \in \mathcal{D} \text{ for all } \theta \in \mathcal{G} \big\}$$

(see [9]).

---

[5] Formally, we need to identify each process X_n with almost all continuous paths with an indistinguishable process with all continuous paths, in order to view X_n as a Borel random element in $\mathcal{C}(\Theta, \mathcal{X})$.

[6] In the sense of Doob.

Lemma 4.3 *Let $Y = \big(Y(\cdot, \theta)\big)_{\theta \in \Theta}$ be a separable $\mathcal{X}$-valued process that admits a continuous modification. Then, $Y = \big(Y(\cdot, \theta)\big)_{\theta \in \Theta}$ is itself continuous* almost surely, *and hence there is an indistinguishable from Y process $\widetilde{Y}$ such that* all *its paths are continuous.*

It is worth noting that, contrary to the general setting in Sect. 4, for this lemma the metric space $\mathcal{X}$ does not need to be complete.

Proof Let $\widetilde{Y} = \big(\widetilde{Y}(\cdot, \theta)\big)_{\theta \in \Theta}$ be a continuous modification of Y, i.e., for all $\theta \in \Theta$ we have $\mathbb{P}(\{\widetilde{Y}(\cdot, \theta) = Y(\cdot, \theta)\}) = 1$ and the paths $\theta \mapsto \widetilde{Y}(\omega, \theta)$ are continuous for all $\omega \in \Omega$. As Y is separable, we can find an at most countable $\Theta_0 \subseteq \Theta$ dense in Θ and $\Omega_0 \in \mathcal{F}$ with $\mathbb{P}(\Omega_0) = 1$ as described prior to Lemma 4.3. Define

$$\Omega_1 = \bigcap_{\theta \in \Theta_0} \{\widetilde{Y}(\cdot, \theta) = Y(\cdot, \theta)\} \cap \Omega_0$$

and observe that $\mathbb{P}(\Omega_1) = 1$. It suffices to show that $Y(\omega, \theta) = \widetilde{Y}(\omega, \theta)$ holds for $\omega \in \Omega_1$ and $\theta \in \Theta$. So let us fix $\omega \in \Omega_1$ and $\theta \in \Theta$.

For $k \in \mathbb{N}$ set $\mathcal{G}_k := \{\vartheta \in \Theta \mid d_\Theta(\theta, \vartheta) < 1/k\}$, and let $\mathcal{D}_k$ denote the closure of the set $\{Y(\omega, \vartheta) \mid \vartheta \in \mathcal{G}_k \cap \Theta_0\}$. Now, separability of Y yields $Y(\omega, \theta) \in \mathcal{D}_k$. In particular, there is some sequence $(\vartheta_n^k)_{n \in \mathbb{N}}$ in $\mathcal{G}_k \cap \Theta_0$ such that $Y(\omega, \vartheta_n^k) \to Y(\omega, \theta)$, as $n \to \infty$. This implies $\widetilde{Y}(\omega, \vartheta_n^k) \to Y(\omega, \theta)$, as $n \to \infty$, due to definition of Ω_1. Moreover, we may select by compactness of Θ a subsequence $(\vartheta_{i(n)}^k)_{n \in \mathbb{N}}$ of $(\vartheta_n^k)_{n \in \mathbb{N}}$ which converges to some $\overline{\vartheta}^k \in \Theta$. Then, by continuity of $\widetilde{Y}$,

$$Y(\omega, \theta) = \lim_{n \to \infty} \widetilde{Y}(\omega, \theta_{i(k)}^k) = \widetilde{Y}(\omega, \overline{\vartheta}^k).$$

As $d_\Theta(\theta, \overline{\vartheta}^k) \leq 1/k$, the sequence $(\overline{\vartheta}^k)_{k \in \mathbb{N}}$ converges to θ. Hence, drawing on the continuity of $\widetilde{Y}$ again, we end up with

$$\widetilde{Y}(\omega, \theta) = \lim_{k \to \infty} \widetilde{Y}(\omega, \overline{\vartheta}^k) = Y(\omega, \theta).$$

This completes the proof. $\square$

5 Central Limit Theorems for Banach-Valued Random Processes

Let (Θ, d_Θ) be a compact metric space, and let $(\mathcal{X}, \|\cdot\|_\mathcal{X})$ be a Banach space. We shall denote by $\mathcal{C}(\Theta, \mathcal{X})$ the space of all continuous mappings from Θ into $\mathcal{X}$. It will be endowed with sup-norm $\|\cdot\|_\infty$ w.r.t. $\|\cdot\|_\mathcal{X}$, and the induced Borel σ-algebra $\mathcal{B}\big(\mathcal{C}(\Theta, \mathcal{X})\big)$.

Consider any i.i.d. sequence $(X_i)_{i \in \mathbb{N}}$ of Bochner-integrable Borel random elements in $\mathcal{C}(\Theta, \mathcal{X})$ on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We want to investigate weak convergence of the sequence $(S_n)_{n \in \mathbb{N}}$ consisting of Borel random elements in $\mathcal{C}(\Theta, \mathcal{X})$

defined by

$$S_n := \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left(X_i - \mathbb{E}^B[X_i] \right) \quad \text{for } n \in \mathbb{N},$$

where $\mathbb{E}^B[X_i]$ denotes the Bochner-integral of X_i. We start with the following observation.

Proposition 5.1 *Let $\|X_1\|_\infty$ be square integrable.*
   (i) The following statements are equivalent:
a) The sequence $(S_n)_{n \in \mathbb{N}}$ is uniformly tight;
b) The sequence $(S_n)_{n \in \mathbb{N}}$ converges weakly to some centered Gaussian random element in $\left(\mathcal{C}(\Theta, \mathcal{X}), \mathcal{B}(\mathcal{C}(\Theta, \mathcal{X})) \right)$.
   (ii) If the equivalent statements in part (i) are satisfied, then the limiting law in b) is tight.

We remark that, as every Borel probability measure in a Polish space is tight, statement (ii) in Proposition 5.1 has a message only when $\mathcal{C}(\Theta, \mathcal{X})$ (equivalently, $\mathcal{X}$) is non-separable.

Proof As the Borel random element X_1 is Bochner integrable, it is almost surely separably valued. Then, we can find a closed separable linear subspace $\widehat{C}$ of $\mathcal{C}(\Theta, \mathcal{X})$ such that $\mathbb{P}(\{X_1 \in \widehat{C}\}) = 1$ (note that $\widehat{C}$ is itself a Polish space and $\widehat{C} \in \mathcal{B}(\mathcal{C}(\Theta, \mathcal{X}))$). It follows that $\mathbb{E}^B[X_1] \in \widehat{C}$. This yields $\mathbb{P}(\{X_1 - \mathbb{E}^B[X_1] \in \widehat{C}\}) = 1$, hence $\mathbb{P}(\{S_n \in \widehat{C}\}) = 1$ for all $n \in \mathbb{N}$. In view of the portmanteau lemma, this yields that every weak limit point of the laws of S_n, $n \in \mathbb{N}$, is concentrated on $\widehat{C}$ (in particular, is tight), thus establishing part (ii). Moreover, the implication b) $\Rightarrow$ a) in part (i) now follows from Prokhorov's theorem, which applies due to the fact that all measures are concentrated on a Polish space.

We turn to the implication a) $\Rightarrow$ b) in part (i). By Prokhorov's theorem, the uniformly tight sequence $(S_n)_{n \in \mathbb{N}}$ is relatively weakly sequentially compact. It remains to prove uniqueness of a limit point and its Gaussianity. To this end, let $r \in \mathbb{N}$ and $\Lambda_j : \mathcal{C}(\Theta, \mathcal{X}) \to \mathbb{R}$, $j = 1, \ldots, r$, be continuous linear functionals. Classical multivariate central limit theorem applies to the sequence $\left((\Lambda_1 \circ S_n, \ldots \Lambda_r \circ S_n) \right)_{n \in \mathbb{N}}$ because

$$\mathbb{E}\left[|\Lambda_j \circ X_1|^2 \right] \leq \|\Lambda_j\|^2 \, \mathbb{E}\left[\|X_1\|_\infty^2 \right] < \infty, \quad j = 1, \ldots, r$$

($\|\Lambda_j\|$ denotes the operator norm of Λ_j) and yields weak convergence to a centered Gaussian law in $\mathbb{R}^r$. This identifies every weak limit point of the laws of S_n, $n \in \mathbb{N}$, as a Gaussian measure and uniquely determines every weak limit point on the σ-algebra $\mathcal{E}$ generated by continuous linear functionals $\mathcal{C}(\Theta, \mathcal{X}) \to \mathbb{R}$. Notice that $\mathcal{E} \subseteq \mathcal{B}(\mathcal{C}(\Theta, \mathcal{X}))$, and the inclusion can be strict (when $\mathcal{C}(\Theta, \mathcal{X})$ is non-separable). However, restricted to $\widehat{C}$ both σ-algebras coincide:

$$\widehat{C} \cap \mathcal{E} = \widehat{C} \cap \mathcal{B}(\mathcal{C}(\Theta, \mathcal{X}))$$

(see [24, Theorem I.2.1]). Recalling that every weak limit point is concentrated on $\widehat{C}$ completes the proof.   □

For application of Proposition 5.1, we can utilize our criterion in Proposition 4.1 and obtain the following result.

Corollary 5.2 *Let* Θ *satisfy condition* (1) *with constants* $C, t > 0$, *and let* $\|X_1\|_\infty$ *be square integrable. Assume that there is a dense subset* $\Theta' \subseteq \Theta$ *such that*

$$\left(S_n(\cdot, \theta)\right)_{n \in \mathbb{N}} \text{ is a uniformly tight sequence of random elements in } \mathcal{X}, \text{ for all } \theta \in \Theta', (12)$$

and that there exist $M, p > 0$ *as well as* $q > t$ *with*

$$\sup_{n \in \mathbb{N}} \mathbb{E}\left[\|S_n(\cdot, \theta) - S_n(\cdot, \vartheta)\|_{\mathcal{X}}^p\right] \le M \, d_\Theta(\theta, \vartheta)^q \quad \text{for } \theta, \vartheta \in \Theta. \tag{13}$$

Then, the sequence $(S_n)_{n \in \mathbb{N}}$ *converges weakly to a tight centered Gaussian random element in* $\left(\mathcal{C}(\Theta, \mathcal{X}), \mathcal{B}\left(\mathcal{C}(\Theta, \mathcal{X})\right)\right)$.

We want to discuss the requirements of Corollary 5.2 for special choices of the Banach space $\mathcal{X}$. Let us start with type 2-Banach spaces. To recall, the Banach space $\mathcal{X}$ is called a *type 2 - Banach space* if there is a constant $C > 0$ such that, for all $n \in \mathbb{N}$ and $\mathcal{X}$-valued independent centered Borel random elements $W_1, \ldots, W_n$ such that $\|W_i\|_{\mathcal{X}}$ are square integrable, we have the following inequality

$$\mathbb{E}\left[\left\|\sum_{i=1}^n W_i\right\|_{\mathcal{X}}^2\right] \le C \sum_{i=1}^n \mathbb{E}\left[\|W_i\|_{\mathcal{X}}^2\right]$$

(see, e.g., [11, Theorem 2.1]). Prominent examples of type 2 - Banach space are the following:

- $\mathcal{X}$ is a finite-dimensional vector space,
- $\mathcal{X}$ is an L^p-space on some σ-finite measure space $(\mathfrak{X}, \mathcal{A}, \nu)$ with L^p-norm $\|\cdot\|_p$ for $p \in [2, \infty[$ (see [18, Section 9.2]).

If $\mathcal{X}$ is a type 2-Banach space, then conditions (12) and (13) can be simplified in the following way.

Proposition 5.3 *Let* $\mathcal{X}$ *be a type 2-Banach space, let* Θ *satisfy condition* (1) *with constants* $C, t > 0$, *and let* $\|X_1\|_\infty$ *be square integrable. Then, it holds:*

(1) *The sequence* $(S_n)_{n \in \mathbb{N}}$ *always satisfies condition* (12), *even with* $\Theta' = \Theta$.
(2) *The sequence* $(S_n)_{n \in \mathbb{N}}$ *satisfies condition* (13) *with* $p = 2$ *whenever there exist* $M > 0$ *and* $q > t$ *such that*

$$\mathbb{E}\left[\|X_1(\cdot, \theta) - X_1(\cdot, \vartheta)\|_{\mathcal{X}}^2\right] \le M \, d_\Theta(\theta, \vartheta)^q \quad \text{for } \theta, \vartheta \in \Theta. \tag{14}$$

In particular, under (14), the sequence $(S_n)_{n \in \mathbb{N}}$ converges weakly to a tight centered Gaussian random element in $\left(\mathcal{C}(\Theta, \mathcal{X}), \mathcal{B}\left(\mathcal{C}(\Theta, \mathcal{X})\right)\right)$.

It is worth noting that, even in the separable case, we cannot get this result from the general central limit theorem in type 2 - Banach spaces (see, e.g., [18, Theorem 10.5]) because, in Proposition 5.3, it is only the space $\mathcal{X}$ and not $\mathcal{C}(\Theta, \mathcal{X})$ that has type 2.

Proof Consider for $\theta \in \Theta$ the continuous linear operator $\pi_\theta \colon \mathcal{C}(\Theta, \mathcal{X}) \to \mathcal{X}$ defined by $\pi_\theta(f) := f(\theta)$. Then by Bochner-integrability of the Borel random element X_1 in $\mathcal{C}(\Theta, \mathcal{X})$, we may conclude that the Borel random element $X_1(\cdot, \theta) = \pi_\theta \circ X_1$ of $\mathcal{X}$ is Bochner-integrable with Bochner-integral $\mathbb{E}^B\left[X_1(\cdot, \theta)\right] = \pi_\theta\left(\mathbb{E}^B\left[X_1\right]\right)$. In particular, it is almost surely separably valued. Hence, the Borel random element $X_1(\cdot, \theta) - \mathbb{E}^B\left[X_1(\cdot, \theta)\right]$ is almost surely separably valued too. This means that $X_1(\cdot, \theta) - \mathbb{E}^B\left[X_1(\cdot, \theta)\right]$ is concentrated on some separable closed subset of $\mathcal{X}$. Due to completeness of $\|\cdot\|_\infty$, this implies that $X_1(\cdot, \theta) - \mathbb{E}^B\left[X_1(\cdot, \theta)\right]$ is a Radon Borel random element of $\mathcal{X}$ (see [24, p. 29, Corollary]). Now, statement 1) follows from the general central limit theorem in type 2-Banach spaces (see [11, Theorem 3.6] or [18, Theorem 10.5]) along with the version of Prokhorov's theorem for Radon measures (see, e.g., [24, Theorem I.3.6]).

Concerning statement 2), by the above definition of type 2-Banach spaces, we can find some constant $C > 0$ such that

$$\sup_{n \in \mathbb{N}} \mathbb{E}\left[\|S_n(\cdot, \theta) - S_n(\cdot, \vartheta)\|_{\mathcal{X}}^2\right] \leq C \, \mathbb{E}\left[\left\|X_1(\cdot, \theta) - X_1(\cdot, \vartheta) - \mathbb{E}^B\left[X_1(\cdot, \theta) - X_1(\cdot, \vartheta)\right]\right\|_{\mathcal{X}}^2\right].$$

We now observe that

$$\mathbb{E}\left[\left\|X_1(\cdot, \theta) - X_1(\cdot, \vartheta) - \mathbb{E}^B\left[X_1(\cdot, \theta) - X_1(\cdot, \vartheta)\right]\right\|_{\mathcal{X}}^2\right]$$
$$\leq 2\,\mathbb{E}\left[\left\|X_1(\cdot, \theta) - X_1(\cdot, \vartheta)\right\|_{\mathcal{X}}^2\right] + 2\left\|\mathbb{E}^B\left[X_1(\cdot, \theta) - X_1(\cdot, \vartheta)\right]\right\|_{\mathcal{X}}^2$$
$$\leq 4\,\mathbb{E}\left[\left\|X_1(\cdot, \theta) - X_1(\cdot, \vartheta)\right\|_{\mathcal{X}}^2\right],$$

where in the last step we use Jensen's inequality. This completes the proof. $\square$

Let us turn to cotype 2-Banach spaces. The Banach space $\mathcal{X}$ is called a *cotype 2-Banach space* if there is a constant $C > 0$ such that, for all $n \in \mathbb{N}$ and $\mathcal{X}$-valued independent centered Borel random elements $W_1, \ldots, W_n$ such that $\|W_i\|_{\mathcal{X}}$ are square integrable, we have the following inequality

$$\mathbb{E}\left[\left\|\sum_{i=1}^n W_i\right\|_{\mathcal{X}}^2\right] \geq C \sum_{i=1}^n \mathbb{E}\left[\|W_i\|_{\mathcal{X}}^2\right]$$

(see, e.g., [5]). For a further preparation, let us also recall that a centered tight Borel random element W in $\mathcal{X}$ is called *pre-gaussian* if there is some centered tight Gaussian

random element G in $\mathcal{X}$ such that

$$\mathbb{E}\left[L_1(W)\, L_2(W)\right] = \mathbb{E}\left[L_1(G)\, L_2(G)\right]$$

holds for every pair L_1, L_2 of continuous linear forms on $\mathcal{X}$.

If $\mathcal{X}$ is cotype 2-Banach space, we can obtain the following criterion for property (12).

Proposition 5.4 *Let $\mathcal{X}$ be a cotype 2-Banach space, let Θ satisfy condition* (1) *with constants C, $t > 0$, and let $\|X_1\|_\infty$ be square integrable. Assume that there is a dense subset $\Theta' \subseteq \Theta$ such that*

$$X_1(\cdot, \theta) - \mathbb{E}^B[X_1(\cdot, \theta)] \quad \text{is pre-gaussian for all } \theta \in \Theta'.$$

Then, the sequence $(S_n)_{n \in \mathbb{N}}$ satisfies property (12) *(with this Θ').*

Proof First note that $X_1(\cdot, \theta)$ is a tight Borel random element in $\mathcal{X}$ for every $\theta \in \Theta$ (cf. the proof of Proposition 5.3). Now the claim of Proposition 5.4 follows from the general central limit theorem in cotype 2-Banach spaces (see [5, Theorem 4.1] or [18, Theorem 10.7]) along with the version of Prokhorov's theorem for Radon measures (see, e.g., [24, Theorem I.3.6]). □

Remark 7 As a prominent example let $\mathcal{X}$ be an L^p-space on some σ-finite measure space $(\mathfrak{X}, \mathcal{A}, \nu)$ with L^p-norm $\|\cdot\|_p$ for $p \in [1, 2]$. Then, it is a cotype 2-Banach space (see [3, p. 188]). Moreover, for any $\theta \in \Theta$, the tight Borel random element $X_1(\cdot, \theta) - \mathbb{E}^B[X_1(\cdot, \theta)]$ is pre-gaussian if and only if $L \circ \left(X_1(\cdot, \theta) - \mathbb{E}^B[X_1(\cdot, \theta)]\right)$ is square integrable for every continuous linear form L on $\mathcal{X}$, and

$$\int_{\mathfrak{X}} \left(\mathbb{E}\left[\left(X_1(\cdot, \theta)_{|x} - \mathbb{E}^B[X_1(\cdot, \theta)]_{|x}\right)^2\right]\right)^{p/2} \nu(dx) < \infty$$

(see [13, Theorem 11]).

6 Proofs

Let us retake general assumptions and notations from Sect. 1. One key of our proofs is the following auxiliary technical result which extends Lemma B.2.7 in [23]. For a finite set B, we shall use notation $card(B)$ to denote its cardinality.

Lemma 6.1 *Let $\overline{\Theta}$ be some nonvoid finite subset of Θ, and let $A \geq 1$ as well as $r \in \mathbb{N}$ such that $A^r \geq card(\overline{\Theta})$. Then, for $c > 0$ there exists some $U \subseteq \overline{\Theta} \times \overline{\Theta}$ satisfying*

$$card(U) \leq A \cdot card(\overline{\Theta}). \tag{15}$$

$$(\theta, \vartheta) \in U \quad \Rightarrow \quad d_\Theta(\theta, \vartheta) \leq c\, r. \tag{16}$$

$$\sup_{\substack{\theta, \vartheta \in \overline{\Theta} \\ d_\Theta(\theta, \vartheta) \leq c}} d_\mathcal{X}(X_\theta, X_\vartheta) \leq 2 \sup_{(\theta, \vartheta) \in U} d_\mathcal{X}(X_\theta, X_\vartheta). \tag{17}$$

Proof According to the proof of Lemma B.2.7 in [23], we may find a sequence $(V_l)_{l\in\mathbb{N}}$ of subsets of $\overline{\Theta}$, a sequence $(\theta_l)_{l\in\mathbb{N}}$ in Θ as well as a sequence $(r_l)_{\in\mathbb{N}}$ in $\{1, \ldots, r\}$ such that the following properties are satisfied

- $V_1 = \overline{\Theta}$ and $\exists\, l_0 \in \mathbb{N}\ \forall\, l \in \mathbb{N},\ l \geq l_0 : V_l = \emptyset$.
- $\theta_l \in V_l$ if $V_l \neq \emptyset$.
- $card\big(\{\theta \in V_l \mid d_\Theta(\theta, \theta_l) \leq r_l c\}\big) \leq A^{r_l}$ if $V_l \neq \emptyset$.
- $V_{l+1} = V_l \setminus \{\theta \in V_l \mid d_\Theta(\theta, \theta_l) \leq (r_l - 1)c\} = \{\theta \in V_l \mid d_\Theta(\theta, \theta_l) > (r_l - 1)c\}$
  if $V_l \neq \emptyset$.
- $\displaystyle\sum_{\substack{l=1 \\ V_l \neq \emptyset}}^{\infty} A^{r_l} \leq A \cdot card(\overline{\Theta})$.

We shall show that the set

$$U := \bigcup_{\substack{l=1 \\ V_l \neq \emptyset}}^{\infty} \{(\theta_l, \theta) \mid \theta \in V_l,\ d_\Theta(\theta_l, \theta) \leq cr_l\}$$

is as required.

First of all

$$card(U) \leq \sum_{\substack{l=1 \\ V_l \neq \emptyset}}^{\infty} card\left(\{(\theta_l, \theta) \mid \theta \in V_l,\ d_\Theta(\theta_l, \theta) \leq cr_l\}\right)$$

$$= \sum_{\substack{l=1 \\ V_l \neq \emptyset}}^{\infty} card\left(\{\theta \in V_l \mid d_\Theta(\theta_l, \theta) \leq cr_l\}\right) \leq \sum_{\substack{l=1 \\ V_l \neq \emptyset}}^{\infty} A^{r_l} \leq A \cdot card(\overline{\Theta})$$

so that U fulfills (15).

Secondly, let $(\theta, \vartheta) \in U$. Then $\theta = \theta_l$ and $\vartheta \in V_l$ with $d_\Theta(\theta_l, \vartheta) \leq cr_l$ for some $l \in \mathbb{N}$ with $V_l \neq \emptyset$. This means $d_\Theta(\theta, \vartheta) \leq cr$ because $r_l \leq r$. Thus, (16) holds for U. So it remains to show that (17) is valid for U.

Let $\theta, \vartheta \in \overline{\Theta}$ with $d_\Theta(\theta, \vartheta) \leq c$. By construction, $\theta, \vartheta \in V_1$, whereas neither θ nor ϑ belongs to V_l for $l \geq l_0$. So we may choose $l_* := \max\{l \in \mathbb{N} \mid \theta, \vartheta \in V_l\}$. Then, $\theta \notin V_{l_*+1}$ or $\vartheta \notin V_{l_*+1}$, without loss of generality $\vartheta \notin V_{l_*+1}$. This means $d_\Theta(\theta_{l_*}, \vartheta) \leq (r_{l_*} - 1)c$ so that also

$$d_\Theta(\theta_{l_*}, \theta) \leq d_\Theta(\theta_{l_*}, \vartheta) + d_\Theta(\vartheta, \theta) \leq r_{l_*} c.$$

Hence, $(\theta_{l_*}, \theta), (\theta_{l_*}, \vartheta) \in U$, and thus,

$$d_{\mathcal{X}}(X_\theta, X_\vartheta) \leq d_{\mathcal{X}}(X_\theta, X_{\theta_{l_*}}) + d_{\mathcal{X}}(X_{\theta_{l_*}}, X_\vartheta) \leq 2 \sup_{(\theta, \vartheta) \in U} d_{\mathcal{X}}(X_\theta, X_\vartheta).$$

This shows (17) and completes the proof. $\qquad\square$

6.1 Proof of Lemma 1.3

In the first step, we want to point out the central chaining argument that we shall use for the proof of Lemma 1.3.

Lemma 6.2 *Let $\overline{\Theta} \subseteq \Theta$ be finite with at least two elements. Let n_0 be the largest element in $\mathbb{Z}$ such that $\Delta(\overline{\Theta}) \leq 2^{-n_0}$, and let*

$$n_1 := \min \left\{ n \in \mathbb{Z} \mid 2^{-n} < \inf_{\substack{\theta, \vartheta \in \overline{\Theta} \\ \theta \neq \vartheta}} d_\Theta(\theta, \vartheta) \right\}.$$

Then, $n_0 < n_1$, and the following statements are valid.

(1) *There exists a family $\{\Theta_n \mid n = n_0, \dots, n_1\}$ of subsets of $\overline{\Theta}$ satisfying*

$$card(\Theta_n) = N(\overline{\Theta}, d_\Theta, 2^{-n}) \quad for\ n \in \{n_0, \dots, n_1\}, \tag{18}$$

$$\inf_{\vartheta \in \Theta_n} d_\Theta(\theta, \vartheta) \leq 2^{-n} \quad for\ n \in \{n_0, \dots, n_1\}\ and\ \theta \in \overline{\Theta}. \tag{19}$$

(2) *The family $\{\Theta_n \mid n = n_0, \dots, n_1\}$ from statement 1) may be associated with a family $\{\varphi_n \mid n = n_0, \dots, n_1\}$ of mappings $\varphi_n : \overline{\Theta} \to \Theta_n$ which fulfill the following properties:*

$$\varphi_{n_1} : \overline{\Theta} \to \overline{\Theta}, \ \theta \mapsto \theta, \tag{20}$$

$$\varphi_{n_0} \equiv \theta_0 \quad for\ some\ \vartheta_0 \in \overline{\Theta}. \tag{21}$$

$$d_\Theta(\varphi_{n+1}(\theta), \varphi_n(\theta)) \leq 2^{-n} \quad for\ n \in \{n_0, \dots, n_1 - 1\}\ and\ \theta \in \overline{\Theta}, \tag{22}$$

$$card\left(\left\{(\varphi_{n+1}(\theta), \varphi_n(\theta)) \mid \theta \in \overline{\Theta}\right\}\right) \leq N(\overline{\Theta}, d_\Theta, 2^{-(n+1)}) \quad if\ n \in \{n_0, \dots, n_1 - 1\}, \tag{23}$$

$$d_\Theta(\varphi_n(\theta), \varphi_n(\vartheta)) \leq 2^{-n+2} + d_\Theta(\theta, \vartheta) \quad for\ n \in \{n_0, \dots, n_1\}\ and\ \theta, \vartheta \in \overline{\Theta}. \tag{24}$$

(3) *The chaining inequality*

$$d_\mathcal{X}(X_\theta, X_{\varphi_n(\theta)}) \leq \sum_{k=n}^{n_1 - 1} d_\mathcal{X}(X_{\varphi_{k+1}(\theta)}, X_{\varphi_k(\theta)}) \quad for\ \theta \in \overline{\Theta} \tag{25}$$

is satisfied if $n \in \{n_0, \dots, n_1 - 1\}$.

(4) *Under assumptions (1) and (3) from Theorem 1.1 with $C > 0$, $q > t > 0$, the inequality*

$$\mathbb{E}\left[\sup_{\theta \in \overline{\Theta}} d_\mathcal{X}(X_\theta, X_{\varphi_n(\theta)})^p\right] \leq M \left(\sum_{k=n}^{n_1 - 1} \frac{N(\overline{\Theta}, d_\Theta, 2^{-(k+1)})^{1/p}}{2^{kq/p}}\right)^p \tag{26}$$

holds for every $n \in \{n_0, \ldots, n_1 - 1\}$. Furthermore,

$$
\mathbb{E}\left[\sup_{\theta \in \overline{\Theta}} d_\mathcal{X}(X_\theta, X_{\varphi_n(\theta)})^p\right] \leq
\begin{cases}
M \, C \, 2^{2t} \dfrac{2^{(-n+1)(q-t)}}{\left(2^{(q-t)/p} - 1\right)^p} & , \quad n_1 \leq 0 \\[2ex]
M \, C \, 2^{2t} \left(\dfrac{2^{(-n+1)(q-t)/p} + 2^{(q-t)/p}}{2^{(q-t)/p} - 1}\right)^p & , \ n < 0 < n_1 \\[2ex]
M \, C \, 2^{q+t} \dfrac{2^{-n(q-t)}}{\left(2^{(q-t)/p} - 1\right)^p} & , \quad n \geq 0
\end{cases}
\tag{27}
$$

for $n \in \{n_0, \ldots, n_1 - 1\}$.

Proof Statement (1) follows immediately from the definition of covering numbers. Furthermore, by construction we have

$$
n_0 < n_1 \quad \text{and} \quad N(\overline{\Theta}, d_\Theta, 2^{-n_0}) = 1, \ N(\overline{\Theta}, d_\Theta, 2^{-n_1}) = card(\overline{\Theta}). \tag{28}
$$

Then, the proof of statement 2) can be found in [23, pp. 608f.]. In view of (20), statement 3) may be verified easily by backward induction along with triangle inequality. So it remains to show statement 4). Let $n \in \{n_0, \ldots, n_1 - 1\}$. By chaining inequality (25), we have

$$
\sup_{\theta \in \overline{\Theta}} d_\mathcal{X}(X_\theta, X_{\varphi_n(\theta)})^p \leq \sup_{\theta \in \overline{\Theta}} \left(\sum_{k=n}^{n_1 - 1} d_\mathcal{X}(X_{\varphi_{k+1}(\theta)}, X_{\varphi_k(\theta)})\right)^p
$$

$$
\leq \left(\sum_{k=n}^{n_1 - 1} \sup_{\theta \in \overline{\Theta}} d_\mathcal{X}(X_{\varphi_{k+1}(\theta)}, X_{\varphi_k(\theta)})\right)^p
$$

This implies by Minkowski's inequality

$$
\left(\mathbb{E}\left[\sup_{\theta \in \overline{\Theta}} d_\mathcal{X}(X_\theta, X_{\varphi_n(\theta)})^p\right]\right)^{1/p} \leq \sum_{k=n}^{n_1 - 1} \left(\mathbb{E}\left[\sup_{\theta \in \overline{\Theta}} d_\mathcal{X}(X_{\varphi_{k+1}(\theta)}, X_{\varphi_k(\theta)})^p\right]\right)^{1/p}.
\tag{29}
$$

Next, set for abbreviation

$$
I_k := \left\{\left(\varphi_{k+1}(\theta), \varphi_k(\theta)\right) \mid \theta \in \overline{\Theta}\right\} \quad (k \in \{n_0, \ldots, n_1 - 1\}).
$$

Then, we obtain in view of (3) along with (22) and (23)

$$
\mathbb{E}\left[\sup_{(\alpha, \tilde{\alpha}) \in I_k} d_\mathcal{X}(X_\alpha, X_{\tilde{\alpha}})^p\right] \leq \sum_{(\alpha, \tilde{\alpha}) \in I_k} \mathbb{E}\left[d_\mathcal{X}(X_\alpha, X_{\tilde{\alpha}})^p\right]
$$

$$
\overset{(22),(3)}{\leq} \frac{M \, card(I_k)}{2^{kq}} \overset{(23)}{\leq} \frac{M \, N(\overline{\Theta}, d_\Theta, 2^{-(k+1)})}{2^{kq}}.
$$

By (29), we end up with

$$
\mathbb{E}\left[\sup_{\theta \in \overline{\Theta}} d_{\mathcal{X}}(X_\theta, X_{\varphi_n(\theta)})^p\right] \overset{(29)}{\leq} \left(\sum_{k=n}^{n_1-1}\left(\mathbb{E}\left[\sup_{(\alpha,\tilde{\alpha}) \in I_k} d_{\mathcal{X}}(X_\alpha, X_{\tilde{\alpha}})^p\right]\right)^{1/p}\right)^p
$$

$$
\leq M \left(\sum_{k=n}^{n_1-1} \frac{N(\overline{\Theta}, d_\Theta, 2^{-(k+1)})^{1/p}}{2^{kq/p}}\right)^p.
$$

This shows (26) of statement 4. For the remaining part of the proof, we additionally assume that property (1) is satisfied with constants $C > 0$, $t \in]0, q[$. Then, we have

$$
N(\overline{\Theta}, d_\Theta, 2^{-(k+1)}) \leq N(\Theta, d_\Theta, 2^{-(k+2)}) \leq C\, 2^{t(k+2)} \quad \text{for } k \in \{n_0, \ldots, n_1 - 1\}.
$$

Note that $2^{-(k+1)} < \Delta(\overline{\Theta}) \leq \Delta(\Theta)$ holds for every $k \in \{n_0, \ldots, n_1 - 1\}$ due to choice of n_0. Now, (27) can be derived easily by routine calculations using geometric summation formulas. This concludes the proof. $\qquad\square$

Proof of Lemma 1.3 If $\delta < \inf\{d_\Theta(\theta, \vartheta) \mid \theta, \vartheta \in \overline{\Theta}, \theta \neq \vartheta\}$, then

$$
\{(\theta, \vartheta) \in \overline{\Theta} \times \overline{\Theta} \mid d_\Theta(\theta, \vartheta) \leq \delta\} = \{(\theta, \theta) \mid \theta \in \overline{\Theta}\}.
$$

In this case, the statement of Lemma 1.3 is trivial. From now on, let us assume $\delta \geq \inf\{d_\Theta(\theta, \vartheta) \mid \theta, \vartheta \in \overline{\Theta}, \theta \neq \vartheta\}$. In addition, let n_0 be the largest element in $\mathbb{Z}$ such that $\Delta(\overline{\Theta}) \leq 2^{-n_0}$, and let

$$
n_1 := \min\left\{n \in \mathbb{Z} \mid 2^{-n} < \inf_{\substack{\theta, \vartheta \in \overline{\Theta} \\ \theta \neq \vartheta}} d_\Theta(\theta, \vartheta)\right\}.
$$

We may find a family $\{\Theta_n \mid n = n_0, \ldots, n_1\}$ of subsets of $\overline{\Theta}$ and a family $\{\varphi_n \mid n = n_0, \ldots, n_1\}$ of mappings $\varphi_n : \overline{\Theta} \to \Theta_n$ as in Lemma 6.2.

If $N(\overline{\Theta}, d_\Theta, \delta/2) = 1$, then $\Delta(\overline{\Theta}) \leq \delta$ so that

$$
\mathbb{E}\left[\sup_{\substack{\theta, \vartheta \in \overline{\Theta} \\ d_\Theta(\theta, \vartheta) \leq \delta}} d_{\mathcal{X}}(X_\theta, X_\vartheta)^p\right] = \mathbb{E}\left[\sup_{\theta, \vartheta \in \overline{\Theta}} d_{\mathcal{X}}(X_\theta, X_\vartheta)^p\right],
$$

and in view of (21) along with (27)

$$
\mathbb{E}\left[\sup_{\theta,\vartheta\in\overline{\Theta}} d_{\mathcal{X}}(X_\theta, X_\vartheta)^p\right] \leq 2^p\,\mathbb{E}\left[\sup_{\theta\in\overline{\Theta}} d_{\mathcal{X}}(X_\theta, X_{\varphi_{n_0}(\theta)})^p\right]
$$

$$
\leq \begin{cases}
M\,C\,2^{2t+p}\,\dfrac{2^{(-n_0+1)(q-t)}}{\left(2^{(q-t)/p}-1\right)^p} & ,\quad n_1 \leq 0 \\[12pt]
M\,C\,2^{2t+p}\left(\dfrac{2^{(-n_0+1)(q-t)/p}+2^{(q-t)/p}}{2^{(q-t)/p}-1}\right)^p, & n_0 < 0 < n_1 \\[12pt]
M\,C\,2^{q+p+t}\,\dfrac{2^{-n_0(q-t)}}{\left(2^{(q-t)/p}-1\right)^p} & ,\quad n_0 \geq 0
\end{cases}.
$$

Moreover, by choice of n_0 we have $2^{-n_0+1} < 4\Delta(\overline{\Theta}) \leq 4\delta$ so that routine calculations yield

$$
\mathbb{E}\left[\sup_{\theta,\vartheta\in\overline{\Theta}} d_{\mathcal{X}}(X_\theta, X_\vartheta)^p\right] \leq \frac{2^{2q+2p}\,M\,C}{\left(2^{(q-t)/p}-1\right)^p}\,\delta^{q-t}.
$$

This shows Lemma 1.3 in case of $N(\overline{\Theta}, d_\Theta, \delta/2) = 1$. Next, let us assume that $N(\overline{\Theta}, d_\Theta, \delta/2) \geq 2$ is valid, and let us choose

$$
n_2 := \max\{n \in \mathbb{Z} \mid \delta \leq 2^{-n+2}\},\ n_3 := n_1 \wedge n_2 \text{ and } \overline{r} := \min\{r \in \mathbb{N} \mid 2^r \geq N(\overline{\Theta}, d_\Theta, \delta/4)\}.
$$

We have $2^{-n_2+1} < \delta < 2\Delta(\overline{\Theta}) \leq 2^{-n_0+1}$ so that $n_2 > n_0$. By choice of n_2, we obtain

$$
\sup_{\substack{\theta,\vartheta\in\overline{\Theta} \\ d_\Theta(\theta,\vartheta)\leq\delta}} d_{\mathcal{X}}(X_\theta, X_\vartheta) \leq \sup_{\substack{\theta,\vartheta\in\overline{\Theta} \\ d_\Theta(\theta,\vartheta)\leq 2^{-n_2+2}}} d_{\mathcal{X}}(X_\theta, X_\vartheta).
$$

Moreover, for $\theta, \vartheta \in \overline{\Theta}$ with $d_\Theta(\theta, \vartheta) \leq 2^{-n_2+2}$ we may further observe

$$
\begin{aligned}
d_{\mathcal{X}}(X_\theta, X_\vartheta) &\leq d_{\mathcal{X}}(X_\theta, X_{\varphi_{n_3}(\theta)}) + d_{\mathcal{X}}(X_{\varphi_{n_3}(\theta)}, X_{\varphi_{n_3}(\vartheta)}) + d_{\mathcal{X}}(X_\vartheta, X_{\varphi_{n_3}(\vartheta)}) \\
&\leq d_{\mathcal{X}}(X_{\varphi_{n_3}(\theta)}, X_{\varphi_{n_3}(\vartheta)}) + 2\sup_{\theta\in\overline{\Theta}} d_{\mathcal{X}}(X_\theta, X_{\varphi_{n_3}(\theta)}).
\end{aligned}
$$

Then, invoking (24), we obtain

$$
d_\Theta\big(\varphi_{n_3}(\theta), \varphi_{n_3}(\vartheta)\big) \leq 2^{-n_3+2} + d_\Theta(\theta, \vartheta) \leq 2^{-n_3+2} + 2^{-n_2+2} \leq 2^{-n_3+3}.
$$

Hence,

$$
\sup_{\substack{\theta,\vartheta\in\overline{\Theta} \\ d_\Theta(\theta,\vartheta)\leq\delta}} d_{\mathcal{X}}(X_\theta, X_\vartheta) \leq \sup_{\substack{\theta,\vartheta\in\overline{\Theta}_{n_3} \\ d_\Theta(\theta,\vartheta)\leq 2^{-n_3+3}}} d_{\mathcal{X}}(X_\theta, X_\vartheta) + 2\sup_{\theta\in\overline{\Theta}} d_{\mathcal{X}}(X_\theta, X_{\varphi_{n_3}(\theta)}).
$$

$$(30)$$

Furthermore, by (18) we may observe

$$2^{\overline{r}} \geq N(\overline{\Theta}, d_\Theta, \delta/4) \geq N(\overline{\Theta}, d_\Theta, 2^{-n_2}) \geq N(\overline{\Theta}, d_\Theta, 2^{-n_3}) = card(\Theta_{n_3}).$$

Therefore, we may apply Lemma 6.1 to Θ_{n_3} and $\overline{r}$, choosing $c := 2^{-n_3+3}$ and $A = 2$. Hence, we may find some $U \subseteq \Theta_{n_3} \times \Theta_{n_3}$ satisfying conditions (15), (16) and (17). Combination of (30) with (17) yields

$$\sup_{\substack{\theta, \vartheta \in \overline{\Theta} \\ d_\Theta(\theta, \vartheta) \leq \delta}} d_{\mathcal{X}}(X_\theta, X_\vartheta) \leq 2 \sup_{(\theta, \vartheta) \in U} d_{\mathcal{X}}(X_\theta, X_\vartheta) + 2 \sup_{\theta \in \overline{\Theta}} d_{\mathcal{X}}(X_\theta, X_{\varphi_{n_3}(\theta)})$$

so that

$$\sup_{\substack{\theta, \vartheta \in \overline{\Theta} \\ d_\Theta(\theta, \vartheta) \leq \delta}} d_{\mathcal{X}}(X_\theta, X_\vartheta)^p \leq 4^p \left(\sup_{(\theta, \vartheta) \in U} d_{\mathcal{X}}(X_\theta, X_\vartheta)^p + \sup_{\theta \in \overline{\Theta}} d_{\mathcal{X}}(X_\theta, X_{\varphi_{n_3}(\theta)})^p \right).$$

Hence,

$$\mathbb{E} \left[\sup_{\substack{\theta, \vartheta \in \overline{\Theta} \\ d_\Theta(\theta, \vartheta) \leq \delta}} d_{\mathcal{X}}(X_\theta, X_\vartheta)^p \right]$$

$$\leq 4^p \, \mathbb{E} \left[\sup_{(\theta, \vartheta) \in U} d_{\mathcal{X}}(X_\theta, X_\vartheta)^p \right] + 4^p \, \mathbb{E} \left[\sup_{\theta \in \overline{\Theta}} d_{\mathcal{X}}(X_\theta, X_{\varphi_{n_3}(\theta)})^p \right]. \tag{31}$$

If $n_3 = n_1$, then $\varphi_{n_3}(\theta) = \theta$ for $\theta \in \overline{\Theta}$ due to (20). Hence,

$$4^p \, \mathbb{E} \left[\sup_{\theta \in \overline{\Theta}} d_{\mathcal{X}}(X_\theta, X_{\varphi_{n_3}(\theta)})^p \right] = 0 \quad \text{if } n_3 = n_1. \tag{32}$$

So let us assume for a moment $n_3 < n_1$. Then invoking property (1) and assumption (3), we may conclude from Lemma 6.2, (27)

$$\mathbb{E} \left[\sup_{\theta \in \Theta} d_{\mathcal{X}}(X_\theta, X_{\varphi_{n_3}(\theta)})^p \right] \leq \begin{cases} M\,C\,2^{2t}\,\dfrac{2^{(-n_3+1)(q-t)}}{\left(2^{(q-t)/p} - 1\right)^p} & , \quad n_1 \leq 0 \\[2ex] M\,C\,2^{2t}\left(\dfrac{2^{(-n_3+1)(q-t)/p} + 2^{(q-t)/p}}{2^{(q-t)/p} - 1}\right)^p, & n_3 < 0 < n_1 \\[2ex] M\,C\,2^{q+t}\,\dfrac{2^{-n_3(q-t)}}{\left(2^{(q-t)/p} - 1\right)^p} & , \quad n_3 \geq 0 \end{cases} \tag{33}$$

We also have $n_3 = n_2$ so that the inequality $2^{-n_3+1} \le \delta$ is valid. Hence, in view of (33) by easy calculations, we end up with

$$4^p \, \mathbb{E}\left[\sup_{\theta \in \overline{\Theta}} d_{\mathcal{X}}(X_\theta, X_{\varphi_{n_3}(\theta)}) \right] \le \frac{2^{2t+3p} \, M \, C}{\left(2^{(q-t)/p} - 1\right)^p} \, \delta^{q-t} \quad \text{if } n_3 < n_1. \quad (34)$$

Furthermore, applying sequentially (3), (16), (15) and (18) we may observe

$$\begin{aligned}
\mathbb{E}\left[\sup_{(\theta,\vartheta) \in U} d_{\mathcal{X}}(X_\theta, X_\vartheta)^p \right] &\le \sum_{(\theta,\vartheta) \in U} \mathbb{E}\left[d_{\mathcal{X}}(X_\theta, X_\vartheta)^p \right] \\
&\overset{(3)}{\le} M \sum_{(\theta,\vartheta) \in U} d_\Theta(\theta, \vartheta)^q \\
&\overset{(16)}{\le} M \, card(U) \, 2^{q \, (-n_3+3)} \, \overline{r}^q \\
&\overset{(15)}{\le} 2M \, card(\Theta_{n_3}) \, 2^{q \, (-n_3+3)} \, \overline{r}^q \\
&\overset{(18)}{\le} 2M \, N(\overline{\Theta}, d_\Theta, 2^{-n_3}) \, 2^{q \, (-n_3+3)} \, \overline{r}^q \\
&\le 2M \, N(\overline{\Theta}, d_\Theta, 2^{-n_2}) \, 2^{q \, (-n_3+3)} \, \overline{r}^q
\end{aligned}$$

By choice of n_1 and δ, we have $2^{-n_3+3} = 2^{-n_1+3} < 16\delta$ if $n_3 < n_2$. Otherwise, we obtain $2^{-n_3+3} = 2^{-n_2+1}4 < 4\delta$ due to definition of n_2. In addition, $2^{-n_2} \ge \delta/4$. Hence,

$$\mathbb{E}\left[\sup_{(\theta,\vartheta) \in U} d_{\mathcal{X}}(X_\theta, X_\vartheta)^p \right] \le 2M \, N(\overline{\Theta}, d_\Theta, \delta/4) \, \delta^q \, (16\overline{r})^q$$

The choice of $\overline{r}$ implies $2^{\overline{r}-1} < N(\overline{\Theta}, d_\Theta, \delta/4)$ so that

$$\overline{r} \le \frac{2 \ln\left(N(\overline{\Theta}, d_\Theta, \delta/4) \right)}{\ln(2)}.$$

Therefore,

$$4^p \, \mathbb{E}\left[\sup_{(\theta,\vartheta) \in U} d_{\mathcal{X}}(X_\theta, X_\vartheta)^p \right] \le 4^{p+3q+1} \, M \, N(\overline{\Theta}, d_\Theta, \delta/4) \left[\ln\left(N(\overline{\Theta}, d_\Theta, \delta/4) \right) \right]^q \, \delta^q. \quad (35)$$

Putting (31), (32), (34) and (35) together, we now easily derive the statement of Lemma 1.3 if $N(\overline{\Theta}, d_\Theta, \delta/2) \ge 2$. The proof is complete. □

6.2 Proof of Theorem 1.1

Let (1) be satisfied with constants $C, t > 0$, and let $(X_\theta)_{\theta \in \Theta}$ fulfill inequality (3) with constants $M > 0, q > t$. Moreover, let us fix $\beta \in]0, (q - t)/p[$. First, we want to show inequality (4) for finite subsets of Θ.

Proposition 6.3 *There exists a finite constant $L(\Theta, C, t, M, p, q, \beta)$ that depends on $\Delta(\Theta)$, C, t, M, p, q, and β only such that, for any finite subset $\overline{\Theta} \subseteq \Theta$ with at least two elements, it holds*

$$\mathbb{E}\left[\sup_{\substack{\theta, \vartheta \in \overline{\Theta} \\ \theta \neq \vartheta}} \frac{d_\mathcal{X}(X_\theta, X_\vartheta)^p}{d_\Theta(\theta, \vartheta)^{\beta p}} \right] \leq L(\Theta, C, t, M, p, q, \beta).$$

Moreover, $L(\Theta, C, t, M, p, q, \beta)$ can be chosen to depend linearly on M: $L(\Theta, C, t, M, p, q, \beta) = M\overline{L}(\Theta, C, t, p, q, \beta)$.

Proof Let $\overline{\Theta}$ be any finite subset of Θ with at least two elements. Set $\eta_k := 2^{-k}(\Delta(\Theta) + 1)$ for $k \in \mathbb{N}$, and let the set J be defined to consist of all $k \in \mathbb{N}$ with $\eta_k < d_\Theta(\theta, \vartheta) \leq 2\eta_k$ for some $\theta, \vartheta \in \overline{\Theta}$. Note $J \neq \emptyset$. Then,

$$\mathbb{E}\left[\sup_{\substack{\theta, \vartheta \in \overline{\Theta} \\ \theta \neq \vartheta}} \frac{d_\mathcal{X}(X_\theta, X_\vartheta)^p}{d_\Theta(\theta, \vartheta)^{\beta p}} \right] \leq \sum_{k \in J} \mathbb{E}\left[\sup \left\{ \frac{d_\mathcal{X}(X_\theta, X_\vartheta)^p}{d_\Theta(\theta, \vartheta)^{\beta p}} \,\middle|\, \theta, \vartheta \in \overline{\Theta}, \, \eta_k < d_\Theta(\theta, \vartheta) \leq 2\eta_k \right\} \right]$$

$$\leq \sum_{k \in J} \eta_k^{-\beta p} \, \mathbb{E}\left[\sup \left\{ d_\mathcal{X}(X_\theta, X_\vartheta)^p \,\middle|\, \theta, \vartheta \in \overline{\Theta}, d_\Theta(\theta, \vartheta) \leq 2\eta_k \right\} \right]$$

$$\leq \sum_{k \in J} 2^{k\beta p} \, \mathbb{E}\left[\sup \left\{ d_\mathcal{X}(X_\theta, X_\vartheta)^p \,\middle|\, \theta, \vartheta \in \overline{\Theta}, d_\Theta(\theta, \vartheta) \leq 2\eta_k \right\} \right] \quad (36)$$

For $k \in J$, the application of Lemma 1.3 yields

$$\mathbb{E}\left[\sup \left\{ d_\mathcal{X}(X_\theta, X_\vartheta)^p \,\middle|\, \theta, \vartheta \in \overline{\Theta}, d_\Theta(\theta, \vartheta) \leq 2\eta_k \right\} \right]$$

$$\leq 4^{2p+4q+2} \, M \left(V_k(\Theta, \overline{\Theta}) \, (2\eta_k)^q \; + \frac{C \, (2\eta_k)^{q-t}}{(2^{(q-t)/p} - 1)^p} \right)$$

$$\leq 4^{2p+4q+2} \, M \, (\Delta(\Theta) + 1)^q \left(V_k(\Theta, \overline{\Theta}) \, 2^{(-k+1)q} \; + \frac{C \, 2^{(-k+1)(q-t)}}{(2^{(q-t)/p} - 1)^p} \right), \quad (37)$$

where

$$V_k(\Theta, \overline{\Theta}) := N(\overline{\Theta}, d_\Theta, \eta_{k+1}) \left[\ln \left(N(\overline{\Theta}, d_\Theta, \eta_{k+1}) \right) \right]^q.$$

Moreover, the set $\{k \in \mathbb{N} \mid \eta_{k+1} \leq \Delta(\Theta)\}$ is nonvoid so that we may select its minimum say k_0. In view of (1), this means

$$N(\overline{\Theta}, d_\Theta, \eta_{k+1}) \leq C \left(\frac{2^{k+1}}{\Delta(\Theta) + 1}\right)^t \leq C \, 2^{(k+1)t} \quad \text{for } k \in \mathbb{N}, k \geq k_0.$$

Hence, for $k \in J$ with $k \geq k_0$ we may give a further upper estimate of inequality (37) by

$$\mathbb{E}\left[\sup \left\{d_\mathcal{X}(X_\theta, X_\vartheta)^p \mid \theta, \vartheta \in \overline{\Theta}, d_\Theta(\theta, \vartheta) \leq 2\eta_k\right\}\right]$$
$$\leq 4^{2p+4q+2} \, 2^{(-k+1)(q-t)} \, M \, C \, (\Delta(\Theta) + 1)^q \, \left(4^t \left[\ln\left(C \cdot 2^{(k+1)t}\right)\right]^q + \frac{1}{\left(2^{(q-t)/p} - 1\right)^p}\right).$$

Then,

$$2^{k\beta p} \, \mathbb{E}\left[\sup \left\{d_\mathcal{X}(X_\theta, X_\vartheta)^p \mid \theta, \vartheta \in \overline{\Theta}, d_\Theta(\theta, \vartheta) \leq 2\eta_k\right\}\right]$$
$$\leq \frac{4^{2p+5q+2} \, M \, C \, (\Delta(\Theta) + 1)^q}{\left(2^{(q-t)/p} - 1\right)^p} \, 2^{(\beta p - (q-t))k} \, \left(4^t \left[\ln\left(C \cdot 2^{(k+1)t}\right)\right]^q \cdot \left(2^{(q-t)/p} - 1\right)^p + 1\right)$$

$$(38)$$

holds for $k \in J$ with $k \geq k_0$. Next, setting

$$a_k := \frac{4^{2p+5q+2} \, M \, C \, (\Delta(\Theta) + 1)^q}{\left(2^{(q-t)/p} - 1\right)^p} \, 2^{(\beta p - (q-t))k} \, \left(4^t \left[\ln\left(C \cdot 2^{(k+1)t}\right)\right]^q \cdot \left(2^{(q-t)/p} - 1\right)^p + 1\right)$$

we may observe

$$\lim_{k \to \infty} \frac{|a_{k+1}|}{|a_k|} = 2^{\beta p - (q-t)} \lim_{k \to \infty} \frac{4^t \left[\ln\left(C \cdot 2^{(k+2)t}\right)\right]^q \cdot \left(2^{(q-t)/p} - 1\right)^p + 1}{4^t \left[\ln\left(C \cdot 2^{(k+1)t}\right)\right]^q \cdot \left(2^{(q-t)/p} - 1\right)^p + 1} = 2^{\beta p - (q-t)} < 1.$$

Therefore,

$$L_1(\Theta, C, t, M, p, q, \beta)$$
$$:= \frac{4^{2p+5q+2} \, M \, C \, (\Delta(\Theta) + 1)^q}{\left(2^{(q-t)/p} - 1\right)^p} \sum_{k=k_0}^{\infty} 2^{(\beta p - (q-t))k} \left(4^t \left[\ln\left(C \cdot 2^{(k+1)t}\right)\right]^q \cdot \left(2^{(q-t)/p} - 1\right)^p + 1\right)$$
$$< \infty. \tag{39}$$

Moreover, by choice of k_0 we have

$$N(\overline{\Theta}, d_\Theta, \eta_{k+1}) = 1 \text{ if } k \in \mathbb{N}, k < k_0$$

which implies that $V_k(\Theta, \overline{\Theta}) = 0$ is valid for $k \in J$ with $k < k_0$. Then with $\sum_{\emptyset} := 0$, the application of (37) yields

$$
\sum_{\substack{k=1 \\ k \in J}}^{k_0-1} 2^{k\beta p} \, \mathbb{E}\left[\sup \left\{d_{\mathcal{X}}(X_\theta, X_\vartheta)^p \,\middle|\, \theta, \vartheta \in \overline{\Theta}, d_\Theta(\theta, \vartheta) \le 2^{-k+1}\big(\Delta(\Theta) + 1\big)\right\}\right]
$$

$$
\le \frac{4^{2p+4q+2} \, M \, \big(\Delta(\Theta) + 1\big)^q}{\big(2^{(q-t)/p} - 1\big)^p} \sum_{k=1}^{k_0-1} C \, 2^{(-k+1)(q-t)} \, 2^{k\beta p}
$$

$$
\le \frac{4^{2p+5q+2} \, M \, \big(\Delta(\Theta) + 1\big)^q}{\big(2^{(q-t)/p} - 1\big)^p} \, C \sum_{k=1}^{\infty} 2^{(\beta p - (q-t))k}. \tag{40}
$$

Since $\beta p < q - t$, we obtain that

$$
L_2(\Theta, C, t, M, p, q, \beta) := \frac{4^{2p+5q+2} \, M \, \big(\Delta(\Theta) + 1\big)^q}{\big(2^{(q-t)/p} - 1\big)^p} \, C \sum_{k=1}^{\infty} 2^{(\beta p - (q-t))k} < \infty. \tag{41}
$$

Combining (36), (38) and (40) with (39) and (41), we end up with

$$
\mathbb{E}\left[\sup_{\substack{\theta, \vartheta \in \overline{\Theta} \\ \theta \ne \vartheta}} \frac{d_{\mathcal{X}}(X_\theta, X_\vartheta)^p}{d_\Theta(\theta, \vartheta)^{\beta p}}\right] \le \sum_{j=1}^{2} L_j(\Theta, C, t, M, p, q, \beta) =: L(\Theta, C, t, M, p, q, \beta).
$$

This yields the first claim of Proposition 6.3. The second claim is a direct consequence of the expressions in (39) and (41). □

Proof of Theorem 1.1 We first fix any $\beta \in]0, (q - t)/p[$. Let the constant $L(\Theta, C, t, M, p, q, \beta)$ be chosen according to Proposition 6.3, and let us consider any at most countable subset $\overline{\Theta}$ of Θ which consists of at least two elements $\overline{\theta}, \vartheta$. We may select some sequence $(\overline{\Theta}_k)_{k \in \Theta}$ of nonvoid finite subsets of $\overline{\Theta}$ with at least two elements satisfying

$$
\overline{\theta}, \vartheta \in \overline{\Theta}_k \subseteq \overline{\Theta}_{k+1} \quad \text{for } k \in \mathbb{N} \quad \text{and} \quad \bigcup_{k=1}^{\infty} \overline{\Theta}_k = \overline{\Theta}.
$$

Then,

$$
\sup_{\substack{\theta, \vartheta \in \overline{\Theta}_k \\ \theta \ne \vartheta}} \frac{d_{\mathcal{X}}(X_\theta, X_\vartheta)^p}{d_\Theta(\theta, \vartheta)^{\beta p}} \nearrow \sup_{k \in \mathbb{N}} \sup_{\substack{\theta, \vartheta \in \overline{\Theta}_k \\ \theta \ne \vartheta}} \frac{d_{\mathcal{X}}(X_\theta, X_\vartheta)^p}{d_\Theta(\theta, \vartheta)^{\beta p}} = \sup_{\substack{\theta, \vartheta \in \overline{\Theta} \\ \theta \ne \vartheta}} \frac{d_{\mathcal{X}}(X_\theta, X_\vartheta)^p}{d_\Theta(\theta, \vartheta)^{\beta p}}
$$

and thus by monotone convergence theorem along with Proposition 6.3

$$\mathbb{E}\left[\sup_{\substack{\theta,\vartheta\in\Theta \\ \theta\neq\vartheta}}\frac{d_{\mathcal{X}}(X_\theta,X_\vartheta)^p}{d_\Theta(\theta,\vartheta)^{\beta p}}\right] = \lim_{k\to\infty}\mathbb{E}\left[\sup_{\substack{\theta,\vartheta\in\Theta_k \\ \theta\neq\vartheta}}\frac{d_{\mathcal{X}}(X_\theta,X_\vartheta)^p}{d_\Theta(\theta,\vartheta)^{\beta p}}\right] \leq L(\Theta,C,t,M,p,q,\beta). \quad (42)$$

This shows (4) due to the second statement of Proposition 6.3.

For the remaining part of the proof let us assume that $d_{\mathcal{X}}$ is complete, and let $\overline{\Theta}$ be some at most countable subset of Θ which is dense w.r.t. d_Θ. As a further consequence of (42), we have $\mathbb{P}(A) = 1$, where

$$A := \left\{\sup_{\substack{\theta,\vartheta\in\overline{\Theta} \\ \theta\neq\vartheta}}\frac{d_{\mathcal{X}}(X_\theta,X_\vartheta)^p}{d_\Theta(\theta,\vartheta)^{\beta p}} < \infty\right\}.$$

This implies that on A the random process $(X_\theta)_{\theta\in\overline{\Theta}}$ has Hölder-continuous paths of order β. By completeness of $d_{\mathcal{X}}$, we may define a new random process $(\overline{X}_\theta)_{\theta\in\Theta}$ via

$$\overline{X}_\theta(\omega) := \begin{cases} \lim_{\substack{\vartheta\to\theta \\ \vartheta\in\overline{\Theta}}} X_\vartheta(\omega), & \omega\in A, \\ \bar{x}, & \omega\notin A, \end{cases}$$

where $\bar{x}\in\mathcal{X}$ is arbitrary. Clearly, this process has Hölder-continuous paths of order β. Furthermore, it can be shown by standard arguments that this random process satisfies (2). We now show that it is a modification of $(X_\theta)_{\theta\in\Theta}$. For this purpose, let us fix any $\theta\in\Theta$, and let $(\vartheta_k)_{k\in\mathbb{N}}$ be a sequence from $\overline{\Theta}$ which converges to θ w.r.t. d_Θ. By construction of $(\overline{X}_\theta)_{\theta\in\Theta}$, we may invoke inequality (3) to conclude

$$\mathbb{E}\left[d_{\mathcal{X}}(X_\theta,\overline{X}_{\vartheta_k})^p\right] \leq \mathbb{E}\left[\mathbb{1}_A\cdot d_{\mathcal{X}}(X_\theta,X_{\vartheta_k})^p\right] \leq M\,d_\Theta(\theta,\vartheta_k)^q \to 0 \quad\text{for } k\to\infty.$$

In particular, on the one hand the sequence $\left(d_{\mathcal{X}}(X_\theta,\overline{X}_{\vartheta_k})\right)_{k\in\mathbb{N}}$ converges in probability to 0. On the other hand by definition of $(\overline{X}_\theta)_{\theta\in\Theta}$, the sequence $\left(d_{\mathcal{X}}(\overline{X}_\theta,\overline{X}_{\vartheta_k})\right)_{k\in\mathbb{N}}$ converges in probability to 0. Then, if $l\in\mathbb{N}$

$$\begin{aligned}
0 &\leq \limsup_{k\to\infty}\mathbb{P}\left(\{d_{\mathcal{X}}(X_\theta,\overline{X}_\theta) > l\}\right) \\
&\leq \limsup_{k\to\infty}\mathbb{P}\left(\{d_{\mathcal{X}}(X_\theta,X_{\vartheta_k}) + d_{\mathcal{X}}(X_{\vartheta_k},\overline{X}_\theta) > l\}\right) \\
&\leq \limsup_{k\to\infty}\left[\mathbb{P}\left(\{d_{\mathcal{X}}(X_\theta,X_{\vartheta_k}) > l/2\}\right) + \mathbb{P}\left(\{d_{\mathcal{X}}(\overline{X}_{\vartheta_k},\overline{X}_\theta) > l/2\}\right)\right] = 0,
\end{aligned}$$

and thus,

$$\mathbb{P}\left(\{d_{\mathcal{X}}(X_\theta,\overline{X}_\theta) > 0\}\right) = \lim_{l\to\infty}\mathbb{P}\left(\{d_{\mathcal{X}}(X_\theta,\overline{X}_\theta) > l\}\right) = 0.$$

Hence, $\mathbb{P}\big(\{X_\theta \neq \overline{X}_\theta\}\big) = 0$, i.e., $(\overline{X}_\theta)_{\theta \in \Theta}$ is a modification of $(X_\theta)_{\theta \in \Theta}$.

Finally, consider an increasing sequence $(\beta_n)_{n \in \mathbb{N}} \subset {]}0, (q-t)/p[$ such that $\beta_n \to (q-t)/p$, as $n \to \infty$. The argument above shows that, for any $n \in \mathbb{N}$, the process $(X_\theta)_{\theta \in \Theta}$ has a modification $(\overline{X}_\theta^n)_{\theta \in \Theta}$ with Hölder-continuous paths of order β_n and satisfying (2). Let us fix for a moment an arbitrary $n \in \mathbb{N}$. The processes $(\overline{X}_\theta^n)_{\theta \in \Theta}$ and $(\overline{X}_\theta^{n+1})_{\theta \in \Theta}$ are indistinguishable because they are modifications of each other, both continuous, and Θ is separable (as a totally bounded metric space). We can, therefore, find an event $\Omega_n \in \mathcal{F}$ with $\mathbb{P}(\Omega_n) = 1$ such that, for all $\omega \in \Omega_n$ and $\theta \in \Theta$, it holds $\overline{X}_\theta^n(\omega) = \overline{X}_\theta^{n+1}(\omega)$. We then define the set

$$\Omega_\infty = \bigcap_{n \in \mathbb{N}} \Omega_n$$

and notice that $\mathbb{P}(\Omega_\infty) = 1$ and, for all $\omega \in \Omega_\infty$, $\theta \in \Theta$ and $n \in \mathbb{N} \setminus \{1\}$, it holds

$$\overline{X}_\theta^1(\omega) = \overline{X}_\theta^n(\omega).$$

Consequently, the process $(\widetilde{X}_\theta)_{\theta \in \Theta}$ defined via

$$\widetilde{X}_\theta(\omega) = \begin{cases} \overline{X}_\theta^1(\omega), & \omega \in \Omega_\infty, \\ \bar{x}, & \omega \notin \Omega_\infty, \end{cases}$$

where $\bar{x} \in \mathcal{X}$ is arbitrary, is a modification of $(X_\theta)_{\theta \in \Theta}$ such that *all* its paths are Hölder-continuous of *all* orders $\beta \in {]}0, (q-t)/p[$. Note that $(\widetilde{X}_\theta)_{\theta \in \Theta}$ also satisfies (2). This concludes the proof. $\qquad\square$

Data Availability No datasets were generated or analyzed during the study

Declarations

Conflict of interest The authors have no competing interests to declare that are relevant to the content of this article.

Appendix A: Criterion for Uniform Tightness

Let (Θ, d_Θ) be a compact metric space and $(\mathcal{X}, d_\mathcal{X})$ a complete metric space. We use the notation $w(\cdot, \cdot)$ for the modulus of continuity on $\mathcal{C}(\Theta, \mathcal{X})$, i.e.,

$$w \colon \mathcal{C}(\Theta, \mathcal{X}) \times\,]0, \infty[\, \to \mathbb{R}, \quad (f, \delta) \mapsto \sup_{\substack{d_\Theta(\theta, \vartheta) \leq \delta \\ \theta, \vartheta \in \Theta}} d_\mathcal{X}\big(f(\theta), f(\vartheta)\big).$$

To make the paper self-contained, we present here the precise formulation of the criterion for uniform tightness, which was applied in the proof of Proposition 4.1.

Theorem A.1 *Let $(X_n)_{n \in \mathbb{N}}$ be a sequence of Borel random elements X_n in $\mathcal{C}(\Theta, \mathcal{X})$ on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\Theta' \subseteq \Theta$ be dense in Θ. The sequence $(X_n)_{n \in \mathbb{N}}$ is uniformly tight if and only if*

$$\big(X_n(\cdot, \theta)\big)_{n \in \mathbb{N}} \textit{ is a uniformly tight sequence of random elements in } \mathcal{X}, \textit{ for all } \theta \in \Theta', \tag{A1}$$

and, for every $\varepsilon > 0$,

$$\lim_{\delta \to 0+} \limsup_{n \to \infty} \mathbb{P}\left(w(X_n, \delta) \geq \varepsilon\right) = 0. \tag{A2}$$

We remark that since, for any $\delta > 0$, the mapping $w(\cdot, \delta)$ is continuous, $w(X_n, \delta)$ is a random variable for every $n \in \mathbb{N}$ (in particular, the probability in (A2) is well defined).

Proof The result is well known for $\mathcal{X} = \mathbb{R}$ (see [26, Theorem 1.5.7]), and a similar one is shown in the proof of Theorem 14.5 from [15] in the case of separable and complete $d_\mathcal{X}$. We shall use arguments from the proof of Theorem 14.5 in [15].

Firstly, let us assume that $(X_n)_{n \in \mathbb{N}}$ is a uniformly tight sequence in $\mathcal{C}(\Theta, \mathcal{X})$, and let $\varepsilon, \eta \in\,]0, 1[$. Then, there exists some compact subset $\mathcal{K} \subseteq \mathcal{C}(\Theta, \mathcal{X})$ such that

$$\sup_{n \in \mathbb{N}} \mathbb{P}(\{X_n \in \mathcal{C}(\Theta, \mathcal{X}) \setminus \mathcal{K}\}) \leq \eta. \tag{A3}$$

By a general version of the Arzela–Ascoli theorem (see [20, Theorem 47.1]), the set $\mathcal{K}$ is equicontinuous which means that we may find for any $\theta \in \Theta$ some $\delta_\theta > 0$ such that $d_\mathcal{X}\big(f(\vartheta), f(\theta)\big) < \varepsilon/2$ for every $f \in \mathcal{K}$ whenever $d_\theta(\vartheta, \theta) < \delta_\theta$. Since Θ is compact, we may apply Lebesgue's number lemma (see [20, Lemma 27.5]) to the open cover of Θ consisting of the open metric balls $B_{\delta_\theta}(\theta)$ around θ with radius δ_θ. In this way, we may select some $\delta > 0$ such that $w(f, \delta) < \varepsilon$ holds for every $f \in \mathcal{K}$. Then, for all $n \in \mathbb{N}$,

$$\mathbb{P}(\{w(X_n, \delta) \geq \varepsilon\}) \leq \mathbb{P}(\{X_n \in \mathcal{C}(\Theta, \mathcal{X}) \setminus \mathcal{K}\}) \leq \eta,$$

and thus,

$$\lim_{\delta \to 0+} \limsup_{n \to \infty} \mathbb{P}\big(\big\{w(X_n, \delta) \geq \varepsilon\big\}\big) \leq \eta. \tag{A4}$$

Moreover, for $\theta \in \Theta$ the projection mapping

$$\pi_\theta \colon \mathcal{C}(\Theta, \mathcal{X}) \to \mathcal{X}, \quad f \mapsto f(\theta)$$

is continuous so that $\pi_\theta(\mathcal{K})$ is a compact subset of $\mathcal{X}$. Furthermore, by (A3)

$$\sup_{n \in \mathbb{N}} \mathbb{P}\big(\big\{X_n(\cdot, \theta) \in \mathcal{X} \setminus \pi_\theta(\mathcal{K})\big\}\big) \leq \sup_{n \in \mathbb{N}} \mathbb{P}\big(\{X_n \in \mathcal{C}(\Theta, \mathcal{X}) \setminus \mathcal{K}\}\big) \leq \eta. \tag{A5}$$

Then, we conclude from (A4) and (A5) that $(X_n)_{n \in \mathbb{N}}$ satisfies conditions (A1) and (A2) because ε, η were chosen arbitrarily in $]0, 1[$. This finishes the proof of the only if part.

For the if part, let $(X_n)_{n \in \mathbb{N}}$ fulfill conditions (A1) and (A2). Fix any $\gamma \in]0, 1[$. Since Θ is compact, the mappings $X_n(\omega, \cdot)$ are uniformly continuous for $\omega \in \Omega$ and $n \in \mathbb{N}$. Hence, for $\omega \in \Omega$ and $n \in \mathbb{N}$, we have $w(X_n, \delta) \to 0$, as $\delta \to 0$. Combining this observation with condition (A2), we may find for any $k \in \mathbb{N}$ some $\overline{\delta}_k > 0$ such that

$$\sup_{n \in \mathbb{N}} \mathbb{P}\big(\big\{w(X_n, \overline{\delta}_k) > 2^{-k}\big\}\big) \leq 2^{-k-1}\,\gamma. \tag{A6}$$

Since Θ is compact, the metric on Θ' is separable. In addition, Θ' is dense. Then, there is some sequence $(\theta_k)_{k \in \mathbb{N}}$ in Θ' which is a dense subset of Θ. Hence, in view of (A1) we may find for every $k \in \mathbb{N}$ some compact subset K_k of $\mathcal{X}$ such that

$$\sup_{n \in \mathbb{N}} \mathbb{P}\big(\big\{X_n(\cdot, \theta_k) \notin K_k\big\}\big) \leq 2^{-k-1}\,\gamma. \tag{A7}$$

Now define

$$B := \bigcap_{k=1}^{\infty} \Big\{f \in \mathcal{C}(\Theta, \mathcal{X}) \mid f(\theta_k) \in K_k, \, w(f, \overline{\delta}_k) \leq 2^{-k}\Big\},$$

and denote the topological closure of B w.r.t. d_∞ by $cl(B)$. Then, we have

$$\sup_{n \in \mathbb{N}} \mathbb{P}\left(\{X_n \in \mathcal{C}(\Theta, \mathcal{X}) \setminus cl(B)\}\right)$$
$$\leq \sup_{n \in \mathbb{N}} \mathbb{P}\left(\{X_n \in \mathcal{C}(\Theta, \mathcal{X}) \setminus B\}\right)$$
$$\leq \sup_{n \in \mathbb{N}} \sum_{k=1}^{\infty} \left[\mathbb{P}(\{X_n(\cdot, \theta_k) \in \mathcal{X} \setminus K_k\}) + \mathbb{P}(\{w(X_n, \overline{\delta}_k) > 2^{-k}\})\right].$$

So invoking (A6) along with (A7), we end up with

$$\sup_{n\in\mathbb{N}} \mathbb{P}\left(\{X_n \in \mathcal{C}(\Theta, \mathcal{X}) \setminus cl(B)\}\right) \leq \sum_{k=1}^{\infty} \left[2^{-k-1}\,\gamma + 2^{-k-1}\,\gamma\right] = \gamma.$$

Hence, it is left to show that B is a relatively compact subset of $\mathcal{C}(\Theta, \mathcal{X})$.

For an arbitrary $\varepsilon > 0$ and for every $\theta \in \Theta$, choose some $k \in \mathbb{N}$ such that $2^{-k} < \varepsilon$. Then, by construction of B, we obtain $w(f, \overline{\delta}_k) < \varepsilon$ for $f \in B$. In particular, $d_{\mathcal{X}}\left(f(\vartheta), f(\theta)\right) < \varepsilon$ for every $f \in B$ and any $\vartheta \in \Theta$ with $d_{\Theta}(\vartheta, \theta) < \overline{\delta}_k$. Thus, we have shown that B is some equicontinuous subset of $\mathcal{C}(\Theta, \mathcal{X})$. Therefore, by a general version of the Arzela–Ascoli theorem [20, Theorem 47.1] it remains to show that the set $\{f(\theta) \mid f \in B\}$ is a relatively compact subset of $\mathcal{X}$ for any $\theta \in \Theta$. This means to show that this set is totally bounded w.r.t. $d_{\mathcal{X}}$ due to completeness of $d_{\mathcal{X}}$.

Let us fix any $\theta \in \Theta$. Choose, for an arbitrary $\varepsilon > 0$, some $k_0 \in \mathbb{N}$ such that $2^{-k_0} < \varepsilon/2$. Since $\{\theta_k \mid k \in \mathbb{N}\}$ is dense, we may find some $k_1 \in \mathbb{N}$ such that $d_{\Theta}(\theta_{k_1}, \theta) < \overline{\delta}_{k_0}$. This implies by the construction of the set B that

$$f(\theta_{k_1}) \in K_{k_1} \quad \text{and} \quad d_{\mathcal{X}}\left(f(\theta_{k_1}), f(\theta)\right) < \varepsilon/2 \quad \text{for every } f \in B. \qquad (A8)$$

The set K_{k_1} is assumed to be compact, in particular, it is totally bounded w.r.t. $d_{\mathcal{X}}$, so that there exist $m \in \mathbb{N}$ and $x_1, \ldots, x_m \in \mathcal{X}$ satisfying

$$K_{k_1} \subseteq \bigcup_{i=1}^{m} B_{\varepsilon/2}(x_i),$$

where, for $r > 0$, $B_r(x_i)$ denotes the open $d_{\mathcal{X}}$-metric ball around x_i with radius r. Then, we may conclude from (A8)

$$\{f(\theta) \mid f \in B\} \subseteq \bigcup_{i=1}^{m} B_{\varepsilon}(x_i).$$

This shows that the set $\{f(\theta) \mid f \in B\}$ is totally bounded w.r.t. $d_{\mathcal{X}}$, which completes the proof. $\qquad \square$

References

1. Aliprantis, C.D., Border, K.C.: Infinite Dimensional Analysis, 3rd edn. Springer, Berlin (2006)
2. Ankirchner, S., Kruse, T., Urusov, M.: Wasserstein convergence rates for random bit approximations of continuous Markov processes. J. Math. Anal. Appl. **493**(2), 124543 (2021)
3. Araujo, A., Gine, E.: The Central Limit Theorem for Real and Banach Valued Random Variables. Wiley, New York (1980)
4. Chentsov, N.N.: Weak convergence of stochastic processes whose trajectories have no discontinuities of the second kind and the "heuristic" approach to the Kolmogorov-Smirnov tests. Theor. Prob. Appl. **1**, 140–144 (1957)
5. Chobanjan, S.A., Tarieladze, V.I.: Gaussian characterizations of certain Banach spaces. J. Multivar. Anal. **7**, 183–203 (1977)

6. Cox, S.G., Hutzenthaler, M., and Jentzen, A: Local Lipschitz continuity in the initial value and strong completeness for nonlinear stochastic differential equations. Preprint, arXiv:1309.5595v3, 1–96, Accepted in Memoires of the American Mathematical Society (2021)
7. Dieudonné, J.: Treatise on Analysis, vol. III. Academic Press, New York, San Francisco, London (1972)
8. do Carmo, M.P.: Riemannian Geometry. Birkhäuser, Boston, Basel, Berlin (1992)
9. Gikhman, I.I., Skorokhod, A.V.: Theory of Random Processes, vol. 1. Springer, Berlin, Heidelberg, New York (1974)
10. Grohs, P., Hornung, F., Jentzen, A., and von Wurstemberger. P.: A proof that artificial neural networks overcome the curse of dimensionality in the numerical approximation of Black-Scholes partial differential equations. Preprint, arXiv:1809.02362v1, 1–124, (2018)
11. Hoffmann-Jorgensen, J., Pisier, G.: The law of large numbers and the central limit theorem in Banach spaces. Ann. Probab. **4**, 587–599 (1976)
12. Hudde, A., Hutzenthaler, M., and Mazzonetto, S.: A stochastic Gronwall inequality and applications to moments, strong completeness, strong local Lipschitz continuity, and perturbations. Preprint, arXiv:1903.08727v2, 1–26, (2019)
13. Jain, N.C.: Central limit theorem and related questions in Banach spaces, In: J. L. Doob (ed.), Proc. Symp. in Pure Mathematics XXXI, 55–66, Am. Math. Soc., Providence (1977)
14. Jost, J.: Riemannian Geometry and Geometric Analysis. Springer, Berlin, Heidelberg, New York (2011)
15. Kallenberg, O.: Foundation of Modern Probability. Springer, New York (1997)
16. Kobayashi, S., Nomizu, K.: Foundations of Differential Geometry I. Interscience Publishers, John Wiley & Sons, New York, London (1963)
17. Lang, A., Potthoff, J., Schlather, M., Schwab, D.: Continuity of random fields on Riemannian manifolds. Commun. Stoch. Anal. **10**(2), 185–193 (2016)
18. Ledoux, M., Talagrand, M.: Probability in Banach Spaces. Springer, Berlin (1991)
19. Mittmann, K., Steinwart, I.: On the existence of continuous modifications of vector-valued random fields. Georgian Math. J. **10**, 311–317 (2003)
20. Munkres, J.R.: Topology, 2nd edn. Prentice Hall, Englewood Cliffs, N.J. (2000)
21. Potthoff, J.: Sample properties of random fields. II. Continuity. Commun. Stoch. Anal. **3**(3), 331–348 (2009)
22. Revuz, D., Yor, M.: Continuous Martingales and Brownian Motion, 3rd edn. Springer, Berlin (1999)
23. Talagrand, M.: Upper and Lower Bounds for Stochastic Processes. Springer, Heidelberg (2014)
24. Vakhania, N.N., Tarieladze, V.I., Chobanyan, S.A.: Probability Distributions on Banach Spaces. Kluwer, Dordrecht (1987)
25. van de Geer, S.: Empirical Processes in M-Estimation. Cambridge University Press, Cambridge (2000)
26. van der Vaart, A.W., Wellner, J.A.: Weak Convergence and Empirical Processes. Springer, New York (1996)