On Self-Similar Bernstein Functions and Corresponding Generalized Fractional Derivatives

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Abstract

We use the theory of Bernstein functions to analyze power law tail behavior with log-periodic perturbations which corresponds to self-similarity of the Bernstein functions. Such tail behavior appears in the context of semistable Lévy processes. The Bernstein approach enables us to solve some open questions concerning semi-fractional derivatives recently introduced in Kern et al. (Fract Calc Appl Anal 22(2):326–357, 2019) by means of the generators of certain semistable Lévy processes. In particular, it is shown that semi-fractional derivatives can be seen as generalized fractional derivatives in the sense of Kochubei (Integr Equ Oper Theory 71:583–600, 2011) and generalized fractional derivatives can be constructed by means of arbitrary Bernstein functions vanishing at the origin.

Keywords Power law tails · Log-periodic behavior · Laplace exponent · Bernstein functions · Self-similarity · Discrete scale invariance · Semistable Lévy process · Semi-fractional derivative · Semi-fractional diffusion · Sonine kernel · Sibuya distribution · Space–time duality

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Dedicated to the memory of Mark Meerschaert (1955–2020).

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1 Introduction

A non-negative function $\tilde{\psi} : (0, \infty) \rightarrow (0, \infty)$ is called a *Bernstein function* if it is of class $C^\infty(0, \infty)$ and

$$(-1)^{n-1} \tilde{\psi}^{(n)}(x) \geq 0 \quad \text{for all } n \in \mathbb{N} \text{ and } x > 0. \quad (1.1)$$

Its first derivative $f = \tilde{\psi}'$ is a *completely monotone* function, i.e.

$$(-1)^n f^{(n)}(x) \geq 0 \quad \text{for all } n \in \mathbb{N}_0 \text{ and } x > 0. \quad (1.2)$$

Due to a celebrated result of Bernstein, the completely monotone function f is the Laplace transform of a unique Borel measure μ on $[0, \infty)$

$$f(x) = \tilde{\mu}(x) := \int_0^\infty e^{-xt} d\mu(t). \quad (1.3)$$

As a consequence, the Bernstein function admits the representation

$$\tilde{\psi}(x) = a + bx + \int_0^\infty (1 - e^{-xt}) d\phi(t) \quad (1.4)$$

for a unique triplet $[a, b, \phi]$, where $a, b \geq 0$ and ϕ is a Borel measure on $(0, \infty)$ satisfying $\int_0^\infty \min\{1, t\} d\phi(t) < \infty$, also called the *Lévy measure*. For details on Bernstein functions, completely monotone functions and their connection to stochastic processes we refer to the monograph [31]. It is well known that in case $a = b = 0$ the Bernstein function $\tilde{\psi}(x) = \int_0^\infty (1 - e^{-xt}) d\phi(t)$ is the Laplace exponent of a *Lévy subordinator* $(X_t)_{t \geq 0}$, i.e. $\mathbb{E}[\exp(-sX_t)] = \exp(-t \cdot \tilde{\psi}(s))$ for all $t \geq 0, s > 0$, where $(X_t)_{t \geq 0}$ is a Lévy process with almost surely non-decreasing sample paths.

In Sect. 2 we will introduce a self-similarity property for Bernstein functions which is intimately connected to the following class of functions.

Definition 1.1 We call a function $\theta : \mathbb{R} \rightarrow \mathbb{R}$ *admissible* with respect to the parameters $\alpha \in (0, 2) \setminus \{1\}$ and $c > 1$ if the following three conditions are fulfilled:

$$\theta(x) > 0 \text{ for all } x \in \mathbb{R}, \quad (1.5)$$

$$\text{the mapping } t \mapsto t^{-\alpha} \theta(\log t) \text{ is non-increasing for } t > 0, \quad (1.6)$$

$$\theta \text{ is } \log(c^{1/\alpha})\text{-periodic.} \quad (1.7)$$

We will frequently require that an admissible function θ is *smooth*, meaning that it is continuous and piecewise continuously differentiable.

In case $\alpha \in (0, 1)$ we use an admissible function θ to define

$$\phi(t, \infty) := t^{-\alpha} \theta(\log t) \quad \text{for all } t > 0 \quad (1.8)$$

as the positive tail of a Lévy measure ϕ concentrated on $(0, \infty)$ which belongs to a semistable distribution ν with log-characteristic function

$$\psi(x) = \int_0^\infty (e^{ixy} - 1) d\phi(y) \quad \text{for all } x \in \mathbb{R} \quad (1.9)$$

given uniquely by the Fourier transform $\widehat{\nu}(x) = \int_{\mathbb{R}} e^{ixy} d\nu(y) = \exp(\psi(x))$. The corresponding Lévy process $(X_t)_{t \geq 0}$ given by $\mathbb{E}[\exp(ix \cdot X_t)] = \exp(t \cdot \psi(x))$ for all $t \geq 0$ and $x \in \mathbb{R}$ is called a *semistable subordinator*. For details on semistable distributions and Lévy processes we refer to the monographs [21, 30]. The power law tail behavior with log-periodic perturbations of an admissible function θ naturally appears in various applications of natural sciences and other areas; see [34] and section 5.4 in [35]. In recent years the asymptotic fine structure of the corresponding measure ϕ and the function ψ has drawn some attention; cf. [14, 15]. Given an admissible function θ with respect to $\alpha \in (0, 1)$ and $c > 1$ with corresponding semistable Lévy measure ϕ given by (1.8), by Definition 2.2 in [13] the *semi-fractional derivative* $\frac{\partial^\alpha}{\partial_{c,\theta} x^\alpha}$ of order $\alpha \in (0, 1)$ is given by the non-local operator

$$\frac{\partial^\alpha}{\partial_{c,\theta} x^\alpha} f(x) := -Lf(x) = \int_0^\infty (f(x) - f(x-y)) d\phi(y), \quad (1.10)$$

where L is the generator of the corresponding semistable Lévy process and at least functions f in the Sobolev space $W^{2,1}(\mathbb{R})$ belong to the domain of the semi-fractional derivative. In terms of the Fourier transform we can equivalently rewrite (1.10) as

$$\widehat{\frac{\partial^\alpha}{\partial_{c,\theta} x^\alpha} f}(x) := -\widehat{L}f(x) = -\psi(x) \widehat{f}(x) \quad \text{for all } x \in \mathbb{R}, \quad (1.11)$$

where the Fourier transform of f is given by $\widehat{f}(x) = \int_{\mathbb{R}} e^{ixy} f(y) dy$; see [13] for details. In Sect. 2, we will start with the elementary observation that for $\alpha \in (0, 1)$ there is a one-to-one correspondence between self-similar Bernstein functions given as the Laplace exponent $\widetilde{\psi}(x) = -\psi(ix)$ for $x > 0$ and semistable Lévy measures ϕ of the form (1.8). In particular, this enables us to show that semi-fractional derivatives of order $\alpha \in (0, 1)$ can be seen as a special case of generalized fractional derivatives in the sense of [17]. A constant function θ corresponds to the complete Bernstein function $\widetilde{\psi}(x) = x^\alpha$ and an ordinary fractional derivative of order $\alpha \in (0, 1)$. For details on classical fractional derivatives we refer to the monographs [16, 26, 28]. In fact we will show that the construction of generalized fractional derivatives is not restricted to self-similar Bernstein functions and holds for arbitrary Bernstein functions with $a = b = 0$ in (1.4).

In Sect. 3 we will prove a discrete approximation formula of the generator in (1.10) involving a generalized Sibuya distribution given in terms of the self-similar Bernstein function.

In case $\alpha \in (1, 2)$ we use an admissible function θ to define

$$\phi(-\infty, -t) := t^{-\alpha} \theta(\log t) \quad \text{for all } t > 0 \quad (1.12)$$

as the negative tail of a Lévy measure ϕ concentrated on $(-\infty, 0)$ which belongs to a different semistable distribution ν with log-characteristic function

$$\psi(x) = \int_{-\infty}^0 (e^{ixy} - 1 - ixy) d\phi(y) \quad \text{for all } x \in \mathbb{R}. \quad (1.13)$$

Given an admissible function θ with respect to $\alpha \in (1, 2)$ and $c > 1$ with corresponding semistable Lévy measure ϕ given by (1.12), according to Definition 2.5 in [13] the negative semi-fractional derivative $\frac{\partial^\alpha}{\partial_{c,\theta}(-x)^\alpha}$ of order $\alpha \in (1, 2)$ is given by the non-local operator

$$\frac{\partial^\alpha}{\partial_{c,\theta}(-x)^\alpha} f(x) := Lf(x) = \int_0^\infty (f(x+y) - f(x) - y f'(x)) d\phi(-y), \quad (1.14)$$

where L is again the generator of the corresponding semistable Lévy process and at least functions f in the Sobolev space $W^{2,1}(\mathbb{R})$ belong to the domain of the semi-fractional derivative. For $\alpha \in (1, 2)$ the function $x \mapsto \psi(-ix)$ cannot be a Bernstein function, but we will show in Sect. 4 that it has an inverse which is a self-similar Bernstein function. This will enable us to solve an open question from [12] concerning space-time duality for semi-fractional differential equations.

2 Self-Similar Bernstein Functions and Semi-Fractional Derivatives

In this section we suppose that $\alpha \in (0, 1)$ in which case the Laplace exponent $\tilde{\psi}$ of the corresponding semistable distribution ν , uniquely given by the Laplace transform $\tilde{\nu}(x) = \int_0^\infty e^{-xy} d\nu(y) = \exp(-\tilde{\psi}(x))$ for $x > 0$, in view of (1.9) can be represented as

$$\tilde{\psi}(x) = -\psi(ix) = \int_0^\infty (1 - e^{-xy}) d\phi(y). \quad (2.1)$$

Clearly, $\tilde{\psi}$ is a Bernstein function since ϕ integrates $\min\{1, t\}$ on $(0, \infty)$.

Definition 2.1 We call a Bernstein function $\tilde{\psi}$ *self-similar* with respect to $\alpha \in (0, 1)$ and $c > 1$ if it admits the discrete scale invariance

$$\tilde{\psi}(c^{1/\alpha}x) = c \cdot \tilde{\psi}(x) \quad \text{for all } x > 0.$$

Clearly, self-similarity can be iterated leading to $\tilde{\psi}(c^{m/\alpha}x) = c^m \cdot \tilde{\psi}(x)$ for all $x > 0$ and $m \in \mathbb{Z}$. Moreover, the set of self-similarity factors

$$\mathbb{S} = \left\{ c > 0 : \tilde{\psi}(c^{1/\alpha}x) = c \cdot \tilde{\psi}(x) \text{ for all } x > 0 \right\} \quad (2.2)$$

is a closed multiplicative subgroup of $(0, \infty)$ due to continuity of $\tilde{\psi}$, such that we either have the discrete scale invariance $\mathbb{S} = c^{\mathbb{Z}}$ for some $c > 1$, or continuous scaling $\mathbb{S} = (0, \infty)$. Our focus is on the discrete scale behavior and we will give an

impression of how corresponding self-similar Bernstein functions look like after the following elementary observation, which is our key result.

Lemma 2.2 *For fixed $\alpha \in (0, 1)$ and $c > 1$ the following statements are equivalent.*

- (i) $\tilde{\psi}$ is a self-similar Bernstein function with respect to $\alpha \in (0, 1)$ and $c > 1$.
- (ii) $\tilde{\psi}(x) = \int_0^\infty (1 - e^{-xy}) d\phi(y)$, where the Lévy measure ϕ is given by (1.12) for some admissible function θ with respect to $\alpha \in (0, 1)$ and $c > 1$.

In either case we have $\tilde{\psi}(x) = x^\alpha \gamma(-\log x)$, where

$$\gamma(x) = \int_0^\infty e^{-t} t^{-\alpha} \theta(\log t + x) dt \quad (2.3)$$

is an admissible $C^\infty(\mathbb{R})$ -function with respect to $\alpha \in (0, 1)$ and $c > 1$.

Proof “(i) $\Rightarrow$ (ii)”: Since $\tilde{\psi}$ is a Bernstein function, by (1.4) we have

$$\tilde{\psi}(x) = a + bx + \int_0^\infty (1 - e^{-xt}) d\phi(t)$$

for some $a, b \geq 0$ and a Lévy measure ϕ on $(0, \infty)$ integrating $\min\{1, t\}$. Iterating the self-similarity relation shows that $\tilde{\psi}(c^{m/\alpha}x) = c^m \tilde{\psi}(x)$ for all $x > 0$ and $m \in \mathbb{Z}$. As $m \rightarrow -\infty$, we see that $\lim_{x \downarrow 0} \tilde{\psi}(x) = 0$ and hence $a = 0$. By the transformation rule self-similarity now reads as

$$\begin{aligned} \tilde{\psi}(c^{1/\alpha}x) &= b c^{1/\alpha}x + \int_0^\infty (1 - e^{-xt}) d(c^{1/\alpha}\phi)(t) \\ &= b cx + c \cdot \int_0^\infty (1 - e^{-xt}) d\phi(t) = c \cdot \tilde{\psi}(x), \end{aligned} \quad (2.4)$$

where $(c^{1/\alpha}\phi)$ denotes the image measure under scale multiplication with $c^{1/\alpha}$. Since the Bernstein triplet $[a, b, \phi]$ is unique, we must have $b = 0$ and $(c^{1/\alpha}\phi) = c \cdot \phi$. Hence by Lemma 7.1.6 and Corollary 7.4.4 in [21] we have that ϕ is a $(c^{1/\alpha}, c)$ -semistable Lévy measure on $(0, \infty)$ fulfilling (1.12) for some admissible function θ with respect to $\alpha \in (0, 1)$ and $c > 1$. The connection of Corollary 7.4.4 in [21] to admissible functions is made precise by Lemma A.1 in the Appendix.

“(ii) $\Rightarrow$ (i)”: Clearly, $\tilde{\psi}$ is a Bernstein function by Theorem 3.2 in [31] and self-similarity follows from (2.4) with $b = 0$ which is valid by Lemma 7.1.6 in [21].

Now define $\gamma(x) := e^{\alpha x} \tilde{\psi}(e^{-x})$ which is of class $C^\infty(\mathbb{R})$ by the product rule and $\gamma(x) > 0$ for all $x \in \mathbb{R}$. Then γ is $\log(c^{1/\alpha})$ -periodic, since

$$\gamma(x + \log(c^{1/\alpha})) = e^{\alpha x} c \cdot \tilde{\psi}(c^{-1/\alpha} e^{-x}) = e^{\alpha x} \tilde{\psi}(e^{-x}) = \gamma(x),$$

and $t \mapsto t^{-\alpha} \gamma(\log t) = \tilde{\psi}(t^{-1})$ is non-increasing. Hence γ is an admissible function with respect to $\alpha \in (0, 1)$ and $c > 1$. Moreover, by partial integration and a change

of variables $t = xy$ we get

$$\begin{aligned}\tilde{\psi}(x) &= \int_0^\infty (1 - e^{-xy}) \, d\phi(y) = x \int_0^\infty e^{-xy} y^{-\alpha} \theta(\log y) \, dy \\ &= x^\alpha \int_0^\infty e^{-t} t^{-\alpha} \theta(\log t - \log x) \, dt = x^\alpha \gamma(-\log x),\end{aligned}$$

which shows that (2.3) holds, concluding the proof. $\square$

Example 2.3 Since γ in (2.3) is admissible with respect to the same parameters $\alpha \in (0, 1)$ and $c > 1$ as θ , we may iterate the procedure. Starting with $\gamma_0 := \theta$ and setting

$$\gamma_k(x) := \int_0^\infty e^{-t} t^{-\alpha} \gamma_{k-1}(\log t + x) \, dt$$

for $k \in \mathbb{N}$, we have $\gamma_1 = \gamma$ and the sequence $(\tilde{\psi}_k)_{k \in \mathbb{N}}$ given by $\tilde{\psi}_k(x) = x^\alpha \gamma_k(-\log x)$ is a sequence of self-similar Bernstein functions by Lemma 2.2.

Example 2.4 If $\mathbb{S} = (0, \infty)$ in (2.2), then by Lemma 2.2 the function θ and hence also the function γ must be constant. Thus $\tilde{\psi}$ is the well known prototype of a Bernstein function $\tilde{\psi}(x) = C x^\alpha$ for some $C > 0$ and $\alpha \in (0, 1)$. To give an explicit example with $\mathbb{S} = c\mathbb{Z}$, for fixed $\alpha \in (0, 1)$ we consider the simple periodic function $\theta(x) = 1 + \frac{1}{\alpha} + \sin x$, which is positive and $\log(c^{1/\alpha})$ -periodic for $c = e^{2\pi\alpha}$. Moreover, θ is a smooth admissible function by Lemma A.1(v) in the Appendix, since $\theta'(x) = \cos x \leq 1 + \alpha(1 + \sin x) = \alpha\theta(x)$. For the corresponding function γ we have

$$\begin{aligned}\gamma(x) &= \int_0^\infty e^{-t} t^{-\alpha} \left(1 + \frac{1}{\alpha} + \sin(\log t + x)\right) \, dt \\ &= \left(1 + \frac{1}{\alpha}\right) \Gamma(1 - \alpha) + \int_0^\infty e^{-t} t^{-\alpha} \sin(\log t + x) \, dt\end{aligned}$$

and hence the self-similar Bernstein function has the representation

$$\tilde{\psi}(x) = x^\alpha \left(\left(1 + \frac{1}{\alpha}\right) \Gamma(1 - \alpha) + \int_0^\infty e^{-t} t^{-\alpha} \sin(\log t - \log x) \, dt \right).$$

Now, for $c = e^{2\pi\alpha}$ we have

$$\begin{aligned}\gamma\left(-\log\left(\sqrt{c^{1/\alpha}}x\right)\right) &= \left(1 + \frac{1}{\alpha}\right) \Gamma(1 - \alpha) + \int_0^\infty e^{-t} t^{-\alpha} \sin(\log t - \log x - \pi) \, dt \\ &= \left(1 + \frac{1}{\alpha}\right) \Gamma(1 - \alpha) - \int_0^\infty e^{-t} t^{-\alpha} \sin(\log t - \log x) \, dt \\ &= \gamma(-\log x) - 2 \int_0^\infty e^{-t} t^{-\alpha} \sin(\log t - \log x) \, dt,\end{aligned}$$

which cannot coincide with $\gamma(-\log x)$ for all $x > 0$. Hence $\tilde{\psi}(\sqrt{c}^{1/\alpha}x) \neq \sqrt{c}\tilde{\psi}(x)$ and we have $\mathbb{S} = c^{\mathbb{Z}}$. Iterating the procedure as in Example 2.3 gives us

$$\gamma_k(x) = (1 + \frac{1}{\alpha}) (\Gamma(1 - \alpha))^k + f_k(x)$$

for some non-constant and periodic function f_k with $|f_k(x)| \leq (\Gamma(1 - \alpha))^k$ for all $x \in \mathbb{R}$ and $k \in \mathbb{N}$. Since $\Gamma(1 - \alpha) > 1$ this shows that in general we cannot expect convergence of the sequence $(\tilde{\psi}_k)_{k \in \mathbb{N}}$ in Example 2.3.

Remark 2.5 If we assume that the admissible function θ is smooth, then it admits a Fourier series representation

$$\theta(x) = \sum_{k \in \mathbb{Z}} c_k e^{ik\tilde{c}x} \quad \text{with} \quad \tilde{c} = \frac{2\pi\alpha}{\log c}. \quad (2.5)$$

In this case the function γ appearing in Lemma 2.2 is given by the modified Fourier series

$$\gamma(x) = \sum_{k \in \mathbb{Z}} c_k \Gamma(ik\tilde{c} - \alpha + 1) e^{ik\tilde{c}x} \quad \text{for } x \in \mathbb{R} \quad (2.6)$$

which can be seen as follows. By Theorem 3.1 in [13] the coefficients of this series appear in a representation of the log-characteristic function

$$\psi(x) = - \sum_{k \in \mathbb{Z}} c_k \Gamma(ik\tilde{c} - \alpha + 1) (-ix)^{\alpha - ik\tilde{c}} \quad \text{for } x \in \mathbb{R} \quad (2.7)$$

and the relation $\gamma(x) = e^{\alpha x} \tilde{\psi}(e^{-x}) = -e^{\alpha x} \psi(ie^{-x})$ easily shows (2.6).

A natural question which arises is, if for two admissible functions θ_1, θ_2 with respect to the parameters $\alpha_1 \in (0, 1)$ and $c_1 > 1$, respectively $\alpha_2 \in (0, 1)$ and $c_2 > 1$, the composition of the corresponding semi-fractional derivatives given by (1.10) can again be a semi-fractional derivative of order $\alpha := \alpha_1 + \alpha_2$. We concentrate on the easiest case when $\alpha \in (0, 1)$ and θ_1, θ_2 have the same periodicity, i.e. $c_1^{1/\alpha_1} = c_2^{1/\alpha_2}$. In view of (1.11) for the corresponding log-characteristic functions we need to show that

$$\psi_1(x) \cdot \psi_2(x) = -\psi(x) \quad \text{for all } x \in \mathbb{R}, \quad (2.8)$$

where ψ is the log-characteristic function of a semistable distribution corresponding to an admissible function θ with respect to the parameters $\alpha \in (0, 1)$ and $c := c_1^{\alpha/\alpha_1} = c_2^{\alpha/\alpha_2} > 1$. Since (2.8) is equivalent to $\tilde{\psi}_1 \cdot \tilde{\psi}_2 = \tilde{\psi}$ for the corresponding log-Laplace exponents and we have self-similarity

$$\begin{aligned} \tilde{\psi}(c^{1/\alpha}x) &= \tilde{\psi}_1(c_1^{1/\alpha_1}x) \cdot \tilde{\psi}_2(c_2^{1/\alpha_2}x) = c_1c_2 \cdot \tilde{\psi}_1(x)\tilde{\psi}_2(x) \\ &= c^{\alpha_1/\alpha}c^{\alpha_2/\alpha} \cdot \tilde{\psi}(x) = c \cdot \tilde{\psi}(x), \end{aligned}$$

in view of Lemma 2.2 this is equivalent to require that $\tilde{\psi}$ is a Bernstein function.

Corollary 2.6 $\tilde{\psi}_1 \cdot \tilde{\psi}_2$ is a Bernstein function iff there exists an admissible function θ with respect to the parameters $\alpha = \alpha_1 + \alpha_2 \in (0, 1)$ and $c = c_1^{\alpha/\alpha_1} = c_2^{\alpha/\alpha_2} > 1$ such that

$$\frac{\partial^{\alpha_2}}{\partial_{c_2, \theta_2} x^{\alpha_2}} \frac{\partial^{\alpha_1}}{\partial_{c_1, \theta_1} x^{\alpha_1}} f = \frac{\partial^{\alpha}}{\partial_{c, \theta} x^{\alpha}} f \quad (2.9)$$

for suitable functions $f \in W^{2,1}(\mathbb{R})$ with $\frac{\partial^{\alpha_1}}{\partial_{c_1, \theta_1} x^{\alpha_1}} f$ belonging to the domain of $\frac{\partial^{\alpha_2}}{\partial_{c_2, \theta_2} x^{\alpha_2}}$.

Remark 2.7 In case θ_1, θ_2 are smooth admissible functions with Fourier series representations as in (2.5)

$$\theta_1(x) = \sum_{k \in \mathbb{Z}} c_{k,1} e^{ik\tilde{c}x} \quad \text{and} \quad \theta_2(x) = \sum_{\ell \in \mathbb{Z}} c_{\ell,2} e^{i\ell\tilde{c}x},$$

where $\tilde{c} = \frac{2\pi\alpha_1}{\log c_1} = \frac{2\pi\alpha_2}{\log c_2} = \frac{2\pi\alpha}{\log c}$, then by (2.7) and the Cauchy product rule we easily get

$$\tilde{\psi}(x) = \tilde{\psi}_1(x) \cdot \tilde{\psi}_2(x) = \sum_{m \in \mathbb{Z}} d_m \Gamma(im\tilde{c} - (\alpha_1 + \alpha_2) + 1) x^{\alpha_1 + \alpha_2 - im\tilde{c}}, \quad (2.10)$$

where

$$d_m = \sum_{\ell \in \mathbb{Z}} c_{m-\ell,1} c_{\ell,2} \frac{\Gamma(i(m-\ell)\tilde{c} - \alpha_1 + 1) \Gamma(i\ell\tilde{c} - \alpha_2 + 1)}{\Gamma(im\tilde{c} - (\alpha_1 + \alpha_2) + 1)}.$$

Uniqueness of the Fourier coefficients then gives us the representation

$$\theta(x) = \sum_{m \in \mathbb{Z}} d_m e^{im\tilde{c}x} \quad (2.11)$$

for the admissible function θ from Corollary 2.6, provided that (2.10) is a Bernstein function.

Finally, we want to show that the semi-fractional derivative of order $\alpha \in (0, 1)$ can be seen as a special case of a generalized fractional derivative introduced in [17]. Starting with (1.10) for a smooth admissible function θ , integration by parts shows that

$$\frac{\partial^{\alpha}}{\partial_{c, \theta} x^{\alpha}} f(x) = \int_0^{\infty} f'(x-y) y^{-\alpha} \theta(\log y) dy$$

as laid out in [13]. Introducing the kernel function $k(y) := y^{-\alpha} \theta(\log y)$ and restricting considerations to functions with support on the positive real line, this can be interpreted as a semi-fractional derivative of Caputo type

$$\frac{\partial^{\alpha}}{\partial_{c, \theta} x^{\alpha}} f(x) = \int_0^x f'(x-y) k(y) dy = (k * f')(x).$$

On the other hand, interchanging the order of integration and differentiation gives a semi-fractional derivative of Riemann–Liouville type

$$\mathbb{D}_{c,\theta}^\alpha f(x) = \frac{d}{dx} \int_0^x f(x-y) k(y) dy = (k * f)'(x),$$

which is also called of convolution type in [36]. The relationship between these forms is given by the formula

$$\frac{\partial^\alpha}{\partial_{c,\theta} x^\alpha} f(x) = \mathbb{D}_{c,\theta}^\alpha f(x) - f(0)k(x) =: \mathbb{D}_{(k)} f(x)$$

which can be derived as (2.33) in [22] and for more general kernel functions $\mathbb{D}_{(k)} f$ is called a generalized fractional derivative in [17]. For further approaches into this direction see [3, 18, 20, 25, 29, 36]. Of particular interest are non-negative locally integrable kernel functions k such that the operator $\mathbb{D}_{(k)}$ possesses a right inverse $\mathbb{I}_{(k)}$ such that $\mathbb{D}_{(k)} \mathbb{I}_{(k)} f = f$. Using the theory of complete Bernstein functions and the relationship to the Stieltjes class, it is shown in [17] that this is possible with

$$\mathbb{I}_{(k)} f(x) = \int_0^x f(y) k^*(x-y) dy$$

for locally bounded measurable functions f if (k, k^*) forms a Sonine pair of kernels, i.e. $k * k^* \equiv 1$; cf. also [27, 33, 37]. In this case $\mathbb{I}_{(k)} f$ is called a generalized fractional integral of order α and it also holds that $\mathbb{I}_{(k)} \mathbb{D}_{(k)} f(x) = f(x) - f(0)$ for absolutely continuous functions f .

We will now show that a Sonine kernel k^* may exist for our specific kernel function $k(y) = y^{-\alpha} \theta(\log y)$ working in the more general framework of Bernstein functions and their relation to completely monotone functions. As argued in [9], necessarily the kernel function k should have an integrable singularity at the origin to be a meaningful generalization of a fractional derivative, but we are not going into this discussion. Hence we aim for possible extensions of the list of specific kernel functions given in section 6 of [20]. Our approach works similar to the method laid out in [17] but, to work in full generality, we have to relax the definition of a Sonine pair in the following sense.

Definition 2.8 Given two Borel measures η, ρ on $[0, \infty)$ we say that (η, ρ) forms a *generalized Sonine pair* if $\eta * \rho = \lambda$, where λ denotes Lebesgue measure restricted to $[0, \infty)$.

Theorem 2.9 Let $\tilde{\psi}$ be a Bernstein function with triplet $[0, 0, \phi]$ and define η as the measure on $[0, \infty)$ with $\eta(\{0\}) = 0$ and density $t \mapsto \phi(t, \infty)$ with respect to Lebesgue measure on $(0, \infty)$, i.e. $d\eta(t) = \phi(t, \infty) dt$. Then η is a Borel measure and there exists another Borel measure ρ such that (η, ρ) forms a generalized Sonine pair.

Proof First note that η is locally finite, since by the Fubini-Tonelli theorem for all $y > 0$ we have

$$\eta([0, y]) = \int_0^y \int_t^\infty d\phi(x) dt = \int_0^\infty \min\{x, y\} d\phi(x) < \infty.$$

Hence η is a Borel measure. We have that $\tilde{\psi}(x) = \int_0^\infty (1 - e^{-xt}) d\phi(t)$ is a Bernstein function and thus $G(x) := \frac{1}{x} \tilde{\psi}(x)$ is completely monotone by Corollary 3.8(iv) in [31]. The tail function of the Lévy measure is a right-continuous, non-increasing function and integration by parts yields the Laplace transform

$$G(x) = \int_0^\infty \frac{1}{x} (1 - e^{-xt}) d\phi(t) = \int_0^\infty e^{-xt} \phi(t, \infty) dt = \tilde{\eta}(x) \quad (2.12)$$

for $x > 0$. Further, by Theorem 3.7 in [31], $G^*(x) := 1/\tilde{\psi}(x)$ is completely monotone, since it is the composition of a Bernstein function and the completely monotone function $x \mapsto \frac{1}{x}$. Hence there exists a Borel measure ρ on $[0, \infty)$ such that $G^*(x) = \int_0^\infty e^{-xt} d\rho(t) = \tilde{\rho}(x)$, which serves as the generalized Sonine partner of η as follows. We easily calculate the Laplace transform of $\eta * \rho$ as

$$\tilde{\eta}(x) \cdot \tilde{\rho}(x) = G(x) \cdot G^*(x) = \frac{\tilde{\psi}(x)}{x} \cdot \frac{1}{\tilde{\psi}(x)} = \frac{1}{x} \quad (2.13)$$

for $x > 0$, which is the Laplace transform of the Lebesgue measure λ on $[0, \infty)$. Now the assertion directly follows from uniqueness of the Laplace transform for probability measures and the translation principle; see Theorem 1a in chapter XIII.1 of [7]. $\square$

By virtue of Theorem 2.9 we now define a generalized fractional derivative and a corresponding generalized fractional integral of suitable functions $f : [0, \infty) \rightarrow \mathbb{R}$ as

$$\mathbb{D}_{(\eta)} f(x) = \int_0^x f'(x-y) d\eta(y) = (f' * \eta)(x) \quad (2.14)$$

for absolutely continuous functions f with locally bounded density f' , respectively

$$\mathbb{I}_{(\rho)} f(x) = \int_0^x f(x-y) d\rho(y) = (f * \rho)(x) \quad (2.15)$$

for locally bounded measurable functions f .

Lemma 2.10 *For absolutely continuous functions $f : [0, \infty) \rightarrow \mathbb{R}$ with locally bounded density f' we have*

$$\begin{aligned} \mathbb{D}_{(\eta)} f(x) &= \frac{d}{dx} \int_0^x f(x-y) d\eta(y) - f(0) \phi(x, \infty) \\ &= (f * \eta)'(x) - f(0) \phi(x, \infty) \end{aligned} \quad (2.16)$$

for almost all $x > 0$.

We may also use (2.16) as the definition of the generalized fractional derivative which has some advantage, since it can be defined for the broader class of locally bounded measurable functions f due to $\int_0^x f(x-y) d\eta(y) = \int_0^x f(x-y) \phi(y, \infty) dy$ being absolutely continuous.

Proof Let $k(x) = \phi(x, \infty)$ be the tail function of the Lévy measure. Then by partial integration and the Fubini–Tonelli theorem the Laplace transform on the right-hand side of (2.16) can be written as

$$\begin{aligned} \int_0^\infty e^{-sx} (f * k)'(x) dx - f(0)\tilde{k}(s) &= s \int_0^\infty e^{-sx} (f * k)(x) dx - f(0)\tilde{k}(s) \\ &= s \int_0^\infty e^{-sx} \int_0^x f(x-y)k(y) dy dx - f(0)\tilde{k}(s) \\ &= s \int_0^\infty k(y) \int_y^\infty e^{-sx} f(x-y) dx dy - f(0)\tilde{k}(s) \\ &= s \tilde{f}(s) \int_0^\infty e^{-sy} k(y) dy - f(0)\tilde{k}(s) \\ &= s \tilde{f}(s)\tilde{k}(s) - f(0)\tilde{k}(s) \end{aligned}$$

On the other hand, by the Fubini–Tonelli theorem and partial integration the Laplace transform of (2.14) is given by

$$\begin{aligned} \int_0^\infty e^{-sx} \int_0^x f'(x-y)k(y) dy dx &= \int_0^\infty k(y) \int_y^\infty e^{-sx} f'(x-y) dx dy \\ &= \int_0^\infty k(y) \left(-e^{-sy} f(0) + s \int_y^\infty e^{-sx} f(x-y) dx \right) dy \\ &= -f(0)\tilde{k}(s) + s \tilde{f}(s)\tilde{k}(s), \end{aligned}$$

which proves the assertion. $\square$

Theorem 2.11 *Let the assumptions of Theorem 2.9 hold.*

- (a) *For absolutely continuous functions f with locally bounded density f' we have $\mathbb{I}_{(\rho)}\mathbb{D}_{(\eta)}f(x) = f(x) - f(0)$ for all $x > 0$.*
- (b) *For locally bounded measurable functions f we have $\mathbb{D}_{(\eta)}\mathbb{I}_{(\rho)}f(x) = f(x)$ for almost all $x > 0$. If f is continuous then the relation holds for all $x > 0$.*

Proof (a) For absolutely continuous functions f with density f' we get

$$\begin{aligned} \mathbb{I}_{(\rho)}\mathbb{D}_{(\eta)}f(x) &= \int_0^x \mathbb{D}_{(\eta)}f(x-y) d\rho(y) = \int_0^x (f' * \eta)(x-y) d\rho(y) \\ &= ((f' * \eta * \rho)(x)) = (f' * \lambda)(x) = \int_0^x f'(x-t) dt \\ &= \int_0^x f'(t) dt = f(x) - f(0). \end{aligned}$$

(b) Since $\mathbb{I}_{(\rho)} f(0) = 0$ and $\int_0^x f(t) dt$ is absolutely continuous, we get

$$\begin{aligned} \mathbb{D}_{(\eta)} \mathbb{I}_{(\rho)} f(x) &= \frac{d}{dx} \int_0^x \mathbb{I}_{(\rho)} f(x-y) d\eta(y) = \frac{d}{dx} (\mathbb{I}_{(\rho)} f * \eta)(x) \\ &= \frac{d}{dx} (f * \rho * \eta)(x) = \frac{d}{dx} (f * \lambda)(x) = \frac{d}{dx} \int_0^x f(x-t) dt \\ &= \frac{d}{dx} \int_0^x f(t) dt = f(x) \end{aligned}$$

for almost all $x > 0$. If f is continuous, this relation holds for all $x > 0$, since the last equality holds by the fundamental theorem of calculus. $\square$

In our specific context of self-similar Bernstein functions, the semi-fractional integral given by the Sonine partner ρ in (2.15) is also determined by a self-similar Bernstein function as follows.

Lemma 2.12 *Let $\tilde{\psi}$ be a self-similar Bernstein function with respect to $\alpha \in (0, 1)$ and $c > 1$. Let ρ be the associated Sonine partner from above, then the primitive $G_I^*(x) := \int_0^x G^*(y) dy = \int_0^x \tilde{\rho}(y) dy$ is a self-similar Bernstein function with respect to $1 - \alpha \in (0, 1)$ and $d := c^{\frac{1-\alpha}{\alpha}} > 1$.*

Proof From Lemma 2.2 we get the scaling relation

$$G^*(c^{1/\alpha}x) = \frac{1}{\tilde{\psi}(c^{1/\alpha}x)} = \frac{1}{c \cdot \tilde{\psi}(x)} = c^{-1} G^*(x) \quad \text{for all } x > 0,$$

which implies $(c^{1/\alpha}\rho) = c^{-1} \cdot \rho$ for the measure ρ in Theorem 2.9. In particular it follows that $\rho(\{0\}) = 0$ and by the Fubini–Tonelli theorem we get

$$\begin{aligned} G_I^*(x) &= \int_0^x G^*(y) dy = \int_0^x \int_{0+}^{\infty} e^{-yt} d\rho(t) dy \\ &= \int_{0+}^{\infty} \frac{1 - e^{-xt}}{t} d\rho(t) = \int_0^{\infty} (1 - e^{-xt}) d\mu(t), \end{aligned}$$

where the Borel measure μ on $(0, \infty)$ is given by $d\mu(t) := \frac{1}{t} d\rho(t)$ and integrates $\min\{1, t\}$ as shown in the proof of Theorem 3.2 in [31]; cf. also Proposition 3.5 in [31]. Thus the primitive $G_I^*(x)$ is a Bernstein function and the scaling relation gives us

$$\begin{aligned} G_I^*(c^{1/\alpha}x) &:= \int_0^{\infty} (1 - e^{-xc^{1/\alpha}t}) d\mu(t) = c^{1/\alpha} \int_{0+}^{\infty} \frac{1 - e^{-xc^{1/\alpha}t}}{c^{1/\alpha}t} d\rho(t) \\ &= c^{1/\alpha} \int_{0+}^{\infty} \frac{1 - e^{-xt}}{t} d(c^{1/\alpha}\rho)(t) \\ &= c^{\frac{1}{\alpha}-1} \int_0^{\infty} (1 - e^{-xt}) d\mu(t) = c^{\frac{1-\alpha}{\alpha}} G_I^*(x). \end{aligned}$$

Now let $d = c^{\frac{1-\alpha}{\alpha}} > 1$ then $d^{\frac{1}{1-\alpha}} = c^{1/\alpha}$ and we have $G_I^*(d^{\frac{1}{1-\alpha}}x) = d \cdot G_I^*(x)$ for all $x > 0$. Hence G_I^* is a self-similar Bernstein function with respect to $1 - \alpha \in (0, 1)$ and $d > 1$. $\square$

Remark 2.13 By Lemma 2.2 the Lévy measure μ corresponding to G_I^* is given by $\mu(t, \infty) = t^{\alpha-1}\sigma(\log t)$ for an admissible function σ with respect to the parameters $1 - \alpha \in (0, 1)$ and $d > 1$. If this admissible function σ is smooth, we get a Sonine pair (k, k^*) in the original sense as follows. Integration by parts yields

$$G_I^*(x) = x \int_0^\infty \frac{1}{x}(1 - e^{-xt}) d\mu(t) = x \int_0^\infty e^{-xt} t^{\alpha-1} \sigma(\log t) dt$$

and hence we get as the derivative

$$\begin{aligned} G^*(x) &= \int_0^\infty e^{-xt} t^{\alpha-1} \sigma(\log t) dt - x \int_0^\infty e^{-xt} t^\alpha \sigma(\log t) dt \\ &= \int_0^\infty e^{-xt} \left(t^{\alpha-1} \sigma(\log t) - \frac{d}{dt} (t^\alpha \sigma(\log t)) \right) dt \\ &= \int_0^\infty e^{-xt} t^{\alpha-1} ((1 - \alpha)\sigma(\log t) - \sigma'(\log t)) dt \\ &=: \int_0^\infty e^{-xt} k^*(t) dt, \end{aligned}$$

where the kernel k^* is non-negative by Lemma A.1. Now the calculation of the Laplace transform in the proof of Theorem 2.9 shows that (k, k^*) is a Sonine pair. Note that in general we cannot expect k^* to be completely monotone as in the approach of [17] with complete Bernstein functions. Nor can we expect that the admissible function σ is smooth in general.

3 Discrete Approximation of the Generator

Recall that by (1.10) the semi-fractional derivative operator of order $\alpha \in (0, 1)$ is given by the negative generator of the continuous convolution semigroup $(\nu^{*t})_{t \geq 0}$, where ν is the semistable distribution with log-characteristic function (1.9). If the tail of the Lévy measure is given by $\phi(t, \infty) = \frac{1}{\Gamma(1-\alpha)} t^{-\alpha}$, i.e. the admissible function $\theta \equiv \frac{1}{\Gamma(1-\alpha)}$ is constant, it is well known that the semi-fractional derivative coincides with the ordinary Riemann–Liouville fractional derivative of order $\alpha \in (0, 1)$ and can be approximated by means of the Grünwald-Letnikov formula

$$\frac{\partial^\alpha}{\partial x^\alpha} f(x) = \lim_{h \downarrow 0} h^{-\alpha} \sum_{j=0}^{\infty} \binom{\alpha}{j} (-1)^j f(x - jh) \quad (3.1)$$

for functions f in the Sobolev space $W^{2,1}(\mathbb{R})$; see section 2.1 in [22] for details. The coefficients in the Grünwald-Letnikov approximation formula appear in a discrete

distribution on the positive integers called *Sibuya distribution*, which first appeared in [32]. A discrete random variable X_α on $\mathbb{N}$ is Sibuya distributed with parameter $\alpha \in (0, 1)$ if

$$\mathbb{P}(X_\alpha = j) = (-1)^{j-1} \binom{\alpha}{j} = (-1)^{j-1} \frac{\alpha(\alpha-1) \cdots (\alpha-j+1)}{j!} \quad \text{for } j \in \mathbb{N}. \quad (3.2)$$

For further details and extensions of the Sibuya distribution we refer to [5, 6, 19] and the literature mentioned therein. Using (3.2) we may rewrite (3.1) as

$$\begin{aligned} \frac{\partial^\alpha}{\partial x^\alpha} f(x) &= \lim_{h \downarrow 0} h^{-\alpha} \sum_{j=0}^{\infty} \binom{\alpha}{j} (-1)^j f(x-jh) \\ &= \lim_{h \downarrow 0} h^{-\alpha} \left(f(x) - \sum_{j=1}^{\infty} \mathbb{P}(X_\alpha = j) f(x-jh) \right) \\ &= \lim_{h \downarrow 0} h^{-\alpha} \left(f(x) - \int_{\mathbb{R}} f(x-hy) d\mathbb{P}_{X_\alpha}(y) \right) \\ &= \lim_{h \downarrow 0} h^{-\alpha} (f * (\varepsilon_0 - \mathbb{P}_{X_\alpha}))(x) \end{aligned}$$

which shows that the Grünwald–Letnikov formula is in fact a discrete approximation of the generator. For further relations of the Sibuya distribution to fractional diffusion equations see [23, 24].

Our aim is to generalize this formula for semi-fractional derivatives by means of the corresponding self-similar Bernstein functions. Note that for the Bernstein function $\tilde{\psi}(x) = x^\alpha$ the nominator on the right-hand side of (3.2) is given by $\tilde{\psi}^{(j)}(1)$ and hence we may define a *semi-fractional Sibuya distribution* in the following way.

Definition 3.1 Given an admissible function θ with respect to $\alpha \in (0, 1)$ and $c > 1$, a semi-fractional Sibuya distributed random variable X_θ on $\mathbb{N}$ is given by

$$\mathbb{P}(X_\theta = j) = \frac{(-1)^{j-1}}{j!} \frac{\tilde{\psi}^{(j)}(1)}{\tilde{\psi}(1)} \quad \text{for } j \in \mathbb{N}, \quad (3.3)$$

where $\tilde{\psi}$ is the corresponding self-similar Bernstein function.

Clearly, the expression in (3.3) is non-negative by (1.1). Moreover, by monotone convergence and a Taylor series approach justified by Proposition 3.6 in [31] we get

$$\begin{aligned} \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j!} \frac{\tilde{\psi}^{(j)}(1)}{\tilde{\psi}(1)} &= \frac{1}{\tilde{\psi}(1)} \sum_{j=1}^{\infty} \frac{\tilde{\psi}^{(j)}(1)}{j!} \lim_{\varepsilon \downarrow 0} (\varepsilon - 1)^{j-1} \\ &= \frac{1}{\tilde{\psi}(1)} \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon - 1} \left(\sum_{j=0}^{\infty} \frac{\tilde{\psi}^{(j)}(1)}{j!} (\varepsilon - 1)^j - \tilde{\psi}(1) \right) \end{aligned}$$

$$= \frac{1}{\tilde{\psi}(1)} \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon - 1} (\tilde{\psi}(\varepsilon) - \tilde{\psi}(1)) = \frac{\tilde{\psi}(1) - \tilde{\psi}(0)}{\tilde{\psi}(1)} = 1.$$

This shows that indeed (3.3) defines a proper distribution on $\mathbb{N}$ with pgf

$$\begin{aligned} G(z) &= \sum_{j=1}^{\infty} \mathbb{P}(X_{\theta} = j) z^j = -\frac{1}{\tilde{\psi}(1)} \sum_{j=1}^{\infty} \frac{\tilde{\psi}^{(j)}(1)}{j!} (-z)^j \\ &= 1 - \frac{1}{\tilde{\psi}(1)} \sum_{j=0}^{\infty} \frac{\tilde{\psi}^{(j)}(1)}{j!} ((1-z) - 1)^j = 1 - \frac{\tilde{\psi}(1-z)}{\tilde{\psi}(1)} \end{aligned}$$

for $|z| \leq 1$. Note that in the above arguments self-similarity of the Bernstein function is not needed. Hence (3.3) defines a proper distribution on $\mathbb{N}$ for every Bernstein function $\tilde{\psi}$ with $\tilde{\psi}(0) = 0$.

Now let θ be a smooth admissible function having Fourier series representation

$$\theta(x) = \sum_{k=-\infty}^{\infty} c_k e^{ik\tilde{c}x} \quad \text{with} \quad \tilde{c} = \frac{2\pi\alpha}{\log c}.$$

Then by Theorem 3.1 in [13] the corresponding self-similar Bernstein function can be expressed in terms of the log-characteristic function ψ as

$$\tilde{\psi}(x) = -\psi(ix) = \sum_{k=-\infty}^{\infty} \omega_k x^{\alpha - ik\tilde{c}} \quad \text{with} \quad \omega_k = c_k \Gamma(ik\tilde{c} - \alpha + 1)$$

and hence for $j \in \mathbb{N}_0$ we have

$$\frac{\tilde{\psi}^{(j)}(1)}{j!} = \sum_{k=-\infty}^{\infty} \omega_k \binom{\alpha - ik\tilde{c}}{j}.$$

Note that these coefficients appear in a Grünwald–Letnikov-type approximation of the semi-fractional derivative given in Theorem 4.1 of [13] which enables us to prove the following approximation formula.

Theorem 3.2 *Let θ be a smooth admissible function with respect to $\alpha \in (0, 1)$ and $c > 1$ with corresponding self-similar Bernstein function $\tilde{\psi}$ and semi-fractional Sibuya distributed random variable X_{θ} given by (3.3). Then for $f \in W^{2,1}(\mathbb{R})$ the semi-fractional derivative can be approximated along the subsequence $h_m = c^{-m/\alpha}$ by*

$$\frac{\partial^{\alpha}}{\partial_{c,\theta} x^{\alpha}} f(x) = \lim_{m \rightarrow \infty} \tilde{\psi}(h_m^{-1}) (f * (\varepsilon_0 - \mathbb{P}_{h_m X_{\theta}})) (x).$$

Proof First note that $h_m^{ik\tilde{c}} = c^{-imk\tilde{c}/\alpha} = e^{-2\pi imk} = 1$ for $\tilde{c} = \frac{2\pi\alpha}{\log c}$ and by self-similarity we have $\tilde{\psi}(h_m^{-1}) = h_m^{-\alpha}\tilde{\psi}(1)$ for all $m \in \mathbb{N}$. Hence we get

$$\begin{aligned} \tilde{\psi}(h_m^{-1}) \left(f * (\varepsilon_0 - \mathbb{P}_{h_m X_\theta}) \right) (x) &= h_m^{-\alpha} \tilde{\psi}(1) \left(f(x) - \int_{\mathbb{R}} f(x - h_m y) d\mathbb{P}_{X_\theta}(y) \right) \\ &= h_m^{-\alpha} \tilde{\psi}(1) \left(f(x) - \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j!} \frac{\tilde{\psi}^{(j)}(1)}{\tilde{\psi}(1)} f(x - jh_m) \right) \\ &= h_m^{-\alpha} \left(\sum_{k=-\infty}^{\infty} \omega_k f(x) + \sum_{j=1}^{\infty} (-1)^j \sum_{k=-\infty}^{\infty} \omega_k \binom{\alpha - ik\tilde{c}}{j} f(x - jh_m) \right) \\ &= h_m^{-\alpha} \sum_{j=0}^{\infty} (-1)^j \sum_{k=-\infty}^{\infty} \omega_k h_m^{ik\tilde{c}} \binom{\alpha - ik\tilde{c}}{j} f(x - jh_m) \rightarrow \frac{\partial^\alpha}{\partial_{c,\theta} x^\alpha} f(x), \end{aligned}$$

where the last convergence follows from Theorem 4.1 in [13]. $\square$

4 Space-Time Duality for Semi-Fractional Diffusions

We first give a sufficient condition for an inverse function to be a Bernstein function.

Lemma 4.1 *Let $f : (0, \infty) \rightarrow (0, \infty)$ be a $C^\infty(0, \infty)$ -function such that f' is a Bernstein function and $f^{(n)}(x) \neq 0$ for all $x > 0$ and $n \in \mathbb{N}$. Then its inverse f^{-1} is a Bernstein function with $(f^{-1})^{(n)}(x) \neq 0$ for all $x > 0$ and $n \in \mathbb{N}$.*

Proof We know that $f^{-1}(x) > 0$ and $(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} > 0$ for all $x > 0$. Moreover, as in Remark A.3 of the Appendix $(f^{-1})''(x) = -\frac{f''(f^{-1}(x))}{(f'(f^{-1}(x)))^3} < 0$ for all $x > 0$. For $n \geq 3$ we inductively use the formula for $(f^{-1})^{(n)}$ given in Lemma A.2 of the Appendix. By induction we have

$$\text{sign} \left(\prod_{j=1}^{n-1} \left(\frac{(f^{-1})^{(j)}(x)}{j!} \right)^{k_j} \right) = \prod_{j=1}^{n-1} (-1)^{(j-1)k_j} = (-1)^{\sum_{j=1}^{n-1} (j-1)k_j}$$

and $\text{sign}(f^{(k_1+\dots+k_{n-1})}(f^{-1}(x))) = (-1)^{k_1+\dots+k_{n-1}}$ since $k_1 + \dots + k_{n-1} \geq 2$ for $n \geq 3$ as $k_1 + 2k_2 + \dots + (n-1)k_{n-1} = n$. This shows that

$$\text{sign} \left(f^{(k_1+\dots+k_{n-1})}(f^{-1}(x)) \prod_{j=1}^{n-1} \left(\frac{(f^{-1})^{(j)}(x)}{j!} \right)^{k_j} \right) = (-1)^{\sum_{j=1}^{n-1} jk_j} = (-1)^n$$

for every summand of the formula in Lemma A.2. Thus $\text{sign}((f^{-1})^{(n)}(x)) = (-1)^{n-1}$ for all $x > 0$ and $n \geq 3$ showing that f^{-1} is a Bernstein function. $\square$

We want to apply Lemma 4.1 to the function

$$\zeta(x) = \psi(-ix) = \int_{-\infty}^0 (e^{xy} - 1 - xy) d\phi(y) > 0 \quad \text{for } x > 0, \quad (4.1)$$

where ϕ is the semistable Lévy measure for $\alpha \in (1, 2)$ from (1.8) concentrated on the negative axis and ψ is the corresponding log-characteristic function from (1.9). Note that $\zeta(x) \rightarrow \infty$ as $x \rightarrow \infty$ due to the tail behavior in (1.8) and by dominated convergence we have

$$\zeta'(x) = \int_{-\infty}^0 y (e^{xy} - 1) d\phi(y) = \int_0^{\infty} (1 - e^{-xy}) y d\phi(-y).$$

Since ϕ integrates $\min\{1, y^2\}$ and fulfills (1.8), the measure μ with $d\mu(y) = y d\phi(-y)$ integrates $\min\{1, y\}$ and thus ζ' is a Bernstein function. This also follows from the higher order derivatives

$$\zeta^{(n)}(x) = \int_{-\infty}^0 y^n e^{xy} d\phi(y) \begin{cases} > 0 & \text{if } n \geq 2 \text{ is even,} \\ < 0 & \text{if } n \geq 3 \text{ is odd,} \end{cases}$$

which additionally shows that these derivatives do not vanish. Thus from Lemma 4.1 we conclude that the inverse ζ^{-1} is a Bernstein function. This will be the key to solve an open problem concerning the following result on space-time duality from [12].

According to [30, Example 28.2], the semistable Lévy process $(X_t)_{t \geq 0}$ with $P\{X_t \in A\} = \nu^{*t}(A)$ for Borel sets $A \in \mathcal{B}(\mathbb{R})$, where ν is the semistable distribution with log-characteristic function ψ from (1.9), possesses $C^\infty(\mathbb{R})$ -densities $x \mapsto p(x, t)$ for every $t > 0$ with $P\{X_t \in A\} = \int_A p(x, t) dx$ and for $t = 0$ we may write $p(x, 0) = \delta(x)$ corresponding to $X_0 = 0$ almost surely. It is shown in [13] that these densities are the point source solution to the semi-fractional diffusion equation

$$\frac{\partial^\alpha}{\partial_{c,\theta}(-x)^\alpha} p(x, t) = \frac{\partial}{\partial t} p(x, t) \quad (4.2)$$

with the negative semi-fractional derivative of order α from (1.14) acting on the space variable. Since the semi-fractional derivative is a non-local operator, this equation is hard to interpret from a physical point of view, whereas non-locality in time may correspond to long memory effects [10]. As a generalization of a space-time duality result for fractional diffusions [4, 11] based on a corresponding result for stable densities by Zolotarev [39, 40], it was shown in [12] that space-time duality may also hold for semi-fractional diffusions. Theorem 3.3 in [12] states that for $x > 0$ and $t > 0$ we have $p(x, t) = \alpha^{-1} h(x, t)$, where $h(x, t)$ is the point source solution to the semi-fractional differential equation

$$\frac{\partial^{1/\alpha}}{\partial_{d,\tau} t^{1/\alpha}} h(x, t) + \frac{\partial}{\partial x} h(x, t) = t^{-1/\alpha} \varrho(\log t) \delta(x) \quad (4.3)$$

with a semi-fractional derivative of order $1/\alpha$ acting on the time variable, provided that τ and ϱ are admissible functions with respect to $1/\alpha \in (\frac{1}{2}, 1)$ and $d = c^{1/\alpha} > 1$ which remained an open problem in [12]. We will now show that indeed τ is admissible and $\varrho(x) = -\alpha\tau'(x)$, provided that τ is smooth. Note that in general ϱ will not be admissible as conjectured in [12], but we will justify the inhomogeneity in (4.3) by different arguments. In [12] the function τ appears in the following way. The above inverse ζ^{-1} is called ξ in [12] and its existence is shown in Lemma 4.1 of [12]. It was further shown in Lemma 4.2 of [12] that $\xi(t) = t^{1/\alpha}g(\log t)$ for a continuously differentiable and $\log(c)$ -periodic function g . Since we now know that $\xi = \zeta^{-1}$ is a Bernstein function, in fact g is a $C^\infty(\mathbb{R})$ -function. Since g is a smooth $\log(c)$ -periodic function, it is representable by its Fourier series

$$g(x) = \sum_{n \in \mathbb{Z}} d_n e^{-in\tilde{d}x} \quad \text{with } \tilde{d} = \frac{2\pi}{\log c} = \frac{2\pi \frac{1}{\alpha}}{\log d} \text{ for } d = c^{1/\alpha}, \quad (4.4)$$

where by Lemma 1 in §12 of [2] we have $|d_n| \leq C \cdot e^{-\frac{\pi}{2}|n|\tilde{d}}$ for some $C > 0$ and all $n \in \mathbb{Z}$. If we require a little more quality, namely that the Fourier coefficients even decay as

$$|d_n| \leq C \cdot e^{-\frac{\pi}{2}|n|\tilde{d}} |n|^{-\frac{3}{2}-\frac{1}{\alpha}-\varepsilon} \quad \text{for some } \varepsilon > 0 \text{ and all } n \in \mathbb{Z} \setminus \{0\}, \quad (4.5)$$

then we can define τ by the Fourier series

$$\tau(x) = \sum_{n \in \mathbb{Z}} \frac{d_n}{\Gamma(in\tilde{d} - \frac{1}{\alpha} + 1)} e^{-in\tilde{d}x}. \quad (4.6)$$

Lemma 4.2 *If (4.5) holds, then the function τ in (4.6) is well-defined and a smooth admissible function with respect to $1/\alpha \in (\frac{1}{2}, 1)$ and $d = c^{1/\alpha}$. Moreover, ζ^{-1} is a self-similar Bernstein function with respect to the same parameters.*

Proof As shown above $\xi = \zeta^{-1}$ is a Bernstein function and with $d = c^{1/\alpha}$ and $\xi(t) = t^{1/\alpha}g(\log t)$ for a $\log(c)$ -periodic function g we get

$$d \cdot \xi(t) = (ct)^{1/\alpha} g(\log(ct)) = \xi(ct) = \xi(d^\alpha t)$$

showing that ξ is a self-similar Bernstein function with respect to $1/\alpha \in (\frac{1}{2}, 1)$ and $d = c^{1/\alpha}$. By Lemma 2.2 we have

$$\xi(x) = \int_0^\infty (1 - e^{-xy}) \, d\mu(y), \quad (4.7)$$

where μ is a semistable Lévy measure with $\mu(t, \infty) = t^{-1/\alpha}\tau(\log t)$ for an admissible function τ with respect to $1/\alpha \in (\frac{1}{2}, 1)$ and $d = c^{1/\alpha}$. It remains to show that τ is

indeed the function we are looking for. Let us assume for a moment that τ is smooth and thus admits the Fourier series

$$\tau(x) = \sum_{n \in \mathbb{Z}} a_n e^{-in\tilde{d}x} \quad \text{with } \tilde{d} = \frac{2\pi}{\log c} = \frac{2\pi \frac{1}{\alpha}}{\log d}.$$

In this case the function γ appearing in Lemma 2.2 fulfills

$$\gamma(-x) = e^{-\frac{1}{\alpha}x} \xi(e^x) = g(x)$$

and by (2.6) has the Fourier series representation

$$g(x) = \gamma(-x) = \sum_{n \in \mathbb{Z}} a_n \Gamma(in\tilde{d} - \frac{1}{\alpha} + 1) e^{-in\tilde{d}x}.$$

A comparison with (4.4) and uniqueness of the Fourier coefficients shows that a_n coincides with $d_n / \Gamma(in\tilde{d} - \frac{1}{\alpha} + 1)$ and thus τ is indeed the function in (4.6). Finally, we have to show that the series in (4.6) converges. Using the asymptotic behavior of the gamma function in Corollary 1.4.4 of [1], the Fourier coefficients fulfill

$$\left| \frac{d_n}{\Gamma(in\tilde{d} - \frac{1}{\alpha} + 1)} \right| \leq K |d_n| \cdot |n|^{-\frac{1}{2} + \frac{1}{\alpha}} e^{\frac{\pi}{2} |n| \tilde{d}}$$

for a constant $K > 0$ and all $n \in \mathbb{Z} \setminus \{0\}$. According to our assumption (4.5) we obtain

$$\left| \frac{d_n}{\Gamma(in\tilde{d} - \frac{1}{\alpha} + 1)} \right| \leq K C |n|^{-2-\varepsilon}$$

for some $\varepsilon > 0$ and all $n \in \mathbb{Z} \setminus \{0\}$ showing that the series in (4.6) converges and the resulting function is continuously differentiable by Theorem 2.6 in [8]. $\square$

Admissibility of τ in combination with Theorem 3.3 in [12] finally enables us to completely solve space-time duality for semi-fractional diffusions. The Eq. (4.3) is derived in [12] by Laplace inversion of the equation

$$\xi(s) \tilde{h}(x, s) - \frac{1}{s} \xi(s) \delta(x) + \frac{\partial}{\partial x} \tilde{h}(x, s) = -\frac{1}{s} \frac{f(s)}{s + f(s)} \xi(s) \delta(x), \quad (4.8)$$

where f is defined in the proof of Theorem 3.1 in [12] as

$$f(s) = \frac{1}{\alpha} \xi(s)^\alpha m'(\log \xi(s)) \quad (4.9)$$

and m is a $\log(c^{1/\alpha})$ -periodic function given by

$$\zeta(s) = x^\alpha m(\log x). \quad (4.10)$$

Theorem 4.3 Assume that (4.5) holds, τ is given as in Lemma 4.2 and define $q(t) = -\alpha\tau'(t)$. Then the point source solutions $p(x, t)$ of the semi-fractional diffusion Eq. (4.2) of order $\alpha \in (1, 2)$ in space and $h(x, t)$ of the semi-fractional Eq. (4.3) of order $1/\alpha \in (\frac{1}{2}, 1)$ in time are equivalent, i.e. $p(x, t) = \alpha^{-1}h(x, t)$ for all $x > 0$ and $t > 0$.

Proof As shown in [12] it remains to carry out Laplace inversion of (4.8). Clearly, $\frac{\partial}{\partial x} \tilde{h}(x, s)$ is the Laplace transform of $\frac{\partial}{\partial x} h(x, t)$. Moreover, the first part on the left-hand side of (4.8) is the Laplace transform of the semi-fractional derivative in time $\frac{\partial^{1/\alpha}}{\partial_{d,\tau} t^{1/\alpha}} h(x, t)$ as argued in [12]. Since both derivatives are continuous functions of the time variable, Laplace inversion of the left-hand side of (4.8) yields the sum of these derivatives.

We may rewrite the right-hand side of (4.8) as follows. From (4.10) we obtain

$$\zeta'(t) = \alpha x^{\alpha-1} m(\log x) + x^{\alpha-1} m'(\log x)$$

and thus we have by (4.10) and $\zeta(\xi(s)) = s$

$$\zeta'(\xi(s)) = \alpha \frac{1}{\xi(s)} s + \frac{1}{\xi(s)} \xi(s)^\alpha m'(\log \xi(s)) = \alpha \frac{1}{\xi(s)} s + \alpha \frac{1}{\xi(s)} f(s),$$

where the last equality follows from (4.9). This shows that

$$f(s) = \frac{1}{\alpha} \zeta'(\xi(s)) \xi(s) - s$$

and the right-hand side of (4.8) can be rewritten as

$$-\frac{1}{s} \frac{f(s)}{s + f(s)} \xi(s) \delta(x) = -\frac{\alpha}{s} \frac{f(s)}{\zeta'(\xi(s))} \delta(x) = -\frac{1}{s} \xi(s) \delta(x) + \alpha \frac{1}{\zeta'(\xi(s))} \delta(x). \quad (4.11)$$

Note that the first part $-\frac{1}{s} \xi(s) \delta(x)$ also appears on the left-hand side of (4.8); cf. Remark 4.4. Moreover, since $\frac{1}{s}$ is the Laplace transform of $1_{(0,\infty)}(t)$, the function $\frac{1}{s} \xi(s)$ is the Laplace transform of the Riemann–Liouville semi-fractional derivative of $1_{(0,\infty)}(t)$ which is

$$\frac{d}{dt} \int_0^t 1_{(0,\infty)}(t-s) s^{-1/\alpha} \tau(\log(s)) ds = t^{-1/\alpha} \tau(\log(t)).$$

Now from $\zeta(\xi(s)) = s$ we get $\zeta'(\xi(s)) \cdot \xi'(s) = 1$ and hence it follows from (4.7) and integration by parts

$$\begin{aligned} \frac{1}{\zeta'(\xi(s))} &= \xi'(s) = \int_0^\infty e^{-st} t d\mu(t) \\ &= \left[-e^{-st} t^{1-1/\alpha} \tau(\log t) \right]_{t=0}^\infty + \int_0^\infty (-s e^{-st} t + e^{-st}) t^{-1/\alpha} \tau(\log t) dt \end{aligned}$$

$$= -s \int_0^\infty e^{-st} t^{1-1/\alpha} \tau(\log t) dt + \int_0^\infty e^{-st} t^{-1/\alpha} \tau(\log t) dt$$

which is the Laplace transform of

$$\begin{aligned} & -\frac{d}{dt} \left(t^{1-1/\alpha} \tau(\log t) \right) + t^{-1/\alpha} \tau(\log t) \\ &= -(1 - \frac{1}{\alpha}) t^{-1/\alpha} \tau(\log t) - t^{-1/\alpha} \tau'(\log t) + t^{-1/\alpha} \tau(\log t) \\ &= t^{-1/\alpha} \left(\frac{1}{\alpha} \tau(\log t) - \tau'(\log t) \right). \end{aligned}$$

Putting things together, due to continuous differentiability of τ proven in Lemma 4.2, Laplace inversion of the right-hand side of (4.8) yields

$$\left(-t^{-1/\alpha} \tau(\log t) + \alpha t^{-1/\alpha} \left(\frac{1}{\alpha} \tau(\log t) - \tau'(\log t) \right) \right) \delta(x) = -\alpha t^{-1/\alpha} \tau'(\log t) \delta(x)$$

concluding the proof. $\square$

Remark 4.4 Note that in the stable case we have that τ is constant and thus $\tau' \equiv 0$. Hence we recover space-time duality for fractional diffusions in [11] as a special case. Further note that with (4.11) the term $-\frac{1}{s} \xi(s) \delta(x)$ appears on both sides of (4.8) and can be cancelled. After cancellation the first part on the left-hand side of (4.8) is the Laplace transform of the Riemann–Liouville semi-fractional derivative of $h(x, t)$ and as in the proof of Theorem 4.3 we may rewrite (4.3) as

$$\mathbb{D}_{d,\tau}^{1/\alpha} h(x, t) + \frac{\partial}{\partial x} h(x, t) = t^{-1/\alpha} \left(\tau(\log t) - \alpha \tau'(\log t) \right) \delta(x),$$

where the semi-fractional derivative acts on the time variable and now the inhomogeneity on the right-hand side is non-negative by Lemma A.1.

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Appendix

We first show an elementary result connecting admissible functions with the representation in Corollary 7.4.4 of [21].

Lemma A.1 *Let $\theta : \mathbb{R} \rightarrow (0, \infty)$ be a periodic function and $\alpha > 0$. Then the following statements are equivalent.*

- (i) $\theta(y + \delta) \leq e^{\alpha\delta}\theta(y)$ for all $y > 0$ and $\delta > 0$.
- (ii) $\theta(y + \delta) \leq e^{\alpha\delta}\theta(y)$ for all $y \in \mathbb{R}$ and $\delta \geq 0$.
- (iii) $\theta(y - \delta) \geq e^{-\alpha\delta}\theta(y)$ for all $y \in \mathbb{R}$ and $\delta \geq 0$.
- (iv) *The mapping $t \mapsto t^{-\alpha}\theta(\log t)$ is non-increasing for $t > 0$.*

If θ is additionally differentiable, then each statement (i)–(iv) is equivalent to

- (v) $\theta'(y) \leq \alpha\theta(y)$ for all $y \in \mathbb{R}$.

Proof “(i) $\Rightarrow$ (ii)”: Note that (ii) is trivially true for $\delta = 0$. If θ has period $p > 0$ and $y \leq 0$, choose $k \in \mathbb{N}$ such that $y + kp > 0$. Then $\theta(y + \delta) = \theta(y + kp + \delta) \leq e^{\alpha\delta}\theta(y + kp) = e^{\alpha\delta}\theta(y)$.

“(ii) $\Rightarrow$ (iii)”: Write $z = y - \delta$ then $\theta(y - \delta) = \theta(z) \geq e^{-\alpha\delta}\theta(z + \delta) = e^{-\alpha\delta}\theta(y)$.

“(iii) $\Rightarrow$ (iv)”: For $0 < s < t$ write $t = e^y$ and $s = e^{y-\delta}$ for some $y \in \mathbb{R}$ and $\delta > 0$. Then $s^{-\alpha}\theta(\log s) = e^{-\alpha(y-\delta)}\theta(y - \delta) \geq e^{-\alpha y}\theta(y) = t^{-\alpha}\theta(\log t)$.

“(iv) $\Rightarrow$ (i)”: Write $y = \log t$ and $\delta = \log \gamma$ for some $t, \gamma > 1$. Then $\theta(y + \delta) = \theta(\log(\gamma t)) = (\gamma t)^{\alpha}(\gamma t)^{-\alpha}\theta(\log(\gamma t)) \leq (\gamma t)^{\alpha}t^{-\alpha}\theta(\log t) = e^{\alpha\delta}\theta(y)$.

“(ii) $\Rightarrow$ (v)”: For fixed $y \in \mathbb{R}$ we have $\frac{\theta(y+\delta)-\theta(y)}{\delta} \leq \frac{e^{\alpha\delta}-1}{\delta}\theta(y)$ for all $\delta > 0$ and for differentiable θ as $\delta \downarrow 0$ we get $\theta'(y) \leq \alpha\theta(y)$.

“(v) $\Rightarrow$ (iv)”: We have $\frac{d}{dt}(t^{-\alpha}\theta(\log t)) = t^{-\alpha-1}(-\alpha\theta(\log t) + \theta'(\log t)) \leq 0$. $\square$

Now we derive a formula for higher order derivatives of inverse functions used in Sect. 3 for which we couldn’t find a suitable reference.

Lemma A.2 *Let $A, B \subseteq \mathbb{R}$ be open and let $f : A \rightarrow B$ be an invertible $C^n(A)$ -function for some $n \in \mathbb{N}$ such that $f'(x) \neq 0$ for all $x \in A$. Then $f^{-1} : B \rightarrow A$ is of class $C^n(B)$ and for $n \geq 2$ we have*

$$(f^{-1})^{(n)}(x) = -\frac{1}{f'(f^{-1}(x))} \sum_{k_1+2k_2+\dots+(n-1)k_{n-1}=n} \frac{n!}{k_1! \cdots k_{n-1}!} f^{(k_1+\dots+k_{n-1})}(f^{-1}(x)) \cdot \prod_{j=1}^{n-1} \left(\frac{(f^{-1})^{(j)}(x)}{j!} \right)^{k_j}.$$

Remark A.3 For $n = 1$ it is well known that $(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$. For $n = 2$ we have $k_1 = 2$ in Lemma A.3 so that

$$(f^{-1})''(x) = -\frac{1}{f'(f^{-1}(x))} f''(f^{-1}(x)) \cdot ((f^{-1})'(x))^2 = -\frac{f''(f^{-1}(x))}{(f'(f^{-1}(x)))^3}.$$

Iterating this procedure leads to a formula for $(f^{-1})^{(n)}$ given in [38] not involving the derivatives of lower order $(f^{-1})', \dots, (f^{-1})^{(n-1)}$.

Proof of Lemma A.2 We use Faà di Bruno's formula of higher order chain rule

$$\frac{d^n}{dx^n} f(g(x)) = \sum_{k_1+2k_2+\dots+nk_n=n} \frac{n!}{k_1! \dots k_n!} f^{(k_1+\dots+k_n)}(g(x)) \prod_{j=1}^n \left(\frac{(g^{(j)}(x))}{j!} \right)^{k_j}.$$

For $j = n$ we must have $k_n = 1$ and $k_1 = \dots = k_{n-1} = 0$, otherwise $k_n = 0$ if $k_j \geq 1$ for some $j \in \{1, \dots, n-1\}$. Hence we get

$$\begin{aligned} \frac{d^n}{dx^n} f(g(x)) &= \sum_{k_1+2k_2+\dots+(n-1)k_{n-1}=n} \frac{n!}{k_1! \dots k_{n-1}!} f^{(k_1+\dots+k_{n-1})}(g(x)) \prod_{j=1}^{n-1} \left(\frac{(g^{(j)}(x))}{j!} \right)^{k_j} \\ &\quad + f'(g(x)) g^{(n)}(x). \end{aligned}$$

If $g = f^{-1}$, we know that $\frac{d^n}{dx^n} f(f^{-1}(x)) = 0$ for $n \geq 2$ which directly leads to the stated formula for $(f^{-1})^{(n)}$. $\square$

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