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On the Correlation of Critical Points and Angular Trispectrum for Random Spherical Harmonics

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Abstract

We prove a Central Limit Theorem for the critical points of random spherical harmonics, in the high-energy limit. The result is a consequence of a deeper characterization of the total number of critical points, which are shown to be asymptotically fully correlated with the sample trispectrum, i.e. the integral of the fourth Hermite polynomial evaluated on the eigenfunctions themselves. As a consequence, the total number of critical points and the nodal length are fully correlated for random spherical harmonics, in the high-energy limit.

Keywords Random fields · Critical points · Wiener chaos expansion · Spherical harmonics · Berry's cancellation phenomenon

Mathematics Subject Classification (2020) 60G60 · 62M15 · 53C65 · 42C10 · 33C55

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1 Introduction and Main Results

1.1 Random Spherical Harmonics and Sample Polyspectra

It is well-known that the eigenvalues $\{-\lambda_\ell\}_{\ell=0,1,2,\dots}$ of the Helmholtz equation

$$\Delta_{\mathbb{S}^2} f + \lambda_\ell f = 0, \quad \Delta_{\mathbb{S}^2} = \frac{\partial^2}{\partial\theta^2} + \cot\theta \frac{\partial}{\partial\varphi} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}, \quad \varphi \in [0, 2\pi), \ \theta \in [0, \pi],$$

on the two-dimensional sphere $\mathbb{S}^2$, are of the form $\lambda_\ell = \ell(\ell+1)$ for some integer $\ell \geq 1$. For any given eigenvalue $-\lambda_\ell$, the corresponding eigenspace is the $(2\ell+1)$-dimensional space of spherical harmonics of degree ℓ; we can choose an arbitrary L^2-orthonormal basis $\{Y_{\ell m}(.)\}_{m=-\ell,\dots,\ell}$ and consider random eigenfunctions of the form

$$f_\ell(x) = \frac{\sqrt{4\pi}}{\sqrt{2\ell+1}} \sum_{m=-\ell}^{\ell} a_{\ell m} Y_{\ell m}(x), \quad x \in \mathbb{S}^2,$$

where the coefficients $\{a_{\ell m}\}$ are independent, standard Gaussian variables if the basis is chosen to be real-valued; the standardization is such that $\mathrm{Var}(f_\ell(x)) = 1$, and the representation is invariant with respect to the choice of any specific basis $\{Y_{\ell m},\ m = -\ell, \dots, \ell\}$. The random fields $\{f_\ell(x),\ x \in \mathbb{S}^2\}$ are isotropic, meaning that the probability laws of $f_\ell(\cdot)$ and $f_\ell^g(\cdot) := f_\ell(g\cdot)$ are the same for any rotation $g \in SO(3)$; they are also centred and Gaussian, and from the addition theorem for spherical harmonics (see [18], Equation (3.42)) the covariance function is given by,

$$\mathbb{E}[f_\ell(x) f_\ell(y)] = P_\ell(\cos d(x, y)),$$

where P_ℓ are the usual Legendre polynomials, $\cos d(x, y) = \cos\theta_x \cos\theta_y + \sin\theta_x \sin\theta_y \cos(\varphi_x - \varphi_y)$ is the spherical geodesic distance between x and y and (θ_x, φ_x), (θ_y, φ_y) are the spherical coordinates of x and y, respectively.
In this paper, we shall be concerned with the number of critical points of $f_\ell(\cdot)$, defined as usual as

$$\mathcal{N}_\ell^c = \left\{ x \in \mathbb{S}^2 : \nabla f_\ell(x) = 0 \right\};$$

it was shown in [24] (see also [8]) that we have

$$\mathbb{E}[\mathcal{N}_\ell^c] = \frac{2}{\sqrt{3}}\ell(\ell+1) + O(1),$$

whereas (see [10]) the variance of $\mathcal{N}_\ell^c$ is such that

$$\mathrm{Var}(\mathcal{N}_\ell^c) = \frac{\ell^2 \log \ell}{3^3\pi^2} + O(\ell^2), \text{ as } \ell \to \infty. \tag{1.1}$$

We now study the limiting distribution of the fluctuations around the expected value. First recall that the sequence of Hermite polynomials $H_q(u)$ is defined by

$$H_q(u) := (-1)^q \frac{1}{\phi(u)} \frac{d^q \phi(u)}{du^q}, \quad \phi(u) := \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{u^2}{2}\right\},$$

so that

$$H_0(u) = 1, \quad H_1(u) = u, \, H_2(u) = u^2 - 1, \quad H_3(u) = u^3 - 3u,$$
$$H_4(u) = u^4 - 6u^2 + 3, \dots$$

We refer to [25] for a detailed discussion of Hermite polynomials, their properties and their ubiquitous role in the analysis of Gaussian processes. Below we shall also exploit the sequence of (random) sample polyspectra, which we define as (see, e.g. [8,17,20,22,23,27])

$$h_{\ell;q} := \int_{\mathbb{S}^2} H_q(f_\ell(x))\mathrm{d}x.$$

It is readily checked that $h_{\ell;0} = 4\pi$ and $h_{\ell;1} = 0$, for all ℓ; we also have $\mathbb{E}\left[h_{\ell;q}\right] = 0$, for all $q = 1, 2, \dots$ As far as variances are concerned, we have that (see [22,23,27])

$$\mathrm{Var}\left(h_{\ell;2}\right) = (4\pi)^2 \frac{2}{2\ell+1}, \quad \mathrm{Var}\left(h_{\ell;4}\right) = \frac{576 \log \ell}{\ell^2} + O(\ell^{-2}),$$
$$\mathrm{Var}\left(h_{\ell;q}\right) = \frac{c_q}{\ell^2} + o(\ell^{-2}),$$

for $q = 3, 5, 6 \dots$, where

$$c_q := \int_0^\infty J_0(\psi)^q \psi \mathrm{d}\psi, \quad J_0(\psi) = \sum_{k=0}^\infty \frac{(-1)^k x^{2k}}{(k!)^2 2^{2k}},$$

and $J_0(.)$ is the usual Bessel function of the first kind.

1.2 Main Results

Our first main result in this paper is to show that the number of critical points and the sample trispectrum $\left\{h_{\ell;4}\right\}$ are asymptotically fully correlated: as $\ell \to \infty$

$$\lim_{\ell\to\infty} \rho^2(\mathcal{N}_\ell^c, h_{\ell;4}) := \lim_{\ell\to\infty} \frac{\mathrm{Cov}^2(\mathcal{N}_\ell^c, h_{\ell;4})}{\mathrm{Var}(\mathcal{N}_\ell^c)\mathrm{Var}(h_{\ell;4})} = 1. \tag{1.2}$$

In fact, our result is slightly sharper than that, as shown in the statement of Theorem 1.1. Recall first that the variance for the total number of critical points was computed in [10] to be asymptotic to (1.1). Let us now introduce the random sequence

$$\mathcal{A}_\ell = -\frac{\lambda_\ell}{2^3 3^2 \sqrt{3}\pi} \int_{\mathbb{S}^2} H_4(f_\ell(x)) \mathrm{d}x = -\frac{\lambda_\ell}{2^3 3^2 \sqrt{3}\pi} h_{\ell;4},$$

for which it is readily seen that

$$\mathbb{E}\left[\mathcal{A}_\ell\right] = 0, \qquad \lim_{\ell \to \infty} \frac{\mathrm{Var}(\mathcal{N}_\ell^c)}{\mathrm{Var}(\mathcal{A}_\ell)} = 1,$$

because

$$\begin{aligned}\mathrm{Var}(\mathcal{A}_\ell) &= \frac{\lambda_\ell^2}{2^6 3^5 \pi^2} \mathrm{Var}(h_{\ell;4}) = \frac{\lambda_\ell^2}{2^6 3^5 \pi^2} \left\{ \frac{576 \log \ell}{\ell^2} + O(\ell^{-2}) \right\} \\ &= \frac{\ell^2 \log \ell}{3^3 \pi^2} + O(\ell^2), \quad \text{as } \ell \to 0.\end{aligned}$$

It is convenient to write

$$\widetilde{\mathcal{A}}_\ell = \frac{\mathcal{A}_\ell}{\sqrt{\mathrm{Var}(\mathcal{A}_\ell)}}.$$

We can now formulate the following

Theorem 1.1 *As* $\ell \to \infty$

$$\rho(\mathcal{N}_\ell^c, \mathcal{A}_\ell) = \frac{\mathrm{Cov}(\mathcal{N}_\ell^c, \mathcal{A}_\ell)}{\sqrt{\mathrm{Var}(\mathcal{N}_\ell^c)\mathrm{Var}(\mathcal{A}_\ell)}} \to 1,$$

and hence

$$\frac{\mathcal{N}_\ell^c - \mathbb{E}\left[\mathcal{N}_\ell^c\right]}{\sqrt{\mathrm{Var}(\mathcal{N}_\ell^c)}} = \widetilde{\mathcal{A}}_\ell + o_p(1).$$

As a consequence of the previous theorem, for $\ell \to \infty$, we have that (1.2) holds, so that the total number of critical points is fully correlated in the limit with $\left\{h_{\ell;4}\right\}$. The limiting distribution of $\left\{h_{\ell;4}\right\}$ was already studied in [23], where it was shown that a (quantitative version of the) Central Limit Theorem holds. Our next main result hence follows immediately; recall first that the Wasserstein distance between the probability distributions of two random variables (X, Y) is defined by

$$d_W(X, Y) := \sup_{h \in \mathrm{Lip}(1)} |\mathbb{E}h(X) - \mathbb{E}h(Y)|,$$

where

$$\mathrm{Lip}(1) := \left\{ h : \mathbb{R} \to \mathbb{R} \text{ such that } \left| \frac{h(x) - h(y)}{x - y} \right| \leq 1 \text{ for all } x \neq y \right\}.$$

Theorem 1.2 *As $\ell \to \infty$, for Z a standard Gaussian variable, we have that*

$$\lim_{\ell\to\infty} d_W \left(\frac{\mathcal{N}_\ell^c - \mathbb{E}\left[\mathcal{N}_\ell^c\right]}{\sqrt{\mathrm{Var}\left(\mathcal{N}_\ell^c\right)}}, Z \right) = 0,$$

and hence

$$\frac{\mathcal{N}_\ell^c - \mathbb{E}\left[\mathcal{N}_\ell^c\right]}{\sqrt{\mathrm{Var}\left(\mathcal{N}_\ell^c\right)}} \overset{law}{\to} Z.$$

Remark 1.3 The previous theorems include actually two separate results, namely:

(i) The asymptotic behaviour of the total number of critical points is dominated by its projection on the fourth-order chaos term (see Sect. 2);
(ii) The projection on the fourth-order chaos can be expressed simply in terms of the fourth-order Hermite polynomial, evaluated on the eigenfunctions $\{f_\ell\}$, without the need to compute Hermite polynomials evaluated on the first and second derivatives of $\{f_\ell\}$, despite the fact that the latter do appear in the Kac–Rice formula and they are not negligible in terms of asymptotic variance.

As we shall discuss in the following section, both these findings have analogous counterparts in the behaviour of the boundary and nodal length, as investigated, i.e. in [21]. Due to the nature of our proof, we do not see any easy path to extend our results to non-smooth distances such as the Kolmogorov one, apart from the bound one obtains by standard inequalities between probability metrics such as (C.2.6) in the book [25]. We recall here also that the Wasserstein distance can be equivalently expressed in terms of couplings as

$$d_W(X, Y) = \inf_{\gamma\in\Gamma(X,Y)} \mathbb{E}_\gamma\left[|X - Y|\right],$$

where $\Gamma(X, Y)$ is the set of all bivariate probability measures having the same marginal laws as X and Y.

1.3 Discussion: Correlation Between Critical Points and Nodal Length

The results in our paper should be compared with a recent stream of the literature which has investigated the relationship between geometric features of random spherical harmonics and sample polyspectra. The first results in this area are due to [22], which studied the excursion area of $\{f_\ell\}$ above a threshold $u \in \mathbb{R}$ (which we label $\mathcal{L}_2(u; \ell)$), and showed that it is asymptotically dominated (after centering) by a term of the form $-u\phi(u)h_{\ell;2}/2$; in particular, they showed that

(i) There is full correlation, in the high-energy limit, between $h_{\ell;2}$ and the excursion area, for all $u \neq 0$;

(ii) For $u = 0$ (the case of the so-called Defect) this leading term vanishes, and the asymptotic behaviour is radically different: all the odd-order chaoses of order greater or equal to 3 are correlated with the excursion area.

The same pattern of behaviour was later established for the boundary length $\mathcal{L}_1(u;\ell)$ (for $u \neq 0$) (see also [33]) and the Euler characteristic $\mathcal{L}_0(u;\ell)$ (for $u \neq 0, \pm 1$, see, i.e. [11] and the references therein), thus covering the behaviour of all three Lipschitz–Killing Curvatures (see [1]). More explicitly, we have that, as $\ell \to \infty$, (see, i.e. [11])

$$\begin{aligned}
\mathcal{L}_0(u;\ell) - \mathbb{E}[\mathcal{L}_0(u;\ell)] &= \frac{1}{2}\frac{\lambda_\ell}{2} H_2(u)H_1(u)\phi(u)\frac{1}{2\pi}h_{\ell;2} + o_p(\ell^{3/2}) \\
&= \frac{1}{2}\frac{\lambda_\ell}{2}(u^3 - u)\phi(u)\frac{1}{2\pi}h_{\ell;2} + o_p(\ell^{3/2}), \\
\mathcal{L}_1(u;\ell) - \mathbb{E}[\mathcal{L}_1(u;\ell)] &= \frac{1}{2}\sqrt{\frac{\lambda_\ell}{2}}\sqrt{\frac{\pi}{8}} H_1^2(u)\phi(u)h_{\ell;2} + o_p(\ell^{1/2}) \\
&= \frac{1}{2}\sqrt{\frac{\lambda_\ell}{2}}\sqrt{\frac{\pi}{8}} u^2\phi(u)h_{\ell;2} + o_p(\ell^{1/2}), \\
\mathcal{L}_2(u;\ell) - \mathbb{E}[\mathcal{L}_2(u;\ell)] &= \frac{1}{2}H_1(u)\phi(u)h_{\ell;2} + o_p(\ell^{-1/2}) \\
&= \frac{1}{2}u\phi(u)h_{\ell;2} + o_p(\ell^{-1/2}),
\end{aligned}$$

whence

$$\begin{aligned}
&\lim_{\ell\to\infty} \rho^2(h_{\ell;2}; \mathcal{L}_2(u;\ell)) = \lim_{\ell\to\infty} \rho^2(h_{\ell;2}; \mathcal{L}_1(u;\ell)) = 1, \text{ for } u \neq 0, \\
&\lim_{\ell\to\infty} \rho^2(h_{\ell;2}; \mathcal{L}_0(u;\ell)) = 1, \text{ for } u \neq 0, \pm 1,
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{\ell\to\infty} \rho^2(\mathcal{L}_0(u;\ell), \mathcal{L}_1(u;\ell)) = \lim_{\ell\to\infty} \rho^2(\mathcal{L}_0(u;\ell), \mathcal{L}_2(u;\ell)) \\
&\quad = \lim_{\ell\to\infty} \rho^2(\mathcal{L}_1(u;\ell), \mathcal{L}_2(u;\ell)) = 1, \text{ for } u \neq 0, \pm 1.
\end{aligned}$$

Loosely speaking, it can be concluded that these three Lipschitz–Killing curvatures are asymptotically proportional to $h_{\ell;2}$ in the high-energy limit for $u \neq 0$ (and also for $u \neq \pm 1$ in the case of Euler characteristics), and thus, they are fully correlated at different thresholds and among themselves.

These results were extended in [12] to critical values over the interval I. More precisely, let $I \subseteq \mathbb{R}$ be any interval in the real line; we are interested in the number of critical points of f_ℓ with value in I:

$$\mathcal{N}_\ell^c(I) = \#\{x \in \mathbb{S}^2 : f_\ell(x) \in I, \nabla f_\ell(x) = 0\}.$$

For the expectation, it was shown in [8] that for every interval $I \subseteq \mathbb{R}$ we have, as $\ell \to \infty$,

$$\mathbb{E}[\mathcal{N}_\ell^c(I)] = \frac{2}{\sqrt{3}}\lambda_\ell \int_I \pi_1^c(t)\mathrm{d}t + O(1), \quad \pi_1^c(t) = \frac{\sqrt{3}}{\sqrt{8\pi}}(2e^{-t^2} + t^2 - 1)e^{-\frac{t^2}{2}};$$

moreover, for I such that

$$\nu^c(I) := \left[\int_I p_3^c(t)\mathrm{d}t\right]^2 \neq 0, \quad p_3^c(t) = \frac{1}{\sqrt{8\pi}}e^{-\frac{3}{2}t^2}\left[2 - 6t^2 - e^{t^2}(1 - 4t^2 + t^4)\right],$$

we have that

$$\lim_{\ell\to\infty} \rho^2(h_{\ell;2}; \mathcal{N}_\ell^c(I)) = 1.$$

More precisely, it was shown in [12], that

$$\mathcal{N}_\ell^c(I) - \mathbb{E}[\mathcal{N}_\ell^c(I)] = *\left[\ell^{3/2}\int_I p_3^c(t)\mathrm{d}t\right] \times \frac{h_{\ell;2}}{\sqrt{\mathrm{Var}(h_{\ell;2})}} + o_p\left(\sqrt{\mathrm{Var}(\mathcal{N}_\ell^c(I))}\right), \text{ as } \ell \to \infty.$$

We call I nondegenerate if and only if

$$\int_I p_3^c(t)\mathrm{d}t \neq 0.$$

For instance, semi-intervals $I = [u, \infty)$ with $u \neq 0$ are nondegenerate. As a consequence, for the same range of values of u, we have that

$$\lim_{\ell\to\infty} \rho^2(\mathcal{L}_a(u;\ell), \mathcal{N}_\ell^c([u,\infty))) = 1, \text{ for } a = 0, 1, 2.$$

For $I = [0, \infty)$ or $\mathbb{R}$ (corresponding to the total number of critical points), the leading constant $\nu^c(I)$ vanishes, and, accordingly, the order of magnitude of the variance is smaller than ℓ^3; indeed, as $\ell \to \infty$ (see [10]),

$$\mathrm{Var}(\mathcal{N}_\ell^c) = \frac{1}{3^3\pi^2}\ell^2\log\ell + O(\ell^2).$$

This behaviour is again similar to what was found for $\mathcal{L}_1(0;\ell)$ (the *nodal length* of random spherical harmonics), for which it was shown in [32] that

$$\mathrm{Var}(\mathcal{L}_1(0;\ell)) = \frac{1}{128}\log\ell + O(1);$$

actually our expression here differs from the one in [32] by a factor 1/4, because $\mathcal{L}_1(0;\ell)$ is equivalent to half the *nodal length* of random spherical harmonics considered in that paper. It was later shown in [21] that the following asymptotic equivalence holds:

$$\mathcal{L}_1(0;\ell) - \mathbb{E}[\mathcal{L}_1(0;\ell)] = -\frac{1}{4}\sqrt{\frac{\lambda_\ell}{2}}\frac{1}{4!}h_{\ell;4} + o_p(\sqrt{\log \ell}),$$

consistent with the computation of the variance in [32], because (see [23])

$$\mathrm{Var}(h_{\ell;4}) = \frac{576\log\ell}{\ell^2} + O(\ell^{-2}), \text{ as } \ell \to \infty.$$

In particular, we have that

$$\lim_{\ell\to\infty} \rho^2(\mathcal{L}_1(0;\ell); h_{\ell;4}) = 1.$$

Our results in this paper show that the asymptotic behaviour of the total number of critical points (i.e. $I = \mathbb{R}$) is dominated by exactly the same component as the nodal length, and indeed

$$\lim_{\ell\to\infty} \rho^2(\mathcal{L}_1(0;\ell); \mathcal{N}_\ell^c) = \lim_{\ell\to\infty} \rho^2(\mathcal{N}_\ell^c; h_{\ell;4}) = 1.$$

Summing up, the literature so far has established the full correlation of Lipschitz–Killing curvatures and critical values among themselves and with the sequence $\{h_{\ell;2}\}$ for nondegenerate values of the threshold parameter u. Here, we show that in the degenerate cases ($u = -\infty$, 0 for critical points) full-correlation still exists between nodal length and critical points, as both are proportional to the sample trispectrum $h_{\ell;4} = \int_{\mathbb{S}^2} H_4(f_\ell(x))\mathrm{d}x$. The correlation is positive, which is to say that the realization that corresponds to a higher number of critical points are those where longer nodal lines are going to be observed. Heuristically, it can be conjectured that a higher number of critical points will typically correspond to a higher number of nodal components, and hence, nodal length will be as well larger than average. One cautious note is needed here: whereas the correlation converges to unity, it does so only at a logarithmic rate, so it may not be simple to visualize this effect by simulations with values of ℓ in the order of a few hundreds. On the contrary, the correlation for values of the threshold u different from zero occurs with rate ℓ^{-1} and shows up very neatly in simulations.

A number of other papers have investigated the geometry of random eigenfunctions on the sphere and on the torus in the last few years. Among these, we recall [22] and [23] for the excursion area and the Defect; [7,15,19] and [5] for the nodal length/volume of arithmetic random waves; [13] for the number of intersections of random eigenfunctions; [21] for the nodal length of random spherical harmonics; [26] for the nodal length of Berry's random waves on the plane; [28] and [29] for nodal intersections. Zeroes of random trigonometric polynomials have been considered, for instance, by [2–4] and the references therein. Moreover, [6,31] and [30] study the fluctuations of nodal length and excursion area over subsets of the torus and of the sphere (see also [14] for random eigenfunctions on more general manifolds); the asymptotic behaviour of the number of critical points of spherical harmonics restricted to subsets or shrinking domains is currently under investigation. Similarly to results which were established in the above-mentioned papers, we expect asymptotic full correlation to hold between the "local" number of critical points and the "local sample trispectrum", as introduced in [31]; the investigation of this conjecture is still ongoing.

1.4 Plan of the Paper

In Sect. 2, we present some background material on Kac–Rice techniques, Wiener chaos expansions and the relevant covariant matrices for our (covariant) gradient and Hessian. The proof of our main results is given in Sect. 3, where we show that the total number of critical points is asymptotically fully correlated with the integral of the fourth-Hermite polynomial evaluated on the eigenfunctions themselves. The projection coefficients on Wiener chaoses that we shall need are only three, and their computation is collected in Sect. 4. In Sect. 5, we consider the terms in the chaos expansion with odd index Hermite polynomials. The technical computations are in "Appendix A".

2 Chaos Expansion

As discussed in [8,9,11] and [12], by means of Kac–Rice formula, the number of critical points can be formally written as

$$\mathcal{N}_\ell^c = \int_{\mathbb{S}^2} |\det \nabla^2 f_\ell(x)| \boldsymbol{\delta}(\nabla f_\ell(x)) \mathrm{d}x,$$

where the identity holds both almost surely (using, i.e. the Federer's coarea formula, see [1]), and in the L^2 sense, i.e.

$$\mathcal{N}_\ell^c = \lim_{\varepsilon \to 0} \int_{\mathbb{S}^2} |\det \nabla^2 f_\ell(x)| \boldsymbol{\delta}_\varepsilon(\nabla f_\ell(x)) \mathrm{d}x = \lim_{\varepsilon \to 0} \mathcal{N}_{\ell,\varepsilon}^c$$

for

$$\mathcal{N}_{\ell,\varepsilon}^c := \int_{\mathbb{S}^2} |\det \nabla^2 f_\ell(x)| \boldsymbol{\delta}_\varepsilon(\nabla f_\ell(x)) \mathrm{d}x, \quad \boldsymbol{\delta}_\varepsilon(.) := \frac{1}{(2\varepsilon)^2} \mathbb{I}_{[-\varepsilon,\varepsilon]^2}(\cdot, \cdot).$$

The validity of this limit, in the $L^2(\Omega)$ sense, was shown in [11,12]. The approach for the proof is to start from the Wiener chaos expansion

$$\mathcal{N}_\ell^c = \sum_{q=0}^{\infty} \mathrm{Proj}[\mathcal{N}_\ell^c | q] =: \sum_{q=0}^{\infty} \mathcal{N}_\ell^c[q], \tag{2.1}$$

where $\{\mathcal{N}_\ell^c[q]\}$ denotes the *chaos-component of order* q, or equivalently the projection of $\mathcal{N}_\ell^c$ on the qth order chaos components, which we shall describe below. In order to define and compute more explicitly these chaos components, let us introduce the differential operators

$$\partial_{1;x} = \frac{\partial}{\partial\theta}\bigg|_{\theta=\theta_x,\varphi=\varphi_x}, \quad \partial_{2;x} = \frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}\bigg|_{\theta=\theta_x,\varphi=\varphi_x},$$
$$\partial_{11;x} = \frac{\partial^2}{\partial\theta^2}\bigg|_{\theta=\theta_x,\varphi=\varphi_x}, \quad \partial_{12;x} = \frac{1}{\sin\theta}\frac{\partial^2}{\partial\theta\partial\varphi}\bigg|_{\theta=\theta_x,\varphi=\varphi_x}, \quad \partial_{22;x}$$
$$= \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\bigg|_{\theta=\theta_x,\varphi=\varphi_x}.$$

Covariant gradient and Hessian follow the standard definitions, discussed for instance in [11]; here, we simply recall that

$$\nabla f_\ell(x) = (\partial_1 f_\ell(x), \partial_2 f_\ell(x)),$$
$$\nabla^2 f_\ell(x) = \begin{pmatrix} \partial_{11} f_\ell(x) & \partial_{12} f_\ell(x) - \cot\theta_x \partial_2 f_\ell(x) \\ \partial_{12} f_\ell(x) - \cot\theta_x \partial_2 f_\ell(x) & \partial_{22} f_\ell(x) + \cot\theta_x \partial_1 f_\ell(x) \end{pmatrix},$$
$$vec\nabla^2 f_\ell(x) = (\partial_{11} f_\ell(x), \partial_{12} f_\ell(x) - \cot\theta_x \partial_2 f_\ell(x), \partial_{22} f_\ell(x) + \cot\theta_x \partial_1 f_\ell(x)).$$

We can then introduce the 5×1 vector $(\nabla f_\ell(x), vec\nabla^2 f_\ell(x))$; its covariance matrix σ_ℓ is constant with respect to x and it is computed in [12]. It can be written in the partitioned form

$$\sigma_\ell = \begin{pmatrix} a_\ell & b_\ell \\ b_\ell^{\mathrm{T}} & c_\ell \end{pmatrix},$$

where the superscript T denotes transposition, and

$$a_\ell = \begin{pmatrix} \frac{\lambda_\ell}{2} & 0 \\ 0 & \frac{\lambda_\ell}{2} \end{pmatrix}, \quad b_\ell = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad c_\ell = \frac{\lambda_\ell^2}{8}\begin{pmatrix} 3 - \frac{2}{\lambda_\ell} & 0 & 1 + \frac{2}{\lambda_\ell} \\ 0 & 1 - \frac{2}{\lambda_\ell} & 0 \\ 1 + \frac{2}{\lambda_\ell} & 0 & 3 - \frac{2}{\lambda_\ell} \end{pmatrix}.$$

Let us recall that the *Cholesky decomposition* of a Hermitian positive-definite matrix A takes the form $A = \Lambda\Lambda^{\mathrm{T}}$, where Λ is a lower triangular matrix with real and positive diagonal entries, and Λ^{T} denotes the conjugate transpose of Λ. It is well-known that every Hermitian positive-definite matrix (and thus also every real-valued symmetric positive-definite matrix) admits a unique Cholesky decomposition.

By an explicit computation, it is possible to show that the Cholesky decomposition of σ_ℓ takes the form $\sigma_\ell = \Lambda_\ell \Lambda_\ell^t$, where

$$\Lambda_\ell = \begin{pmatrix} \frac{\sqrt{\lambda_\ell}}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & \frac{\sqrt{\lambda_\ell}}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & \frac{\sqrt{\lambda_\ell}\sqrt{3\lambda_\ell-2}}{2\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & \frac{\sqrt{\lambda_\ell}\sqrt{\lambda_\ell-2}}{2\sqrt{2}} & 0 \\ 0 & 0 & \frac{\sqrt{\lambda_\ell}(\lambda_\ell+2)}{2\sqrt{2}\sqrt{3\lambda_\ell-2}} & 0 & \frac{\lambda_\ell\sqrt{\lambda_\ell-2}}{\sqrt{3\lambda_\ell-2}} \end{pmatrix} =: \begin{pmatrix} \tau_1 & 0 & 0 & 0 & 0 \\ 0 & \tau_1 & 0 & 0 & 0 \\ 0 & 0 & \tau_3 & 0 & 0 \\ 0 & 0 & 0 & \tau_4 & 0 \\ 0 & 0 & \tau_2 & 0 & \tau_5 \end{pmatrix};$$

in the last expression, for notational simplicity we have omitted the dependence of the τ_is on ℓ. matrix is block diagonal, because under isotropy the gradient components are independent from the Hessian when evaluated at the same point. We can hence define a five-dimensional standard Gaussian vector $Y(x) = (Y_1(x), Y_2(x), Y_3(x), Y_4(x), Y_5(x))$ with independent components such that

$$\begin{aligned} &(\partial_1 f_\ell(x), \partial_2 f_\ell(x), \partial_{11} f_\ell(x), \partial_{12} f_\ell(x) - \cot\theta_x \partial_2 f_\ell(x), \partial_{22} f_\ell(x) + \cot\theta_x \partial_1 f_\ell(x)) \\ &\quad = \Lambda_\ell Y(x) \\ &\quad = (\tau_1 Y_1(x), \tau_1 Y_2(x), \tau_3 Y_3(x), \tau_4 Y_4(x), \tau_5 Y_5(x) + \tau_2 Y_3(x)). \end{aligned}$$

Note that asymptotically

$$\tau_1 \sim \frac{\ell}{\sqrt{2}}, \quad \tau_2 \sim \frac{\ell^2}{\sqrt{24}}, \quad \tau_3 \sim \sqrt{\frac{3}{8}}\ell^2, \quad \tau_4 \sim \frac{\ell^2}{\sqrt{8}}, \quad \tau_5 \sim \frac{\ell^2}{\sqrt{3}},$$

where (as usual) $a_\ell \sim b_\ell$ means that the ratio between the left- and right-hand side tends to unity as $\ell \to \infty$. Hence,

$$\begin{aligned} Y_a(x) &= \frac{\sqrt{2}}{\sqrt{\lambda_\ell}} \partial_{a;x} f_\ell(x), \quad a = 1, 2, \\ Y_3(x) &= \frac{2\sqrt{2}}{\sqrt{\lambda_\ell}\sqrt{3\lambda_\ell - 2}} \partial_{11;x} f_\ell(x), \\ Y_4(x) &= \frac{2\sqrt{2}}{\sqrt{\lambda_\ell}\sqrt{\lambda_\ell - 2}} \partial_{21;x}, \\ Y_5(x) &= \frac{\sqrt{3\lambda_\ell - 2}}{\lambda_\ell\sqrt{\lambda_\ell - 2}} \partial_{22;x} f_\ell(x) - \frac{\lambda_\ell + 2}{\lambda_\ell\sqrt{\lambda_\ell - 2}\sqrt{3\lambda_\ell - 2}} \partial_{11;x} f_\ell(x). \end{aligned}$$

Thus, we obtain

$$\begin{aligned} \mathcal{N}_\ell^c &= \lim_{\varepsilon\to 0} \int_{\mathbb{S}^2} |\det \nabla^2 f_\ell(x)| \boldsymbol{\delta}_\varepsilon(\nabla f_\ell(x)) \mathrm{d}x \\ &= \lim_{\varepsilon\to 0} \int_{\mathbb{S}^2} |\partial_{11;x} f_\ell(x) \partial_{22;x} f_\ell(x) - (\partial_{12;x} f_\ell(x))^2| \boldsymbol{\delta}_\varepsilon(\partial_{1;x} f_\ell(x), \partial_{2;x} f_\ell(x)) \mathrm{d}x \end{aligned}$$

$$= \lim_{\varepsilon\to 0} \int_{\mathbb{S}^2} |\tau_3 Y_3(x)(\tau_5 Y_5(x) + \tau_2 Y_3(x)) - (\tau_4 Y_4(x))^2| \boldsymbol{\delta}_\varepsilon(\tau_1 Y_1(x), \tau_1 Y_2(x)) \mathrm{d}x$$

$$= \lim_{\varepsilon\to 0} \lambda_\ell^2 \int_{\mathbb{S}^2} \left| \frac{\tau_3\tau_5}{\lambda_\ell^2} Y_3(x) Y_5(x) + \frac{\tau_2\tau_3}{\lambda_\ell^2} Y_3^2(x) - \frac{\tau_4^2}{\lambda_\ell^2} Y_4^2(x) \right| \boldsymbol{\delta}_\varepsilon(\tau_1 Y_1(x), \tau_1 Y_2(x)) \mathrm{d}x,$$

where

$$\frac{\tau_3\tau_5}{\lambda_\ell^2} \sim \frac{1}{\sqrt{8}}, \quad \frac{\tau_2\tau_3}{\lambda_\ell^2} \sim \frac{1}{8}, \quad \frac{\tau_4^2}{\lambda_\ell^2} \sim \frac{1}{8}.$$

The qth order chaos is the space generated by the L^2-completion of linear combinations of the form $H_{q_1}(Y_1) \cdots H_{q_5}(Y_5)$, with $q_1 + q_2 + \cdots + q_5 = q$ (see, i.e. [25]); in other words, it is the linear span of cross-product of Hermite polynomials computed in the independent random variables Y_i, $i = 1, 2, \ldots 5$, which generate the gradient and Hessian of f_ℓ. In particular, the fourth-order chaos can be written in the following form:

$$\begin{aligned} &\mathcal{N}_\ell^c[4] \\ &= \lambda_\ell \Bigg[\frac{1}{2!2!} \sum_{i=2}^{5} \sum_{j=1}^{i-1} h_{ij} \int_{\mathbb{S}^2} H_2(Y_i(x)) H_2(Y_j(x)) \mathrm{d}x + \frac{1}{4!} \sum_{i=1}^{5} k_i \int_{\mathbb{S}^2} H_4(Y_i(x)) \mathrm{d}x \\ &\quad + \frac{1}{3!} \sum_{\substack{i,j=1 \\ i\neq j}}^{5} g_{ij} \int_{\mathbb{S}^2} H_3(Y_i(x)) H_1(Y_j(x)) \mathrm{d}x \\ &\quad + \frac{1}{2} \sum_{\substack{i,j,k=1 \\ i\neq j\neq k}}^{5} p_{ijk} \int_{\mathbb{S}^2} H_2(Y_i(x)) H_1(Y_j(x)) H_1(Y_k(x)) \mathrm{d}x \\ &\quad + \sum_{\substack{i,j,k,l=1 \\ i\neq j\neq k\neq l}}^{5} q_{ijkl} \int_{\mathbb{S}^2} H_1(Y_i(x)) H_1(Y_j(x)) H_1(Y_k(x)) H_1(Y_l(x)) \mathrm{d}x \Bigg], \end{aligned} \tag{2.2}$$

where

$$\begin{aligned} h_{ij} &= \lim_{\varepsilon\to 0} \lambda_\ell \, \mathbb{E}\left[\left| \frac{\tau_3\tau_5}{\lambda_\ell^2} Y_3 Y_5 + \frac{\tau_2\tau_3}{\lambda_\ell^2} Y_3^2 - \frac{\tau_4^2}{\lambda_\ell^2} Y_4^2 \right| \boldsymbol{\delta}_\varepsilon(\tau_1 Y_1, \tau_1 Y_2) H_2(Y_i) H_2(Y_j) \right], \\ k_i &= \lim_{\varepsilon\to 0} \lambda_\ell \, \mathbb{E}\left[\left| \frac{\tau_3\tau_5}{\lambda_\ell^2} Y_3 Y_5 + \frac{\tau_2\tau_3}{\lambda_\ell^2} Y_3^2 - \frac{\tau_4^2}{\lambda_\ell^2} Y_4^2 \right| \boldsymbol{\delta}_\varepsilon(\tau_1 Y_1, \tau_1 Y_2) H_4(Y_i) \right], \\ g_{ij} &= \lim_{\varepsilon\to 0} \lambda_\ell \, \mathbb{E}\Bigg[\left| \frac{\tau_3\tau_5}{\lambda_\ell^2} Y_3 Y_5 + \frac{\tau_2\tau_3}{\lambda_\ell^2} Y_3^2 - \frac{\tau_4^2}{\lambda_\ell^2} Y_4^2 \right| \boldsymbol{\delta}_\varepsilon(\tau_1 Y_1, \tau_1 Y_2) \\ &\qquad H_3(Y_i(x)) H_1(Y_j(x)) \Bigg], \end{aligned}$$

$$p_{ijk} = \lim_{\varepsilon\to 0} \lambda_\ell \, \mathbb{E}\left[\left| \frac{\tau_3\tau_5}{\lambda_\ell^2} Y_3 Y_5 + \frac{\tau_2\tau_3}{\lambda_\ell^2} Y_3^2 - \frac{\tau_4^2}{\lambda_\ell^2} Y_4^2 \right| \boldsymbol{\delta}_\varepsilon(\tau_1 Y_1, \tau_1 Y_2) \right.$$
$$\left. H_2(Y_i(x)) H_1(Y_j(x)) H_1(Y_k(x)) \right],$$
$$q_{ijkl} = \lim_{\varepsilon\to 0} \lambda_\ell \, \mathbb{E}\left[\left| \frac{\tau_3\tau_5}{\lambda_\ell^2} Y_3 Y_5 + \frac{\tau_2\tau_3}{\lambda_\ell^2} Y_3^2 - \frac{\tau_4^2}{\lambda_\ell^2} Y_4^2 \right| \boldsymbol{\delta}_\varepsilon(\tau_1 Y_1, \tau_1 Y_2) \right.$$
$$\left. H_1(Y_i(x)) H_1(Y_j(x)) H_1(Y_k(x)) H_1(Y_l(x)) \right].$$

The projection coefficients k_i, h_{ij}, g_{ij}, p_{ijk}, and q_{ijkl} are constant with respect to ℓ.

3 Proof of Theorems 1.1 and 1.2

3.1 Proof of Theorem 1.1

In this section, we give the proof of our main result. Let us start with the $L^2(\Omega)$, ε-approximation to the number of critical points [12]

$$\mathcal{N}_\ell^c = \lim_{\varepsilon\to 0} \mathcal{N}_{\ell,\varepsilon}^c, \quad \mathcal{N}_{\ell,\varepsilon}^c = \int_{\mathbb{S}^2} |\det\nabla^2 f_\ell(x)| \boldsymbol{\delta}_\varepsilon(\nabla f_\ell(x)) \mathrm{d}x,$$

for every $x \in \mathbb{S}^2$ we define

$$|\det\nabla^2 f_\ell(x)| \boldsymbol{\delta}_\varepsilon(\nabla f_\ell(x)) = \sum_{q=0}^{\infty} \psi_\ell^\varepsilon(x;q) =: \psi_\ell^\varepsilon(x).$$

By continuity of the inner product in $L^2(\Omega)$, we write

$$\begin{aligned} \operatorname{Cov}(\mathcal{N}_\ell^c, h_{\ell;4}) &= \lim_{\varepsilon\to 0} \operatorname{Cov}(\mathcal{N}_{\ell,\varepsilon}^c, h_{\ell;4}) \\ &= \lim_{\varepsilon\to 0} \mathbb{E}\left[\int_{\mathbb{S}^2} |\det\nabla^2 f_\ell(x)| \boldsymbol{\delta}_\varepsilon(\nabla f_\ell(x)) \mathrm{d}x \int_{\mathbb{S}^2} H_4(f_\ell(y)) \mathrm{d}y \right] \\ &= \lim_{\varepsilon\to 0} \mathbb{E}\left[\int_{\mathbb{S}^2} \sum_{q=0}^{\infty} \psi_\ell^\varepsilon(x;q) \mathrm{d}x \int_{\mathbb{S}^2} H_4(f_\ell(y)) \mathrm{d}y \right]. \end{aligned}$$

Now, note that both $\psi_\ell^\varepsilon(x)$ and $H_4(f_\ell(y))$ are isotropic processes on $\mathbb{S}^2$, and hence, we have

$$\begin{aligned} &\mathbb{E}\left[\int_{\mathbb{S}^2} \sum_{q=0}^{\infty} \psi_\ell^\varepsilon(x;q) \mathrm{d}x \int_{\mathbb{S}^2} H_4(f_\ell(y)) \mathrm{d}y \right] \\ &= \mathbb{E}\left[\int_{\mathbb{S}^2} \lim_{Q\to\infty} \sum_{q=0}^{Q} \psi_\ell^\varepsilon(x;q) \mathrm{d}x \int_{\mathbb{S}^2} H_4(f_\ell(y)) \mathrm{d}y \right] \end{aligned}$$

$$= \lim_{Q\to\infty} \mathbb{E}\left[\int_{\mathbb{S}^2} \sum_{q=0}^{Q} \psi_\ell^\varepsilon(x;q)\mathrm{d}x \int_{\mathbb{S}^2} H_4(f_\ell(y))\mathrm{d}y\right]$$

by continuity of covariances. Moreover, because all integrands are finite-order polynomials we have

$$\begin{aligned}
&\lim_{Q\to\infty} \mathbb{E}\left[\int_{\mathbb{S}^2} \sum_{q=0}^{Q} \psi_\ell^\varepsilon(x;q)\mathrm{d}x \int_{\mathbb{S}^2} H_4(f_\ell(y))\mathrm{d}y\right] \\
&= \lim_{Q\to\infty} \sum_{q=0}^{Q} \int_{\mathbb{S}^2}\int_{\mathbb{S}^2} \mathbb{E}\left[\psi_\ell^\varepsilon(x;q) H_4(f_\ell(y))\right]\mathrm{d}x\mathrm{d}y \\
&= \int_{\mathbb{S}^2}\int_{\mathbb{S}^2} \mathbb{E}\left[\psi_\ell^\varepsilon(x;4) H_4(f_\ell(y))\right]\mathrm{d}x\mathrm{d}y \\
&= 16\pi^2 \int_0^{\pi/2} \mathbb{E}\left[\psi_\ell^\varepsilon(\overline{x};4) H_4(f_\ell(y(\phi)))\right]\sin\phi\mathrm{d}\phi,
\end{aligned}$$

where in the last steps we used orthogonality of Wiener chaoses and isotropy; we take $\overline{x} = (\frac{\pi}{2}, 0)$ and $y(\phi) = (\frac{\pi}{2}, \phi)$. More explicitly, the previous argument allows us to perform our argument on *the equator*, where θ is fixed to $\pi/2$. Note that

$$\begin{aligned}
\psi_\ell^\varepsilon(\overline{x};4) = \lambda_\ell\Big[&\frac{1}{2!2!}\sum_{i=2}^{5}\sum_{j=1}^{i-1} h_{ij}^\varepsilon H_2(Y_i(\bar{x}))H_2(Y_j(\bar{x})) + \frac{1}{4!}\sum_{i=1}^{5} k_i^\varepsilon H_4(Y_i(\bar{x})) \\
&+ \frac{1}{3!}\sum_{\substack{i,j=1\\ i\neq j}}^{5} g_{ij}^\varepsilon H_3(Y_i(x))H_1(Y_j(x)) \\
&+ \frac{1}{2}\sum_{\substack{i,j,k=1\\ i\neq j\neq k}}^{5} q_{ijk}^\varepsilon H_2(Y_i(x))H_1(Y_j(x))H_1(Y_k(x)) \\
&+ \sum_{\substack{i,j,k,l=1\\ i\neq j\neq k\neq l}}^{5} p_{ijkl}^\varepsilon H_1(Y_i(x))H_1(Y_j(x))H_1(Y_k(x))H_1(Y_l(x))\Big],
\end{aligned}$$

and hence

$$\begin{aligned}
&\mathrm{Cov}(\mathcal{N}_\ell^c, h_{\ell;4}) \\
&= 16\pi^2 \lim_{\varepsilon\to 0}\int_0^{\pi/2} \mathbb{E}\left[\psi_\ell^\varepsilon(\overline{x};4) H_4(f_\ell(y(\phi)))\right]\sin\phi\mathrm{d}\phi \\
&= 16\pi^2\lambda_\ell \frac{1}{2!2!}\sum_{i=2}^{5}\sum_{j=1}^{i-1}\{\lim_{\varepsilon\to 0} h_{ij}^\varepsilon\}
\end{aligned}$$

$$
\begin{aligned}
&\int_0^{\pi/2} \mathbb{E}\left[H_2(Y_i(\bar{x}))H_2(Y_j(\bar{x}))H_4(f_\ell(y(\phi)))\right]\sin\phi \mathrm{d}\phi \\
&+ 16\pi^2\lambda_\ell \frac{1}{4!}\sum_{i=1}^{5}\{\lim_{\varepsilon\to 0} k_i^\varepsilon\} \int_0^{\pi/2} \mathbb{E}\left[H_4(Y_i(\bar{x}))H_4(f_\ell(y(\phi)))\right]\sin\phi \mathrm{d}\phi \\
&+ 16\pi^2\lambda_\ell \frac{1}{3!}\sum_{i\neq j}\{\lim_{\varepsilon\to 0} g_{ij}^\varepsilon\} \int_0^{\pi/2} \mathbb{E}\left[H_3(Y_i(\bar{x}))H_1(Y_j(\bar{x}))H_4(f_\ell(y(\phi)))\right]\sin\phi \mathrm{d}\phi \\
&+ 16\pi^2\lambda_\ell \frac{1}{2}\sum_{i\neq j\neq k}\{\lim_{\varepsilon\to 0} p_{ijk}^\varepsilon\} \\
&\int_0^{\pi/2} \mathbb{E}\left[H_2(Y_i(\bar{x}))H_1(Y_j(\bar{x}))H_1(Y_k(\bar{x}))H_4(f_\ell(y(\phi)))\right]\sin\phi \mathrm{d}\phi \\
&+ 16\pi^2\lambda_\ell \sum_{i\neq j\neq k\neq l}\{\lim_{\varepsilon\to 0} q_{ijkl}^\varepsilon\} \\
&\int_0^{\pi/2} \mathbb{E}\left[H_1(Y_i(\bar{x}))H_1(Y_j(\bar{x}))H_1(Y_k(\bar{x}))H_1(Y_l(\bar{x}))H_4(f_\ell(y(\phi)))\right]\sin\phi \mathrm{d}\phi.
\end{aligned}
$$

We shall show below that the asymptotic behaviour of $\mathrm{Cov}(\mathcal{N}_\ell^c, h_{\ell;4})$ is dominated by three terms corresponding to

$$
\int_0^{\pi/2} \mathbb{E}\left[H_4(Y_2(\bar{x}))H_4(f_\ell(y(\phi)))\right]\sin\phi \mathrm{d}\phi, \quad \int_0^{\pi/2} \mathbb{E}\left[H_4(Y_5(\bar{x}))H_4(f_\ell(y(\phi)))\right] \sin\phi \mathrm{d}\phi,
$$

and

$$
\int_0^{\pi/2} \mathbb{E}\left[H_2(Y_2(\bar{x}))H_2(Y_5(\bar{x}))H_4(f_\ell(y(\phi)))\right]\sin\phi \mathrm{d}\phi.
$$

The computation of these leading covariances is given in the three Lemmas A.1–A.3 to follow, where it is shown that

$$
\begin{aligned}
&\int_0^{\pi/2} \mathbb{E}\left[H_4(Y_2(\bar{x}))H_4(f_\ell(y(\phi)))\right]\sin\phi \mathrm{d}\phi = 4!\frac{2\cdot 3}{\pi^2}\frac{\log \ell}{\ell^2} + O(\ell^{-2}), \\
&\int_0^{\pi/2} \mathbb{E}\left[H_4(Y_5(\bar{x}))H_4(f_\ell(y(\phi)))\right]\sin\phi \mathrm{d}\phi = 4!\frac{3^3}{2\pi^2}\frac{\log \ell}{\ell^2} + O(\ell^{-2}), \\
&\int_0^{\pi/2} \mathbb{E}\left[H_2(Y_2(\bar{x}))H_2(Y_5(\bar{x}))H_4(f_\ell(y(\phi)))\right]\sin\phi \mathrm{d}\phi \\
&\quad = 4!\frac{3}{\pi^2}\frac{\log \ell}{\ell^2} + O(\ell^{-2}).
\end{aligned}
$$

All the remaining terms in $\mathrm{Cov}(\mathcal{N}_\ell^c, h_{\ell;4})$ are shown to be $O(\ell^{-2})$ or smaller in Sect. 5 and Lemmas A.4–A.9 below. From Proposition 4.1, we know that

$$k_2 = \lim_{\varepsilon\to 0} k_2^\varepsilon = \frac{1}{\pi}\frac{\sqrt{3}}{2}, \quad k_5 = \lim_{\varepsilon\to 0} k_5^\varepsilon = -\frac{1}{\pi}\frac{7}{3^3\sqrt{3}}, \qquad h_{25} := \lim_{\varepsilon\to 0} h_{25}^\varepsilon = -\frac{1}{\pi}\frac{1}{3\sqrt{3}}.$$

Substituting and after some straightforward algebra, one obtains

$$\begin{aligned}
&\mathrm{Cov}(\mathcal{N}_\ell^c, h_{\ell;4})\\
&= \lambda_\ell \left\{4\pi^2 h_{25} 4! \frac{3}{\pi^2}\frac{\log\ell}{\ell^2} + \frac{2}{3}\pi^2 k_2 4! 2^2 \frac{3}{\pi^2}\frac{\log\ell}{2\ell^2} + \frac{2}{3}\pi^2 k_5 4! 3^2 \frac{3}{\pi^2}\frac{\log\ell}{2\ell^2} + O(\ell^{-2})\right\}\\
&= \frac{\lambda_\ell}{3\sqrt{3}} 4! \frac{\log\ell}{\ell^2}\frac{1}{\pi} \times \{-12 + 18 - 7\} + O(1)\\
&= -\frac{\lambda_\ell}{3\sqrt{3}} 4! \frac{\log\ell}{\ell^2}\frac{1}{\pi} + O(1).
\end{aligned}$$

Because

$$\mathcal{A}_\ell = -\frac{\lambda_\ell}{2^3 3^2\sqrt{3}\pi}\int_{\mathbb{S}^2} H_4(f_\ell(x))\mathrm{d}x = -\frac{\lambda_\ell}{2^3 3^2 \sqrt{3}\pi} h_{\ell;4},$$

we find

$$\mathrm{Cov}(\mathcal{N}_\ell^c, \mathcal{A}_\ell) = \frac{\lambda_\ell^2}{2^3 3^2 \sqrt{3}\pi}\frac{1}{3\sqrt{3}} 4! \frac{\log\ell}{\ell^2}\frac{1}{\pi} + O(1) = \frac{\ell^2\log\ell}{3^3\pi^2} + O(1),$$

so that our proof of our main theorem is completed, recalling that, as $\ell\to\infty$,

$$\mathrm{Var}(\mathcal{N}_\ell^c) \sim \mathrm{Var}(\mathcal{A}_\ell) = \frac{\ell^2\log\ell}{3^3\pi^2} + O(1).$$

Remark 3.1 A consequence of Theorem 1.1 is that, as $\ell\to\infty$,

$$\mathrm{Var}\left(\mathrm{Proj}[\mathcal{N}_\ell^c|4]\right) = \frac{\ell^2\log\ell}{3^3\pi^2} + O(\ell^2),$$

so that

$$\lim_{\ell\to\infty}\frac{\mathrm{Var}\left(\mathrm{Proj}[\mathcal{N}_\ell^c|4]\right)}{\mathrm{Var}\left(\mathcal{N}_\ell^c\right)} = 1.$$

Note that by orthogonality, we have

$$\mathrm{Var}\left(\mathcal{N}_\ell^c\right) = \sum_{q=0}^{\infty} \mathrm{Var}\left(\mathrm{Proj}[\mathcal{N}_\ell^c|q]\right) = \mathrm{Var}\left(\mathrm{Proj}[\mathcal{N}_\ell^c|4]\right) + \sum_{k=1}^{\infty} \mathrm{Var}\left(\mathrm{Proj}[\mathcal{N}_\ell^c|4+2k]\right),$$

where the odd terms in the expansion vanish by symmetry arguments, $\mathrm{Var}\left(\mathrm{Proj}[\mathcal{N}_\ell^c|0]\right) = 0$ is obvious and $\mathrm{Var}\left(\mathrm{Proj}[\mathcal{N}_\ell^c|2]\right) = 0$ was shown in [12]. Hence, we have the bound

$$\sum_{k=1}^{\infty} \mathrm{Proj}[\mathcal{N}_\ell^c|4+2k] = o(\ell^2 \log \ell).$$

In fact, by a careful investigation of the asymptotic behaviour of higher-order chaotic projections it seems possible to establish the slightly stronger result

$$\sum_{k=1}^{\infty} \mathrm{Proj}[\mathcal{N}_\ell^c|4+2k] = O(\ell^2);$$

we omit this investigation for brevity's sake.

3.2 Proof of Theorem 1.2

It was shown in [23] that

$$\lim_{\ell\to\infty} d_W \left(\frac{h_{\ell;4}}{\sqrt{\mathrm{Var}\left(h_{\ell;4}\right)}}, Z \right) = 0;$$

the result then follows from Theorem 1.1 and the triangle inequality

$$\begin{aligned}
&\lim_{\ell\to\infty} d_W \left(\frac{\mathcal{N}_\ell^c - \mathbb{E}\left[\mathcal{N}_\ell^c\right]}{\sqrt{\mathrm{Var}\left(\mathcal{N}_\ell^c\right)}}, Z \right) \\
&\quad\leq \lim_{\ell\to\infty} d_W \left(\frac{\mathcal{N}_\ell^c - \mathbb{E}\left[\mathcal{N}_\ell^c\right]}{\sqrt{\mathrm{Var}\left(\mathcal{N}_\ell^c\right)}}, \frac{\mathcal{A}_\ell}{\sqrt{\mathrm{Var}\left(\mathcal{A}_\ell\right)}} \right) \\
&\qquad + \lim_{\ell\to\infty} d_W \left(\frac{\mathcal{A}_\ell}{\sqrt{\mathrm{Var}\left(\mathcal{A}_\ell\right)}}, Z \right) \\
&\quad= \lim_{\ell\to\infty} d_W \left(\frac{\mathcal{N}_\ell^c - \mathbb{E}\left[\mathcal{N}_\ell^c\right]}{\sqrt{\mathrm{Var}\left(\mathcal{N}_\ell^c\right)}}, \frac{\mathcal{A}_\ell}{\sqrt{\mathrm{Var}\left(\mathcal{A}_\ell\right)}} \right)
\end{aligned}$$

$$+ \lim_{\ell\to\infty} d_W \left(\frac{h_{\ell;4}}{\sqrt{\mathrm{Var}\left(h_{\ell;4}\right)}}, Z \right)$$
$$= 0.$$

4 Evaluation of the Projection Coefficients h_{52}, k_2, k_5

In this section, we evaluate the three projection coefficients in the Wiener chaos expansion which are required for the completion of our arguments.

Proposition 4.1 *We have that*

$$k_2 = \frac{1}{\pi}\frac{\sqrt{3}}{2}, \quad k_5 = -\frac{1}{\pi}\frac{7}{3^3\sqrt{3}}, \quad h_{25} = -\frac{1}{\pi}\frac{1}{3\sqrt{3}}.$$

Proof Let us recall first the following simple result

$$\varphi_a := \lim_{\varepsilon\to 0} \mathbb{E}[H_a(Y)\delta_\varepsilon(\tau_1 Y)] = \begin{cases} \frac{1}{\sqrt{2\pi}\tau_1} & a=0, \\ 0 & a=1, \\ -\frac{1}{\sqrt{2\pi}\tau_1} & a=2, \\ \frac{3}{\sqrt{2\pi}\tau_1} & a=4. \end{cases}$$

Indeed, for example

$$\lim_{\varepsilon\to 0} \mathbb{E}[H_4(Y)\delta_\varepsilon(\tau_1 Y)] = \lim_{\varepsilon\to 0} \mathbb{E}[(Y^4 - 6Y^2 + 3)\delta_\varepsilon(\tau_1 Y)] = \frac{3}{\sqrt{2\pi}\tau_1},$$

since

$$\lim_{\varepsilon\to 0} \mathbb{E}[Y^n \delta_\varepsilon(\tau_1 Y)] = \lim_{\varepsilon\to 0} \int_{-\infty}^{\infty} y^n \delta_\varepsilon(\tau_1 y)\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}\mathrm{d}y = \begin{cases} \frac{1}{\sqrt{2\pi}\tau_1} & n=0, \\ 0 & n=1,2,3\dots \end{cases}$$

Now, note that

$$\begin{aligned} k_2 &= \lim_{\varepsilon\to 0} k_2^\varepsilon \\ &= \lambda_\ell\, \mathbb{E}\left[\left|\frac{1}{2\sqrt{2}}Y_3Y_5 + \frac{1}{8}Y_3^2 - \frac{1}{8}Y_4^2\right|\right]\varphi_0\,\varphi_4 \\ &= \frac{3}{\pi}\mathbb{E}\left[\left|\frac{1}{2\sqrt{2}}Y_3Y_5 + \frac{1}{8}Y_3^2 - \frac{1}{8}Y_4^2\right|\right], \\ k_5 &= \lim_{\varepsilon\to 0} k_5^\varepsilon \\ &= \lambda_\ell\, \mathbb{E}\left[\left|\frac{1}{2\sqrt{2}}Y_3Y_5 + \frac{1}{8}Y_3^2 - \frac{1}{8}Y_4^2\right| H_4(Y_5)\right]\varphi_0^2 \end{aligned}$$

$$= \frac{1}{\pi}\, \mathbb{E}\left[\left|\frac{1}{2\sqrt{2}}Y_3Y_5 + \frac{1}{8}Y_3^2 - \frac{1}{8}Y_4^2\right| H_4(Y_5)\right],$$

and

$$\begin{aligned} h_{52} &= \lim_{\varepsilon\to 0} h_{25}^{\varepsilon} \\ &= \lambda_\ell\, \mathbb{E}\left[\left|\frac{1}{2\sqrt{2}}Y_3Y_5 + \frac{1}{8}Y_3^2 - \frac{1}{8}Y_4^2\right| H_2(Y_5)\right] \varphi_0\, \varphi_2 \\ &= -\frac{1}{\pi}\mathbb{E}\left[\left|\frac{1}{2\sqrt{2}}Y_3Y_5 + \frac{1}{8}Y_3^2 - \frac{1}{8}Y_4^2\right| H_2(Y_5)\right]. \end{aligned}$$

Let us introduce the change of variables

$$Z_1 = \sqrt{3}Y_3, \quad Z_2 = Y_4, \quad Z_3 = \frac{\sqrt{8}}{\sqrt{3}}Y_5 + \frac{1}{\sqrt{3}}Y_3,$$

so that (Z_1, Z_2, Z_2) is a centred Gaussian vector with covariance matrix

$$\begin{pmatrix} 3 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 3 \end{pmatrix},$$

and we can write

$$\frac{1}{2\sqrt{2}}Y_3Y_5 + \frac{1}{8}Y_3^2 - \frac{1}{8}Y_4^2 = \frac{1}{8}(Z_1Z_3 - Z_2^2).$$

The coefficient k_2 can be computed as follows: write

$$\begin{aligned} \mathbb{E}\left[\left|\frac{1}{\sqrt{8}}Y_3Y_5 + \frac{1}{8}Y_3^2 - \frac{1}{8}Y_4^2\right|\right] &= \frac{1}{8}\mathbb{E}\left[\left|Z_1Z_3 - Z_2^2\right|\right] \\ &= \frac{1}{8}\mathbb{E}\left[\left|(Z_1, Z_2, Z_3)^{\mathrm{T}} A (Z_1, Z_2, Z_3)\right|\right], \end{aligned}$$

where the symmetric matrix A given by

$$A = \begin{pmatrix} 0 & 0 & 1/2 \\ 0 & -1 & 0 \\ 1/2 & 0 & 0 \end{pmatrix};$$

we apply [16], Theorem 2.1, to obtain

$$\begin{aligned} &\mathbb{E}\left[\left|Z_1Z_3 - Z_2^2\right|\right] \\ &= \frac{2}{\pi}\int_0^\infty \frac{1}{t^2}\left\{1 - \frac{1}{2\sqrt{\det(I - 2it\Sigma A)}} - \frac{1}{2\sqrt{\det(I + 2it\Sigma A)}}\right\} \mathrm{d}t \end{aligned}$$

where we have that

$$\det(I - 2it\Sigma A) = 1 + 12t^2 + 16it^3, \quad \det(I + 2it\Sigma A) = 1 + 12t^2 - 16it^3,$$

and computing the integral with Cauchy methods for residuals, we get

$$\mathbb{E}\left[\left|Z_1 Z_3 - Z_2^2\right|\right] = \frac{4}{\sqrt{3}},$$

and

$$k_2 = \frac{3}{\pi}\mathbb{E}\left[\left|\frac{1}{\sqrt{8}}Y_3 Y_5 + \frac{1}{8}Y_3^2 - \frac{1}{8}Y_4^2\right|\right] = \frac{\sqrt{3}}{2\pi},$$

as claimed. We introduce now the following notation

$$\mathcal{I}_r = \mathbb{E}[|Z_1 Z_3 - Z_2^2|(Z_1 - 3Z_3)^r]$$

for $r = 0, 2, 4$, so that,

$$\begin{aligned} h_{52} &= -\frac{1}{\pi}\frac{1}{8}\mathbb{E}\left[\left|Z_1 Z_3 - Z_2^2\right| H_2\left(\frac{1}{\sqrt{8}\sqrt{3}}(3Z_3 - Z_1)\right)\right] \\ &= -\frac{1}{\pi}\frac{1}{8}\mathbb{E}\left[\left|Z_1 Z_3 - Z_2^2\right| \left(\frac{1}{\sqrt{8}\sqrt{3}}(3Z_3 - Z_1)\right)^2\right] + \frac{1}{\pi}\frac{1}{8}\mathbb{E}\left[\left|Z_1 Z_3 - Z_2^2\right|\right] \\ &= -\frac{1}{\pi}\frac{1}{3\cdot 2^6}\mathcal{I}_2 + \frac{1}{\pi}\frac{1}{2^3}\mathcal{I}_0, \end{aligned}$$

and

$$\begin{aligned} k_5 &= \frac{1}{\pi}\;\frac{1}{8}\mathbb{E}\left[\left|Z_1 Z_3 - Z_2^2\right| H_4\left(\frac{1}{\sqrt{8}\sqrt{3}}(3Z_3 - Z_1)\right)\right] \\ &= \frac{1}{\pi}\frac{1}{2^9\cdot 3^2}\mathcal{I}_4 - \frac{1}{\pi}\;\frac{1}{2^5}\mathcal{I}_2 + \frac{1}{\pi}\;\frac{3}{2^3}\mathcal{I}_0. \end{aligned}$$

The statement follows applying the results in [10] where it is proved that

$$\mathcal{I}_2 = \frac{2^5\cdot 5}{\sqrt{3}}, \quad \mathcal{I}_4 = \frac{2^8\cdot 5^2\cdot 7}{3\sqrt{3}}.$$

□

5 Terms with Odd Index Hermite Polynomials

In this section, we prove that the terms in the 4-th chaos formula (2.2) with odd index Hermite polynomials produce in $\mathrm{Cov}(\mathcal{N}_\ell^c, h_{\ell;4})$ terms of order $O(\ell^{-2})$ and terms equal to zero. We first focus, in the following proposition, on the projection coefficients.

Proposition 5.1 *The projection coefficients g_{ij}, p_{ijk} and q_{ijkl} are such that*

- *For $i, j \neq 3, 5$, we have $g_{ij} = 0$,*
- *For $j, k \neq 1, 2, 4$, we have $p_{ijk} = 0$,*
- *We have $q_{ijkl} = 0$.*

Proof Recalling that for a odd we have

$$\lim_{\varepsilon \to 0} \mathbb{E}[H_a(Y)\delta_\varepsilon(\tau_1 Y)] = 0,$$

from this we immediately see that the coefficients g_{ij} with $i, j = 1, 2$ are all equal to zero. We consider now the coefficients g_{ij} with $i = 4$ or $j = 4$, for these coefficients we observe that the expectation with respect to the random variable Y_4 vanishes since it is expressed as the integral of an odd function. The proof of the last two points of the statement is similar. $\square$

In Lemmas A.7–A.9, we prove that the terms in $\mathrm{Cov}(\mathcal{N}_\ell^c, h_{\ell;4})$ that are multiplied by the projection coefficients not discussed in Proposition 5.1 are either zero or of order $O(\ell^{-2})$. In particular, we prove that, for $a, b = 3, 5$, $a \neq b$,

$$\int_0^{\pi/2} \mathbb{E}[H_3(Y_a(\bar{x}))H_1(Y_b(\bar{x}))H_4(f_\ell(y(\phi)))] \sin\phi \mathrm{d}\phi = O(\ell^{-2}),$$

for $a = 1, 2$,

$$\int_0^{\pi/2} \mathbb{E}[H_2(Y_a(\bar{x}))H_1(Y_3(\bar{x}))H_1(Y_5(\bar{x}))H_4(f_\ell(y(\phi)))] \sin\phi \mathrm{d}\phi = 0,$$

and that

$$\int_0^{\pi/2} \mathbb{E}[H_2(Y_4(\bar{x}))H_1(Y_3(\bar{x}))H_1(Y_5(\bar{x}))H_4(f_\ell(y(\phi)))] \sin\phi \mathrm{d}\phi = 0.$$

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Appendix A Auxiliary Lemmas

In this appendix, we collect a number of technical results that were exploited for the correlation results above. We divide the results into two subsections, collecting, respectively, dominant and subdominant terms.

A.1 Dominant Terms

In this subsection, we collect the results concerning the three dominant terms.

Lemma A.1 *As $\ell \to \infty$,*

$$\int_0^{\pi/2} \mathbb{E}\left[H_4(Y_2(\bar{x}))H_4(f_\ell(y(\phi)))\right]\sin\phi \mathrm{d}\phi = 4!2^2\frac{3}{\pi^2}\frac{\log\ell}{2\ell^2} + O(\ell^{-2}).$$

Proof Note first that, by Diagram Formula (see [18] Section 4.3.1),

$$\begin{aligned}\mathbb{E}\left[H_4(Y_2(\bar{x}))H_4(f_\ell(y(\phi)))\right] &= 4!\left\{\mathbb{E}\left[Y_2(\bar{x}))f_\ell(y(\phi))\right]\right\}^4 \\ &= 4!\left\{\mathbb{E}\left[\sqrt{\frac{2}{\ell(\ell+1)}}\partial_{2;x}f_\ell(x)f_\ell(y(\phi))\right]\right\}^4 \\ &= 4!\frac{2^2}{\ell^2(\ell+1)^2}\left\{\mathbb{E}\left[\partial_{2;x}f_\ell(x)f_\ell(y(\phi))\right]\right\}^4.\end{aligned}$$

Now, we have easily

$$\langle x, y\rangle = \cos\theta_x\cos\theta_y + \sin\theta_x\sin\theta_y\cos(\varphi_x-\varphi_y),$$

and

$$\begin{aligned}\mathbb{E}\left[\partial_{2;x}f_\ell(x)f_\ell(y(\phi))\right] &= \frac{1}{\sin\theta}\frac{\partial}{\partial\varphi_x}P_\ell(\langle x, y\rangle)\Big|_{x=\bar{x},y=y(\phi)} \\ &= -\frac{1}{\sin\theta}P'_\ell(\langle x, y\rangle)\sin\theta_x\sin\theta_y\sin(\varphi_x-\varphi_y)\Big|_{x=\bar{x},y=y(\phi)} \\ &= P'_\ell(\cos\phi)\sin(\phi).\end{aligned}$$

Thus, we obtain

$$\begin{aligned}&\int_0^{\pi/2}\mathbb{E}\left[H_4(Y_2(\bar{x}))H_4(f_\ell(y(\phi)))\right]\sin\phi \mathrm{d}\phi \\ &= 4!\frac{2^2}{\ell^2(\ell+1)^2}\int_0^{\pi/2}\left\{P'_\ell(\cos\phi)\sin(\phi)\right\}^4\sin\phi \mathrm{d}\phi \\ &= 4!\frac{2^2}{\ell^2(\ell+1)^2}\frac{3\ell^4}{\pi^2}\frac{\log\ell}{\ell^2} + O(\ell^{-2})\end{aligned}$$

$$= 4!\frac{12}{\pi^2}\frac{\log \ell}{\ell^2} + O(\ell^{-2}),$$

using, see Lemma A.10 below,

$$\int_0^{\pi/2} [P_\ell^{(r)}(\cos\phi)\sin^r \phi]^4 \sin\phi \mathrm{d}\phi = \frac{3\ell^{4r}}{\pi^2}\frac{\log \ell}{2\ell^2} + O(\ell^{4r-2}).$$

□

Lemma A.2 *As* $\ell \to \infty$

$$\int_0^{\pi/2} \mathbb{E}\,[H_4(Y_5(\bar{x}))H_4(f_\ell(y(\phi)))] \sin\phi \mathrm{d}\phi = 4!3^2\frac{3}{\pi^2}\frac{\log \ell}{2\ell^2} + O(\ell^{-2}).$$

Proof As before, note first that

$$\begin{aligned}
&\mathbb{E}\,[H_4(Y_5(\bar{x}))H_4(f_\ell(y(\phi)))] \\
&\quad = 4!\,\{\mathbb{E}\,[Y_5(\bar{x})f_\ell(y(\phi))]\}^4 \\
&\quad = 4!\left\{\mathbb{E}\left[\left(\frac{\sqrt{3\lambda_\ell - 2}}{\lambda_\ell\sqrt{\lambda_\ell - 2}}\partial_{22;x} f_\ell(x)\right.\right.\right. \\
&\qquad \left.\left.\left. - \frac{(\lambda_\ell + 2)}{\lambda_\ell\sqrt{\lambda_\ell - 2}\sqrt{3\lambda_\ell - 2}}\partial_{11;x} f_\ell(x)\right) f_\ell(y(\phi))\right]\right\}^4 \\
&\quad = 4!\left\{\mathbb{E}\left[\left(\alpha_{1\ell}\partial_{22;x} f_\ell(x) - \alpha_{2\ell}\partial_{11;x} f_\ell(x)\right) f_\ell(y(\phi))\right]\right\}^4,
\end{aligned}$$

where we wrote

$$\alpha_{1\ell} := \frac{\sqrt{3\lambda_\ell - 2}}{\lambda_\ell\sqrt{\lambda_\ell - 2}}, \qquad \alpha_{2\ell} := \frac{(\lambda_\ell + 2)}{\lambda_\ell\sqrt{\lambda_\ell - 2}\sqrt{3\lambda_\ell - 2}};$$

note that

$$\alpha_{1\ell} = \frac{\sqrt{3}}{\ell^2} + O(\frac{1}{\ell^3}), \qquad \alpha_{2\ell} = \frac{1}{\sqrt{3}\ell^2} + O(\frac{1}{\ell^3}).$$

Now,

$$\begin{aligned}
\mathbb{E}\left[\partial_{22;x} f_\ell(x) f_\ell(y(\phi))\right] &= \frac{1}{\sin^2\theta_x}\frac{\partial^2}{\partial\varphi_x^2}P_\ell(\langle x, y\rangle)\Big|_{x=\bar{x},y=y(\phi)} \\
&= P_\ell''(\cos\phi)\sin^2\phi - P_\ell'(\cos\phi)\cos\phi.
\end{aligned}$$

Likewise

$$\mathbb{E}\left[\partial_{11;x} f_\ell(x) f_\ell(y(\phi)\right] = \frac{\partial^2}{\partial\theta_x^2}P_\ell(\langle x, y\rangle)\Big|_{x=\bar{x},y=y(\phi)} = -P_\ell'(\cos\phi)\cos\phi.$$

Thus, we obtain

$$
\begin{aligned}
&\left\{\mathbb{E}\left[\left(\alpha_{1\ell}\partial_{22;x} f_\ell(x) - \alpha_{2\ell}\partial_{11;x} f_\ell(x)\right) f_\ell(y(\phi))\right]\right\}^4 \\
&\quad = \alpha_{1\ell}^4 \left\{P_\ell''(\cos\phi)\sin^2\phi + P_\ell'(\cos\phi)\cos\phi\right\}^4 \\
&\qquad + 4\alpha_{1\ell}^3\alpha_{2\ell}\left\{P_\ell''(\cos\phi)\sin^2\phi + P_\ell'(\cos\phi)\cos\phi\right\}^3 P_\ell'(\cos\phi)\cos\phi \\
&\qquad + 6\alpha_{1\ell}^2\alpha_{2\ell}^2\left\{P_\ell''(\cos\phi)\sin^2\phi + P_\ell'(\cos\phi)\cos\phi\right\}^2 \left\{P_\ell'(\cos\phi)\cos\phi\right\}^2 \\
&\qquad + 4\alpha_{1\ell}\alpha_{2\ell}^3\left\{P_\ell''(\cos\phi)\sin^2\phi + P_\ell'(\cos\phi)\cos\phi\right\} \left\{P_\ell'(\cos\phi)\cos\phi\right\}^3 \\
&\qquad + \alpha_{2\ell}^4\left\{P_\ell'(\cos\phi)\cos\phi\right\}^4 .
\end{aligned}
$$

Now, again using Lemma A.10 below,

$$
\int_0^{\pi/2}\left\{P_\ell''(\cos\phi)\sin^2\phi\right\}^4 \sin\phi \mathrm{d}\phi = \frac{3\ell^8}{2\pi^2}\frac{\log\ell}{\ell^2} + O(\ell^6)
$$

and exploiting instead Lemma A.12

$$
\begin{aligned}
&\int_0^{\pi/2}\left\{P_\ell''(\cos\phi)\sin^2\phi\right\}^k \left\{P_\ell'(\cos\phi)\cos\phi\right\}^{4-k}\sin\phi \mathrm{d}\phi \\
&\quad = O(\ell^6), \text{ for all } k = 1,\dots 4.
\end{aligned}
$$

Noting that, for $k = 1,\dots 4$,

$$
\alpha_{1\ell}^4 = \frac{3^2}{\ell^8} + O(\ell^{-7}) \text{ and } \alpha_{1\ell}^k\alpha_{2\ell}^{4-k} = O(\ell^{-8}),
$$

the proof is completed. □

Lemma A.3 *As $\ell \to \infty$,*

$$
\int_0^{\pi/2}\mathbb{E}\left[H_2(Y_2(\bar{x}))H_2(Y_5(\bar{x}))H_4(f_\ell(y(\phi)))\right]\sin\phi \mathrm{d}\phi = 4!(2\cdot 3)\frac{1}{\pi^2}\frac{\log\ell}{2\ell^2} + O(\ell^{-2}).
$$

Proof Again by Diagram Formula, we have that

$$
\begin{aligned}
&\mathbb{E}\left[H_2(Y_2(\bar{x}))H_2(Y_5(\bar{x}))H_4(f_\ell(y(\phi)))\right] \\
&\quad = 24\left\{\mathbb{E}\left[Y_2(\bar{x}) f_\ell(y(\phi))\right]\right\}^2 \left\{\mathbb{E}\left[Y_5(\bar{x}) f_\ell(y(\phi))\right]\right\}^2 \\
&\quad = 24\left\{\mathbb{E}\left[\sqrt{\frac{2}{\lambda_\ell}}\partial_{2;x} f_\ell(x) f_\ell(y(\phi)\right]\right\}^2 \left\{\mathbb{E}\left[\alpha_{1\ell}\partial_{22;x} f_\ell(x)\right.\right. \\
&\qquad \left.\left. -\alpha_{2l}\partial_{11;x} f_\ell(x) f_\ell(y(\phi))\right]\right\}^2
\end{aligned}
$$

$$= 24\frac{2}{\lambda_\ell}\left\{P_\ell'(\cos\phi)\sin(\phi)\right\}^2\Big\{\alpha_{1\ell}(P_\ell''(\cos\phi)\sin^2\phi - P_\ell'(\cos\phi)\cos\phi)$$
$$+\alpha_{2l}P_\ell'(\cos\phi)\cos\phi\Big\}^2 .$$

Now, using repeatedly Lemmas A.10 and A.12 we obtain

$$\int_0^{\pi/2}\left\{P_\ell'(\cos\phi)\sin\phi\right\}^2\Big\{\alpha_{1\ell}(P_\ell''(\cos\phi)\sin^2\phi - P_\ell'(\cos\phi)\cos\phi)$$
$$+\alpha_{2l}P_\ell'(\cos\phi)\cos\phi\Big\}^2\sin\phi d\phi$$
$$= \frac{3}{\pi^2}\frac{\log\ell}{2} + O(1) ,$$

and thus, the conclusion follows. □

A.2 Subdominant Terms

The behaviour of subdominant terms can be characterized rather easily, as follows.

Lemma A.4 *As $\ell \to \infty$, for $a = 1, 3, 4$,*

$$\int_0^{\pi/2}\mathbb{E}\left[H_4(Y_a(\bar{x}))H_4(f_\ell(y(\phi)))\right]\sin\phi d\phi = O(\ell^{-2}).$$

Proof For $a = 1$, we have that

$$\begin{aligned}\mathbb{E}\left[H_4(Y_1(\bar{x}))H_4(f_\ell(y(\phi)))\right] &= 4!\left\{\mathbb{E}\left[Y_1(\bar{x})f_\ell(y(\phi))\right]\right\}^4\\ &= 4!\left\{\mathbb{E}\left[\sqrt{\frac{2}{\ell(\ell+1)}}\partial_{1;x}f_\ell(x)f_\ell(y(\phi)\right]\right\}^4\\ &= 4!\frac{2^2}{\ell^2(\ell+1)^2}\left\{\mathbb{E}\left[\partial_{1;x}f_\ell(x)f_\ell(y(\phi))\right]\right\}^4 .\end{aligned}$$

Now, we have easily

$$\begin{aligned}&\mathbb{E}\left[\partial_{1;x}f_\ell(x)f_\ell(y(\phi))\right]\\ &= \frac{\partial}{\partial\theta}P_\ell(\langle x, y\rangle)\Big|_{x=\bar{x},y=y(\phi)}\\ &= P_\ell'(\langle x, y\rangle)\left\{-\sin\theta_x\cos\theta_y + \cos\theta_x\sin\theta_y\sin(\varphi_x-\varphi_y)\right\}\big|_{x=\bar{x},y=y(\phi)}\\ &= 0.\end{aligned}$$

Similarly

$$\mathbb{E}\left[H_4(Y_3(\bar{x}))H_4(f_\ell(y(\phi)))\right] = 4!\frac{8^2}{\lambda_\ell^2(3\lambda_\ell-2)^2}\left\{\mathbb{E}\left[\partial_{11;x}f_\ell(x)f_\ell(y(\phi)\right]\right\}^4 ,$$

and

$$\begin{aligned}
&\mathbb{E}\left[\partial_{11;x} f_\ell(x) f_\ell(y(\phi))\right] \\
&\quad = P_\ell''(\langle x, y\rangle)\left\{-\sin\theta_x\cos\theta_y + \cos\theta_x\sin\theta_y\sin(\varphi_x-\varphi_y)\right\}^2\Big|_{x=\bar{x},y=y(\phi)} \\
&\quad = + P_\ell'(\langle x, y\rangle)\left\{-\cos\theta_x\cos\theta_y - \sin\theta_x\sin\theta_y\sin(\varphi_x-\varphi_y)\right\}^2\Big|_{x=\bar{x},y=y(\phi)} \\
&\quad = -P_\ell'(\cos\phi)\sin^2\phi,
\end{aligned}$$

whence

$$\begin{aligned}
&\int_0^{\pi/2}\mathbb{E}\left[H_4(Y_a(\bar{x}))H_4(f_\ell(y(\phi)))\right]\sin\phi d\phi \\
&\quad = 4!\frac{8^2}{\lambda_\ell^2(3\lambda_\ell-2)^2}\int_0^{\pi/2}\left\{P_\ell'(\cos\phi)\sin^2\phi\right\}^4\sin\phi d\phi = O(\ell^{-6}).
\end{aligned}$$

Finally,

$$\mathbb{E}\left[H_4(Y_4(\bar{x}))H_4(f_\ell(y(\phi)))\right] = 4!\frac{8^2}{\lambda_\ell^2(\lambda_\ell-2)^2}\left\{\mathbb{E}\left[\partial_{21;x} f_\ell(x) f_\ell(y(\phi))\right]\right\}^4,$$

where

$$\begin{aligned}
&\mathbb{E}\left[\partial_{21;x} f_\ell(x) f_\ell(y(\phi))\right] \\
&\quad = \frac{1}{\sin\theta_x}P_\ell''(\langle x, y\rangle)\left\{-\sin\theta_x\cos\theta_y + \cos\theta_x\sin\theta_y\sin(\varphi_x-\varphi_y)\right\}^2\Big|_{x=\bar{x},y=y(\phi)} \\
&\qquad + \frac{1}{\sin\theta_x}P_\ell'(\langle x, y\rangle)\left\{\cos\theta_x\sin\theta_y\cos(\varphi_x-\varphi_y)\right\}^2\Big|_{x=\bar{x},y=y(\phi)} \\
&\quad = 0.
\end{aligned}$$

□

Lemma A.5 *For $a = 1, 4$, we have that*

$$\int_0^{\pi/2}\mathbb{E}\left[H_2(Y_a(\bar{x}))H_2(Y_c(\bar{x}))H_4(f_\ell(y(\phi)))\right]\sin\phi d\phi = 0, \quad \textit{where } c = 1,\ldots 5,\ c \neq a.$$

Proof It was shown in the proof of Lemma A.4 that $\mathbb{E}\left[Y_1(\bar{x})) f_\ell(y(\phi)\right] = \mathbb{E}\left[Y_4(\bar{x})) f_\ell(y(\phi)\right] = 0$. The result is then an immediate consequence of the Diagram Formula (see (A.2.13) on page 202 of [25]). □

We are then left with only two terms to consider.

Lemma A.6 *For $a = 2, 5$, we have that*

$$\int_0^{\pi/2} \mathbb{E}\left[H_2(Y_a(\bar{x}))H_2(Y_3(\bar{x}))H_4(f_\ell(y(\phi)))\right] \sin\phi \mathrm{d}\phi = O(\ell^{-2}).$$

Proof We have that

$$\begin{aligned}
&\mathbb{E}\left[H_2(Y_2(\bar{x}))H_2(Y_3(\bar{x}))H_4(f_\ell(y(\phi)))\right] \\
&\quad = 4!\, \{\mathbb{E}\left[Y_2(\bar{x}) f_\ell(y(\phi))\right]\}^2 \{\mathbb{E}\left[Y_3(\bar{x}) f_\ell(y(\phi))\right]\}^2 \\
&\quad = 4! \times \frac{2}{\ell(\ell+1)} \left\{\mathbb{E}\left[\partial_{2;x} f_\ell(x) f_\ell(y(\phi))\right]\right\}^2 \\
&\qquad \times \frac{8}{\lambda_\ell(3\lambda_\ell - 2)} \left\{\mathbb{E}\left[\partial_{11;x} f_\ell(x) f_\ell(y(\phi))\right]\right\}^2 \\
&\quad = 4! \times \frac{2}{\ell(\ell+1)} \left\{P'_\ell(\cos\phi)\sin(\phi)\right\}^2 \times \frac{8}{\lambda_\ell(3\lambda_\ell - 2)} \left\{P'_\ell(\cos\phi)\sin^2\phi\right\}^2,
\end{aligned}$$

and therefore

$$\begin{aligned}
&\int_0^{\pi/2} \mathbb{E}\left[H_2(Y_2(\bar{x}))H_2(Y_3(\bar{x}))H_4(f_\ell(y(\phi)))\right] \sin\phi \mathrm{d}\phi \\
&\quad \le \text{const} \times \frac{1}{\ell^6} \int_0^{\pi/2} \left\{P'_\ell(\cos\phi)\sin(\phi)\right\}^4 \sin^3\phi \mathrm{d}\phi \\
&\quad = O(\ell^{-4}).
\end{aligned}$$

Finally,

$$\begin{aligned}
&\mathbb{E}\left[H_2(Y_5(\bar{x}))H_2(Y_3(\bar{x}))H_4(f_\ell(y(\phi)))\right] \\
&\quad = 4!\, \{\mathbb{E}\left[Y_5(\bar{x}) f_\ell(y(\phi))\right]\}^2 \{\mathbb{E}\left[Y_3(\bar{x}) f_\ell(y(\phi))\right]\}^2 \\
&\quad = 4! \left\{\mathbb{E}\left[(\alpha_{1\ell}\partial_{22;x} f_\ell(x) - \alpha_{2\ell}\partial_{11;x} f_\ell(x)) f_\ell(y(\phi))\right]\right\}^2 \\
&\qquad \times \frac{8}{\lambda_\ell(3\lambda_\ell - 2)} \left\{\mathbb{E}\left[\partial_{11;x} f_\ell(x) f_\ell(y(\phi))\right]\right\}^2 \\
&\quad = 4! \left\{\alpha_{1\ell}(P''_\ell(\cos\phi)\sin^2(\phi) - P'_\ell(\cos\phi)\cos\phi) + \alpha_{2\ell} P'_\ell(\cos\phi)\cos\phi\right\}^2 \\
&\qquad \times \frac{8}{\lambda_\ell(3\lambda_\ell - 2)} \left\{P'_\ell(\cos\phi)\cos\phi\right\}^2;
\end{aligned}$$

exploiting again Lemma A.12, the result follows. □

Lemma A.7 *As $\ell \to \infty$, for $a, b = 3, 5$, $a \neq b$*

$$\int_0^{\pi/2} \mathbb{E}[H_3(Y_a(\bar{x}))H_1(Y_b(\bar{x}))H_4(f_\ell(y(\phi)))] \sin\phi \mathrm{d}\phi = O(\ell^{-2}).$$

Proof By applying again the Diagram Formula (see [18] Section 4.3.1), we have

$$\begin{aligned}&\int_0^{\pi/2} \mathbb{E}[H_3(Y_a(\bar{x}))H_1(Y_b(\bar{x}))H_4(f_\ell(y(\phi)))]\sin\phi \mathrm{d}\phi\\&=\int_0^{\pi/2}\{3^2 4\mathbb{E}[Y_a(\bar{x})Y_b(\bar{x})]\mathbb{E}^2[Y_a(\bar{x})f_\ell(y(\phi))]\\&\quad+4!\mathbb{E}^3[Y_a(\bar{x})f_\ell(y(\phi))]\mathbb{E}[Y_b(\bar{x})f_\ell(y(\phi))]\}\sin\phi \mathrm{d}\phi.\end{aligned}$$

We observe that

$$\begin{aligned}\mathbb{E}[Y_3(\bar{x})Y_5(\bar{x})]&=\frac{\sqrt{8}}{\sqrt{\lambda_\ell}\sqrt{3\lambda_\ell-2}}\frac{\sqrt{3\lambda_\ell-2}}{\lambda_\ell\sqrt{\lambda_\ell-2}}\mathbb{E}[\partial_{22;x}f_\ell(\bar{x})\partial_{11;x}f_\ell(\bar{x})]\\&\quad-\frac{\sqrt{8}}{\sqrt{\lambda_\ell}\sqrt{3\lambda_\ell-2}}\frac{\lambda_\ell+2}{\lambda_\ell\sqrt{3\lambda_\ell-2}\sqrt{\lambda_\ell-2}}\mathbb{E}[\partial_{11;x}f_\ell(\bar{x})\partial_{11;x}f_\ell(\bar{x})]\\&=\frac{\sqrt{8}}{\sqrt{\lambda_\ell}\sqrt{3\lambda_\ell-2}}\frac{\sqrt{3\lambda_\ell-2}}{\lambda_\ell\sqrt{\lambda_\ell-2}}\frac{\lambda_\ell}{8}[\lambda_\ell+2]\\&\quad-\frac{\sqrt{8}}{\sqrt{\lambda_\ell}\sqrt{3\lambda_\ell-2}}\frac{\lambda_\ell+2}{\lambda_\ell\sqrt{3\lambda_\ell-2}\sqrt{\lambda_\ell-2}}\frac{\lambda_\ell}{8}[3\lambda_\ell-2]\\&=0,\end{aligned}$$

moreover

$$\mathbb{E}[Y_3(\bar{x})f_\ell(y(\phi))]=-\frac{\sqrt{8}}{\sqrt{\lambda_\ell}\sqrt{3\lambda_\ell-2}}P_\ell'(\cos\phi)\cos\phi,$$

and

$$\begin{aligned}\mathbb{E}[Y_5(\bar{x})f_\ell(y(\phi))]&=\frac{\sqrt{3\lambda_\ell-2}}{\lambda_\ell\sqrt{\lambda_\ell-2}}\mathbb{E}[\partial_{22;x}f_\ell(\bar{x})f_\ell(y(\phi))]\\&\quad-\frac{\lambda_\ell+2}{\lambda_\ell\sqrt{\lambda_\ell-2}\sqrt{3\lambda_\ell-2}}\mathbb{E}[\partial_{11;x}f_\ell(\bar{x})f_\ell(y(\phi))]\\&=\frac{\sqrt{3\lambda_\ell-2}}{\lambda_\ell\sqrt{\lambda_\ell-2}}[P''(\cos\phi)\sin^2\phi-P_\ell'(\cos\phi)\cos\phi]\\&\quad-\frac{\lambda_\ell+2}{\lambda_\ell\sqrt{\lambda_\ell-2}\sqrt{3\lambda_\ell-2}}[-P_\ell'(\cos\phi)\cos\phi].\end{aligned}$$

The statement follows by applying Lemma A.12. □

Lemma A.8 *For $a=1,2$,*

$$\int_0^{\pi/2}\mathbb{E}[H_2(Y_a(\bar{x}))H_1(Y_3(\bar{x}))H_1(Y_5(\bar{x}))H_4(f_\ell(y(\phi)))]\sin\phi \mathrm{d}\phi=0.$$

Proof From Diagram Formula, we have

$$\begin{aligned}&\int_0^{\pi/2} \mathbb{E}[H_2(Y_a(\bar{x}))H_1(Y_3(\bar{x}))H_1(Y_5(\bar{x}))H_4(f_\ell(y(\phi)))]\sin\phi \mathrm{d}\phi\\ &= \int_0^{\pi/2}\{2\mathbb{E}[Y_a(\bar{x})Y_3(\bar{x})]\mathbb{E}[Y_a(\bar{x})Y_5(\bar{x})]\\ &\quad + 4!\mathbb{E}[Y_a(\bar{x})Y_3(\bar{x})]\mathbb{E}[Y_3(\bar{x})f_\ell(y(\phi))]\mathbb{E}[Y_a(\bar{x})f_\ell(y(\phi))]\\ &\quad + 3\cdot 4\mathbb{E}[Y_3(\bar{x})Y_5(\bar{x})]\mathbb{E}^2[Y_a(\bar{x})f_\ell(y(\phi))]\\ &\quad + 4!\mathbb{E}[Y_a(\bar{x})Y_5(\bar{x})]\mathbb{E}[Y_3(\bar{x})f_\ell(y(\phi))]\mathbb{E}[Y_a(\bar{x})f_\ell(y(\phi))]\\ &\quad + 4!\mathbb{E}[Y_3(\bar{x})f_\ell(y(\phi))]\mathbb{E}[Y_5(\bar{x})f_\ell(y(\phi))]\mathbb{E}^2[Y_a(\bar{x})f_\ell(y(\phi))]\}\sin\phi \mathrm{d}\phi.\end{aligned}$$

The statement follows by observing that (for $a = 1, 2$) $\mathbb{E}[Y_a(\bar{x})Y_3(\bar{x})] = 0$, $\mathbb{E}[Y_a(\bar{x})Y_5(\bar{x})] = 0$, $\mathbb{E}[Y_3(\bar{x})Y_5(\bar{x})] = 0$, and $\mathbb{E}[Y_a(\bar{x})f_\ell(y(\phi))] = 0$. □

Lemma A.9 *We have*

$$\int_0^{\pi/2} \mathbb{E}[H_2(Y_4(\bar{x}))H_1(Y_3(\bar{x}))H_1(Y_5(\bar{x}))H_4(f_\ell(y(\phi)))]\sin\phi \mathrm{d}\phi = 0.$$

Proof Once again, the statement follows from Diagram Formula, which gives

$$\begin{aligned}&\int_0^{\pi/2} \mathbb{E}[H_2(Y_4(\bar{x}))H_1(Y_3(\bar{x}))H_1(Y_5(\bar{x}))H_4(f_\ell(y(\phi)))]\sin\phi \mathrm{d}\phi\\ &= \int_0^{\pi/2}\{2\mathbb{E}[Y_4(\bar{x})Y_3(\bar{x})]\mathbb{E}[Y_4(\bar{x})Y_5(\bar{x})]\\ &\quad + 4!\mathbb{E}[Y_4(\bar{x})Y_3(\bar{x})]\mathbb{E}[Y_3(\bar{x})f_\ell(y(\phi))]\mathbb{E}[Y_4(\bar{x})f_\ell(y(\phi))]\\ &\quad + 3\cdot 4\mathbb{E}[Y_3(\bar{x})Y_5(\bar{x})]\mathbb{E}^2[Y_4(\bar{x})f_\ell(y(\phi))]\\ &\quad + 4!\mathbb{E}[Y_4(\bar{x})Y_5(\bar{x})]\mathbb{E}[Y_3(\bar{x})f_\ell(y(\phi))]\mathbb{E}[Y_4(\bar{x})f_\ell(y(\phi))]\\ &\quad + 4!\mathbb{E}[Y_3(\bar{x})f_\ell(y(\phi))]\mathbb{E}[Y_5(\bar{x})f_\ell(y(\phi))]\mathbb{E}^2[Y_4(\bar{x})f_\ell(y(\phi))]\}\sin\phi \mathrm{d}\phi\end{aligned}$$

and $\mathbb{E}[Y_4(\bar{x})Y_3(\bar{x})] = 0$, $\mathbb{E}[Y_4(\bar{x})Y_5(\bar{x})] = 0$, $\mathbb{E}[Y_3(\bar{x})Y_5(\bar{x})] = 0$, $\mathbb{E}[Y_4(\bar{x})f_\ell(y(\phi))] = 0$. □

A.3 Some Useful Integrals

We write as usual

$$P_\ell^{(r)}(u) = \frac{\mathrm{d}^r}{\mathrm{d}u^r}P_\ell(u).$$

For our main arguments to follow, a key step is to recall the following results, which are proved in [8], Lemma C3. For all constants $C > 0$, we have, uniformly over $C/\ell \le \phi \le \pi/\ell$

$$P_\ell^{(r)}(u) = \sqrt{\frac{2}{\pi}} \frac{\ell^{2r-\frac{1}{2}}}{\sin^{r+\frac{1}{2}} \phi} (-1)^{r/2} \cos \psi_\ell^{\pm} + R_\ell^{(r)}(\phi), \quad r = 0, 1, 2, \tag{5.1}$$

where $\psi_\ell^{\pm} = (\ell + 1/2)\phi - \pi/4$ for $r = 0, 2$, and $\psi_\ell^{\pm} = (\ell + 1/2)\phi + \pi/4$ for $r = 1$, and

$$R_\ell^{(0)}(\phi) = O\left(\frac{1}{\sqrt{\ell\phi}}\right), \quad R_\ell^{(1)}(\phi) = O\left(\frac{1}{\sqrt{\ell}\phi^{5/2}}\right), \quad R_\ell^{(2)}(\phi) = O\left(\frac{\sqrt{\ell}}{\phi^{7/2}}\right). \tag{5.2}$$

Our results will then follow from the following two lemmas:

Lemma A.10 *For $r = 0, 1, 2$, we have*

$$\int_0^{\pi/2} [P_\ell^{(r)}(\cos\phi) \sin^r \phi]^4 \sin\phi \mathrm{d}\phi = \frac{3\ell^{4r}}{\pi^2} \frac{\log \ell}{2\ell^2} + O(\ell^{4r-2}).$$

Likewise

$$\int_0^{\pi/2} [P_\ell(\cos\phi)]^2 [P_\ell''(\cos\phi) \sin^2 \phi]^2 \sin\phi \mathrm{d}\phi = \frac{3\ell^2 \log \ell}{2\pi^2} + O(\ell^2),$$
$$\int_0^{\pi/2} [P_\ell(\cos\phi)]^2 [P_\ell'(\cos\phi) \sin \phi]^2 \sin\phi \mathrm{d}\phi = \frac{\log \ell}{2\pi^2} + O(1),$$
$$\int_0^{\pi/2} [P_\ell'(\cos\phi)]^2 [P_\ell''(\cos\phi) \sin \phi]^2 \sin\phi \mathrm{d}\phi = \frac{\ell^4 \log \ell}{2\pi^2} + O(\ell^4).$$

Remark A.11 More compactly, for $r_1, r_2 = 0, 1, 2$, we could have written the single expression

$$\begin{aligned} &\int_0^{\pi/2} [P_\ell^{(r_1)}(\cos\phi) \sin^{r_1} \phi]^2 [P_\ell^{(r_2)}(\cos\phi) \sin^{r_2} \phi]^2 \sin\phi \mathrm{d}\phi \\ &\quad = \frac{(2 + (-1)^{r_1+r_2})\ell^{2(r_1+r_2)}}{\pi^2} \frac{\log \ell}{2\ell^2} \\ &\qquad + O(\ell^{2(r_1+r_2)-2}). \end{aligned}$$

Proof We recall first that $P_\ell^{(r)}(\cos\phi) \le \ell^{2r}$ for all $\phi \in [0, 2\pi)$. Hence,

$$\int_0^{C/\ell} [P_\ell^{(r)}(\cos\phi) \sin^r \phi]^4 \sin\phi \mathrm{d}\phi \le \text{const} \times \ell^{8r} \int_0^{C/\ell} \sin^{4r+1} \phi \mathrm{d}\phi = O(\ell^{4r-2}),$$

and it suffices to consider $\phi > C/\ell$. Hence, we have

$$\int_0^{\pi/2} [P_\ell^{(r)}(\cos\phi) \sin^r \phi]^4 \sin\phi \mathrm{d}\phi$$

$$
\begin{aligned}
&= \frac{2^2}{\pi^2} \int_{C/\ell}^{\pi/2} \left[\frac{\ell^r}{\sqrt{\ell \sin \phi}} \cos \left((\ell + 1/2)\phi \pm \frac{\pi}{4} \right) \right]^4 \sin \phi \mathrm{d}\phi \\
&+ 4 \frac{2^{3/2}}{\pi^{3/2}} \int_{C/\ell}^{\pi/2} \left[\frac{\ell^r}{\sqrt{\ell \sin \phi}} \cos \left((\ell + 1/2)\phi \pm \frac{\pi}{4} \right) \right]^3 \left[R_\ell^{(r)}(\phi) \sin^r \phi \right] \sin \phi \mathrm{d}\phi \\
&+ 6 \frac{2}{\pi} \int_{C/\ell}^{\pi/2} \left[\frac{\ell^r}{\sqrt{\ell \sin \phi}} \cos \left((\ell + 1/2)\phi \pm \frac{\pi}{4} \right) \right]^2 \left[R_\ell^{(r)}(\phi) \sin^r \phi \right]^2 \sin \phi \mathrm{d}\phi \\
&+ 4 \frac{2^{1/2}}{\pi^{1/2}} \int_{C/\ell}^{\pi/2} \left[\frac{\ell^r}{\sqrt{\ell \sin \phi}} \cos \left((\ell + 1/2)\phi \pm \frac{\pi}{4} \right) \right] \left[R_\ell^{(r)}(\phi) \sin^r \phi \right]^3 \sin \phi \mathrm{d}\phi \\
&+ \int_{C/\ell}^{\pi/2} \left[R_\ell^{(r)}(\phi) \sin^r \phi \right]^4 \sin \phi \mathrm{d}\phi.
\end{aligned}
$$

It is not difficult to see that, for $k = 1, \dots, 4$,

$$
\int_{C/\ell}^{\pi/2} \left[\frac{\ell^r}{\sqrt{\ell \sin \phi}} \cos \left((\ell + 1/2)\phi \pm \frac{\pi}{4} \right) \right]^k \left[R_\ell^{(r)}(\phi) \sin^r \phi \right]^{4-k} \sin \phi \mathrm{d}\phi = O(\ell^{4r-2});
$$

indeed the previous integrals are bounded by, for $r = 2$,

$$
\begin{aligned}
&\ell^{kr-k/2} \int_{C/\ell}^{\pi/2} \frac{1}{\sin^{k/2} \phi} \left[R_\ell^{(2)}(\phi) \sin^r \phi \right]^{4-k} \sin \phi \mathrm{d}\phi \\
&\leq \text{const} \times \ell^{3k/2} \int_{C/\ell}^{\pi/2} \frac{1}{\sin^{k/2} \phi} \left[\frac{\ell^{1/2}}{\phi^{3/2}} \right]^{4-k} \sin \phi \mathrm{d}\phi \\
&\leq \text{const} \times \ell^{k+2} \int_{C/\ell}^{\pi/2} \frac{1}{\sin^{k/2} \phi} \frac{1}{\phi^{6-3k/2}} \sin \phi \mathrm{d}\phi \\
&\leq \text{const} \times \ell^{k+2} \int_{C/\ell}^{\pi/2} \phi^{k-5} \mathrm{d}\phi = O(\ell^6).
\end{aligned}
$$

Likewise, for $r = 1$,

$$
\begin{aligned}
&\ell^{k-k/2} \int_{C/\ell}^{\pi/2} \frac{1}{\sin^{k/2} \phi} \left[R_\ell^{(r)}(\phi) \sin^r \phi \right]^{4-k} \sin \phi \mathrm{d}\phi \\
&\leq \text{const} \times \ell^{k/2} \int_{C/\ell}^{\pi/2} \frac{1}{\sin^{k/2} \phi} \left[\frac{1}{\ell^{1/2}\phi^{3/2}} \right]^{4-k} \sin \phi \mathrm{d}\phi \\
&\leq \text{const} \times \ell^{k-2} \int_{C/\ell}^{\pi/2} \frac{1}{\sin^{k/2} \phi} \frac{1}{\phi^{6-3k/2}} \sin \phi \mathrm{d}\phi \\
&\leq \text{const} \times \ell^{k-2} \int_{C/\ell}^{\pi/2} \phi^{k-5} \mathrm{d}\phi = O(\ell^2).
\end{aligned}
$$

Thus,

$$\int_0^{\pi/2} [P_\ell^{(r)}(\cos\phi)\sin^r\phi]^4 \sin\phi \mathrm{d}\phi$$
$$= \frac{2^2}{\pi^2}\int_{C/\ell}^{\pi/2}\left[\frac{\ell^r}{\sqrt{\ell\sin\phi}}\cos\left(\left(\ell+\frac{1}{2}\right)\phi \pm \frac{\pi}{4}\right)\right]^4 \sin\phi \mathrm{d}\phi + O(\ell^{4r-2}).$$

The following equalities can be established by simple trigonometric identities:

$$\cos^4\left(\left(\ell+\frac{1}{2}\right)\phi - \frac{\pi}{4}\right) = \frac{3}{8} + \frac{1}{8}(-\cos(2\phi(2\ell+1)) + 4\sin(\phi(2\ell+1))),$$
$$\cos^4\left(\left(\ell+\frac{1}{2}\right)\phi + \frac{\pi}{4}\right) = \frac{3}{8} + \frac{1}{8}(-\cos(2\phi(2\ell+1)) - 4\sin(\phi(2\ell+1))).$$

Thus, we have

$$\int_0^{\pi/2} [P_\ell^{(r)}(\cos\phi)\sin^r\phi]^4 \sin\phi \mathrm{d}\phi = \ell^{4r-2}\frac{2^2}{\pi^2}\frac{3}{8}\int_{C/\ell}^{\pi/2}\frac{1}{\sin\phi}\mathrm{d}\phi + O(\ell^{4r-2})$$
$$= \frac{3}{2\pi^2}\ell^{4r-2}\log\ell + O(\ell^{4r-2}),$$

since

$$\int_{C/\ell}^{\pi/2}\frac{1}{\sin\phi}\mathrm{d}\phi = \frac{1}{2}\log\left(\frac{1-\cos\phi}{1+\cos\phi}\right)\Bigg|_{C/\ell}^{\pi/2} = \log\ell + O(1).$$

The proof of the first part of the lemma is then concluded. The proof of the second result is very similar, and we can omit some details; in particular, we simply recall the identity

$$\cos^2\left(\frac{2\ell+1}{2}\phi + \frac{\pi}{4}\right)\cos^2\left(\frac{2\ell+1}{2}\phi - \frac{\pi}{4}\right)$$
$$= \left[\frac{\sqrt{2}}{2}\cos\left(\frac{2\ell+1}{2}\phi\right) - \frac{\sqrt{2}}{2}\sin\left(\frac{2\ell+1}{2}\phi\right)\right]^2\left[\frac{\sqrt{2}}{2}\cos\left(\frac{2\ell+1}{2}\phi\right)\right.$$
$$\left.+\frac{\sqrt{2}}{2}\sin\left(\frac{2\ell+1}{2}\phi\right)\right]^2$$
$$= \frac{1}{4}\left[\cos^2\left(\frac{2\ell+1}{2}\phi\right) - \sin^2\left(\frac{2\ell+1}{2}\phi\right)\right]^2$$
$$= \frac{1}{4}\cos^2((2\ell+1)\phi).$$

Because $\cos 2x + 1 = 2\cos^2 x$, it is not difficult to see that

$$\int_{C/\ell}^{\pi/2}\frac{\cos^2((2\ell+1)\phi)}{\sin\phi}\mathrm{d}\phi = \frac{1}{2}\int_{C/\ell}^{\pi/2}\frac{1}{\sin\phi}\mathrm{d}\phi$$

$$+\int_{C/\ell}^{\pi/2} \frac{\cos(2(2\ell+1)\phi)}{2\sin\phi} d\phi = \frac{1}{2}\log\ell + O(1).$$

Dealing with the lower order terms as in the first part of the lemma, we can now conclude with our second statement, i.e.

$$\int_0^{\pi/2} [P_\ell(\cos\phi)\sin^{r_1}\phi]^2 [P_\ell^{(4)}(\cos\phi)\sin^{r_2}\phi]^2 \sin\phi d\phi = \frac{3\ell^8}{2\pi^2}\frac{\log\ell}{\ell^2} + O(\ell^6),$$
$$\int_0^{\pi/2} [P_\ell(\cos\phi)\sin^{r_1}\phi]^2 [P_\ell^{(2)}(\cos\phi)\sin^{r_2}\phi]^2 \sin\phi d\phi = \frac{\ell^8}{\pi^2}\frac{\log\ell}{2\ell^2} + O(\ell^6),$$
$$\int_0^{\pi/2} [P_\ell^{(2)}(\cos\phi)\sin^{r_1}\phi]^2 [P_\ell^{(4)}(\cos\phi)\sin^{r_2}\phi]^2 \sin\phi d\phi = \frac{\ell^8}{\pi^2}\frac{\log\ell}{2\ell^2} + O(\ell^6).$$

□

In our second auxiliary result, an upper bound is given.

Lemma A.12 *As $\ell \to \infty$, we have that*

$$\int_0^{\pi/2} \left|P_\ell'(\cos\phi)\right|^k \left|P_\ell''(\cos\phi)\sin^2\phi\right|^{4-k} \sin\phi d\phi = O(\ell^6), \quad \textit{for all} \;\; k = 1,\dots 4.$$

Proof As before, for the "local" component where $\phi < C/\ell$ (some fixed constant C) we have

$$\int_0^{C/\ell} \left\{P_\ell''(\cos\phi)\sin^2\phi\right\}^k \left\{P_\ell'(\cos\phi)\cos\phi\right\}^{4-k} \sin\phi$$
$$\leq \text{const} \times \ell^{4k} \times \ell^{8-2k} \int_0^{C/\ell} \sin^{2k}\phi \sin\phi d\phi$$
$$= O(\ell^{8+2k-(2k+2)}) = O(\ell^6)\;.$$

On the other hand, using again formulas (5.1) and (5.2), and computations analogous to Lemma A.10, we find easily that

$$\int_{C/\ell}^{\pi/2} \left\{P_\ell'(\cos\phi)\cos\phi\right\}^k \left\{P_\ell''(\cos\phi)\sin^2\phi\right\}^{4-k} \sin\phi d\phi$$
$$\leq \text{const} \times \ell^{k/2} \times \ell^{6-3k/2} \int_{C/\ell}^{\pi/2} \frac{1}{\sin^{3k/2}\phi}\frac{1}{\sin^{2-k/2}\phi} \sin\phi d\phi + O(\ell^6)$$
$$\leq \text{const} \times \ell^{6-k} \int_{C/\ell}^{\pi/2} \frac{1}{\sin^{2+k}\phi} \sin\phi d\phi + O(\ell^6) = O(\ell^6).$$

□

Note in particular that for $k = 4$, we obtain the bound

$$\int_0^{\pi/2} [P_\ell'(\cos\phi)]^4 \sin\phi \, d\phi = O(\ell^6).$$

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