On a Certain Operator Related to Blackwell's Markov Chain

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Abstract

We present an example of a densely defined, linear operator on the l^1 space with the property that each basis vector of the standard Schauder basis of l^1 does not belong to its domain. Our example is based on the construction of a Markov chain with all states instantaneous given by D. Blackwell in 1958. In addition, it turns out that the closure of this operator is the generator of a strongly continuous semigroup of Markov operators associated with Blackwell's chain.

Keywords Blackwell's example · Markov semigroup · unbounded operator

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1 Introduction

W. Feller and H.P. McKean showed in 1956, see [3], that there exists a Markov chain with all states instantaneous. They considered a diffusion x_t with the state space $K = [0, 1]$ and with some specified transition function, and proved that this diffusion is in fact equivalent to a Markov chain $\hat{x}_t$ with the state space $\mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers. In addition, the Q matrix of the chain has $-\infty$ down the diagonal and zeroes elsewhere. Thus, the Feller–McKean chain is a continuous process which spends almost all of its time on $\mathbb{N}$ even though it leaves each state instantly.

In 1958, D. Blackwell, see [1], gave a more tractable example of a Markov chain with all states instantaneous. In this situation, the connection between intensity matrices of Markov chains and semigroups of Markov operators in l^1 corresponding to them, is of special interest. To be precise, let $X(t)$, $t \geq 0$ be a Markov chain with values in $\mathbb{N}$ and $\{P(t), t \geq 0\}$ be its corresponding Markov semigroup in l^1 with generator A . Lemma 2 in [5] says that if a given state $i \in \mathbb{N}$ is instantaneous then $e_i \notin \mathcal{D}(A)$, where e_i is an element of the standard Schauder basis in l^1 and $\mathcal{D}(A)$ is the domain of A , see Sect. 2

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for definitions. Since all states of Blackwell's chain are instantaneous, $e_i \notin \mathcal{D}(A_B)$ for every $i \in \mathbb{N}$, where A_B is the generator of the corresponding Blackwell semigroup. Because A_B is not explicitly given in the literature (as far as I am aware), I became interested in finding a formula for this generator. However, due to difficulties with a precise description of the domains of such generators, I slightly modified my problem. It can be stated as follows: find a formula for a linear operator $A : \mathcal{D}(A) \rightarrow l^1$ that has the following properties

$$(i) \mathcal{D}(A) \text{ is dense in } l^1, \quad (ii) e_i \notin \mathcal{D}(A) \text{ for every } i \in \mathbb{N}, \quad (1)$$

and such that explicit calculations are possible. So now we do not require A to be a generator.

Theorem 1 answers our question and an example of an operator with such properties is given by (12)–(13). Moreover, Theorem 1 implies that the closure of A equals A_B , i.e. $\overline{A} = A_B$, see Sect. 4 for details.

In the next section, we introduce basic notions and present the construction of Blackwell's semigroup. This construction is needed for us to define the operator A in Sect. 3.

2 Preliminaries

2.1 The Blackwell Semigroup

We begin with some definitions. Let l^1 be the Banach space of all absolutely summable sequences $x = (\xi_i)_{i \in \mathbb{N}}$, i.e. sequences such that $\sum_{i \in \mathbb{N}} |\xi_i| < \infty$, with the norm $\|x\| = \sum_{i \in \mathbb{N}} \xi_i$. In this article, $\mathbb{N} = \{1, 2, 3, \dots\}$. By $\{e_i\}_{i \in \mathbb{N}}$, we denote the standard Schauder basis in l^1 , i.e. $e_i = (0, \dots, 0, 1, 0, \dots)$ with 1 in the i th coordinate. Any $x \in l^1$ can be then written as $x = \sum_{i \in \mathbb{N}} \xi_i e_i$.

Throughout we assume that $\alpha_n, \beta_n, n \geq 1$ are positive numbers satisfying

$$\sum_{n=1}^{\infty} \frac{\beta_n}{\alpha_n + \beta_n} < \infty, \quad \sum_{n=1}^{\infty} \beta_n = \infty. \quad (2)$$

Incidentally, this assumption implies that $\lim_{n \rightarrow \infty} \frac{\beta_n}{\alpha_n} = 0$ and $\sum_{n=1}^{\infty} \frac{\beta_n}{\alpha_n} < \infty$.

Now we follow the construction of the Blackwell semigroup as described in [2]. Firstly, let $\mathbb{I}$ be the set of functions $\tilde{i} : \mathbb{N} \rightarrow \{0, 1\}$ admitting value 1 finitely many times. Elements of $\mathbb{I}$ may be considered as sequences with a finite number of nonzero terms. Since $\mathbb{I}$ is countable, the definition of $l^1(\mathbb{I})$ is analogous to that of $l^1(\mathbb{N})$. By $\{e_{\tilde{i}}\}_{\tilde{i} \in \mathbb{I}}$ we denote the natural basis in $l^1(\mathbb{I})$. For $n \geq 1$, let F_n be the map $F_n : \mathbb{I} \rightarrow \mathbb{I}$ changing the n th coordinate of an $\tilde{i}$ from 0 to 1 and *vice versa*. The map $G_n : \mathbb{I} \rightarrow \{0, 1\}$ assigns to an $\tilde{i}$ its n th coordinate.

For $n \geq 1$, let B_n be the bounded linear operator on $l^1(\mathbb{I})$ determined by its values on $e_{\tilde{i}}, \tilde{i} \in \mathbb{I}$, by

$$B_n e_{\tilde{i}} = \begin{cases} -\beta_n e_{\tilde{i}} + \beta_n e_{F_n(\tilde{i})}, & \text{if } G_n(\tilde{i}) = 0, \\ -\alpha_n e_{\tilde{i}} + \alpha_n e_{F_n(\tilde{i})}, & \text{if } G_n(\tilde{i}) = 1. \end{cases} \quad (3)$$

Then for any $x = \sum_{\tilde{i} \in \mathbb{I}} \delta_{\tilde{i}} e_{\tilde{i}}$ from $l^1(\mathbb{I})$ we have $B_n x = \sum_{\tilde{i} \in \mathbb{I}} \delta_{\tilde{i}} B_n e_{\tilde{i}}$.

As explained in [2], B_n is the generator of a Markov chain on $\mathbb{I}$ in which the n th coordinate of an $\tilde{i} \in \mathbb{I}$ jumps between 0 and 1. In terms of semigroups, B_n generates a Markov semigroup $\{e^{tB_n}, t \geq 0\}$ which values on $\{e_{\tilde{i}}\}_{\tilde{i} \in \mathbb{I}}$ are given by

$$e^{tB_n} e_{\tilde{i}} = \begin{cases} p_n(t) e_{\tilde{i}} + (1 - p_n(t)) e_{F_n(\tilde{i})}, & \text{if } G_n(\tilde{i}) = 0, \\ q_n(t) e_{\tilde{i}} + (1 - q_n(t)) e_{F_n(\tilde{i})}, & \text{if } G_n(\tilde{i}) = 1, \end{cases}$$

where

$$p_n(t) = \frac{\alpha_n}{\alpha_n + \beta_n} + \frac{\beta_n}{\alpha_n + \beta_n} e^{-(\alpha_n + \beta_n)t},$$

$$q_n(t) = \frac{\beta_n}{\alpha_n + \beta_n} + \frac{\alpha_n}{\alpha_n + \beta_n} e^{-(\alpha_n + \beta_n)t}.$$

Since B_n 's commute and are bounded, $T_n(t) = \prod_{k=1}^n e^{tB_k}$ defines a strongly continuous semigroup $\{T_n(t), t \geq 0\}$, with generator $A_n = \sum_{k=1}^n B_k$. The semigroup $\{T_n(t), t \geq 0\}$ describes n combined independent Markov chains, each changing one of the first n coordinates of $\tilde{i} \in \mathbb{I}$.

Theorem 4.2 of [2] says that semigroups $\{T_n(t), t \geq 0\}$ converge, when $n \rightarrow \infty$, to a strongly continuous semigroup $\{T(t), t \geq 0\}$ composed of Markov operators. In such a case, we say that the infinite product $\prod_{k=1}^{\infty} e^{tB_k}$ exists and denote

$$\prod_{k=1}^{\infty} e^{tB_k} := T(t). \quad (4)$$

This limit semigroup is just the Blackwell semigroup defined on $l^1(\mathbb{I})$. It is also denoted by e^{tA_B} , where A_B is the generator of $\{T(t), t \geq 0\}$. Although a formula for $T(t)$ is given in Lemma 4.1 of [2] it is not useful here, since it is expressed as an infinite sum of terms which are difficult to handle.

The fact that semigroups $T_n(t)$ converge strongly is also a conclusion from Theorem 1, see Remark 1 in Sect. 4.

2.2 The Binary Bijection

Here we specify a bijection between $\mathbb{N}$ and $\mathbb{I}$. By doing so, we are able to write explicit formulae for isomorphic images of B_n 's and A_n 's.

Let $f(1) = \tilde{1} := (0, 0, 0, \dots) \in \mathbb{I}$ and for $i \geq 2$ define

$$f(i) = \tilde{i} := (j_1, \dots, j_m, 0, 0, \dots), \quad (5)$$

where $j_k \in \{0, 1\}$, $k = 1, 2, \dots, m$, are such that $\sum_{k=1}^m j_k 2^{k-1} = i - 1$. This is simply to say that we assign the binary representation of $i - 1$ to a number $i \geq 2$. For instance, we have

$$f(2) = \tilde{2} = (1, 0, 0, 0, \dots), \quad f(8) = \tilde{8} = (1, 1, 1, 0, 0, \dots), \quad \text{etc.}$$

It is obvious that $\tilde{i} \in \mathbb{I}$ and that each element of $\mathbb{I}$ is a binary representation of a natural number. In consequence, we have a one-to-one correspondence between $l^1(\mathbb{N})$ and $l^1(\mathbb{I})$. This correspondence, written between bases $\{e_i\}_{i \in \mathbb{N}}$ and $\{e_{\tilde{i}}\}_{\tilde{i} \in \mathbb{I}}$, can be defined as follows

$$f(e_i) := e_{f(i)}, \quad i \in \mathbb{N},$$

where $f(i)$ is given by (5). Then we extend f from $\{e_i\}_{i \in \mathbb{N}}$ to all elements of $l^1(\mathbb{N})$ by linearity, i.e. $f(x) = \sum_{i \in \mathbb{N}} \xi_i f(e_i)$ if $x = \sum_{i \in \mathbb{N}} \xi_i e_i$.

Using (3), we define operators $\mathcal{B}_n$, $n \geq 1$, on $l^1(\mathbb{N})$ as follows. Firstly, define

$$\mathcal{B}_n e_i := f^{-1}(\mathcal{B}_n f(e_i)), \quad \text{for } i \in \mathbb{N}, \quad (6)$$

and $\mathcal{B}_n x = \sum_{i \in \mathbb{N}} \xi_i \mathcal{B}_n e_i$ for any $x \in l^1(\mathbb{N})$.

Lemma 1 *If $(\eta_i)_{i \in \mathbb{N}} = \mathcal{B}_n(\xi_i)_{i \in \mathbb{N}}$, then*

$$\eta_i = \begin{cases} -\beta_n \xi_i + \alpha_n \xi_{i+2^{n-1}}, & \text{if } i \bmod 2^n \in \{1, 2, \dots, 2^{n-1}\}, \\ -\alpha_n \xi_i + \beta_n \xi_{i-2^{n-1}}, & \text{if } i \bmod 2^n \in \{0, 2^{n-1} + 1, \dots, 2^n - 1\}. \end{cases} \quad (7)$$

Proof Fix $n \geq 1$. To find an explicit formula for $\mathcal{B}_n$ notice that for $\tilde{i} = f(i)$, where $i \in \mathbb{N}$, we have

$$F_n(\tilde{i}) = \begin{cases} \left(\dots, \underbrace{1}_{n\text{-th}}, \dots \right), & \text{if } \tilde{i} = \left(\dots, \underbrace{0}_{n\text{-th}}, \dots \right), \\ \left(\dots, \underbrace{0}_{n\text{-th}}, \dots \right), & \text{if } \tilde{i} = \left(\dots, \underbrace{1}_{n\text{-th}}, \dots \right). \end{cases}$$

From this, we immediately obtain

$$f^{-1}(F_n(f(i))) = \begin{cases} i + 2^{n-1}, & \text{if } f(i) = \left(\dots, \underbrace{0}_{n-th}, \dots \right), \\ i - 2^{n-1}, & \text{if } f(i) = \left(\dots, \underbrace{1}_{n-th}, \dots \right). \end{cases} \quad (8)$$

Next, we find $G_n(f(i))$. The binary arithmetic with some algebra shows that $G_n(f(i)) = 0$ if $i \bmod 2^n \in \{1, 2, \dots, 2^{n-1}\}$ and $G_n(f(i)) = 1$ otherwise. Therefore, we conclude from (3) and (8) that the operator $\mathcal{B}_n$ defined by (6) has the form

$$\mathcal{B}_n e_i = \begin{cases} -\beta_n e_i + \beta_n e_{i+2^{n-1}}, & \text{if } i \bmod 2^n \in \{1, 2, \dots, 2^{n-1}\}, \\ -\alpha_n e_i + \alpha_n e_{i-2^{n-1}}, & \text{if } i \bmod 2^n \in \{0, 2^{n-1} + 1, \dots, 2^n - 1\}, \end{cases} \quad (9)$$

where $i \geq 1$. Since $\mathcal{B}_n(\xi_i)_{i \in \mathbb{N}} = \sum_{i \in \mathbb{N}} \xi_i \mathcal{B}_n e_i$, (9) implies directly (7). $\square$

For instance, $\mathcal{B}_1$ has the form

$$\mathcal{B}_1(\xi_i)_{i \in \mathbb{N}} = (-\beta_1 \xi_1 + \alpha_1 \xi_2, \beta_1 \xi_1 - \alpha_1 \xi_2, -\beta_1 \xi_3 + \alpha_1 \xi_4, \beta_1 \xi_3 - \alpha_1 \xi_4, \dots).$$

Denote

$$\mathcal{A}_n = \sum_{k=1}^n \mathcal{B}_k, \quad n \geq 1. \quad (10)$$

The following corollary is a direct conclusion from Lemma 1.

Corollary 1 *If $(\eta_i)_{i \in \mathbb{N}} = \mathcal{A}_n(\xi_i)_{i \in \mathbb{N}}$, then $\eta_i = \sum_{k=1}^n \zeta_k$, where ζ_k , $k = 1, \dots, n$, are given by*

$$\zeta_k = \begin{cases} -\beta_k \xi_i + \alpha_k \xi_{i+2^{k-1}}, & \text{if } i \bmod 2^k \in \{1, 2, \dots, 2^{k-1}\}, \\ -\alpha_k \xi_i + \beta_k \xi_{i-2^{k-1}}, & \text{if } i \bmod 2^k \in \{0, 2^{k-1} + 1, \dots, 2^k - 1\}. \end{cases}$$

It turns out that $\mathcal{B}_n$'s and $\mathcal{A}_n$'s have the following interesting property.

Corollary 2 *Suppose that $x \in l^1(\mathbb{N})$ and denote $(\eta_i)_{i \in \mathbb{N}} = \mathcal{B}_n x$, $(\eta'_i)_{i \in \mathbb{N}} = \mathcal{A}_n x$. Then*

$$\sum_{i=1}^{\infty} \eta_i = \sum_{i=1}^{\infty} \eta'_i = 0. \quad (11)$$

Proof Due to (10), it is enough to prove $\sum_{i=1}^{\infty} \eta_i = 0$. Since $\mathcal{B}_n$ is bounded, i.e. $\|\mathcal{B}_n x\| \leq \max\{\alpha_n, \beta_n\} \|x\|$, we have

$$\sum_{i=1}^{\infty} \eta_i = \lim_{m \rightarrow \infty} \sum_{i=1}^m \eta_i = \lim_{m \rightarrow \infty} \left(\sum_{i=1}^{k2^n} \eta_i + \sum_{i=k2^n+1}^m \eta_i \right)$$

for any $k \in \mathbb{N}$. The first sum in the above limit is 0 by Lemma 1. Furthermore, for any $\varepsilon > 0$ there exists k_0 such that $\sum_{i=k_0}^{\infty} |\eta_i| < \varepsilon$. In consequence, $|\sum_{i=k_0}^{\infty} \eta_i| < M\varepsilon$ for a certain constant $M > 0$ and $\varepsilon > 0$. Since ε is arbitrary (11) is proved. $\square$

3 The Operator

We define $\mathcal{A}$ as

$$\mathcal{A}x := \sum_{k=1}^{+\infty} \mathcal{B}_k x = \lim_{n \rightarrow +\infty} \mathcal{A}_n x \quad (12)$$

with domain

$$\mathcal{D}(\mathcal{A}) = \left\{ x \in l^1(\mathbb{N}) : \sum_{k=1}^{+\infty} \|\mathcal{B}_k x\| < \infty \right\}. \quad (13)$$

Let $x = (\xi_i)_{i \in \mathbb{N}}$ be an element of $\mathcal{D}(\mathcal{A})$ and denote $(\eta_i)_{i \in \mathbb{N}} = \mathcal{A}x$. Since the norm convergence in $l^1(\mathbb{N})$ implies the coordinate-wise convergence, we conclude from Corollary 1 that

$$\eta_1 = \lim_{n \rightarrow +\infty} \sum_{k=1}^n (-\beta_k \xi_1 + \alpha_k \xi_{1+2^{k-1}}), \quad (14)$$

and if $i = 2^l + m$ for some $l \geq 0$ and $m \in \{1, 2, \dots, 2^l\}$, we have

$$\eta_i = \sum_{k=1}^{l+1} \zeta_k + \lim_{n \rightarrow +\infty} \sum_{k=l+2}^n (-\beta_k \xi_i + \alpha_k \xi_{i+2^{k-1}}), \quad (15)$$

where $\zeta_k, k = 1, \dots, l+1$ are given by

$$\zeta_k = \begin{cases} -\beta_k \xi_i + \alpha_k \xi_{i+2^{k-1}}, & \text{if } i \bmod 2^k \in \{1, 2, \dots, 2^{k-1}\}, \\ -\alpha_k \xi_i + \beta_k \xi_{i-2^{k-1}}, & \text{if } i \bmod 2^k \in \{0, 2^{k-1} + 1, \dots, 2^k - 1\}. \end{cases}$$

Notice that a necessary condition for an x to be in $\mathcal{D}(\mathcal{A})$ is

$$\lim_{n \rightarrow +\infty} (-\beta_n \xi_i + \alpha_n \xi_{i+2^{n-1}}) = 0,$$

for every $i \geq 1$.

For now, one can merely infer that $0 \in \mathcal{D}(\mathcal{A})$. However, in the proof of Theorem 1 we will produce some non-trivial elements of $\mathcal{D}(\mathcal{A})$. The following theorem is the main result of this article.

Theorem 1 *For the operator $\mathcal{A}$ defined by (12)–(13), the conditions in (1) are satisfied.*

Proof First we show (ii). Fix $i \in \mathbb{N}$ and notice that by (2) we have

$$\lim_{n \rightarrow +\infty} \sum_{k=l+2}^n (-\beta_k \xi_i + \alpha_k \xi_{i+2k-1}) = - \lim_{n \rightarrow +\infty} \sum_{k=l+2}^n \beta_k = -\infty,$$

proving that $e_i \notin \mathcal{D}(\mathcal{A})$.

Next we prove (i). Let $x = (\xi_1, \xi_2, \dots)$ be defined as follows. First we put

$$\xi_1 = 1, \quad \xi_{1+2k-1} = \frac{\beta_k}{\alpha_k}, \quad k \geq 1, \quad (16)$$

and if $i = 2^l + m$ for some $l \geq 0$ and $m \in \{1, 2, \dots, 2^l\}$, we define recursively

$$\xi_{i+2k-1} = \frac{\beta_k}{\alpha_k} \xi_i, \quad k \geq l+2. \quad (17)$$

From this, we infer that for $i = 1 + 2^{l_1-1} + \dots + 2^{l_m-1}$ with $l_1 < l_2 < \dots < l_m$ one can write

$$\xi_{i+2k-1} = \frac{\beta_k}{\alpha_k} \cdot \frac{\beta_{l_m} \beta_{l_m-1} \dots \beta_{l_1}}{\alpha_{l_m} \alpha_{l_m-1} \dots \alpha_{l_1}}, \quad k \geq l_m + 1.$$

Then, the norm of x can be expressed as

$$\|x\| = 1 + \sum_{l=1}^{\infty} \frac{\beta_l}{\alpha_l} + \sum_{l_1 < l_2} \frac{\beta_{l_1} \beta_{l_2}}{\alpha_{l_1} \alpha_{l_2}} + \sum_{l_1 < l_2 < l_3} \frac{\beta_{l_1} \beta_{l_2} \beta_{l_3}}{\alpha_{l_1} \alpha_{l_2} \alpha_{l_3}} + \dots \quad (18)$$

From the Problem 3.8.26 in [4] and the fact that $\sum_{l=1}^{\infty} \frac{\beta_l}{\alpha_l} < \infty$ (which follows from (2)), we have

$$\|x\| = \prod_{l=1}^{\infty} \left(1 + \frac{\beta_l}{\alpha_l} \right) < \infty,$$

meaning that $x \in l^1(\mathbb{N})$. To conclude that $x \in \mathcal{D}(\mathcal{A})$ we need to show that condition (13) is satisfied. To this end, denote $(\eta_i)_{i \in \mathbb{N}} = \mathcal{B}_k x$ and notice that (7) implies $\eta_i = -\beta_k \xi_i + \alpha_k \frac{\beta_k}{\alpha_k} \xi_i = 0$ or $\eta_i = -\alpha_k \xi_i + \beta_k \frac{\alpha_k}{\beta_k} \xi_i = 0$. In other words,

$$\sum_{k=1}^{+\infty} \|\mathcal{B}_k x\| = 0.$$

The above means that $x \in \mathcal{D}(\mathcal{A})$ and $\mathcal{A}x = 0$.

Now we construct a sequence of elements of $\mathcal{D}(\mathcal{A})$ converging to e_1 . Fix $p \geq 2$ and define $y_p = (\xi_1, \xi_2, \dots)$ as follows

$$\xi_1 = 1, \quad \xi_{1+2^{k-1}} = \begin{cases} 0 & , \quad k = 1, 2, \dots, p-1, \\ \frac{\beta_k}{\alpha_k} & , \quad k \geq p, \end{cases} \quad (19)$$

and if $i = 2^l + m$ for some $l \geq 0$ and $m \in \{1, 2, \dots, 2^l\}$, we define $\xi_{i+2^{k-1}}$ recursively as in (17). Observe that $\|y_p\| \leq \|x\|$ for every $p \geq 2$. Additionally, from (19) we have $\xi_i = 0$ for $i = 2, 3, \dots, 1 + 2^{p-1}$. It is clear that the coordinate-wise limit of y_p is e_1 .

Now choose any $\varepsilon > 0$ and fix $M \in (0, 1)$. We conclude from (2) that there exists l_0 such that $\sum_{l=l_0}^{\infty} \frac{\beta_l}{\alpha_l} < \varepsilon$ and $\frac{\beta_l}{\alpha_l} < M$, for all $l \geq l_0$. This, together with (18), implies

$$\|y_p - e_1\| \leq \varepsilon \sum_{k=0}^{\infty} M^k = \frac{\varepsilon}{1-M}, \quad \text{for } p \geq l_0.$$

Since ε is arbitrary, $\lim_{p \rightarrow +\infty} \|y_p - e_1\| = 0$.

What remains to show is $\mathcal{A}y_p \in l^1(\mathbb{N})$. From Lemma 1, we have

$$\mathcal{B}_k y_p = \left(-\beta_k, 0, \dots, 0, \underbrace{\beta_k}_{1+2^{k-1}\text{-th}}, 0, 0, \dots \right), \quad k = 1, 2, \dots, p-1$$

and $\mathcal{B}_k y_p = 0$ for $k \geq p$. This implies

$$\sum_{k=1}^{+\infty} \|\mathcal{B}_k y_p\| = 2 \sum_{k=1}^{p-1} \beta_k < \infty, \quad p \geq 2.$$

So $y_p \in \mathcal{D}(\mathcal{A})$ for every $p \geq 2$ and

$$\mathcal{A}y_p = (\eta_i)_{i \in \mathbb{N}} = \begin{cases} -\sum_{k=1}^{p-1} \beta_k, & i = 1, \\ \beta_k, & \text{for } i = 1 + 2^{k-1} \text{ and } k = 1, 2, \dots, p-1, \\ 0, & \text{otherwise.} \end{cases}$$

Incidentally, notice that $\mathcal{A}y_p$ has only a finite number of nonzero components and $\sum_{k=1}^{\infty} \eta_k = 0$. To summarize, we have constructed a sequence y_p , $p \geq 2$, of elements of $\mathcal{D}(\mathcal{A})$ converging to e_1 .

Now we show that a similar construction can be carried out for any e_i , $i \geq 2$. Fix $i \geq 2$ and observe that for x defined by (16)–(17) we have

$$\|\frac{1}{\xi_i}x\| = \left\| \left(\frac{1}{\xi_i}, \frac{\xi_2}{\xi_i}, \dots, \frac{\xi_{i-1}}{\xi_i}, \underbrace{1}_{i\text{-th}}, \frac{\xi_{i+1}}{\xi_i}, \dots \right) \right\| = \frac{1}{\xi_i}\|x\| < \infty. \quad (20)$$

Suppose that $i = 2^l + m$ for some $l \geq 0$ and $m \in \{1, 2, \dots, 2^l\}$. We define $z_p = (\xi_1, \xi_2, \dots)$ as follows. At first, let

$$\xi_i = 1, \quad \xi_{i+2^{k-1}} = \begin{cases} 0 & , \quad k = l+2, l+3, \dots, p-1, \\ \frac{\beta_k}{\alpha_k} & , \quad k \geq p, \end{cases} \quad (21)$$

where $p > i + 2^{l+1}$. For $k = 1, 2, \dots, l+1$, define

$$\begin{cases} \xi_{i+2^{k-1}} = 0, & \text{if } i \bmod 2^k \in \{1, 2, \dots, 2^{k-1}\}, \\ \xi_{i-2^{k-1}} = 0, & \text{if } i \bmod 2^k \in \{0, 2^{k-1}+1, \dots, 2^k-1\}. \end{cases}$$

In all other cases, elements of z_p are defined recursively by (17).

Notice that $\xi_j = 0$ for $j \in \{1, 2, \dots, i + 2^{p-1}\} \setminus \{i\}$ and that the coordinate-wise limit of z_p is e_i . Furthermore, it is clear from (20) that $\|z_p\| \leq \frac{1}{\xi_i}\|x\|$ and that $\lim_{p \rightarrow \infty} \|z_p - e_i\| = 0$. Computing $\mathcal{A}z_p$ in a similar way as we did $\mathcal{A}y_p$, we see that it has only a finite number of nonzero terms, meaning that $z_p \in \mathcal{D}(\mathcal{A})$. This completes the proof. $\square$

4 Conclusions

Recall that $(\mathcal{B}_n)_{n \in \mathbb{N}}$ is a sequence of bounded operators defined on $l^1(\mathbb{N})$ and $(e^{t\mathcal{B}_n})_{n \in \mathbb{N}}$ is the corresponding sequence of commuting Markov semigroups, i.e. for any $m, n \in \mathbb{N}$ and $x \in l^1(\mathbb{N})$, we have

$$e^{t\mathcal{B}_n}e^{s\mathcal{B}_m}x = e^{s\mathcal{B}_m}e^{t\mathcal{B}_n}x, \quad t, s \geq 0.$$

The above follows from the fact that $\mathcal{B}_n$'s are isomorphic images of B_n 's, see (6). Hence, for every $n \in \mathbb{N}$, $\mathcal{T}_n(t) = \prod_{k=1}^n e^{t\mathcal{B}_k}$ is a strongly continuous semigroup with the generator $\mathcal{A}_n$ given by (10).

The following is a consequence of Theorem 1.

Corollary 3 *The semigroups $\{\mathcal{T}_n(t), t \geq 0\}$ converge, when $n \rightarrow \infty$, to a strongly continuous Markov semigroup $\{\mathcal{T}(t), t \geq 0\}$. This limit semigroup is the Blackwell semigroup on $l^1(\mathbb{N})$. Furthermore, the generator $\mathcal{A}_B$ of $\{\mathcal{T}(t), t \geq 0\}$ is the closure of $\mathcal{A}$ given by (12)–(13). In other words, $\overline{\mathcal{A}} = \mathcal{A}_B$.*

Proof Theorem 2.1 of [2] specifies conditions under which an infinite product of commuting contraction semigroups exists. All those conditions are satisfied in our case (thanks to Theorem 1), especially the key one that $\mathcal{D}(\mathcal{A})$ is dense in $l^1(\mathbb{N})$. $\square$

Remark 1 Theorem 1 may be also considered as an alternative proof of existence of (4) to the proof presented in [2]. Since semigroups $\{\mathcal{T}_n(t), t \geq 0\}$ and $\{T_n(t), t \geq 0\}$ are isomorphic, the strong convergence of one of them implies the strong convergence of the other and vice versa. The proof of convergence of $T_n(t)$ presented in [2] can be termed as “a direct proof”.

Corollary 4 Suppose that $x \in \mathcal{D}(\mathcal{A})$ and denote $(\eta_i)_{i \in \mathbb{N}} = \mathcal{A}x$, Then

$$\sum_{i=1}^{\infty} \eta_i = 0. \quad (22)$$

Proof Notice that η_i ’s are the coordinate-wise limits of $\mathcal{A}_n x$, see (14) and (15). Hence, (22) is a consequence of (11). $\square$

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