Backbone Decomposition of Multitype Superprocesses

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Abstract

In this paper, we provide a construction of the so-called backbone decomposition for multitype supercritical superprocesses. While backbone decompositions are fairly well known for both continuous-state branching processes and superprocesses in the one-type case, so far no such decompositions or even description of prolific genealogies have been given for the multitype cases. Here we focus on superprocesses, but by turning the movement off, we get the prolific backbone decomposition for multitype continuous-state branching processes as an easy consequence of our results.

Keywords Multitype superprocesses · Multitype continuous-state branching processes · Non-local branching mechanism · Backbone · Conditioning on extinction · Prolific individuals

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1 Introduction

Let us consider a supercritical Galton–Watson process Z with offspring distribution given by $\{\pi(i), i \in \mathbb{N}\}$, where $\mathbb{N} = \{0, 1, 2, \dots\}$, satisfying $\sum_{i \geq 0} i\pi(i) \in (1, \infty]$ and

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with $\pi(0) > 0$. We write

$$g(s) = \sum_{i \geq 0} s^i \pi(i),$$

for the generating function of the offspring distribution. It is well known that the equation $g(s) = s$ has a unique root $\rho \in (0, 1)$ which coincides with the probability of extinction of Z starting from a single ancestor. It is also known that individuals can be classified into two categories: prolific and non-prolific. The prolific individuals are those who have at least one descendant in every subsequent generation, that is they are part of the surviving genealogies and form a Galton–Watson process Z_p with offspring distribution

$$\pi_p(i) = \sum_{j \geq i} \binom{j}{i} (1 - \rho)^{i-1} \rho^{j-i} \pi(j).$$

It is a well-understood phenomenon that, on the survival event, the process Z can be pathwise decomposed into the Galton–Watson process Z_p which is then decorated with independent copies of the original Galton–Watson process conditioned to become extinct, i.e., a subcritical Galton–Watson process with offspring distribution given by

$$\pi_e(i) = \rho^{i-1} \pi(i).$$

The process Z_p is known as the backbone, and this pathwise construction is called the backbone decomposition.

A natural generalisation of Galton–Watson processes is multitype branching particle systems, in other words a Markov process which encodes not only the size of the population, but also the location of each particle in space; additionally, it allows to distinguish between a finite number of particle types, where each type has a distinct probabilistic behaviour. More precisely, each particle, depending on its type, performs a random motion in space, and at the end of its lifetime, it gives birth to a random number of offspring in each type. The offspring start to evolve from their common birth site independently from each other, following the behaviour determined by their types.

Multitype superprocesses arise as scaling limits of multitype branching particle systems. Intuitively, one can think of a multitype superprocess as some mass (or cloud of infinitesimal particles) whose size, proportion of types and position evolves randomly in time. The dynamic of a superprocess depends on two quantities, its branching mechanism that explains how the mass is created and the semigroups associated with the movement of mass of each type. In other words, the infinitesimal particles move around in space according to a motion semigroup determined by their type. At the end of their lifetime, they create mass not only in their own type, but they also have the potential to immigrate mass into other types. This branching event is governed by a branching mechanism whose parameters depend on the type of the infinitesimal particle. If the branching mechanism is spatially independent, by turning the movement

off, we obtain the so-called multitype continuous-state branching processes (MCB processes). That is to say an MCB process describes how the total mass in each type evolves in a multitype superprocess.

The aim of this manuscript is to study a similar pathwise construction, as the one described by supercritical Galton–Watson process, for supercritical multitype superprocesses which have a positive probability of finite time extinction. In other words, we are interested on the prolific genealogies and provide a backbone decomposition for the aforementioned class of processes. Here, we focus on the case where the branching mechanisms are spatially independent and the spatial motions are described by conservative diffusions. Therefore, if we do not take into account the spatial component, we obtain the prolific backbone decomposition for MCB processes as a direct consequence of our results.

Intuitively, the main result of this paper can be described as follows: For a multitype superprocess, the *backbone* is a multitype branching particle system formed by all the prolific individuals as in the Galton–Watson case. The multitype superprocess can be recovered by dressing the backbone with three different sorts of immigration, all of them corresponding to independent copies of the original process conditioned to become extinct. The *continuous immigration* is described by a Poisson point process of independent conditioned processes along the backbone, where the rate of immigration depends on the type of the particle in the backbone and a random measure, known as Dynkin–Kuznetsov measure, which assigns zero initial mass to the immigration process. The *discontinuous immigration* is described again by a Poisson point process of independent conditioned processes along the backbone, where the rate of immigration is given by the law of the conditioned process with an initial randomised mass that depends on the type of the particle in the backbone. Finally, *at each branching time of the backbone*, independent copies of the conditioned process immigrate, with initial randomised mass that depends on the number of offspring of each type of the particle at the branching point.

The backbone decomposition for superprocesses has been studied before but only for the one-type case. Indeed, Evans and O’Connell [9] were the first in describing such decomposition for supercritical superprocesses with quadratic spatially independent branching mechanism and Markovian movement. Later, Engländer and Pinsky [7] provided a similar decomposition for the spatially dependent case. However, in [7] and [9] no pathwise construction was offered; they only provide a distributional one. The first pathwise backbone decomposition seems to be given in Salisbury and Verzani [18], who consider the case of conditioning a super-Brownian motion as it exits a given domain such that the exit measure contains at least n pre-specified points in its support. Another pathwise backbone decomposition for branching particle systems is given in Etheridge and Williams [8]. In Duquesne and Winkel [5], this decomposition was given for continuous-state branching processes satisfying Grey’s condition. Afterwards, Kyprianou and Ren [15] obtain a backbone decomposition for general continuous-state branching processes with immigration. Beresticky et. al. [3] worked with a general spatially independent branching mechanism and a diffusion process as the spatial movement. This result was generalized by Kyprianou et. al. [14] for a spatially dependent mechanism and by Chen. et. al. [4] for a spatially dependent mechanism and a symmetric Hunt movement. Finally, a backbone decomposition for

superdiffusions with non-local branching mechanism can be found in Murillo and Pérez [16].

Recently, some works have shown the robustness of the backbone decomposition for supercritical superprocesses as a mechanism for transferring results from branching particle systems into the setting of superprocesses. Kyprianou et. al. [13] use the backbone decomposition to characterize the growth in the right-most point in the support of a super-Brownian motion with a barrier, obtained analytical properties of the associated one-sided Fisher–Kolmogorov–Petrovskii–Piscounov wave equation, as well as the distribution of mass on the exit measure associated with the barrier. Another example can be found in Eckhoff et. al. [6], where the authors use the pathwise representation of a supercritical superdiffusion in terms of the backbone decomposition to derive the strong law of large numbers for a wide class of superdiffusions from the corresponding result for branching particle diffusions.

The remainder of the paper is structured as follows. In Sect. 2, we give a description of the objects involved in the backbone decomposition of the multitype superprocess. More precisely, we formally introduce the multitype superprocess and we study the event of extinction in order to describe the law of the process conditioned on becoming extinct. Then, we exhibit the prolific individuals and give a detailed description of the Dynkin–Kuznetsov measure. With all these elements, we present our main result which is the backbone decomposition for multitype superprocesses at the end of Sect. 2. For simplicity of the exposition, the proof of all the results are presented in Sect. 3.

2 Preliminaries and Main Result

In Subsect. 2.1, we first define the multitype superprocess and show that the total mass vector is a multitype continuous-state branching process. With this in hand, we then give a classification of the superprocess in terms of its asymptotic behaviour. In Subsect. 2.2, the event of extinction is studied and we give a simple condition for the process to go extinct in finite time with positive probability, but not almost surely. Then, the law of the multitype superprocess conditioned to become extinct is given. In Subsect. 2.3 and 2.4, we describe the prolific individuals and explain the Dynkin–Kuznetsov measure. In Subsect. 2.5, we put all the pieces together and present the backbone decomposition and our main results. Finally, in Subsect. 2.6 we provide some concluding remarks with directions for future research.

2.1 Multitype Superprocesses

Before we introduce multitype superprocesses and some of their properties, we first recall some basic notation. Let $\ell \in \mathbb{N}$ be a natural number, and set $S = \{1, 2, \dots, \ell\}$. We denote by $\mathcal{M}(\mathbb{R}^d)$, $\mathcal{B}(\mathbb{R}^d)$ and $\mathcal{B}^+(\mathbb{R}^d)$ the respective spaces of finite Borel measures, bounded Borel functions and positive bounded Borel functions on $\mathbb{R}^d$. The space $\mathcal{M}(\mathbb{R}^d)$ is endowed with the topology of weak convergence.

For $\mathbf{u}, \mathbf{v} \in \mathbb{R}^\ell$, we introduce $[\mathbf{u}, \mathbf{v}] = \sum_{j=1}^{\ell} u_j v_j$, and $\mathbf{u} \cdot \mathbf{v}$ as the vector with entries $(\mathbf{u} \cdot \mathbf{v})_j = u_j v_j$. For a matrix A , we denote by A^t its transpose. For any

$\mathbf{f} = (f_1, \dots, f_\ell)^\top \in \mathcal{B}(\mathbb{R}^d)^\ell$ and $\boldsymbol{\mu} = (\mu_1, \dots, \mu_\ell)^\top \in \mathcal{M}(\mathbb{R}^d)^\ell$, we define

$$\langle \mathbf{f}, \boldsymbol{\mu} \rangle := \sum_{i=1}^{\ell} \int_{\mathbb{R}^d} f_i(x) \mu_i(dx).$$

Furthermore, we also use $\|\mathbf{u}\| := [\mathbf{u}, \mathbf{u}]^{1/2}$ for the Euclidian norm of any $\mathbf{u} \in \mathbb{R}^\ell$, and $\|\boldsymbol{\mu}\| := \langle \mathbf{1}, \boldsymbol{\mu} \rangle$ for the total mass of the measure $\boldsymbol{\mu}$.

Suppose that for any $i \in S$, the process $\xi^{(i)} = (\xi_t^{(i)}, t \geq 0)$ is a conservative diffusion with transition semigroup $\mathbb{P}^{(i)} = (\mathbb{P}_t^{(i)}, t \geq 0)$ on $\mathbb{R}^d$. The collection of transition semigroups is denoted by $\mathbf{P} = \{\mathbb{P}^{(i)}, i \in S\}$. We also introduce a vectorial function $\boldsymbol{\psi} : S \times \mathbb{R}_+^\ell \rightarrow \mathbb{R}^\ell$ such that

$$\begin{aligned} \psi(i, \boldsymbol{\theta}) &:= -[\boldsymbol{\theta}, \mathbf{B}\mathbf{e}_i] + \beta_i \theta_i^2 + \int_{\mathbb{R}_+^\ell} \left(e^{-[\boldsymbol{\theta}, \mathbf{y}]} - 1 + \theta_i y_i \right) \Pi(i, d\mathbf{y}), \\ \boldsymbol{\theta} &\in \mathbb{R}_+^\ell, i \in S, \end{aligned} \quad (2.1)$$

where $\mathbf{B}$ is an $\ell \times \ell$ real valued matrix such that $B_{ij} \mathbf{1}_{\{i \neq j\}} \in \mathbb{R}_+$, $\{\mathbf{e}_1, \dots, \mathbf{e}_\ell\}$ is the natural basis in $\mathbb{R}^\ell$, $\beta_i \in \mathbb{R}_+$, and Π is a measure satisfying the following integrability condition

$$\int_{\mathbb{R}_+^\ell \setminus \{\mathbf{0}\}} \left((|\mathbf{y}| \wedge |\mathbf{y}|^2) + \sum_{j \in S} \mathbf{1}_{\{j \neq i\}} y_j \right) \Pi(i, d\mathbf{y}) < \infty, \quad \text{for } i \in S.$$

We call the vectorial function $\boldsymbol{\psi}$ the branching mechanism, and we also refer to Π as its associated Lévy measure. Now we are ready to state the definition of a multitype superprocess.

Definition 1 A strong Markov process $X = (X_t, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}_\mu)$ with state space $\mathcal{M}(\mathbb{R}^d)^\ell$ is called a $(\mathbf{P}, \boldsymbol{\psi})$ -multitype superprocess (with ℓ types) if it satisfies

$$\mathbb{E}_\mu \left[e^{-\langle \mathbf{f}, X_t \rangle} \right] = \exp \left\{ -\langle \mathbf{V}_t \mathbf{f}, \boldsymbol{\mu} \rangle \right\}, \quad \text{for any } \mathbf{f} \in \mathcal{B}^+(\mathbb{R}^d)^\ell \text{ and } \boldsymbol{\mu} \in \mathcal{M}(\mathbb{R}^d)^\ell, \quad (2.2)$$

where $\mathbf{V}_t \mathbf{f}(x) = (V_t^{(1)} \mathbf{f}(x), \dots, V_t^{(\ell)} \mathbf{f}(x))^\top : \mathbb{R}^d \rightarrow \mathbb{R}_+^\ell$ is the unique locally bounded solution to the vectorial integral equation

$$V_t^{(i)} \mathbf{f}(x) = \mathbb{P}_t^{(i)} f_i(x) - \int_0^t ds \int_{\mathbb{R}^d} \psi(i, \mathbf{V}_{t-s} \mathbf{f}(y)) \mathbb{P}_s^{(i)}(x, d\mathbf{y}), \quad i \in S. \quad (2.3)$$

Intuitively, equation (2.2) describes the evolution of a cloud of infinitesimal particles that move around in space. Each particle has a type, where this type determines not only the particle's movement semigroup, but also its lifetime and its branching mechanism. An infinitesimal particle of type i moves around according to the semigroup $\mathbb{P}^{(i)}$, and

at the end of its lifetime it branches according to the branching mechanism $\psi(i, \cdot)$. This branching event creates mass not only in type i , but it also has the potential to immigrate mass into the other types. The infinitesimal particles of the created mass evolve independently of each other, each following the behaviour determined by its type.

The following proposition shows the existence of a multitype superprocesses associated to $(\mathbf{P}, \psi)$. Even though its proof is based on similar arguments as those found in the proof of Theorem 6.4 in Li [17], for completeness we present the proof in Subsect. 3.1.

Proposition 1 *Let ψ be a branching mechanism given by (2.1) and $\mathbf{P}$ be a collection of the transition semigroups of some conservative diffusions. Then, there exists a unique $(\mathbf{P}, \psi)$ -multitype superprocess.*

Next we want to understand the asymptotic behaviour of the multitype superprocess. In order to do so, we will make use of the fact that if the branching mechanism is given as in (2.1), then the total mass vector of the superprocess is a MCB process (see Lemma 1).

First, recall that a *multitype continuous-state branching process*, $\mathbf{Y} = (\mathbf{Y}_t, t \geq 0)$, with branching mechanism ψ can be characterised through its Laplace transform. If we denote by $\mathbf{P}_y$ the law of such a process with initial state $y \in \mathbb{R}_+^\ell$, then

$$\mathbf{E}_y \left[e^{-[\theta, \mathbf{Y}_t]} \right] = \exp \left\{ -[\mathbf{y}, \mathbf{v}_t(\theta)] \right\}, \quad \text{for } \theta \in \mathbb{R}_+^\ell, t \geq 0, \quad (2.4)$$

where $\mathbf{v}_t(\theta) = (v_t(1, \theta), \dots, v_t(\ell, \theta))^\top$ is the unique locally bounded solution, with nonnegative entries, to the system of integral equations

$$\mathbf{v}_t(i, \theta) = \theta_i - \int_0^t \psi(i, \mathbf{v}_{t-s}(\theta)) ds, \quad i \in S. \quad (2.5)$$

Lemma 1 *Suppose that $(X_t, \mathbb{P}_\mu)_{t \geq 0}$ is a $(\mathbf{P}, \psi)$ -multitype superprocess and define the total mass vector as $\mathbf{Y} = (\mathbf{Y}_t, t \geq 0)$ with entries*

$$Y_t(i) = X_t(i, \mathbb{R}^d) = \int_{\mathbb{R}^d} X_t(i, dx), \quad t \geq 0,$$

and initial vector $\mathbf{y} = (y_1, \dots, y_\ell)^\top$, where $y_i = \mu_i(\mathbb{R}^d)$, $i \in S$. Then, $(\mathbf{Y}, \mathbf{P}_y)$ is a MCB process.

The proof of this result will be given in Subsect. 3.1. Using the previous lemma, we can now give a classification of the multitype superprocess through its first moments and thus describe its asymptotic behaviour. More precisely, denote by $\mathbf{M}(t)$ the $\ell \times \ell$ matrix with elements

$$M(t)_{ij} = \mathbb{E}_{e_i \delta_x} [\langle e_j, X_t \rangle] = \mathbf{E}_{e_i} [\langle e_j, \mathbf{Y}_t \rangle], \quad i, j \in S,$$

where $\mathbf{e}_i \delta_x$ denotes a measure valued vector that has unit mass at position $x \in \mathbb{R}^d$, in the i -th coordinate, and zero mass everywhere else. Barczy et al. [1] (see Lemma 3.4) proved that for all $t > 0$,

$$\mathbf{M}(t) = \mathbf{e}^t \tilde{\mathbf{B}}^t,$$

where the matrix $\tilde{\mathbf{B}}$ is the linear part of $\boldsymbol{\psi}$ in (2.1), i.e.,

$$\tilde{B}_{i,j} = B_{i,j} + \int_{\mathbb{R}_+^\ell} (y_i - \delta_{i,j} y_j) \Pi(j, d\mathbf{y}). \quad (2.6)$$

Moreover, after straightforward computations (see for instance the computations after identity (2.15) in [1]) we observe that the branching mechanism $\boldsymbol{\psi}$ can be rewritten as follows

$$\psi(i, \boldsymbol{\theta}) = -[\boldsymbol{\theta}, \tilde{\mathbf{B}} \mathbf{e}_i] + \beta_i \theta_i^2 + \int_{\mathbb{R}_+^\ell} \left(e^{-[\boldsymbol{\theta}, \mathbf{y}]} - 1 + [\boldsymbol{\theta}, \mathbf{y}] \right) \Pi(i, d\mathbf{y}), \quad \boldsymbol{\theta} \in \mathbb{R}_+^\ell, i \in S.$$

Suppose that the matrix $\tilde{\mathbf{B}}$ is irreducible. Then, a Perron–Frobenius-type result (see Appendix A of [2]) guarantees that there exists a unique leading eigenvalue Γ , and right and left eigenvectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}_+^\ell$, whose coordinates are strictly positive such that, for $t \geq 0$,

$$\tilde{\mathbf{B}}^t \mathbf{u} = \Gamma \mathbf{u}, \quad \mathbf{v}^t \tilde{\mathbf{B}}^t = \Gamma \mathbf{v}^t, \quad \mathbf{M}(t) \mathbf{u} = \mathbf{e}^{\Gamma t} \mathbf{u}, \quad \text{and} \quad \mathbf{v}^t \mathbf{M}(t) = \mathbf{e}^{\Gamma t} \mathbf{v}^t.$$

Moreover, Γ determines the long-term behaviour of X . In Kyprianou and Palau [11], the authors show that when $\Gamma \leq 0$ the total mass goes to zero almost surely. On the other hand, Barczy and Pap [2] show that if $\Gamma > 0$, then the total mass process satisfies

$$\lim_{t \rightarrow \infty} e^{-\Gamma t} \mathbf{E}_{\mathbf{e}_i} [\mathbf{Y}_t] = \mathbf{e}_i^t \mathbf{u} \mathbf{v}^t, \quad \text{for } i \in S,$$

which is a nonzero vector. Therefore, employing the same terminology as in the one-type case, we call the process *supercritical*, *critical* or *subcritical* accordingly as Γ is strictly positive, equal to zero or strictly negative.

In the remainder of this manuscript, we assume that the process is supercritical.

2.2 The supercritical multitype superprocess conditioned to become extinct

In order to construct the supercritical multitype superprocess conditioned to become extinct, we first study the event of extinction. Recall that $\mathcal{E} := \{\|\mathbf{X}_t\| = 0 \text{ for some } t > 0\}$, denotes the event of *extinction* and take $w_i : \mathbb{R}^d \mapsto \mathbb{R}_+$ as

$$w_i(x) := -\log \mathbb{P}_{\mathbf{e}_i \delta_x}(\mathcal{E}), \quad i \in S. \quad (2.7)$$

Since the branching mechanism is spatially independent, and the total mass vector of X is a MCB process, we get that $w_i(x) = w_i$, for all $x \in \mathbb{R}^d$, for some constant w_i .

In what follows, we assume

$$0 < w_i < \infty, \quad \text{for all } i \in S. \quad (2.8)$$

This condition allows us to have a positive probability (but less than one) of extinction at any finite time, which is necessary in the construction of the multitype superprocess conditioned on becoming extinct, and also in the construction of the Dynkin-Kuznetsov measure, as we will see later.

Proposition 2 *Suppose that condition (2.8) holds. Then $\psi(w) = 0$. Moreover, for $x \in \mathbb{R}^d$, $i \in S$ and $t > 0$, we have that*

$$0 < \mathbb{P}_{e_i \delta_x}(\|X_t\| = 0) < 1.$$

For simplicity of the exposition, the proof of this result is presented in Subsect. 3.2. Note that similar assumptions have been used in most of the cases where backbones have been constructed. For instance in [3] and [5], the authors assume Grey's condition which is equivalent to w_i being finite. In [4,6,12,16], a very similar condition appears for the spatially dependent case. Note that assumption (2.8) is not very restrictive, as we can see in the next lemma, where we give a relatively large class of multitype superprocesses satisfying assumption (2.8). The proof of this lemma is given in Subsect. 3.2.

Lemma 2 *Suppose that $(X_t, \mathbb{P}_\mu)_{t \geq 0}$ is a supercritical multitype superprocess (i.e., $\Gamma > 0$) with branching mechanism given by (2.1). If $\beta := \inf_{i \in S} \beta_i > 0$ and the following “ $x \log x$ ”-condition*

$$\sum_{i=1}^{\ell} \int_{\{1 \leq \langle \mathbf{1}, \mathbf{y} \rangle < \infty\}} \langle \mathbf{1}, \mathbf{y} \rangle \log(\langle \mathbf{1}, \mathbf{y} \rangle) \Pi(i, d\mathbf{y}) < \infty, \quad (2.9)$$

are satisfied, then assumption (2.8) holds.

Now, we are ready to present the main result of this subsection. It states that the multitype superprocess conditioned on becoming extinct is also a multitype superprocess, whose branching mechanism can be given in terms of ψ .

Proposition 3 *Suppose that X is a multitype superprocess that satisfies assumption (2.8). For each $\mu \in \mathcal{M}(\mathbb{R}^d)^\ell$, define the law of X with initial configuration μ conditioned on becoming extinct by $\mathbb{P}_\mu^\dagger$, and let $\mathcal{F}_t := \sigma(X_s, s \leq t)$. Specifically, for all events A , measurable with respect to $\mathcal{F}$,*

$$\mathbb{P}_\mu^\dagger(A) = \mathbb{P}_\mu(A | \mathcal{E}).$$

Then, for all $\mathbf{f} \in \mathcal{B}^+(\mathbb{R}^d)^\ell$

$$\mathbb{E}_\mu^\dagger \left[e^{-\langle \mathbf{f}, \mathbf{X}_t \rangle} \right] = \exp \left\{ -\langle \mathbf{V}_t^\dagger \mathbf{f}, \boldsymbol{\mu} \rangle \right\},$$

where

$$V_t^{\dagger, (i)} \mathbf{f}(x) := V_t^{(i)}(\mathbf{f} + \mathbf{w})(x) - w_i, \quad i \in S,$$

is the unique locally bounded solution to

$$V_t^{\dagger, (i)} \mathbf{f}(x) = P_t^{(i)} f_i(x) - \int_0^t ds \int_{\mathbb{R}^d} \psi^\dagger(i, V_{t-s}^\dagger \mathbf{f}(y)) P_s^{(i)}(x, dy), \quad i \in S, \quad (2.10)$$

where $\psi^\dagger(\lambda) := \psi(\lambda + \mathbf{w})$ and $\mathbf{w}$ is given by (2.7). In other words, $(X, \mathbb{P}_\mu^\dagger)$ is a $(\mathbf{P}, \psi^\dagger)$ -multitype superprocess.

The proof of this result is presented in Subsect. 3.2.

2.3 Prolific individuals

In the previous subsection, we described the multitype superprocess conditioned to become extinct, which is going to be used as the dressing of the backbone. Here we define the backbone itself, in other words we consider those individuals of the superprocess who are responsible for the infinite growth of the process. In our case, the so-called prolific individuals, i.e., those with an infinite genealogical line of descent, form a multitype branching particle system where the particles move according to the same motion semigroups as the superprocess itself, and their branching generator can be expressed in terms of the branching mechanism of the superprocess.

In order to describe the branching particle system that forms the backbone of the multitype superprocess, we first provide the offspring distribution of these prolific individuals. The proof of this result is deferred to Subsect. 3.3.

Lemma 3 For every $1 \leq i \leq \ell$, define the family $\{p_{j_1, \dots, j_\ell}^{(i)}, (j_1, \dots, j_\ell) \in \mathbb{N}^\ell\}$ as

$$\begin{aligned} p_{j_1, \dots, j_\ell}^{(i)} = & \frac{1}{w_i q_i} \left(\beta_i w_i^2 \mathbf{1}_{\{j=2e_i\}} + (B_{ki} w_k \right. \\ & + \left. \int_{\mathbb{R}_+^\ell} w_k y_k e^{-[\mathbf{w}, \mathbf{y}]} \Pi(i, d\mathbf{y}) \right) \mathbf{1}_{\{j=e_k\}} \mathbf{1}_{\{i \neq k\}} \\ & + \left. \int_{\mathbb{R}_+^\ell} \frac{(w_1 y_1)^{j_1} \dots (w_\ell y_\ell)^{j_\ell}}{j_1! \dots j_\ell!} e^{-[\mathbf{w}, \mathbf{y}]} \Pi(i, d\mathbf{y}) \mathbf{1}_{\{j_1 + \dots + j_\ell \geq 2\}} \right), \end{aligned} \quad (2.11)$$

where $\mathbf{j} = (j_1, \dots, j_\ell)$. Then, $\{p_j^{(i)}, \mathbf{j} \in \mathbb{N}^\ell\}$ is a probability distribution, for every $1 \leq i \leq \ell$.

Let $\mathbf{Z} = (\mathbf{Z}_t, t \geq 0)$ be a multitype branching particle system (MBPS) with ℓ types, where the movement of particles is given by the collection of semigroups $\mathbb{P} = \{\mathbb{P}^{(i)}, i \in S\}$, the offspring distributions are given by $\{p_j^{(i)}, j \in \mathbb{N}^\ell\}$, for every $1 \leq i \leq \ell$, while the branching rate $\mathbf{q} \in \mathbb{R}_+^\ell$ takes the form

$$q_i = \left. \frac{\partial}{\partial x_i} \psi(i, \mathbf{x}) \right|_{\mathbf{x}=\mathbf{w}}, \quad i \in S, \quad (2.12)$$

where $\mathbf{w}$ was defined in (2.7). More precisely, a particle of type $i \in S$ moves according to the semigroup $\mathbb{P}^{(i)}$, branches at rate q_i , and has offspring distribution given by $\{p_j^{(i)}, j \in \mathbb{N}^\ell\}$.

With these elements, we can describe the branching generator of $\mathbf{Z}$, which is a function $\mathbf{F}(\mathbf{s}) = (F_1(\mathbf{s}), \dots, F_\ell(\mathbf{s}))^\top$, $\mathbf{s} \in [0, 1]^\ell$ given by

$$F_i(\mathbf{s}) = q_i \sum_{j \in \mathbb{N}^\ell} (s_1^{j_1} \dots s_\ell^{j_\ell} - s_i) p_{j_1, \dots, j_\ell}^{(i)} = \frac{1}{w_i} \psi(i, \mathbf{w} \cdot (\mathbf{1} - \mathbf{s})), \quad i \in S, \quad (2.13)$$

where we recall that $\mathbf{1}$ denotes the vector with value 1 in each coordinate and $\mathbf{u} \cdot \mathbf{v}$ is the element-wise multiplication of the vectors $\mathbf{u}$ and $\mathbf{v}$.

The intuition behind the process $\mathbf{Z}$ is as follows. A particle of type i from its birth executes a $\mathbb{P}^{(i)}$ motion and after an independent and exponentially distributed random time with parameter q_i dies and gives birth at its death position to a random number of offspring with distribution $\{p_j^{(i)}, j \in \mathbb{N}^\ell\}$. The offspring particles will then evolve independently of each other, each following the behaviour specified by its type. We call $\mathbf{Z}$ the *backbone* of the multitype superprocess $\mathbf{X}$ and denote its initial distribution by $\mathbf{v} \in \mathcal{M}_a(\mathbb{R}^d)^\ell$, where $\mathcal{M}_a(\mathbb{R}^d)$ denotes the space of atomic measures on $\mathbb{R}^d$. Comparing the form of the offspring distribution between the one-type case and the multitype case, the main difference is that now we are allowed to have one offspring at a branching event. However, in this case, that offspring has to have a different type from its parent.

2.4 Dynkin-Kuznetsov measure

A final key ingredient in the construction of the backbone decomposition is the so-called Dynkin–Kuznetsov measure. It provides a way to dress the backbone when assigning zero initial mass to the immigration process. It is important to note that the existence of such measures was taken for granted in most of the references that appear in the literature, in particular in [3, 6, 12, 16]. Fortunately, from their assumptions and the way that the dressing processes are constructed this omission does not play an important role in the validity of their results. Here, we provide a rigorous argument for their existence. See also Chen et al. [4] for the study of Dynkin–Kuznetsov measures for one-type superprocesses with non-local branching mechanism.

Let us denote by $\mathcal{X}$ the space of càdlàg paths from $[0, \infty)$ to $\mathcal{M}(\mathbb{R}^d)^\ell$.

Proposition 4 *Let X be a $(\mathbf{P}, \psi)$ -multitype superprocess satisfying (2.8). For $x \in \mathbb{R}^d$, there exists a measure $\mathbb{N}_{xe_i}$ on the space $\mathcal{X}$ satisfying*

$$\mathbb{N}_{xe_i} \left(1 - e^{-\langle f, X_t \rangle} \right) = -\log \mathbb{E}_{\delta_x e_i} \left[e^{-\langle f, X_t \rangle} \right], \quad (2.14)$$

for all $f \in \mathcal{B}(\mathbb{R}^d)^\ell$ and $t \geq 0$.

Again, for simplicity of exposition, we provide the proof of this Proposition in Subsect. 3.4.

Following the same terminology as in the literature, we call $\{(\mathbb{N}_{xe_i}, x \in \mathbb{R}^d), i \in S\}$ the Dynkin–Kuznetsov measures. We denote by $\mathbb{N}^\dagger$ the Dynkin–Kuznetsov measures associated to the multitype superprocess conditioned on extinction, which are also well defined (see the discussion after the proof of Proposition 4).

2.5 The backbone decomposition

Now we are ready to put all pieces together. Let X be a $(\mathbf{P}, \psi)$ -multitype superprocess satisfying (2.8). Recall that our primary aim is to give a pathwise decomposition of X along its embedded backbone Z . The main idea is to dress the process Z with immigration, where the processes we immigrate are independent copies of the $(\mathbf{P}, \psi^\dagger)$ -multitype superprocess. The dressing relies on three different types of immigration mechanisms. These are two types of Poissonian immigrations along the life span of each prolific individual, and an additional creation of mass at the branch points of the embedded particle system. In the first dressing, we immigrate independent copies of the $(\mathbf{P}, \psi^\dagger)$ -multitype superprocess, where the immigration rate along a particle of type $i \in S$ is related to a subordinator in $\mathbb{R}_+^\ell$, whose Laplace exponent is given by

$$\begin{aligned} \phi(i, \lambda) &= \left. \frac{\partial}{\partial x_i} \psi^\dagger(i, \mathbf{x}) \right|_{\mathbf{x}=\lambda} - \left. \frac{\partial}{\partial x_i} \psi^\dagger(i, \mathbf{x}) \right|_{\mathbf{x}=\mathbf{0}} \\ &= \left. \frac{\partial}{\partial x_i} \psi(i, \mathbf{x}) \right|_{\mathbf{x}=\lambda+\mathbf{w}} - \left. \frac{\partial}{\partial x_i} \psi(i, \mathbf{x}) \right|_{\mathbf{x}=\mathbf{w}}, \end{aligned}$$

which can be rewritten as

$$\phi(i, \lambda) = 2\beta_i \lambda_i + \int_{\mathbb{R}_+^\ell} \left(1 - e^{-[\lambda, \mathbf{y}]} \right) y_i e^{-[\mathbf{w}, \mathbf{y}]} \Pi(i, d\mathbf{y}). \quad (2.15)$$

When an individual of type $i \in S$ has branched and its offspring is given by $\mathbf{j} = (j_1, \dots, j_\ell) \in \mathbb{N}^\ell$, we immigrate an independent copy of the $(\mathbf{P}, \psi^\dagger)$ -multitype superprocess where the initial mass has distribution

$$\begin{aligned} \eta_j^{(i)}(\mathbf{dy}) = & \frac{1}{w_i q_i p_j} \left(\beta_i w_i^2 \mathbf{1}_{\{j=2e_i\}} \delta_0(\mathbf{dy}) \right. \\ & + \left(B_{ki} w_k \delta_0(\mathbf{dy}) + w_k y_k e^{-[\mathbf{w}, \mathbf{y}]} \Pi(i, \mathbf{dy}) \right) \mathbf{1}_{\{j=e_k\}} \mathbf{1}_{\{i \neq k\}} \\ & \left. + \frac{(w_1 y_1)^{j_1} \dots (w_\ell y_\ell)^{j_\ell}}{j_1! \dots j_\ell!} e^{-[\mathbf{w}, \mathbf{y}]} \Pi(i, \mathbf{dy}) \mathbf{1}_{\{j_1 + \dots + j_\ell \geq 2\}} \right). \end{aligned} \quad (2.16)$$

Before we state our main results, we recall and introduce some notation. Recall that $\mathcal{X}$ denotes the space of càdlàg paths from $[0, \infty)$ to $\mathcal{M}(\mathbb{R}^d)^\ell$. Similarly to the one-type case, we use an Ulam–Harris labelling to reference the particles, and we denote the obtained tree by $\mathcal{T}$. For a particle $u \in \mathcal{T}$, let γ_u denote the type of the particle, τ_u its birth time, σ_u its death time, and $z_u(t)$ its spatial position at time t (whenever $\tau_u \leq t < \sigma_u$).

Definition 2 For $\mathbf{v} \in \mathcal{M}_a(\mathbb{R}^d)^\ell$, let $\mathbf{Z}$ be a MBPS with initial configuration $\mathbf{v}$, and let $\tilde{\mathbf{X}}$ be an independent copy of $\mathbf{X}$ under $\mathbb{P}_\mu^\dagger$. We define the stochastic process $\mathbf{\Lambda} = (\mathbf{\Lambda}_t, t \geq 0)$ on $\mathcal{M}(\mathbb{R}^d)^\ell$ by

$$\mathbf{\Lambda} = \tilde{\mathbf{X}} + \mathbf{I}^{\mathbb{N}^\dagger} + \mathbf{I}^{\mathbb{P}^\dagger} + \mathbf{I}^\eta,$$

where the processes $\mathbf{I}^{\mathbb{N}^\dagger} = (\mathbf{I}_t^{\mathbb{N}^\dagger}, t \geq 0)$, $\mathbf{I}^{\mathbb{P}^\dagger} = (\mathbf{I}_t^{\mathbb{P}^\dagger}, t \geq 0)$, and $\mathbf{I}^\eta = (\mathbf{I}_t^\eta, t \geq 0)$ are independent of $\tilde{\mathbf{X}}$ and, conditionally on $\mathbf{Z}$, are mutually independent. Moreover, these three processes are described pathwise as follows.

i) **Continuous immigration.** The process $\mathbf{I}^{\mathbb{N}^\dagger}$ is $\mathcal{M}(\mathbb{R}^d)^\ell$ -valued such that

$$\mathbf{I}_t^{\mathbb{N}^\dagger} = \sum_{u \in \mathcal{T}} \sum_{t \wedge \tau_u \leq r < t \wedge \sigma_u} \mathbf{X}_{t-r}^{(1,u,r)},$$

where given $\mathbf{Z}$ independently for each $u \in \mathcal{T}$ such that $\tau_u < t$, the processes $\mathbf{X}^{(1,u,r)}$ are countable in number and correspond to $\mathcal{X}$ -valued Poissonian immigration along the space-time trajectory $\{(z_u(r), r), r \in [\tau_u, t \wedge \sigma_u)\}$ with rate $2\beta\gamma_u \mathbf{dr} \times \mathbf{dN}_{z_u(r)}^\dagger e_{\gamma_u}$.

ii) **Discontinuous immigration.** The process $\mathbf{I}^{\mathbb{P}^\dagger}$ is $\mathcal{M}(\mathbb{R}^d)^\ell$ -valued such that

$$\mathbf{I}_t^{\mathbb{P}^\dagger} = \sum_{u \in \mathcal{T}} \sum_{t \wedge \tau_u \leq r < t \wedge \sigma_u} \mathbf{X}_{t-r}^{(2,u,r)}$$

where given $\mathbf{Z}$ independently for each $u \in \mathcal{T}$ such that $\tau_u \leq t$, the processes $\mathbf{X}^{(2,u,r)}$ are countable in number and correspond to $\mathcal{X}$ -valued, Poissonian immigration along the space-time trajectory $\{(z_u(r), r), r \in [\tau_u, t \wedge \sigma_u)\}$ with rate

$$\mathbf{dr} \times \int_{\mathbf{y} \in \mathbb{R}_+^\ell} y_{\gamma_u} e^{-[\mathbf{w}, \mathbf{y}]} \Pi(\gamma_u, \mathbf{dy}) \times \mathbf{dP}_{\mathbf{y}\delta_{z_u(r)}}^\dagger.$$

iii) **Branch point-based immigration.** The process I^η is $\mathcal{M}(\mathbb{R}^d)^\ell$ -valued such that

$$I_t^\eta = \sum_{u \in \mathcal{T}} \mathbf{1}_{\{\sigma_u \leq t\}} X_{t-\sigma_u}^{(3,u)}$$

where given Z independently for each $u \in \mathcal{T}$ such that $\sigma_u \leq t$, the process $X_{t-\sigma_u}^{(3,u)}$ is an independent copy of X issued at time σ_u with law $\mathbb{P}_{Y_u \delta_{zu}(\sigma_u)}$ where Y_u is an independent random variable with distribution $\eta_{\mathcal{N}_1^u, \dots, \mathcal{N}_\ell^u}^{(\gamma_u)}(\mathrm{d}y)$. Here $(\mathcal{N}_1^u, \dots, \mathcal{N}_\ell^u)$ is the offspring of u , i.e., $\mathcal{N}_i^u$ is the number of offspring of type i .

Moreover, we denote the law of the pair (Λ, Z) by $\widehat{\mathbb{P}}_{(\mu, \nu)}$.

Since Z is a MBPS and, given Z , immigrating mass occurs independently according to a Poisson point process or at the splitting times of Z , we can deduce that the process $((\Lambda, Z), \widehat{\mathbb{P}}_{(\mu, \nu)})$ is Markovian. It is important to note that the mass which has immigrated up to a fixed time evolves in a Markovian way thanks to the branching property.

Now we are ready to state the main results of the paper. Our first result determines the law of the couple (Λ, Z) , and in particular shows that Λ is conservative.

Theorem 1 For $\mu \in \mathcal{M}(\mathbb{R}^d)^\ell$, $\nu \in \mathcal{M}_a(\mathbb{R}^d)^\ell$, $f, h \in \mathcal{B}^+(\mathbb{R}^d)^\ell$, and $t \geq 0$ we have

$$\widehat{\mathbb{E}}_{(\mu, \nu)} \left[e^{-\langle f, \Lambda_t \rangle - \langle h, Z_t \rangle} \right] = \exp \left\{ -\langle V_t^\dagger f, \mu \rangle - \langle U_t^{(f)} h, \nu \rangle \right\}, \quad (2.17)$$

where $V^\dagger$ is defined in (2.10), and $\exp\{-U_t^{(f)} h(x)\} = (\exp\{-U_t^{(f,1)} h(x)\}, \dots, \exp\{-U_t^{(f,\ell)} h(x)\})^\top : \mathbb{R}^d \rightarrow \mathbb{R}_+^\ell$ is the unique $[0, 1]^\ell$ -valued solution to the system of integral equations

$$\begin{aligned} e^{-U_t^{(f,i)} h(x)} &= P_t^{(i)} e^{-h_i(x)} + \frac{1}{w_i} \int_0^t \mathrm{d}s \int_{\mathbb{R}^d} \left[\psi^\dagger \left(i, -w \cdot e^{-U_{t-s}^{(f)} h(y)} + V_{t-s}^\dagger f(y) \right) \right. \\ &\quad \left. - \psi^\dagger \left(i, V_{t-s}^\dagger f(y) \right) \right] P_s^{(i)}(x, \mathrm{d}y) \end{aligned} \quad (2.18)$$

for $x \in \mathbb{R}^d$, and $t \geq 0$. In particular, for each $t \geq 0$, Λ_t has almost surely finite mass.

Finally, we state the main result of this paper which, actually, is a consequence of Theorem 1. To be more precise, we consider a randomised version of the law $\widehat{\mathbb{P}}_{(\nu, \mu)}$ by replacing the deterministic choice of ν in such a way that for each $i \in S$, ν_i is a Poisson random measure in $\mathbb{R}^d$ having intensity $w_i \mu_i$. The resulting law is denoted by $\widehat{\mathbb{P}}_\mu$.

Theorem 2 For any $\mu \in \mathcal{M}(\mathbb{R}^d)^\ell$ the process $(\Lambda, \widehat{\mathbb{P}}_\mu)$ is Markovian and has the same law as $(X, \mathbb{P}_\mu)$.

The proofs of both Theorems are presented in Subsect. 3.5.

2.6 Concluding Remarks

To summarise, in this paper we construct the backbone decomposition for supercritical multitype superprocesses which have a positive probability of finite time extinction (see assumption (2.8)). Here, we assume that the branching mechanism is spatially independent and that the spatial motions correspond to conservative diffusions. In particular, by turning the movement off, we obtain the backbone decomposition for the so-called multitype continuous-state branching processes (MCB processes). Up to our knowledge, this seems to be the first study of the prolific genealogies of multitype branching processes.

There are various venues of future research. First of all, a very natural question (but challenging) is to deduce the backbone decomposition for multitype superprocesses with spatially dependent mechanisms. In this case, the total mass is not a MCB process anymore and implicitly there is no intuition of how the prolific genealogies can be studied and how the genealogy tree must be dressed since the event of extinction has not been studied in depth.

Another important and interesting problem regarding the assumptions of this paper is associated with the event of extinction. More precisely, in Lemma 2 we provide a sufficient (but not necessary) condition for assumption (2.8) to hold. In the one-type supercritical case, it turns out that Grey's condition, i.e.,

$$\int^{\infty} \frac{dx}{\psi(x)} < \infty,$$

where ψ is the branching mechanism, is a necessary and sufficient condition for the probability of extinction to be positive. Thus it will be very interesting to find, in the supercritical case, an analogue of Grey's condition which is necessary and sufficient for assumption (2.8).

Finally, it will be very interesting to construct the backbone when condition (2.8) is not satisfied. Actually, this condition is necessary for the construction of the process conditioned to become extinct and the construction of the Dynkin–Kuznetsov measure. An alternative for the former is to construct the process conditioned to become extinguished (i.e., the total mass goes to zero at large times). On the contrary, the Dynkin–Kuznetsov measure seems to be more complicated to construct. For the one-type supercritical case, the Dynkin–Kuznetsov measure is only needed when the quadratic term, i.e., β , is positive which implies condition (2.8). In the multitype supercritical case, the main problem will be to construct the Dynkin–Kuznetsov measure when some of the β_i 's can be zero but not all of them.

3 Proofs

Here we present the proofs of the results in the previous section. To facilitate the exposition of the proofs, we are going to separate them into the same subsections as their statements.

3.1 Proofs for Subsection 2.1

In this section, we prove Proposition 1 and Lemma 1.

Proof of Proposition 1 Recall that $(P_t^{(i)}, t \geq 0)$ denotes the semigroup of the diffusion $(\xi_t^{(i)}, t \geq 0)$. We introduce $\Xi = (\Xi_t, t \geq 0)$, a Markov process in the product space $\mathbb{R}^d \times S$ whose transition semigroup $(T_t, t \geq 0)$ is given by

$$T_t f(x, i) = \int_{\mathbb{R}^d} f(y, i) P_t^{(i)}(x, dy) \quad \text{for } x \in \mathbb{R}^d, \quad (3.1)$$

where f is a bounded Borel function on $\mathbb{R}^d \times S$. We denote the aforementioned set of functions by $\mathcal{B}(\mathbb{R}^d \times S)$ and we use $\mathcal{M}(\mathbb{R}^d \times S)$ for the space of finite Borel measures on $\mathbb{R}^d \times S$, endowed with the topology of weak convergence.

For each $f \in \mathcal{B}(\mathbb{R}^d \times S)$, we introduce the operator

$$\Psi(x, i, f) = \psi(i, (f(x, 1), \dots, f(x, \ell))).$$

Recall that for a measure $\mu \in \mathcal{M}(\mathbb{R}^d \times S)$, we use the notation

$$\langle f, \mu \rangle = \int_{\mathbb{R}^d \times S} f(x, i) \mu(dx, i).$$

Following the theory developed in the monograph of Li [17], we observe that the operator Ψ satisfies equation (2.26) in [17] and that the assumptions of Theorems 2.21 and 5.6, in the same monograph, are fulfilled. Therefore, there exists a strong Markov superprocess $\mathcal{Z} = (\mathcal{Z}_t, \mathcal{G}_t, \mathbb{Q}_\mu)$ with state space $\mathcal{M}(\mathbb{R}^d \times S)$, and transition probabilities determined by

$$\mathbb{Q}_\mu \left[e^{-\langle f, \mathcal{Z}_t \rangle} \right] = \exp \left\{ -\langle \nabla_t f, \mu \rangle \right\}, \quad t \geq 0,$$

where $f \in \mathcal{B}(\mathbb{R}^d \times S)$ and $t \mapsto \nabla_t f$ is the unique locally bounded positive solution to

$$\nabla_t f(x, i) = T_t f(x, i) - \int_0^t ds \int_{\mathbb{R}^d \times S} \Psi(y, j, \nabla_{t-s} f) T_s(x, i, d(y, j)).$$

For $i \in S$ and $\mu \in \mathcal{M}(\mathbb{R}^d \times S)$, we define $U_i \mu \in \mathcal{M}(\mathbb{R}^d)$ by $U_i \mu(B) = \mu(B \times \{i\})$ for $B \in \mathcal{B}(\mathbb{R}^d)$, the Borel sets in $\mathbb{R}^d$. Observe that $\mu \mapsto (U_i \mu)_{i \in S}$ is a homeomorphism between $\mathcal{M}(\mathbb{R}^d \times S)$ and $\mathcal{M}(\mathbb{R}^d)^\ell$. In other words, we can define a strong Markov process $X \in \mathcal{M}(\mathbb{R}^d)^\ell$ associated with $\mathcal{Z}$ and $(U_i)_{i \in S}$ as follows. For each $i \in S$, we define $X_t(i, dx) := U_i \mathcal{Z}_t(dx) = \mathcal{Z}_t(dx \times \{i\})$ with probabilities $\mathbb{P}_\mu := \mathbb{Q}_\mu$, where $\mu = (\mu_1, \dots, \mu_\ell) \in \mathcal{M}(\mathbb{R}^d)^\ell$, and each $\mu_i = U_i \mu$. In a similar way, there is a homeomorphism between $\mathcal{B}(\mathbb{R}^d)^\ell$ and $\mathcal{B}(\mathbb{R}^d \times S)$; that is to say for $f \in \mathcal{B}(\mathbb{R}^d)^\ell$ we define $f(x, i) = f_i(x)$. By applying the aforementioned homeomorphisms, we

deduce that $(X_t, \mathbb{P}_\mu)$ satisfies (2.2), and (2.3) has a unique locally bounded solution. $\square$

Now, we prove that the total mass vector of a multitype superprocess is a MCB process.

Proof of Lemma 1 Let $\theta \in \mathbb{R}_+^\ell$ and take the functions $f_i(x) = \theta_i$ for each $i \in S$, $x \in \mathbb{R}^d$. Since the branching mechanism and the vector θ are spatially independent, the system of functions $V_t \theta$ that satisfies (2.3) does not depend on $x \in \mathbb{R}^d$. In other words

$$\begin{aligned} V_t^{(i)} \theta &= P_t^{(i)} \theta_i - \int_0^t ds \int_{\mathbb{R}^d} \psi(i, V_{t-s} \theta) P_s^{(i)}(x, dy) \\ &= \theta_i - \int_0^t \psi(i, V_{t-s} \theta) ds, \quad i \in S. \end{aligned}$$

Therefore, $V_t \theta$ is a solution to the system (2.5). Since the solution is unique, $V_t \theta = v_t(\theta)$ for any $x \in \mathbb{R}^d$. By (2.2) and the relationship between X and Y , we have shown the total mass vector is indeed a MCB process. $\square$

3.2 Proofs for Subsection 2.2

In this section, we provide the proofs of Proposition 2, Lemma 2 and Proposition 3.

Proof of Proposition 2 By (2.7) and the branching property of X , we have

$$\mathbb{P}_\mu(\mathcal{E}) = e^{-\langle w, \mu \rangle}. \quad (3.2)$$

Furthermore by conditioning the event $\mathcal{E}$ on $\mathcal{F}_t$ and using the Markov property, we obtain that

$$e^{-\langle w, \mu \rangle} = \mathbb{E}_\mu \left[\mathbb{E}[\mathbf{1}_{\mathcal{E}} | \mathcal{F}_t] \right] = \mathbb{E}_\mu \left[\mathbb{E}_{X_t}[\mathbf{1}_{\mathcal{E}}] \right] = \mathbb{E}_\mu \left[e^{-\langle w, X_t \rangle} \right].$$

Thus from (2.3) and the assumption (2.8) we also get that $\psi(w) = 0$.

For the second part of the statement, we recall the definition of the total mass vector $Y = (Y_t, t \geq 0)$ whose entries satisfy $Y_t(i) = X_t(i, \mathbb{R}^d)$. By assumption (2.8), we know that the probability is less than one. Now we prove that it is positive. From identity (2.4) and assumption (2.8), we know that for each $i \in S$, there exists a positive deterministic time T_i such that

$$P_{e_i}(\|Y_t\| = 0) = e^{-\lim_{\theta \hookrightarrow \infty} v_t(i, \theta)} \begin{cases} = 0 & \text{for } t < T_i, \\ > 0 & \text{for } t > T_i, \end{cases}$$

where $v_t(i, \theta)$ is given by (2.5) and $\theta \hookrightarrow \infty$ means that each coordinate of θ goes to ∞ .

Next, we define the sets $S_1 := \{i \in S : T_i = 0\}$ and $S_2 := \{i \in S : T_i > 0\}$. For a vector $y = (y_1, \dots, y_\ell)$, we denote its support by $\text{supp}(y) := \{i \in S : y_i \neq 0\}$. Thus, the proof will be completed if we show that $S_2 = \emptyset$. We proceed by contradiction.

Let us assume that $S_2 \neq \emptyset$ and define $T := \inf\{T_i : i \in S_2\}$ which is strictly positive by definition. Take $i \in S_2$ and observe from the Markov property that

$$\begin{aligned} 0 &= \mathbf{P}_{e_i}(\|Y_{3T/4}\| = 0) \\ &\geq \mathbf{P}_{e_i}(\|Y_{3T/4}\| = 0, \text{supp}(Y_{T/2}) \subset S_1) \\ &= \mathbf{E}_{e_i}[\mathbf{P}_{Y_{T/2}}(\|Y_{T/4}\| = 0), \text{supp}(Y_{T/2}) \subset S_1]. \end{aligned}$$

By the branching property, if $\mathbf{y}$ is a vector such that $\text{supp}(\mathbf{y}) \subset S_1$ then $\mathbf{P}_{\mathbf{y}}(\|Y_t\| = 0) > 0$, for all $t > 0$. Therefore, we necessarily have

$$0 = \mathbf{P}_{e_i}(\text{supp}(Y_{T/2}) \subset S_1),$$

and implicitly

$$1 = \mathbf{P}_{e_i}(\text{supp}(Y_{T/2}) \cap S_2 \neq \emptyset) = \mathbf{P}_{e_i}(\|Y_{T/2}\| > 0), \quad \text{for all } i \in S_2.$$

Hence, using the branching property again, if $\mathbf{y}$ is a vector such that $\text{supp}(\mathbf{y}) \cap S_2 \neq \emptyset$, we have

$$1 = \mathbf{P}_{\mathbf{y}}(\|Y_{T/2}\| > 0) = \mathbf{P}_{\mathbf{y}}(\text{supp}(Y_{T/2}) \cap S_2 \neq \emptyset).$$

Finally, we use the Markov property recursively and the previous equality, to deduce that for all $k \geq 1$,

$$\mathbf{P}_{\mathbf{y}}(\|Y_{kT/2}\| > 0) = 1 \quad \text{for all } i \in S_2,$$

which is inconsistent with the definitions of T and T_i . In other words, $S_2 = \emptyset$. This completes the proof. $\square$

Now we prove Lemma 2, which provides a class of multitype superprocesses that satisfy assumption (2.8).

Proof of Lemma 2 Since the superprocess is supercritical and the “ $x \log x$ ”-condition (2.9) is satisfied, according to Theorem 1.4 in Kyprianou et al. [12], we have

$$\mathbb{P}_{e_i \delta_x} \left(\lim_{t \rightarrow \infty} \|X_t\| = 0 \right) < 1, \quad \text{for } i \in S, \quad x \in \mathbb{R}^d.$$

The latter implies

$$\mathbb{P}_{e_i \delta_x}(\mathcal{E}) \leq \mathbb{P}_{e_i \delta_x} \left(\lim_{t \rightarrow \infty} \|X_t\| = 0 \right) < 1.$$

From (2.4) and the fact that the total mass is a MCB process, it is clear that

$$\mathbb{P}_{e_i \delta_x}(\|X_t\| = 0) = \exp \left\{ - \lim_{\theta \hookrightarrow \infty} v_t(i, \theta) \right\},$$

where $v_t(i, \theta)$ is given by (2.5) and recall that $\theta \hookrightarrow \infty$ means that each coordinate of θ goes to ∞ . In other words, if we show that

$$\lim_{t \rightarrow \infty} \lim_{\theta \hookrightarrow \infty} v_t(i, \theta) < \infty \quad \text{for all } i \in S,$$

then we have that (2.8) holds. In order to prove that the above limit is finite, we introduce

$$A_t(\theta) := \sup_{i \in S} \frac{v_t(i, \theta)}{u_i},$$

where u_i denotes the i -th coordinate of the right eigenvector associated to Γ . Since the supremum of finitely many continuously differentiable functions is differentiable except at most countably many isolated points, we may fix $t \geq 0$ such that $A_t(\theta)$ is differentiable at t and select i in such a way that $A_t(\theta)u_i = v_t(i, \theta)$. Then by using (2.1), (2.5), and (2.6) we can deduce that

$$\begin{aligned} u_i \frac{d}{dt} A_t(\theta) &= \frac{d}{dt} v_t(i, \theta) \\ &= \sum_{j \in S} \tilde{B}_{ji} u_j \frac{v_t(j, \theta)}{u_j} - \beta_i (v_t(i, \theta))^2 \\ &\quad - \int_{\mathbb{R}_+^\ell} \left(e^{-[v_t(\theta), y]} - 1 + [v_t(\theta), y] \right) \Pi(i, dy). \end{aligned}$$

Since $1 - x - e^{-x} \leq 0$, for all $x > 0$, $\tilde{B}_{i,j} \mathbf{1}_{\{i \neq j\}} > 0$ and $A_t(\theta)u_i = v_t(i, \theta)$, we have

$$u_i \frac{d}{dt} A_t(\theta) \leq \sum_{j \in S} \tilde{B}_{ji} u_j A_t(\theta) - \beta_i (A_t(\theta)u_i)^2 = A_t(\theta)(\tilde{B}^t u)_i - \beta_i (A_t(\theta)u_i)^2.$$

Next, we use that u is an eigenvector of $\tilde{B}^t$ to get

$$u_i \frac{d}{dt} A_t(\theta) \leq A_t(\theta)\Gamma u_i - \beta_i (A_t(\theta)u_i)^2.$$

By defining $\underline{u} := \inf_{i \in S} u_i$ and recalling the definition of β , the previous identity implies

$$\frac{d}{dt} A_t(\theta) \leq A_t(\theta)\Gamma - \beta \underline{u} (A_t(\theta))^2.$$

Since, Γ , β and $\underline{u}$ are strictly positive, an integration by parts allow us to deduce that

$$A_t(\theta) \leq \frac{\Gamma e^{\Gamma t}}{\frac{\Gamma}{A_0(\theta)} + \beta \underline{u}(e^{\Gamma t} - 1)}.$$

Finally, if we define $\bar{u} := \sup_{i \in S} u_i$, the previous computations lead to

$$w_i = \lim_{t \rightarrow \infty} \lim_{\theta \hookrightarrow \infty} v_t(i, \theta) \leq \bar{u} \lim_{t \rightarrow \infty} \lim_{\theta \hookrightarrow \infty} A_t(\theta) \leq \frac{\bar{u}\Gamma}{\beta \underline{u}} < \infty.$$

□

Finally, we prove Proposition 3.

Proof of Proposition 3 Using (3.2), assumption (2.8) and the Markov property, we have for $f \in \mathcal{B}^+(\mathbb{R}^d)^\ell$

$$\begin{aligned} \mathbb{E}_\mu^\dagger \left[e^{-\langle f, X_t \rangle} \right] &= e^{\langle w, \mu \rangle} \mathbb{E}_\mu \left[e^{-\langle f, X_t \rangle} \mathbf{1}_{\mathcal{E}} \right] \\ &= e^{\langle w, \mu \rangle} \mathbb{E}_\mu \left[e^{-\langle f, X_t \rangle} \mathbb{P}_{X_t}(\mathcal{E}) \right] \\ &= e^{\langle w, \mu \rangle} \mathbb{E}_\mu \left[e^{-\langle f, X_t \rangle} e^{-\langle w, X_t \rangle} \right] \\ &= e^{-\langle V_t(f+w) - w, \mu \rangle}. \end{aligned}$$

Since $V_t(f+w)$ satisfies (2.3), using the definitions of $V_t^\dagger f$ and $\psi^\dagger$ we obtain that $V_t^\dagger f$ satisfies (2.10). By Proposition 2, $\psi(w) = \mathbf{0}$. Then, computing $\psi(\theta+w) - \psi(w)$, we deduce that

$$\psi^\dagger(i, \theta) = -[\theta, B^\dagger e_i] + \beta_i \theta_i^2 + \int_{\mathbb{R}_+^\ell} \left(e^{-[\theta, y]} - 1 + \theta_i y_i \right) e^{-[w, y]} \Pi(i, dy), \quad (3.3)$$

where

$$B_{ij}^\dagger = B_{ij} - \left(2\beta_i w_i + \int_{\mathbb{R}_+^\ell} \left(1 - e^{-[y, w]} \right) y_i \Pi(i, dy) \right) \mathbf{1}_{\{j=i\}}. \quad (3.4)$$

This implies that $\psi^\dagger$ is a branching mechanism, and therefore, the solution of (2.10) is unique. In other words, X under $\mathbb{P}_\mu^\dagger$ is a multitype superprocess with branching mechanism given by $\psi^\dagger(\theta)$. □

3.3 Proofs for Subsection 2.3

In this subsection, we prove that $\{p_{j_1, \dots, j_\ell}^{(i)}, (j_1, \dots, j_\ell) \in \mathbb{N}^\ell\}$ is a probability distribution, for every $1 \leq i \leq \ell$.

Proof of Lemma 3 Let $i \in S$. Recall that $\psi(\mathbf{w}) = \mathbf{0}$ and the definition of q_i as in (2.12). Then,

$$\begin{aligned}
 w_i q_i &= w_i q_i - \psi(i, \mathbf{w}) \\
 &= w_i \left(-B_{ii} + 2\beta_i w_i + \int_{\mathbb{R}_+^\ell} (1 - e^{-[\mathbf{w}, \mathbf{y}]}) y_i \Pi(i, d\mathbf{y}) \right) \\
 &\quad + [\mathbf{w}, \mathbf{B}e_i] - \beta_i w_i^2 - \int_{\mathbb{R}_+^\ell} (e^{-[\mathbf{w}, \mathbf{y}]} - 1 + w_i y_i) \Pi(i, d\mathbf{y}) \\
 &= \sum_{j \neq i} B_{ji} w_j + \beta_i w_i^2 + \int_{\mathbb{R}_+^\ell} e^{-[\mathbf{w}, \mathbf{y}]} (e^{[\mathbf{w}, \mathbf{y}]} - 1 - w_i y_i) \Pi(i, d\mathbf{y}) \\
 &= \sum_{j \neq i} \left(B_{ji} w_j + \int_{\mathbb{R}_+^\ell} w_j y_j e^{-[\mathbf{w}, \mathbf{y}]} \Pi(i, d\mathbf{y}) \right) \\
 &\quad + \beta_i w_i^2 + \int_{\mathbb{R}_+^\ell} e^{-[\mathbf{w}, \mathbf{y}]} (e^{[\mathbf{w}, \mathbf{y}]} - 1 - [\mathbf{w}, \mathbf{y}]) \Pi(i, d\mathbf{y}) \\
 &= \sum_{j \neq i} \left(B_{ji} w_j + \int_{\mathbb{R}_+^\ell} w_j y_j e^{-[\mathbf{w}, \mathbf{y}]} \Pi(i, d\mathbf{y}) \right) + \beta_i w_i^2 \\
 &\quad + \int_{\mathbb{R}_+^\ell} \sum_{j_1 + \dots + j_\ell \geq 2} \frac{(w_1 y_1)^{j_1} \dots (w_\ell y_\ell)^{j_\ell}}{j_1! \dots j_\ell!} e^{-[\mathbf{w}, \mathbf{y}]} \Pi(i, d\mathbf{y}),
 \end{aligned} \tag{3.5}$$

where in the last row we have used the multinomial theorem, i.e.,

$$\begin{aligned}
 \sum_{n=2}^{\infty} \frac{[\mathbf{x}, \mathbf{y}]^n}{n!} &= \sum_{n=2}^{\infty} \frac{1}{n!} \sum_{j_1 + \dots + j_\ell = n} \binom{n}{j_1, \dots, j_\ell} \prod_{k=1}^{\ell} (x_k y_k)^{j_k} \\
 &= \sum_{j_1 + \dots + j_\ell \geq 2} \frac{(x_1 y_1)^{j_1} \dots (x_\ell y_\ell)^{j_\ell}}{j_1! \dots j_\ell!}.
 \end{aligned} \tag{3.6}$$

Therefore, by summing $p_{j_1, \dots, j_\ell}^{(i)}$ over all $(j_1, \dots, j_\ell) \in \mathbb{N}^\ell$ and using equation (3.5), we deduce that $p^{(i)}$ is a probability distribution. $\square$

3.4 Proofs for Subsection 2.4

We now prove the existence of the Dynkin–Kuznetsov measure.

Proof of Proposition 4 Let us denote by $\mathcal{M}^0(\mathbb{R}^d \times S) := \mathcal{M}(\mathbb{R}^d \times S) \setminus \{0\}$, where 0 is the null measure. Consider the Markov superprocess $\mathcal{Z}$ introduced in the proof of Proposition 1. Let $(Q_t, t \geq 0)$ and $(V_t, t \geq 0)$ be the transition and cumulant semigroups associated with $\mathcal{Z}$. By Theorem 1.36 in [17], V_t has the following repre-

sensation

$$\begin{aligned} \mathbb{V}_t f(x, i) &= \int_{\mathbb{R}^d \times S} f(y, j) \Lambda_t(x, i, \mathrm{d}(y, j)) \\ &\quad + \int_{\mathcal{M}^0(\mathbb{R}^d \times S)} \left(1 - e^{-\langle f, \nu \rangle}\right) L_t(x, i, \mathrm{d}\nu), \quad t \geq 0, \end{aligned}$$

where f is a positive Borel function on $\mathbb{R}^d \times S$, $\Lambda_t(x, i, \mathrm{d}(y, j))$ is a bounded kernel on $\mathbb{R}^d \times S$, and $(1 \wedge \langle 1, \nu \rangle) L_t(x, i, \mathrm{d}\nu)$ is a bounded kernel from $\mathbb{R}^d \times S$ to $\mathcal{M}^0(\mathbb{R}^d \times S)$.

Let $\tilde{\mathcal{X}}^+$ be the space of càdlàg paths $t \rightarrow \tilde{w}_t$ from $[0, \infty)$ to $\mathcal{M}(\mathbb{R}^d \times S)$ having the null measure as a trap. Let $(Q_t^0, t \geq 0)$ be the restriction of $(Q_t, t \geq 0)$ to $\mathcal{M}^0(\mathbb{R}^d \times S)$ and

$$E_0 := \left\{ (x, i) \in \mathbb{R}^d \times S : \Lambda_t(x, i, \mathbb{R}^d \times S) = 0, \text{ for all } t > 0 \right\}.$$

By Proposition 2.8 in [17], for all $(x, i) \in E_0$ the family of measures $(L_t(x, i, \cdot), t \geq 0)$ on $\mathcal{M}^0(\mathbb{R}^d \times S)$ constitutes an entrance law for $(Q_t^0, t \geq 0)$. Therefore, by Theorem A.40 of [17] for all $(x, i) \in E_0$ there exists a unique σ -finite measure $\tilde{\mathbb{N}}_{(x,i)}$ on $\tilde{\mathcal{X}}^+$ such that $\tilde{\mathbb{N}}_{(x,i)}(\{0\}) = 0$, and for any $0 < t_1 < \dots < t_n < \infty$

$$\begin{aligned} \tilde{\mathbb{N}}_{(x,i)}(\mathcal{Z}_{t_1} \in \mathrm{d}v_1, \mathcal{Z}_{t_2} \in \mathrm{d}v_2, \dots, \mathcal{Z}_{t_n} \in \mathrm{d}v_n) \\ = L_{t_1}(x, i, \mathrm{d}v_1) Q_{t_2-t_1}^0(v_1, \mathrm{d}v_2) \dots Q_{t_n-t_{n-1}}^0(v_{n-1}, \mathrm{d}v_n). \end{aligned}$$

It follows that for all $t > 0$, $(x, i) \in E_0$, and $f \in \mathcal{B}(\mathbb{R}^d \times S)$ positive, we have

$$\tilde{\mathbb{N}}_{(x,i)} \left(1 - e^{-\langle f, \mathcal{Z}_t \rangle} \right) = \int_{\mathcal{M}^0(\mathbb{R}^d \times S)} \left(1 - e^{-\langle f, \nu \rangle} \right) L_t(x, i, \mathrm{d}\nu) = \mathbb{V}_t f(x, i).$$

Recall the homeomorphism $\mu \mapsto (U_i \mu)_{i \in S}$ and the definition of the superprocess X from the proof of Proposition 1. By taking the constant function $f(x, i) = \lambda \in \mathbb{R}$, and using the definitions of $\mathbb{V}_t, Q_t$, we deduce that

$$-\log \mathbb{E}_{e_i \delta_x} \left[e^{-\lambda \langle 1, X_t \rangle} \right] = \lambda \Lambda_t(x, i, \mathbb{R}^d \times S) + \int_{\mathcal{M}^0(\mathbb{R}^d \times S)} \left(1 - e^{-\lambda \langle 1, \nu \rangle} \right) L_t(x, i, \mathrm{d}\nu).$$

If we take λ to infinity, the left-hand side of the above identity converges to $-\log \mathbb{P}_{e_i \delta_x}(\|X_t\| = 0)$ which is finite by Proposition 2. Henceforth, $\Lambda_t(x, i, \mathbb{R}^d \times S) = 0$ and $(x, i) \in E_0$.

Next, recall that $\mathcal{X}$ denotes the space of càdlàg paths from $[0, \infty)$ to $\mathcal{M}(\mathbb{R}^d)^\ell$. Then $(U_i)_{i \in S}$ induces a homeomorphism between $\tilde{\mathcal{X}}$ and $\mathcal{X}$. More precisely, the homeomorphism $\mathcal{U} : \tilde{\mathcal{X}} \rightarrow \mathcal{X}$ is given by $\tilde{w}_t \rightarrow \mathbf{w}_t = (w_t(1), \dots, w_t(\ell))$ where for all $i \in S$ the measure in the i th coordinate is given by $w_t(i, B) = \tilde{w}_t(B \times \{i\})$. This implies that for all $(x, i) \in \mathbb{R}^d \times S$ we can define the measures $\mathbb{N}_{xe_i}$ on $\mathcal{X}$ given by

$\mathbb{N}_{xe_i}(B) := \tilde{\mathbb{N}}_{(x,i)}(\mathcal{U}^{-1}(B))$. In other words, we obtain

$$\mathbb{N}_{xe_i} \left(1 - e^{-\langle f, X_t \rangle} \right) = -\log \mathbb{E}_{e_i \delta_x} \left[e^{-\langle f, X_t \rangle} \right],$$

for all $f \in \mathcal{B}(\mathbb{R}^d)^\ell$ and $t \geq 0$. $\square$

It is important to note that the Dynkin–Kuznetsov measures $\mathbb{N}^\dagger$ associated to the multitype superprocess conditioned on extinction are also well defined since $|\log \mathbb{P}_{\delta_x e_i}^\dagger(\mathcal{E})| < \infty$.

3.5 Proofs of the main results

In this subsection, we prove Theorem 1 and Theorem 2, which are the two main results of this manuscript.

Recall that Theorem 1 describes the Laplace functional of the dressed tree. In order to prove this result we will make use of the fact that, given the backbone, the three types of immigration process are independent. This will allow us to break up the total mass based on the way it was immigrated, characterise the pieces separately, then put the building blocks together to deduce the required formula for the Laplace functional. Following this idea, we will use the next two lemmas to prove Theorem 1.

Lemma 4 For each $f \in \mathcal{B}(\mathbb{R}^d)^\ell$, $\nu \in \mathcal{M}_a(\mathbb{R}^d)^\ell$, $\mu \in \mathcal{M}(\mathbb{R}^d)^\ell$, and $t \geq 0$ we have

$$\widehat{\mathbb{E}}_{(\mu, \nu)} \left[e^{-\langle f, I_t^{\mathbb{N}^\dagger} + I_t^{\mathbb{P}^\dagger} \rangle} \middle| (Z_s, s \leq t) \right] = \exp \left\{ - \int_0^t \langle \phi(V_{t-r}^\dagger f), Z_r \rangle dr \right\}, \quad (3.7)$$

where ϕ is given by (2.15) and $V_t^\dagger f$ satisfies (2.10).

Proof As the different immigration mechanisms are independent given the backbone, we may look at the Laplace functional of the continuous and discontinuous immigrations separately. For the continuous immigration, we can condition on Z , use Campbell's formula, then equation (2.14) for $\mathbb{N}^\dagger$, and finally the definition of $V_t^\dagger f(x) = (V_t^{\dagger, (1)} f(x), \dots, V_t^{\dagger, (\ell)} f(x))^t$ to obtain

$$\begin{aligned} & \widehat{\mathbb{E}}_{(\mu, \nu)} \left[\exp \{ - \langle f, I_t^{\mathbb{N}^\dagger} \rangle \} \middle| (Z_s, s \leq t) \right] \\ &= \exp \left\{ - \sum_{u \in \mathcal{T}} 2\beta_{\gamma_u} \int_{t \wedge \tau_u}^{t \wedge \sigma_u} dr \mathbb{N}_{z_u(r) e_{\gamma_u}}^\dagger \left(1 - e^{-\langle f, X_{t-r} \rangle} \right) \right\} \\ &= \exp \left\{ - \sum_{u \in \mathcal{T}} 2\beta_{\gamma_u} \int_{t \wedge \tau_u}^{t \wedge \sigma_u} dr V_{t-r}^{\dagger, (\gamma_u)} f(z_u(r)) \right\}. \end{aligned}$$

In a similar way, for the discontinuous immigration, by conditioning on $\mathbf{Z}$, using Campbell's formula and the definition of $V_t^\dagger f$ we get

$$\begin{aligned} \widehat{\mathbb{E}}_{(\mu, \nu)} \left[\exp \{ - \langle f, \mathbf{I}_t^{\mathbb{P}^\dagger} \rangle \} \middle| (\mathbf{Z}_s, s \leq t) \right] \\ = \exp \left\{ - \sum_{u \in \mathcal{T}} \int_{t \wedge \tau_u}^{t \wedge \sigma_u} dr \int_{\mathbb{R}_+^\ell} y \gamma_u e^{-[w, y]} \mathbb{E}_{y \delta_{z_u(r)}}^\dagger \left[1 - e^{-\langle f, \mathbf{X}_{t-r} \rangle} \right] \Pi(\gamma_u, dy) \right\} \\ = \exp \left\{ - \sum_{u \in \mathcal{T}} \int_{t \wedge \tau_u}^{t \wedge \sigma_u} dr \int_{\mathbb{R}_+^\ell} y \gamma_u e^{-[w, y]} \left(1 - e^{-[V_{t-r}^\dagger f(z_u(r)), y]} \right) \Pi(\gamma_u, dy) \right\}. \end{aligned}$$

Therefore, by putting the pieces together we obtain the following

$$\begin{aligned} \widehat{\mathbb{E}}_{(\mu, \nu)} \left[\exp \left\{ - \langle f, \mathbf{I}_t^{\mathbb{N}^\dagger} + \mathbf{I}_t^{\mathbb{P}^\dagger} \rangle \right\} \middle| (\mathbf{Z}_s, s \leq t) \right] \\ = \exp \left\{ - \sum_{u \in \mathcal{T}} \int_{t \wedge \tau_u}^{t \wedge \sigma_u} \phi(\gamma_u, V_{t-r}^\dagger f(z_u(r))) dr \right\}, \end{aligned} \quad (3.8)$$

where $\phi(i, \lambda)$ is given by formula (2.15). The previous equation is in terms of the tree $\mathcal{T}$. We want to rewrite it in terms of the multitype branching diffusion, thus

$$\begin{aligned} \sum_{u \in \mathcal{T}} \int_{t \wedge \tau_u}^{t \wedge \sigma_u} \phi(\gamma_u, V_{t-r}^\dagger f(z_u(r))) dr &= \sum_{i \in S} \sum_{u \in \mathcal{T}, \gamma_u = i} \int_{t \wedge \tau_u}^{t \wedge \sigma_u} \phi(i, V_{t-r}^\dagger f(z_u(r))) dr \\ &= \int_0^t \sum_{i \in S} \sum_{u \in \mathcal{T}, z_u = i} \phi(i, V_{t-r}^\dagger f(z_u(r))) \mathbf{1}_{\{r \in [t \wedge \tau_u, t \wedge \sigma_u)\}} dr \\ &= \int_0^t \left\langle \phi(V_{t-r}^\dagger, \mathbf{Z}_r) \right\rangle dr. \end{aligned}$$

□

Observe that the processes $\mathbf{I}^{\mathbb{N}^\dagger} = (\mathbf{I}_t^{\mathbb{N}^\dagger}, t \geq 0)$, $\mathbf{I}^{\mathbb{P}^\dagger} = (\mathbf{I}_t^{\mathbb{P}^\dagger}, t \geq 0)$ and $\mathbf{I}^\eta = (\mathbf{I}_t^\eta, t \geq 0)$ are initially zero-valued $\widehat{\mathbb{P}}_{(\mu, \nu)}$ -a.s. In order to study the rest of the immigration along the backbone we have the following result.

Lemma 5 Suppose that $f, h \in \mathcal{B}(\mathbb{R}^d)^\ell$ and $\mathbf{g}_s(x) \in \mathcal{B}(\mathbb{R} \times \mathbb{R}^d)^\ell$. Define the vectorial function $e^{-W_t(x)} = (e^{-W_t^{(1)}(x)}, \dots, e^{-W_t^{(\ell)}(x)})$ as follows

$$e^{-W_t^{(i)}(x)} := \widehat{\mathbb{E}}_{(\mu, e_i \delta_x)} \left[\exp \left\{ - \langle f, \mathbf{I}_t^\eta \rangle - \langle h, \mathbf{Z}_t \rangle - \int_0^t \langle \mathbf{g}_{t-s}, \mathbf{Z}_s \rangle ds \right\} \right].$$

Then, $e^{-W_t(x)}$ is a locally bounded solution to the integral system

$$e^{-W_t^{(i)}(x)} = \mathbb{P}_t^{(i)} e^{-h_i(x)} + \frac{1}{w_i} \int_0^t ds \int_{\mathbb{R}^d} \left[H_{t-s}^{(i)}(y, \mathbf{w} \cdot e^{-W_{t-s}^{(i)}(y)} - w_i g_{t-s}^i(y) e^{-W_{t-s}^{(i)}(y)} \right] \mathbb{P}_s^{(i)}(x, dy), \quad (3.9)$$

where

$$H_s^{(i)}(x, \boldsymbol{\theta}) = [\boldsymbol{\theta}, \mathbf{B}^\dagger \mathbf{e}_i] + \beta_i \theta_i^2 + \int_{\mathbb{R}_+^\ell} \left(e^{[\boldsymbol{\theta}, \mathbf{y}]} - 1 - \theta_i y_i \right) e^{-[\mathbf{w} + \mathbf{V}_s^\dagger \mathbf{f}(x), \mathbf{y}]} \Pi(i, d\mathbf{y}). \quad (3.10)$$

In the latter formula, $\mathbf{B}^\dagger$ is given by (3.4) and $\mathbf{V}_t^\dagger \mathbf{f}$ is the unique solution to (2.10).

It is important to note that $\mathbf{W}$ depends on the functions $\mathbf{f}$, $\mathbf{h}$ and $\mathbf{g}$ but for simplicity on exposition we suppress this dependency.

Proof Recall that $\mathbf{Z}$ is a multitype branching particle system, where the motion of each particle with type $i \in S$ is given by the semigroup $\mathbb{P}^{(i)}$ and its branching generator is given by (2.12). For simplicity, we denote by $\mathbb{P}_x^{(i)}$ the law of the diffusion $\xi^{(i)}$ starting at x . By conditioning on the time of the first branching event of $\mathbf{Z}$ we get

$$\begin{aligned} e^{-W_t^{(i)}(x)} &= \mathbb{E}_x^{(i)} \left[e^{-q_i t} e^{-\int_0^t g_{t-r}^i(\xi_r^{(i)}) dr} e^{-h_i(\xi_t^{(i)})} \right] \\ &\quad + \mathbb{E}_x^{(i)} \left[\int_0^t q_i e^{-q_i s} e^{-\int_0^s g_{s-r}^i(\xi_r^{(i)}) dr} \right. \\ &\quad \left. \sum_{\mathbf{j} \in \mathbb{N}^\ell} p_{\mathbf{j}}^{(i)} e^{-\sum_{k \in S} j_k W_{t-s}^{(k)}(\xi_s^{(i)})} \int_{\mathbb{R}_+^\ell} \eta_{\mathbf{j}}^{(i)}(d\mathbf{y}) e^{-[\mathbf{V}_{t-s}^\dagger \mathbf{f}(\xi_s^{(i)}), \mathbf{y}]} d\mathbf{s} \right], \end{aligned}$$

where $\mathbf{j} = (j_1, \dots, j_\ell)$. On the other hand, by Proposition 2.9 in [17], we see that $e^{-W_t^{(i)}(x)}$ also satisfies

$$\begin{aligned} e^{-W_t^{(i)}(x)} &= \mathbb{E}_x^{(i)} \left[e^{-h_i(\xi_t^{(i)})} \right] - \mathbb{E}_x^{(i)} \left[\int_0^t q_i e^{-W_{t-s}^{(i)}(x)} ds \right] \\ &\quad - \mathbb{E}_x^{(i)} \left[\int_0^t g_{t-s}^i(\xi_s^{(i)}) e^{-W_{t-s}^{(i)}(x)} ds \right] \\ &\quad + \mathbb{E}_x^{(i)} \left[\int_0^t q_i \sum_{\mathbf{j} \in \mathbb{N}^\ell} p_{\mathbf{j}}^{(i)} e^{-\sum_{k \in S} j_k W_{t-s}^{(k)}(\xi_s^{(i)})} \int_{\mathbb{R}_+^\ell} \eta_{\mathbf{j}}^{(i)}(d\mathbf{y}) e^{-[\mathbf{V}_{t-s}^\dagger \mathbf{f}(\xi_s^{(i)}), \mathbf{y}]} d\mathbf{s} \right]. \end{aligned}$$

By substituting the definitions of $p_{\mathbf{j}}^{(i)}$ and $\eta_{\mathbf{j}}^{(i)}$ (see (2.11) and (2.16)), we get that for all $x \in \mathbb{R}^d$

$$R(x) := \sum_{\mathbf{j} \in \mathbb{N}^\ell} p_{\mathbf{j}}^{(i)} e^{-[\mathbf{j}, \mathbf{W}_{t-s}(x)]} \int_{\mathbb{R}_+^\ell} \eta_{\mathbf{j}}^{(i)}(d\mathbf{y}) e^{-[\mathbf{V}_{t-s}^\dagger \mathbf{f}(x), \mathbf{y}]}$$

$$\begin{aligned}
&= \frac{1}{w_i q_i} \sum_{j \in \mathbb{N}^\ell} \left[\beta_i w_i^2 e^{-[j, \mathbf{W}_{t-s}(x)]} \mathbf{1}_{\{j=2\mathbf{e}_i\}} + \left(B_{ki} w_k e^{-[j, \mathbf{W}_{t-s}(x)]} \right. \right. \\
&\quad \left. \left. + \int_{\mathbb{R}_+^\ell} w_k y_k e^{-[\mathbf{w}, \mathbf{y}]} e^{-[j, \mathbf{W}_{t-s}(x)]} e^{-[V_{t-s}^\dagger f(x), \mathbf{y}]} \Pi(i, d\mathbf{y}) \right) \mathbf{1}_{\{j=\mathbf{e}_k\}} \mathbf{1}_{\{k \neq i\}} \right. \\
&\quad \left. + \int_{\mathbb{R}_+^\ell} \frac{(w_1 y_1)^{j_1} \dots (w_\ell y_\ell)^{j_\ell}}{j_1! \dots j_\ell!} e^{-[\mathbf{w}, \mathbf{y}]} e^{-[j, \mathbf{W}_{t-s}(x)]} \right. \\
&\quad \left. e^{-[V_{t-s}^\dagger f(x), \mathbf{y}]} \Pi(i, d\mathbf{y}) \mathbf{1}_{\{j_1 + \dots + j_\ell \geq 2\}} \right] \\
&= \frac{1}{w_i q_i} \left[\beta_i \left(w_i e^{-W_{t-s}^{(i)}(x)} \right)^2 \right. \\
&\quad \left. + \sum_{k \in S, k \neq i} e^{-W_{t-s}^{(k)}(x)} \left(B_{ki} w_k + \int_{\mathbb{R}_+^\ell} w_k y_k e^{-[\mathbf{w}, \mathbf{y}]} e^{-[V_{t-s}^\dagger f(x), \mathbf{y}]} \Pi(i, d\mathbf{y}) \right) \right. \\
&\quad \left. + \int_{\mathbb{R}_+^\ell} \sum_{n \geq 2} \frac{[\mathbf{w} \cdot e^{-W_{t-s}(x)}, \mathbf{y}]^n}{n!} e^{-[\mathbf{w}, \mathbf{y}]} e^{-[V_{t-s}^\dagger f(x), \mathbf{y}]} \Pi(i, d\mathbf{y}) \right],
\end{aligned}$$

where in the last row we have used (3.6). By merging the two integrals, we get

$$\begin{aligned}
R(x) &= \frac{1}{w_i q_i} \left[\beta_i \left(w_i e^{-W_{t-s}^{(i)}(x)} \right)^2 + \sum_{k \in S, k \neq i} B_{ki} w_k e^{-W_{t-s}^{(k)}(x)} \right. \\
&\quad \left. + \int_{\mathbb{R}_+^\ell} \left(e^{[\mathbf{w} \cdot e^{-W_{t-s}(x)}, \mathbf{y}]} - 1 - w_i e^{-W_{t-s}^{(i)}(x)} y_i \right) e^{-[\mathbf{w} + V_{t-s}^\dagger f(x), \mathbf{y}]} \Pi(i, d\mathbf{y}) \right].
\end{aligned}$$

So, putting the pieces together and using the definitions of q_i , $\mathbf{B}^\dagger$ and $H^{(i)}$, (see identities (2.12), (3.4) and (3.10)) we deduce that

$$\begin{aligned}
e^{-W_t^{(i)}(x)} &= \mathbb{E}_x^{(i)} \left[e^{-h_i(\xi_t^{(i)})} - \int_0^t g_{t-s}^i(\xi_s^{(i)}) e^{-W_{t-s}^{(i)}(\xi_s^{(i)})} ds \right. \\
&\quad \left. + \frac{1}{w_i} \int_0^t H_{t-s}^{(i)}(\xi_s^{(i)}, \mathbf{w} \cdot e^{-W_{t-s}(\xi_s^{(i)})}) ds \right],
\end{aligned}$$

as stated. Therefore, $e^{-W_t(x)}$ satisfies (3.9). $\square$

Now, we are ready to give the proof of Theorem 1.

Proof of Theorem 1 Since $\tilde{X}$ is an independent copy of X under $\mathbb{P}_\mu^\dagger$, it is enough to show that for $\mu \in \mathcal{M}(\mathbb{R}^d)^\ell$, $\nu \in \mathcal{M}_a(\mathbb{R}^d)^\ell$, $\mathbf{f}, \mathbf{h} \in \mathcal{B}^+(\mathbb{R}^d)^\ell$, the vectorial function $e^{-U_t^{(f)} h(x)}$ defined by

$$e^{-U_t^{(f)} h(x)} := \widehat{\mathbb{E}}_{\mu, \mathbf{e}_i \delta_x} \left[e^{-\langle \mathbf{f}, \mathbf{I}_t^{\mathbb{N}^\dagger} + \mathbf{I}_t^{\mathbb{P}^\dagger} + \mathbf{I}_t^\eta \rangle - \langle \mathbf{h}, \mathbf{Z}_t \rangle} \right],$$

is a solution to (2.18) and that this solution is unique. By its definition, it is clear that $e^{-U_t^{(f)} h(x)} \in [0, 1]^\ell$ for all $x \in \mathbb{R}^d$ and $t \geq 0$. On the other hand from Lemma 4, we observe that

$$e^{-U_t^{(f,i)} h(x)} = \widehat{\mathbb{E}}_{\mu, e_i \delta_x} \left[\exp \left\{ -\langle f, I_t^\eta \rangle - \langle h, Z_t \rangle - \int_0^t \langle \phi(V_{t-r}^\dagger f), Z_r \rangle dr \right\} \right].$$

Therefore Lemma 5 implies that the vectorial function $e^{-U_t^{(f)} h(x)}$ satisfies

$$e^{-U_t^{(f,i)} h(x)} = \mathbb{E}_x^{(i)} \left[e^{-h_i(\xi_t^{(i)})} + \frac{1}{w_i} \int_0^t \left(H_{t-s}^{(i)}(\xi_s^{(i)}, w \cdot e^{-U_{t-s}^{(f,i)} h(\xi_s^{(i)})} \right) \right. \\ \left. - \phi(i, V_{t-r}^\dagger f(\xi_s^{(i)})) w_i e^{-U_{t-s}^{(f,i)} h(\xi_s^{(i)})} \right] ds \Bigg],$$

where $H^{(i)}$ is given as in (3.10). Using the definitions of $\psi^\dagger$, ϕ and H (see identities (2.15) (3.3) and (3.10)), we observe for all $i \in S$, $x \in \mathbb{R}^d$ and $\theta \in \mathbb{R}_+^l$, that

$$H_t^{(i)}(x, \theta) - \phi(i, V_t^\dagger f(x)) \theta_i = \psi^\dagger(i, -\theta + V_t^\dagger f(x)) - \psi^\dagger(i, V_t^\dagger f(x)).$$

Therefore, $e^{-U_t^{(f)} h(x)}$ is a solution to (2.18).

In order to finish the proof, we show that the solution to (2.18) is unique. Our arguments use Gronwall's lemma and similar ideas to those used in the monograph of Li [17] and in Proposition 1. With this purpose in mind, we first deduce some additional inequalities. Recall that the function $\psi^\dagger(i, \theta)$ defined in (3.3) is a branching mechanism. Using similar notation as in Proposition 1, we introduce the operator

$$\Psi^\dagger(x, i, f) = \psi^\dagger(i, (f(x, 1), \dots, f(x, \ell))),$$

for $f \in \mathcal{B}(\mathbb{R}^d \times S)$, and observe that it satisfies identity (2.26) in [17]. Therefore, following line by line the arguments in the proof of Proposition 2.20 in [17], we may deduce that $\Psi^\dagger$ satisfies Condition 2.11 in [17]. In other words, for all $a \geq 0$, there exists $L_a > 0$ such that

$$\sup_{(x,i) \in \mathbb{R}^d \times S} |\Psi^\dagger(x, i, f) - \Psi^\dagger(x, i, g)| \leq L_a \|f - g\|, \quad \text{for } f, g \in \mathcal{B}_a(\mathbb{R}^d \times S), \quad (3.11)$$

where $\|f\| := \sup_{(x,i) \in \mathbb{R}^d \times S} |f(x, i)|$ and $\mathcal{B}_a(\mathbb{R}^d \times S) := \{f \in \mathcal{B}(\mathbb{R}^d \times S) : \|f\| \leq a\}$.

On the other hand by Proposition 2.21 in [17], for all $f \in \mathcal{B}(\mathbb{R}^d \times S)$, there exists $t \mapsto V_t^\dagger f$ a unique locally bounded positive solution to

$$V_t^\dagger f(x, i) = \mathbb{T}_t f(x, i) - \int_0^t ds \int_{\mathbb{R}^d \times S} \Psi^\dagger(y, j, V_{t-s}^\dagger f) \mathbb{T}_s(x, i, d(y, j)),$$

where the semigroup $\mathbb{T}_t$ is given as in (3.1). Moreover, by Proposition 2.14 in [17], for all $T > 0$ there exists $C(T)$ such that

$$\sup_{0 \leq s \leq T} \sup_{(x,i) \in \mathbb{R}^d \times S} |V_s^\dagger f(x, i)| \leq C(T) \|f\|.$$

Hence using the homeomorphism between $\mathcal{B}(\mathbb{R}^d)^\ell$ and $\mathcal{B}(\mathbb{R}^d \times S)$ which was defined in the proof of Proposition 1 (i.e., for $\mathbf{f} \in \mathcal{B}(\mathbb{R}^d)^\ell$, we define $f(x, i) = f_i(x)$) and the previous inequality, we deduce that

$$\sup_{0 \leq s \leq T} \sup_{x \in \mathbb{R}^d} \sup_{i \in S} \left| V_s^{\dagger, (i)} \mathbf{f}(x) \right| \leq C(T) \|\mathbf{f}\| \quad \text{for } \mathbf{f} \in \mathcal{B}^+(\mathbb{R}^d)^\ell, \quad (3.12)$$

where $\|\mathbf{f}\| = \sup_{x \in \mathbb{R}^d} \sup_{i \in S} |f_i(x)|$ and $V^\dagger \mathbf{f}$ is given by (2.10).

Next, we take $e^{-W_t(x)}$ and $e^{-\tilde{W}_t(x)}$, two solutions of (2.18), and observe that for all $i \in S$

$$\begin{aligned} w_i e^{-W_t^{(i)}(x)} - w_i e^{-\tilde{W}_t^{(i)}(x)} &= \int_0^t ds \int_{\mathbb{R}^d} \left[\psi^\dagger(i, -\mathbf{w} \cdot e^{-W_{t-s}(y)} + V_{t-s}^\dagger \mathbf{f}(y)) \right. \\ &\quad \left. - \psi^\dagger(i, -\mathbf{w} \cdot e^{-\tilde{W}_{t-s}(y)} + V_{t-s}^\dagger \mathbf{f}(y)) \right] \mathbb{P}_s^{(i)}(x, dy). \end{aligned}$$

Since $e^{-W_t(x)} \in [0, 1]^\ell$ and $V^\dagger \mathbf{f}$ satisfies (3.12), we have, for all $s \leq T$, that

$$\left\| -\mathbf{w} \cdot e^{-W_s(x)} + V_s^\dagger \mathbf{f}(x) \right\| \leq \|\mathbf{w}\| + C(T) \|\mathbf{f}\| := a(T),$$

and the same inequality holds for $e^{-\tilde{W}_t(x)}$. Therefore, by the definition of $\Psi^\dagger$ and (3.11), there exists $L_T > 0$ such that we obtain, for all $t \leq T$, the following inequality

$$\begin{aligned} &\left| w_i e^{-W_t^{(i)}(x)} - w_i e^{-\tilde{W}_t^{(i)}(x)} \right| \\ &\leq \int_0^t ds \int_{\mathbb{R}^d} L_T \left\| \mathbf{w} \cdot e^{-W_{t-s}(x)} - \mathbf{w} \cdot e^{-\tilde{W}_{t-s}(x)} \right\| \mathbb{P}_s^{(i)}(x, dy). \end{aligned}$$

The latter implies the following inequality

$$\begin{aligned} &\left\| \mathbf{w} \cdot e^{-W_t(x)} - \mathbf{w} \cdot e^{-\tilde{W}_t(x)} \right\| \\ &\leq L_T \int_0^t \left\| \mathbf{w} \cdot e^{-W_s(x)} - \mathbf{w} \cdot e^{-\tilde{W}_s(x)} \right\| ds, \quad \text{for all } t \leq T. \end{aligned}$$

Thus by Gronwall's inequality, we deduce that

$$\mathbf{w} \cdot e^{-W_s(x)} = \mathbf{w} \cdot e^{-\tilde{W}_s(x)} \quad \text{for all } s \leq T.$$

Using that $T > 0$ was arbitrary, we get the uniqueness of the solution to (2.18). $\square$

Finally, we are ready to prove our main result.

Proof of Theorem 2 Recall that $((\Lambda, \mathbf{Z}), \widehat{\mathbb{P}}_{(\mathbf{v}, \mu)})$ is a Markov process and that $\widehat{\mathbb{P}}_\mu$ is defined as $\widehat{\mathbb{P}}_{(\tilde{\mathbf{v}}, \mu)}$, where $\tilde{\mathbf{v}}$ is such that $\tilde{\mathbf{v}}_i$ is a Poisson random measure with intensity $w_i \mu_i$, for all $i \in S$. Therefore, for $s, t \geq 0$, we see that

$$\widehat{\mathbb{E}}_\mu \left[f(\Lambda_{t+s}) \middle| (\Lambda_u, u \leq s) \right] = \widehat{\mathbb{E}}_{\mu, \tilde{\mathbf{v}}} \left[f(\Lambda_{t+s}) \middle| (\Lambda_u, u \leq s) \right] = \widehat{\mathbb{E}}_{(\Lambda_s, \mathbf{Z}_s)} \left[f(\Lambda_t) \right].$$

Then, in order to deduce that $(\Lambda, \widehat{\mathbb{P}}_\mu)$ is Markovian, we need to show that each coordinate of $\mathbf{Z}_t = (Z_t^1, \dots, Z_t^\ell)$ given $\Lambda_t = (\Lambda_t^1, \dots, \Lambda_t^\ell)$ is a Poisson random measure with intensity $w_i \Lambda_t^i$. From Campbell's formula for Poisson random measures (see for instance Sect. 3.2 of [10]), the latter is equivalent to showing that for all $\mathbf{h} \in \mathcal{B}^+(\mathbb{R}^d)^\ell$

$$\widehat{\mathbb{E}}_\mu \left[e^{-\langle \mathbf{h}, \mathbf{Z}_t \rangle} \middle| \Lambda_t \right] = \exp \left\{ -\langle \mathbf{w} \cdot (\mathbf{1} - e^{\mathbf{h}}), \Lambda_t \rangle \right\},$$

or equivalently, that for all $\mathbf{f}, \mathbf{h} \in \mathcal{B}^+(\mathbb{R}^d)^\ell$

$$\widehat{\mathbb{E}}_\mu \left[e^{-\langle \mathbf{f}, \Lambda_t \rangle - \langle \mathbf{h}, \mathbf{Z}_t \rangle} \right] = \widehat{\mathbb{E}}_\mu \left[e^{-\langle \mathbf{w} \cdot (\mathbf{1} - e^{\mathbf{h}}) + \mathbf{f}, \Lambda_t \rangle} \right]. \quad (3.13)$$

Using (2.17) with $\tilde{\mathbf{v}}$, we find

$$\widehat{\mathbb{E}}_\mu \left[e^{-\langle \mathbf{f}, \Lambda_t \rangle - \langle \mathbf{h}, \mathbf{Z}_t \rangle} \right] = \exp \left\{ -\langle V_t^\dagger \mathbf{f} + \mathbf{w} \cdot (\mathbf{1} - e^{-U_t^{(\mathbf{f})}} \mathbf{h}), \mu \rangle \right\}.$$

Similarly, considering (2.17) again with $\tilde{\mathbf{v}}, \tilde{\mathbf{f}} = \mathbf{w} \cdot (\mathbf{1} - e^{-\mathbf{h}}) + \mathbf{f}$ and $\tilde{\mathbf{h}} = \mathbf{0}$, we get that

$$\begin{aligned} & \widehat{\mathbb{E}}_\mu \left[e^{-\langle \mathbf{w} \cdot (\mathbf{1} - e^{-\mathbf{h}}) + \mathbf{f}, \Lambda_t \rangle} \right] \\ &= \exp \left\{ -\left\langle V_t^\dagger (\mathbf{w} \cdot (\mathbf{1} - e^{-\mathbf{h}}) + \mathbf{f}) + \mathbf{w} \cdot (\mathbf{1} - e^{-U_t^{(\mathbf{w} \cdot (\mathbf{1} - e^{-\mathbf{h}}) + \mathbf{f})} \mathbf{0}}), \mu \right\rangle \right\}. \end{aligned}$$

Hence, if we prove that for any $\mathbf{f}, \mathbf{h} \in \mathcal{B}^+(\mathbb{R}^d)^\ell$, $x \in \mathbb{R}^d$, and $i \in S$, the following identity holds

$$\begin{aligned} V_t^{\dagger(i)} \mathbf{f}(x) + w_i (1 - e^{-U_t^{(\mathbf{f}, i)} \mathbf{h}(x)}) &= V_t^{\dagger(i)} (\mathbf{w} \cdot (\mathbf{1} - e^{-\mathbf{h}}) + \mathbf{f})(x) \\ &\quad + w_i \left(1 - e^{-U_t^{((\mathbf{w} \cdot (\mathbf{1} - e^{-\mathbf{h}}) + \mathbf{f}), i)} \mathbf{0}(x)} \right), \end{aligned} \quad (3.14)$$

we can deduce (3.13).

In order to obtain (3.14), we first observe that identities (2.10) and (2.18) together with the definition of $\psi^\dagger$ allow us to see that both left and right hand sides of (3.14) solve (2.3) with initial condition $\mathbf{f} + \mathbf{w} \cdot (\mathbf{1} - e^{-\mathbf{h}})$. Since (2.3) has a unique solution,

namely $V_t(f + w \cdot (1 - e^{-h}))$, we conclude that (3.14) holds and it is equal to $V_t^{(i)}(f + w \cdot (1 - e^{-h}))(x)$. Hence, we can finally deduce that $(\Lambda, \widehat{\mathbb{P}}_\mu)$ is a Markov process. Moreover, we have

$$\widehat{\mathbb{E}}_\mu \left[e^{-\langle f, \Lambda_t \rangle - \langle h, Z_t \rangle} \right] = e^{-\langle V_t(f + w \cdot (1 - e^{-h})), \mu \rangle} = \mathbb{E}_\mu \left[e^{-\langle f + w \cdot (1 - e^{-h}), X_t \rangle} \right],$$

and if, in particular, we take $h = \mathbf{0}$ the above identity is reduced to

$$\widehat{\mathbb{E}}_\mu \left[e^{-\langle f, \Lambda_t \rangle} \right] = \mathbb{E}_\mu \left[e^{-\langle f, X_t \rangle} \right].$$

This completes the proof. $\square$

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