Sample Path Properties of Generalized Random Sheets with Operator Scaling

Ercan Sönmez¹

Received: 21 November 2018 / Revised: 16 August 2019 / Published online: 3 November 2020
© The Author(s) 2020

Abstract

We consider operator scaling α -stable random sheets, which were introduced in Hoffmann (Operator scaling stable random sheets with application to binary mixtures. Dissertation Universität Siegen, 2011). The idea behind such fields is to combine the properties of operator scaling α -stable random fields introduced in Biermé et al. (Stoch Proc Appl 117(3):312–332, 2007) and fractional Brownian sheets introduced in Kamont (Probab Math Stat 16:85–98, 1996). We establish a general uniform modulus of continuity of such fields in terms of the polar coordinates introduced in Biermé et al. (2007). Based on this, we determine the box-counting dimension and the Hausdorff dimension of the graph of a trajectory over a non-degenerate cube $I \subset \mathbb{R}^d$.

Keywords Fractional random fields · Stable random sheets · Operator scaling · Selfsimilarity · Box-counting dimension · Hausdorff dimension

Mathematics Subject Classification (2020) Primary 60G60; Secondary 28A78 · 28A80 · 60G17 · 60G52

1 Introduction

In this paper, we consider a harmonizable operator scaling α -stable random sheet as introduced in [11]. The main idea is to combine the properties of operator scaling α -stable random fields and fractional Brownian sheets in order to obtain a more general class of random fields. Let us recall that a scalar valued random field $\{X(x) : x \in \mathbb{R}^d\}$ is said to be *operator scaling* for some matrix $E \in \mathbb{R}^{d \times d}$ and some $H > 0$ if

$$\{X(c^E x) : x \in \mathbb{R}^d\} \stackrel{\text{f.d.}}{=} \{c^H X(x) : x \in \mathbb{R}^d\} \quad \text{for all } c > 0, \quad (1.1)$$

✉ Ercan Sönmez
Ercan.soenmez@aau.at

¹ Department of Statistics, University of Klagenfurt, Universitätsstrasse 65-67,
9020 Klagenfurt, Austria

where $\stackrel{\text{f.d.}}{=}$ means equality of all finite-dimensional marginal distributions and, as usual, $c^E = \sum_{k=0}^{\infty} \frac{(\log c)^k}{k!} E^k$ is the matrix exponential. These fields can be regarded as an anisotropic generalization of self-similar random fields (see, e.g., [8]), whereas the fractional Brownian sheet $\{B_{H_1, \dots, H_d}(x) : x \in \mathbb{R}^d\}$ with Hurst indices $0 < H_1, \dots, H_d < 1$ can be seen as an anisotropic generalization of the well-known fractional Brownian field (see, e.g., [13]) and satisfies the scaling property

$$\begin{aligned} & \{B_{H_1, \dots, H_d}(c_1 x_1, \dots, c_d x_d) : x = (x_1, \dots, x_d) \in \mathbb{R}^d\} \\ & \stackrel{\text{f.d.}}{=} \{c_1^{H_1} \dots c_d^{H_d} B_{H_1, \dots, H_d}(x) : x \in \mathbb{R}^d\} \end{aligned}$$

for all constants $c_1, \dots, c_d > 0$. See [3, 10, 27] and the references therein for more information on the fractional Brownian sheet.

Throughout this paper, let $d = \sum_{j=1}^m d_j$ for some $m \in \mathbb{N}$ and $\tilde{E}_j \in \mathbb{R}^{d_j \times d_j}$, $j = 1, \dots, m$ be matrices with positive real parts of their eigenvalues. We define matrices $E_1, \dots, E_m \in \mathbb{R}^{d \times d}$ as

$$E_j = \begin{pmatrix} 0 & & & & 0 \\ & \ddots & & & \\ & & 0 & & \\ & & \tilde{E}_j & & \\ & & & 0 & \\ 0 & & & & \ddots & \\ & & & & & 0 \end{pmatrix}.$$

Further, we define the block diagonal matrix $E \in \mathbb{R}^{d \times d}$ as

$$E = \sum_{j=1}^m E_j = \begin{pmatrix} \tilde{E}_1 & & 0 \\ & \ddots & \\ 0 & & \tilde{E}_m \end{pmatrix}.$$

In analogy to the terminology in [11, Definition 1.1.1], a random field $\{X(x) : x \in \mathbb{R}^d\}$ is called *operator scaling stable random sheet* if for some $H_1, \dots, H_m > 0$ we have

$$\{X(c^{E_j} x) : x \in \mathbb{R}^d\} \stackrel{\text{f.d.}}{=} \{c^{H_j} X(x) : x \in \mathbb{R}^d\} \quad (1.2)$$

for all $c > 0$ and $j = 1, \dots, m$. Note that, by applying (1.2) iteratively, any operator scaling stable random sheet is also operator scaling for the matrix E and the exponent $H = \sum_{j=1}^m H_j$ in the sense of (1.1). Further, note that this definition is indeed a generalization of operator scaling random fields, since for $m = 1$, $d = d_1$ and $E = E_1 = \tilde{E}_1$ (1.2) coincides with the definition introduced in [4]. Another example of a random field satisfying (1.2) is given by the fractional Brownian sheet, where $E_j = d_j = 1$ for $j = 1, \dots, m$ in this case. Operator scaling stable random sheets have been proven to be quite flexible in modeling physical phenomena and can be

applied in order to extend the well-known Cahn–Hilliard phase-field model. See [1] and the references therein for more information.

Random fields satisfying a scaling property such as (1.1) or (1.2) are very popular in modeling, see [14,22] and the references in [5] for some applications. Most of these fields are Gaussian. However, Gaussian fields are not always flexible for example in modeling heavy tail phenomena. For this purpose, α -stable random fields have been introduced. See [17] for a good introduction to α -stable random fields.

Using a moving average and a harmonizable representation, the authors in [4] defined and analyzed two different classes of symmetric α -stable random fields satisfying (1.1). Following the outline in [4,5], these two classes were generalized to random fields satisfying (1.2) in [11]. The fields constructed in [4] have stationary increments, i.e., they satisfy

$$\{X(x+h) - X(h) : x \in \mathbb{R}^d\} \stackrel{\text{f.d.}}{=} \{X(x) : x \in \mathbb{R}^d\} \quad \text{for all } h \in \mathbb{R}^d.$$

This property has been proven to be quite useful in studying the sample path properties. However, the property of stationary increments is no more true for the fields constructed in [11]. The absence of this property is one of the challenging difficulties we face in determining results about their sample paths.

Another main tool in studying sample paths of operator scaling stable random sheets are polar coordinates with respect to the matrices E_j , $j = 1, \dots, m$, introduced in [16] and used in [4,5]. If $\{X(x) : x \in \mathbb{R}^d\}$ is an operator scaling symmetric α -stable random sheet with $\alpha = 2$, using (1.2), one can write the variance of $X(x)$, $x \in \mathbb{R}^d$, as

$$\mathbb{E}[X^2(x)] = \tau_E(x)^{2H} \mathbb{E}[X^2(l_E(x))],$$

where $H = \sum_{j=1}^m H_j$ and $\tau_E(x)$ is the radial part of x with respect to E and $l_E(x)$ is its polar part. Therefore, if the random field has stationary increments in the Gaussian case information about the behavior of the polar coordinates $(\tau_{\tilde{E}_j}(x), l_{\tilde{E}_j}(x))$ contains information about the sample path regularity. This property also holds in the stable case $\alpha \in (0, 2)$. Moreover, this also remains to be true for operator scaling random sheets which do not have stationary increments but satisfy a slightly weaker property, see Corollary 3.3 below.

This paper is organized as follows. In Sect. 2, we introduce the main tools we need for the study in this paper. Section 2.1 is devoted to a spectral decomposition result from [16]. Section 2.2 is about the change to polar coordinates with respect to scaling matrices and we establish a relation between the radius $\tau_E(x)$ and the radii $\tau_{\tilde{E}_j}(x)$, $1 \leq j \leq m$, in Lemma 2.2 below. In Sect. 3, we present the results in [11] about the existence of harmonizable and moving average representations of operator scaling α -stable random sheets. Here, we will only focus on a harmonizable representation. Moreover, we prove that these random sheets fulfill a generalized type of modulus of continuity, which is deduced by showing the applicability of results in [5,6]. Based on this and generalizing a combination of methods used in [2,4,5,24], in Sect. 4 we present our results on the Hausdorff dimension and box-counting dimension of the graph of harmonizable operator scaling stable random sheets.

2 Preliminaries

2.1 Spectral Decomposition

Let $A \in \mathbb{R}^{d \times d}$ be a matrix with p distinct positive real parts of its eigenvalues $0 < a_1 < \dots < a_p$ for some $p \leq d$. Factor the minimal polynomial of A into $f_1, \dots, f_p$, where all roots of f_i have real part equal to a_i , and define $V_i = \text{Ker}(f_i(A))$. Then, by [16, Theorem 2.1.14],

$$\mathbb{R}^d = V_1 \oplus \dots \oplus V_p$$

is a direct sum decomposition, i.e. we can write any $x \in \mathbb{R}^d$ uniquely as

$$x = x_1 + \dots + x_p$$

for $x_i \in V_i$, $1 \leq i \leq p$. Further, we can choose an inner product on $\mathbb{R}^d$ such that the subspaces $V_1, \dots, V_p$ are mutually orthogonal. Throughout this paper, for any $x \in \mathbb{R}^d$ we will choose $\|x\| = \langle x, x \rangle^{1/2}$ as the corresponding Euclidean norm. In view of our methods this will entail no loss of generality, since all norms are equivalent.

2.2 Polar Coordinates

We now recall the results about the change to polar coordinates used in [4,5]. As before, let $A \in \mathbb{R}^{d \times d}$ be a matrix with distinct positive real parts of its eigenvalues $0 < a_1 < \dots < a_p$ for some $p \leq d$. According to [4, Sect. 2] there exists a norm $\|\cdot\|_A$ on $\mathbb{R}^d$ such that for the unit sphere $S_A = \{x \in \mathbb{R}^d : \|x\|_A = 1\}$ the mapping $\Psi_A : (0, \infty) \times S_A \rightarrow \mathbb{R}^d \setminus \{0\}$ defined by $\Psi_A(r, \theta) = r^A \theta$ is a homeomorphism. To be more precise, the norm $\|\cdot\|_A$ is defined by

$$\|x\|_A = \int_0^1 \|t^A x\| \frac{dt}{t}, \quad x \in \mathbb{R}^d. \quad (2.1)$$

Thus, we can write any $x \in \mathbb{R}^d \setminus \{0\}$ uniquely as

$$x = \tau_A(x)^A l_A(x), \quad (2.2)$$

where $\tau_A(x) > 0$ is called the radial part of x with respect to A and $l_A(x) \in \{x \in \mathbb{R}^d : \tau_A(x) = 1\}$ is called the direction. It is clear that $\tau_A(x) \rightarrow \infty$ as $\|x\| \rightarrow \infty$ and $\tau_A(x) \rightarrow 0$ as $\|x\| \rightarrow 0$. Further, one can extend $\tau_A(\cdot)$ continuously to $\mathbb{R}^d$ by setting $\tau_A(0) = 0$. Note that, by (2.2), it is straightforward to see that $\tau_A(\cdot)$ satisfies

$$\tau_A(c^A x) = c \cdot \tau_A(x) \quad \text{for all } c > 0.$$

Such functions are called A -homogeneous.

Let us recall a result about bounds on the growth rate of $\tau_A(\cdot)$ in terms of $a_1, \dots, a_p$ established in [4, Lemma 2.1].

Lemma 2.1 *Let $\varepsilon > 0$ be small enough. Then, there exist constants $K_1, \dots, K_4 > 0$ such that*

$$K_1 \|x\|^{\frac{1}{a_1} + \varepsilon} \leq \tau_A(x) \leq K_2 \|x\|^{\frac{1}{a_p} - \varepsilon}$$

for all x with $\tau_A(x) \leq 1$, and

$$K_3 \|x\|^{\frac{1}{a_p} - \varepsilon} \leq \tau_A(x) \leq K_4 \|x\|^{\frac{1}{a_1} + \varepsilon}$$

for all x with $\tau_A(x) \geq 1$.

We remark that the bounds on the growth rate of $\tau_A(\cdot)$ have been improved in [5, Proposition 3.3], but the bounds given in Lemma 2.1 suffice for our purposes.

The following Lemma will be needed in the next section in order to give an upper bound on the modulus of continuity.

Lemma 2.2 *Let $E, \tilde{E}_1, \dots, \tilde{E}_m$ be as above. Then, there exists a constant $C \geq 1$ such that*

$$C^{-1} \sum_{j=1}^m \tau_{\tilde{E}_j}(x_j) \leq \tau_E(x) \leq C \sum_{j=1}^m \tau_{\tilde{E}_j}(x_j)$$

for any $x = (x_1, \dots, x_m) \in \mathbb{R}^{d_1} \times \dots \times \mathbb{R}^{d_m} = \mathbb{R}^d$.

Proof Let $\mathbb{R}^{\tilde{d}_j} := \{0\} \times \dots \times \{0\} \times \mathbb{R}^{d_j} \times \{0\} \times \dots \times \{0\} \subset \mathbb{R}^d$, $1 \leq j \leq m$, be a subspace and note that

$$\mathbb{R}^d = \mathbb{R}^{\tilde{d}_1} \oplus \dots \oplus \mathbb{R}^{\tilde{d}_m}$$

is a direct sum decomposition with respect to E . Throughout, write $x = (x_1, \dots, x_m) = \bar{x}_1 + \dots + \bar{x}_m$ with respect to this decomposition. From [15, Lemma 2.2], we have for some $c \geq 1$

$$\frac{1}{c} \sum_{i=1}^m \tau_E(\bar{x}_i) \leq \tau_E(x) \leq c \sum_{i=1}^m \tau_E(\bar{x}_i).$$

It remains to prove $\tau_E(\bar{x}_i) = \tau_{\tilde{E}_i}(x_i)$ for $1 \leq i \leq m$. Without loss of generality assume $i = 1$ and for simplicity in this proof let us assume that $m = 2$. Thus, for any vector $x \in \mathbb{R}^d$ let us write $x = (x_1, x_2) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$. Note that by definition

$$\begin{aligned} (x_1, 0) &= \tau_E(x_1, 0)^E l_E(x_1, 0) = \left(\tau_E(x_1, 0)^{\tilde{E}_1} l_E(x_1, 0)_1, \tau_E(x_1, 0)^{\tilde{E}_2} l_E(x_1, 0)_2 \right) \\ &= \left(\tau_E(x_1, 0)^{\tilde{E}_1} l_E(x_1, 0)_1, 0 \right), \end{aligned}$$

where we used the notation $l_E(x) = (l_E(x)_1, l_E(x)_2) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$. But on the other hand one can write

$$x_1 = \tau_{\tilde{E}_1}(x_1)^{\tilde{E}_1} l_{\tilde{E}_1}(x_1)$$

yielding that

$$\tau_{\tilde{E}_1}(x_1)^{\tilde{E}_1} l_{\tilde{E}_1}(x_1) = \tau_E(x_1, 0)^{\tilde{E}_1} l_E(x_1, 0)_1.$$

Further noting that

$$l_E(x_1, 0) = (l_E(x_1, 0)_1, l_E(x_1, 0)_2) = (l_E(x_1, 0)_1, 0)$$

and taking into account the definition of the norm $\|\cdot\|_{\tilde{E}_1}$ given in (2.1) we obtain

$$\begin{aligned} \|l_{\tilde{E}_1}(x_1)\|_{\tilde{E}_1} &= 1 = \|(l_E(x_1, 0)_1, 0)\|_E = \int_0^1 \|t^E(l_E(x_1, 0)_1, 0)\| \frac{dt}{t} \\ &= \int_0^1 \|t^{\tilde{E}_1} l_E(x_1, 0)_1\| \frac{dt}{t} = \|l_E(x_1, 0)_1\|_{\tilde{E}_1} \end{aligned}$$

Thus, by the uniqueness of the representation we have $\tau_{\tilde{E}_1}(x_1) = \tau_E(x_1, 0)$ and $l_{\tilde{E}_1}(x_1) = l_E(x_1, 0)_1$ as desired. This concludes the proof. $\square$

Corollary 2.3 *Let $E, \tilde{E}_1, \dots, \tilde{E}_m$ be as above. Then, there exists a constant $C \geq 1$ such that*

$$C^{-1} \sum_{j=1}^m \tau_{\tilde{E}_j}(x_j)^H \leq \tau_E(x)^H$$

for any $H > 0$ and $x = (x_1, \dots, x_m) \in \mathbb{R}^{d_1} \times \dots \times \mathbb{R}^{d_m} = \mathbb{R}^d$.

3 Harmonizable Operator Scaling Random Sheets

We consider harmonizable operator scaling stable random sheets defined in [11] and present some related results established in [11]. Most of these will also follow from the results derived in [4,5]. Throughout this paper, for $j = 1, \dots, m$ assume that the real parts of the eigenvalues of $\tilde{E}_j$ are given by $0 < a_1^j < \dots < a_{p_j}^j$ for some $p_j \leq d_j$. Let $q_j = \text{trace}(\tilde{E}_j)$. Suppose that $\psi_j : \mathbb{R}^{d_j} \rightarrow [0, \infty)$ are continuous $\tilde{E}_j^T$ -homogeneous functions, which means according to [4, Definition 2.6] that

$$\psi_j(c \tilde{E}_j^T x) = c \psi_j(x) \quad \text{for all } c > 0.$$

Moreover, we assume that $\psi_j(x) \neq 0$ for $x \neq 0$. See [4,5] for various examples of such functions.

Let $0 < \alpha \leq 2$ and $W_\alpha(d\xi)$ be a complex isotropic symmetric α -stable random measure on $\mathbb{R}^d$ with Lebesgue control measure (see [17, Chapter 6.3]).

Theorem 3.1 *For any vector $x \in \mathbb{R}^d$ let $x = (x_1, \dots, x_m) \in \mathbb{R}^{d_1} \times \dots \times \mathbb{R}^{d_m} = \mathbb{R}^d$. The random field*

$$X_\alpha(x) = \operatorname{Re} \int_{\mathbb{R}^d} \prod_{j=1}^m (e^{i\langle x_j, \xi_j \rangle} - 1) \psi_j(\xi_j)^{-H_j - \frac{q_j}{\alpha}} W_\alpha(d\xi), \quad x \in \mathbb{R}^d \quad (3.1)$$

exists and is stochastically continuous if and only if $H_j \in (0, a_1^j)$ for all $j = 1, \dots, m$.

Proof This result has been proven in detail in [11], but it also follows as an easy consequence of [4, Theorem 4.1]. By the definition of stable integrals (see [17]), $X_\alpha(x)$ exists if and only if

$$\Gamma_\alpha(x) = \int_{\mathbb{R}^d} \prod_{j=1}^m |e^{i\langle x_j, \xi_j \rangle} - 1|^\alpha \psi_j(\xi_j)^{-\alpha H_j - q_j} d\xi < \infty,$$

but this is equivalent to

$$\Gamma_\alpha^j(x) = \int_{\mathbb{R}^{d_j}} |e^{i\langle x_j, \xi_j \rangle} - 1|^\alpha \psi_j(\xi_j)^{-\alpha H_j - q_j} d\xi_j < \infty,$$

for all $j = 1, \dots, m$. Since, in [4, Theorem 4.1], it is shown that $\Gamma_\alpha^j(x)$ is finite if and only if $H_j \in (0, a_1^j)$ the statement follows, see [11] for details. The stochastic continuity can be deduced similarly as a consequence of [4, Theorem 4.1]. $\square$

Note that from (3.1) it follows that $X_\alpha(x) = 0$ for all $x = (x_1, \dots, x_m) \in \mathbb{R}^{d_1} \times \dots \times \mathbb{R}^{d_m} = \mathbb{R}^d$ such that $x_j = 0$ for at least one $j \in \{1, \dots, m\}$.

The following result has been established in [11, Corollary 4.2.1]. The proof is carried out as the proof of [4, Corollary 4.2 (a)] via characteristic functions of stable integrals and by noting that $c^{E_j}x = (x_1, \dots, x_{j-1}, c^{E_j}x_j, x_{j+1}, \dots, x_m)$ for all $c > 0$ and $x = (x_1, \dots, x_m) \in \mathbb{R}^{d_1} \times \dots \times \mathbb{R}^{d_m} = \mathbb{R}^d$.

Corollary 3.2 *Under the conditions of Theorem 3.1, the random field $\{X_\alpha(x) : x \in \mathbb{R}^d\}$ is operator scaling in the sense of (1.2), that is, for any $c > 0$*

$$\{X(c^{E_j}x) : x \in \mathbb{R}^d\} \stackrel{\text{f.d.}}{=} \{c^{H_j}X(x) : x \in \mathbb{R}^d\}. \quad (3.2)$$

As we shall see below, fractional Brownian sheets fall into the class of random fields given by (3.1). It is known that a fractional Brownian sheet does not have stationary increments. Thus, in general, a random field given by (3.1) does not possess stationary increments. But it satisfies a slightly weaker property, as the following statement shows.

Corollary 3.3 *Under the conditions of Theorem 3.1, for any $h \in \mathbb{R}^{d_j}$, $j = 1, \dots, m$*

$$\{X_\alpha(x_1, \dots, x_{j-1}, x_j + h, x_{j+1}, \dots, x_m) - X_\alpha(x_1, \dots, x_{j-1}, h, x_{j+1}, \dots, x_m) : x \in \mathbb{R}^d\} \\ \stackrel{\text{f.d.}}{=} \{X_\alpha(x) : x \in \mathbb{R}^d\},$$

where we used the notation $x = (x_1, \dots, x_m) \in \mathbb{R}^{d_1} \times \dots \times \mathbb{R}^{d_m} = \mathbb{R}^d$.

Proof This result has been established in [11, Corollary 4.2.2] and is proven similarly to [4, Corollary 4.2 (b)]. $\square$

As an easy consequence of the results in this paper, we will derive global Hölder critical exponents of the random fields defined in (3.1). Following [7, Definition 5], $\beta \in (0, 1)$ is said to be the Hölder critical exponent of the random field $\{X(x) : x \in \mathbb{R}^d\}$, if there exists a modification X^* of X such that for any $s \in (0, \beta)$ the sample paths of X^* satisfy almost surely a uniform Hölder condition of order s on any compact set $I \subset \mathbb{R}^d$, i.e., there exists a positive and finite random variable Z such that almost surely

$$|X^*(x) - X^*(y)| \leq Z\|x - y\|^s \quad \text{for all } x, y \in I, \quad (3.3)$$

whereas, for any $s \in (\beta, 1)$, (3.3) almost surely fails.

Let us now state our main result of this section. Note that under the assumption $H_j < a_j^1$ and up to considering matrices $\tilde{E}_j = \frac{E_j}{H_j}$ instead of E_j in (1.2), $1 \leq j \leq m$, and with the observation that

$$c_1^j \tau_{\tilde{E}_j}(x_j)^{H_j} \leq \tau_{\tilde{E}_j}(x_j) \leq c_2^j \tau_{\tilde{E}_j}(x_j)^{H_j}, \quad \forall x_j \in \mathbb{R}^{d_j}$$

for some positive and finite constants c_1^j, c_2^j as noted in [6, Remark 5.1], without loss of generality we will assume $H_j = 1 < a_j^1$ in the proof of the following statement. We will make this assumption for notational convenience.

Proposition 3.4 *Under the above assumptions and the assumption that $H_j = 1$ or, equivalently $a_j^1 > 1$ for $j = 1, \dots, m$ there exists a modification X_α^* of the random field in (3.1) such that for any $\varepsilon > 0$ and any non-empty compact set $G_d \subset \mathbb{R}^d$*

$$\sup_{\substack{x, y \in G_d \\ x \neq y}} \frac{|X_\alpha^*(x) - X_\alpha^*(y)|}{\sum_{j=1}^m \tau_{\tilde{E}_j}(x_j - y_j)^{H_j} [\log(1 + \sum_{j=1}^n \tau_{\tilde{E}_j}(x_j - y_j)^{-1})]^{\frac{1}{2}}} < \infty \quad a.s.$$

if $\alpha = 2$ and

$$\sup_{\substack{x, y \in G_d \\ x \neq y}} \frac{|X_\alpha^*(x) - X_\alpha^*(y)|}{\sum_{j=1}^m \tau_{\tilde{E}_j}(x_j - y_j)^{H_j} [\log(1 + \sum_{j=1}^n \tau_{\tilde{E}_j}(x_j - y_j)^{-1})]^{\varepsilon + \frac{1}{2} + \frac{1}{\alpha}}} < \infty \quad a.s.$$

if $\alpha \in (0, 2)$, where we used the notation $x = (x_1, \dots, x_m) \in \mathbb{R}^{d_1} \times \dots \times \mathbb{R}^{d_m} = \mathbb{R}^d$. In particular, for any $0 < \gamma < H_j$ and $x = (x_1, \dots, x_m), y = (y_1, \dots, y_m) \in G_d$ one can find a positive and finite constant C such that

$$|X_\alpha^*(x) - X_\alpha^*(y)| \leq C \sum_{j=1}^m \tau_{E_j}(x_j - y_j)^\gamma \quad (3.4)$$

holds almost surely.

Proof Let us first assume that $\alpha = 2$. In the following let $\|\cdot\|_p$ denote the p -norm for $p \geq 1$, c an unspecified positive constant, $G_d \subset \mathbb{R}^d$ an arbitrary compact set, $r > 0$ and $B_E(r) = \{x \in \mathbb{R}^d : \tau_E(x) \leq r\}$. Moreover, by

$$d_X(x, y) = \mathbb{E}[|X_2(x) - X_2(y)|^2]^{\frac{1}{2}}, \quad x, y \in \mathbb{R}^d,$$

we denote the canonical metric associated to X_2 . We first show for $x, y \in G_d$ that

$$d_X(x, y) \leq c\tau_E(x - y). \quad (3.5)$$

By the equivalence of norms one can find a constant c such that

$$\frac{1}{c} \sum_{i=1}^m |u_i|^2 \leq \left(\sum_{i=1}^m |u_i| \right)^2 \leq c \sum_{i=1}^m |u_i|^2$$

for any $u \in \mathbb{R}^m$. Further let us remark that by definition the variance of the centered Gaussian random variable $X_2(x)$ in (3.1) is given by

$$\Gamma^2(x) = \mathbb{E}[X_2(x)^2] = c \int_{\mathbb{R}^d} \prod_{j=1}^m |e^{i\langle x_j, \xi_j \rangle} - 1|^2 \psi_j(\xi_j)^{-2-q_j} d\xi.$$

Note that for all $1 \leq j \leq m$ and $x = (x_1, \dots, x_m) \in G_d$ one can find a constant $0 < M(x) < \infty$ such that

$$\Gamma^2(x_1, \dots, x_{j-1}, \theta, x_{j+1}, \dots, x_n) \leq M(x) \leq \max_{x \in G_d} M(x) =: M \in (0, \infty),$$

where $\theta \in \mathbb{R}^{d_j}$ with $\tau_{E_j}(\theta) = 1$. Using all this and the elementary inequality

$$\begin{aligned} & |X_2(x) - X_2(y)| \\ & \leq \sum_{i=1}^m |X_2(x_1, \dots, x_{i-1}, x_i, y_{i+1}, \dots, y_m) - X_2(x_1, \dots, x_{i-1}, y_i, y_{i+1}, \dots, y_m)| \end{aligned}$$

with the convention that

$$X_2(x_1, \dots, x_{i-1}, y_i, y_{i+1}, \dots, y_m) = X_2(y)$$

for $i = 1$ and

$$X_2(x_1, \dots, x_{i-1}, x_i, y_{i+1}, \dots, y_m) = X_2(x)$$

for $i = m$ we get for all $x = (x_1, \dots, x_m)$ and $y = (y_1, \dots, y_m) \in G_d$

$$\begin{aligned} & \mathbb{E}[|X_2(x) - X_2(y)|^2] \\ & \leq \mathbb{E}\left[\left(\sum_{i=1}^m |X_2(x_1, \dots, x_{i-1}, x_i, y_{i+1}, \dots, y_m) - X_2(x_1, \dots, x_{i-1}, y_i, y_{i+1}, \dots, y_m)|\right)^2\right] \\ & \leq c\mathbb{E}\left[\sum_{i=1}^m |X_2(x_1, \dots, x_{i-1}, x_i, y_{i+1}, \dots, y_m) - X_2(x_1, \dots, x_{i-1}, y_i, y_{i+1}, \dots, y_m)|^2\right] \\ & = c\mathbb{E}\left[\left(\sum_{i=1}^m |X_2(x_1, \dots, x_{i-1}, x_i - y_i, y_{i+1}, \dots, y_m)|\right)^2\right], \end{aligned}$$

where we used Corollary 3.3 in the equality and the equivalence of norms in the last inequality. Using the operator scaling property and the generalized polar coordinates for $x_i - y_i$ we can further get an upper estimate of the last expression by

$$\begin{aligned} & c \sum_{i=1}^m \tau_{\tilde{E}_i}(x_i - y_i)^2 \mathbb{E}\left[|X_2(x_1, \dots, x_{i-1}, l_{\tilde{E}_i}(x_i - y_i), y_{i+1}, \dots, y_m)|^2\right] \\ & \leq cM \sum_{i=1}^m \tau_{\tilde{E}_i}(x_i - y_i)^2 \leq cM\tau_E(x - y)^2, \end{aligned}$$

where we used Corollary 2.3 with $H = 2$ in the last inequality, which proves (3.5). Now define an auxiliary Gaussian random field $Y = \{Y(t, s) : t \in G_d, s \in B_E(r)\}$ by

$$Y(t, s) = X_2(t + s) - X_2(t), \quad t \in G_d, s \in B_E(r),$$

where $r > 0$ is such that $B_E(r) \subset G_d$. Denote by D the diameter of $G_d \times B_E(r)$ in the metric d_Y associated with Y . Then, using (3.5) it is easy to see that $D \leq cr$ for some positive constant c . Using the latter inequality, by the arguments made in the proof of [15, Theorem 4.2] if $N(\varepsilon)$ denotes the smallest number of open d_Y -balls of radius $\varepsilon > 0$ needed to cover $G_d \times B_E(r)$ we obtain that

$$\int_0^D \sqrt{\log N(\varepsilon)} d\varepsilon \leq cr \sqrt{\log(1 + r^{-1})}.$$

Then, it follows from [21, Lemma 2.1] that for all $u \geq 2cr\sqrt{\log(1+r^{-1})}$

$$P\left(\sup_{(t,s) \in G_d \times B_E(r)} |X_2(t+s) - X_2(t)| \geq u\right) \leq \exp\left(-\frac{u^2}{4D^2}\right).$$

Therefore, by a standard Borel-Cantelli argument we conclude

$$\sup_{\substack{x,y \in G_d \\ x \neq y}} \frac{|X_2^*(x) - X_2^*(y)|}{\tau_E(x-y)\sqrt{\log(1+\tau_E(x-y)^{-1})}} < \infty \quad a.s. \quad (3.6)$$

for a continuous modification X_2^* of X_2 , which by Lemma 2.2 is equivalent to

$$\sup_{\substack{x,y \in G_d \\ x \neq y}} \frac{|X_\alpha^*(x) - X_\alpha^*(y)|}{\sum_{j=1}^m \tau_{\tilde{E}_j}(x_j - y_j)^{H_j} [\log(1 + \sum_{j=1}^n \tau_{\tilde{E}_j}(x_j - y_j)^{-1})]^{\frac{1}{2}}} < \infty \quad a.s.$$

Let us now assume that $\alpha \in (0, 2)$. In this case, the proof is a slight modification and extension of the proof of [6, Proposition 5.1] and the idea is to check the assumptions (i), (ii) and (iii) of Proposition 4.3 of the latter reference. Throughout this proof, we let c be a universal unspecified positive and finite constant and in the following let

$$f_\alpha(u, \xi) = \prod_{j=1}^m (e^{i\langle u_j, \xi_j \rangle} - 1) \psi_\alpha(\xi)$$

with

$$\psi_\alpha(\xi) = \prod_{j=1}^m \psi_j(\xi_j)^{-1 - \frac{q_j}{\alpha}}.$$

As in [6, Example 5.1] one checks that for all $\xi \in \mathbb{R}^d$, $\xi_j \neq 0$,

$$\psi_\alpha(\xi) \leq c \prod_{j=1}^m \tau_{\tilde{E}_j^T}(\xi_j)^{-1 - \frac{q_j}{\alpha}} \quad (3.7)$$

and, in particular there exist constants $A_j > 0$, $1 \leq j \leq m$, such that (3.7) holds for all $\|\xi_j\| > A_j$. For $\zeta > 0$ chosen arbitrarily small we consider the function $\tilde{\mu}$ on $\mathbb{R}^d$ given by $\tilde{\mu}(\xi) = \prod_{j=1}^m \tilde{\mu}_j(\xi_j)$ with

$$\tilde{\mu}_j(\xi_j) = \left(\|\xi_j\| + 1\right)^\alpha \mathbb{1}_{\{\|\xi_j\| \leq A_j\}} + \tau_{\tilde{E}_j^T}(\xi_j)^{-q_j} |\log \tau_{\tilde{E}_j^T}(\xi_j)|^{-1-\zeta} \mathbb{1}_{\{\|\xi_j\| > A_j\}}.$$

We observe that $\tilde{\mu}$ is positive on $\mathbb{R}^d \setminus \{0\}$ and, similarly to the calculations made in the proof of [6, Proposition 5.1], we obtain that

$$\int_{\mathbb{R}^{d_j}} \tilde{\mu}_j(\xi_j) d\xi_j = c \in (0, \infty), \quad \forall 1 \leq j \leq m.$$

Define $\mu_j = \frac{\tilde{\mu}_j}{c}$. Moreover, note that

$$\int_{\mathbb{R}^d} \tilde{\mu}(\xi) d\xi = c \in (0, \infty).$$

Hence, $\mu = \frac{\tilde{\mu}}{c}$ is well defined and now, as in the proof of [6, Proposition 5.1], we are going to check the assumptions (i), (ii) and (iii) of [6, Proposition 4.3] for

$$V_1(u) = f_\alpha(u, \Xi) \mu(\Xi)^{-\frac{1}{\alpha}},$$

where $u \in \mathbb{R}^d$ and Ξ is assumed to be a random vector on $\mathbb{R}^d$ with density μ .

We choose a constant $c \in (0, \infty)$ such that

$$\left| \prod_{j=1}^m (e^{i\langle x_j, \xi_j \rangle} - 1) - \prod_{j=1}^m (e^{i\langle y_j, \xi_j \rangle} - 1) \right| \leq \prod_{j=1}^m c \|\tau_E(x - y)^{\tilde{E}_j^T} \xi_j\|_{\tilde{E}_j^T}.$$

Note that this is possible. Then, it follows

$$|V_1(x) - V_1(y)| \leq |\psi_\alpha(\Xi)| \prod_{j=1}^m \min \left(c \|\rho(x, y)^{\tilde{E}_j^T} \Xi_j\|_{\tilde{E}_j^T}, 1 \right)$$

for the quasi-metric ρ on $\mathbb{R}^d$ defined by

$$\rho(x, y) = \tau_E(x - y), \quad \forall x, y \in \mathbb{R}^d.$$

Hence, we have

$$|V_1(x) - V_1(y)| \leq g(\rho(x, y), \Xi)$$

with g defined by

$$g(h, \xi) = |\psi_\alpha(\xi)| \prod_{j=1}^m \min \left(c \|h^{\tilde{E}_j^T} \xi_j\|_{\tilde{E}_j^T}, 1 \right),$$

so that we precisely recover assumption (i) in [6, Proposition 4.3] for the random field $\mathcal{G} = (g(h, \xi))_{h \in [0, \infty)}$. Moreover, assumption (ii) immediately follows as in the proof of [6, Proposition 5.1] from the definition of the norms $\|\cdot\|_{\tilde{E}_j^T}$ and by noting that the

product of monotonic functions again is monotonic. It remains to prove assumption (iii) in [6, Proposition 4.3]. To this end we write

$$\begin{aligned} I(h) &:= \mathbb{E}[\mathcal{G}^2(h)] = \int_{\mathbb{R}^d} g(h, \xi)^2 \mu(\xi)^{1-\frac{2}{\alpha}} d\xi \\ &= \prod_{j=1}^m \int_{\mathbb{R}^{d_j}} \min\left(c \|h^{\tilde{E}_j^T} \xi_j\|_{\tilde{E}_j^T}, 1\right)^2 \psi_j(\xi_j)^{-2-2\frac{q_j}{\alpha}} \mu_j(\xi_j)^{1-\frac{2}{\alpha}} d\xi_j. \end{aligned}$$

Using equality (3.7) similarly as shown in the calculations made in the proof of [6, Proposition 5.1] we obtain that

$$\int_{\mathbb{R}^{d_j}} \min\left(c \|h^{\tilde{E}_j^T} \xi_j\|_{\tilde{E}_j^T}, 1\right)^2 \psi_j(\xi_j)^{-2-2\frac{q_j}{\alpha}} \mu_j(\xi_j)^{1-\frac{2}{\alpha}} d\xi_j \leq ch^2 |\log(h)|^{2(1+\zeta)(\frac{1}{\alpha}-\frac{1}{2})}$$

which yields that

$$\begin{aligned} I(h) &= \prod_{j=1}^m \int_{\mathbb{R}^{d_j}} \min\left(c \|h^{\tilde{E}_j^T} \xi_j\|_{\tilde{E}_j^T}, 1\right)^2 \psi_j(\xi_j)^{-2-2\frac{q_j}{\alpha}} \mu_j(\xi_j)^{1-\frac{2}{\alpha}} d\xi_j \\ &\leq ch^{2m} |\log(h)|^{2m(1+\zeta)(\frac{1}{\alpha}-\frac{1}{2})}, \end{aligned}$$

so that assumption (iii) in [6, Proposition 4.3] with pm instead of p is fulfilled. Following the lines of the proof of [6, Proposition 5.1], we obtain that there exists a modification X_α^* of X_α such that

$$\sup_{\substack{x, y \in G_d \\ x \neq y}} \frac{|X_\alpha^*(x) - X_\alpha^*(y)|}{\tau_E(x - y) [\log(1 + \tau_E(x - y)^{-1})]^{\varepsilon + \frac{1}{2} + \frac{1}{\alpha}}} < \infty \quad a.s.$$

for any $\varepsilon > 0$ and any non-empty compact set $G_d \subset \mathbb{R}^d$, which by Lemma 2.2 is equivalent to

$$\sup_{\substack{x, y \in G_d \\ x \neq y}} \frac{|X_\alpha^*(x) - X_\alpha^*(y)|}{\sum_{j=1}^m \tau_{\tilde{E}_j}(x_j - y_j) [\log(1 + \sum_{j=1}^m \tau_{\tilde{E}_j}(x_j - y_j)^{-1})]^{\varepsilon + \frac{1}{2} + \frac{1}{\alpha}}} < \infty \quad a.s.$$

This completes the proof. $\square$

Corollary 3.5 *Under the assumptions of Theorem 3.1, there exist a positive and finite random variable Z and a continuous modification of X_α such that for any $s \in (0, \min_{1 \leq j \leq m} \frac{H_j}{a_{p_j}})$ the uniform Hölder condition (3.3) holds almost surely.*

We remark that Corollary 3.5 is not a statement about critical Hölder exponents. However, as a consequence of Theorem 4.1 below, we will see that any continuous version of X_α admits $\min_{1 \leq j \leq m} \frac{H_j}{a_{p_j}}$ as the critical exponent.

4 Hausdorff Dimension

We now state our main result on the Hausdorff and box-counting dimension of the graph of X_α defined in (3.1). In the following, for a set $B \subset \mathbb{R}^d$ we denote by $\underline{\dim}_B B$, $\overline{\dim}_B B$ and $\dim_{\mathcal{H}} B$ its lower, upper box-counting and Hausdorff dimension, respectively. We refer to [9] for a definition of these objects.

Theorem 4.1 *Suppose that the conditions of Theorem 3.1 hold. Then, for any continuous version of X_α , almost surely*

$$\dim_{\mathcal{H}} G_{X_\alpha}([0, 1]^d) = \dim_B G_{X_\alpha}([0, 1]^d) = d + 1 - \min_{1 \leq j \leq m} \frac{H_j}{a_{p_j}^j}, \quad (4.1)$$

where

$$G_{X_\alpha}([0, 1]^d) = \left\{ (x, X_\alpha(x)) : x \in [0, 1]^d \right\}$$

is the graph of X_α over $[0, 1]^d$.

Proof Let us choose a continuous version of X_α by Corollary 3.5. From Corollary 3.5, for any $0 < s < \min_{1 \leq j \leq m} \frac{H_j}{a_{p_j}^j}$, the sample paths of X_α satisfy almost surely a uniform Hölder condition of order s on $[0, 1]^d$. Thus, by a d -dimensional version of [9, Corollary 11.2] we have

$$\dim_{\mathcal{H}} G_{X_\alpha}([0, 1]^d) \leq \overline{\dim}_B G_{X_\alpha}([0, 1]^d) \leq d + 1 - s, \quad a.s.$$

Letting $s \uparrow \min_{1 \leq j \leq m} \frac{H_j}{a_{p_j}^j}$ along rational numbers yields the upper bound in (4.1).

It remains to prove the lower bound in (4.1). Since the inequality

$$\underline{\dim}_B B \geq \dim_{\mathcal{H}} B$$

holds for every $B \subset \mathbb{R}^d$ (see [9, Chapter 3.1]), it suffices to show

$$\dim_{\mathcal{H}} G_{X_\alpha}([0, 1]^d) \geq d + 1 - \min_{1 \leq j \leq m} \frac{H_j}{a_{p_j}^j}, \quad a.s.$$

Further, note that, since $Q = [\frac{1}{2}, 1]^d \subset [0, 1]^d$, we have

$$\dim_{\mathcal{H}} G_{X_\alpha}([0, 1]^d) \geq \dim_{\mathcal{H}} G_{X_\alpha}(Q)$$

by monotonicity of the Hausdorff dimension. Thus, it is even enough to show that

$$\dim_{\mathcal{H}} G_{X_\alpha}(Q) \geq d + 1 - \min_{1 \leq j \leq m} \frac{H_j}{a_{p_j}^j}, \quad a.s. \quad (4.2)$$

We will show this by combining the methods used in [2,4,5]. From now on, without loss of generality, we will assume that

$$\min_{1 \leq j \leq m} \frac{H_j}{a_{p_j}^j} = \frac{H_1}{a_{p_1}^1}.$$

Let $\gamma > 1$. Following the argument in [4, Theorem 5.6], in view of the Frostman criterion [9, Theorem 4.13 (a)], it suffices to show that

$$I_\gamma := \int_{Q \times Q} \mathbb{E} \left[(\|x - y\|^2 + |X_\alpha(x) - X_\alpha(y)|^2)^{-\frac{\gamma}{2}} \right] dx dy < \infty$$

in order to obtain $\dim_{\mathcal{H}} G_{X_\alpha}(Q) \geq \gamma$ almost surely.

Using the characteristic function of the symmetric α -stable random field X_α , as in the proof of [5, Proposition 5.7], it can be shown that there is a constant $C_1 > 0$ such that

$$I_\gamma \leq C_1 \int_{Q \times Q} \|x - y\|^{1-\gamma} \sigma^{-1}(x, y) dx dy,$$

where

$$\sigma(x, y) = \|X_\alpha(x) - X_\alpha(y)\|_\alpha.$$

Combining this with Theorem 4.2 below we get

$$I_\gamma \leq \tilde{C}_1 \int_{Q_1 \times Q_1} \|x_1 - y_1\|^{1-\gamma} \tau_{\tilde{E}_1}(x_1 - y_1)^{-H_1} dx_1 dy_1,$$

for some $\tilde{C}_1 > 0$ and $Q_1 = [\frac{1}{2}, 1]^{d_1}$. With this inequality the assertion readily follows from the proof of [4, Theorem 5.6]. $\square$

The following Theorem is crucial for proving Theorem 4.1 and its proof is based on [23, Theorem 1]. See also [24–26]. Let us remark that a similar method of the following proof has been applied in [24, Theorem 3.4] for certain α -stable random fields if $1 \leq \alpha \leq 2$. In the following, we are able to extend this method for $0 < \alpha < 1$ and, in particular this shows that the statement of [24, Theorem 3.5] can be formulated for $0 < \alpha < 1$ as well.

Theorem 4.2 *There exists a constant $C_4 > 0$, depending on $H_1, \dots, H_m, q_1, \dots, q_m$ and d only, such that for all $x = (x_1, \dots, x_m), y = (y_1, \dots, y_m) \in [\frac{1}{2}, 1]^{d_1} \times \dots \times [\frac{1}{2}, 1]^{d_m}$ we have*

$$\sigma(x, y) \geq C_4 \cdot \tau_{\tilde{E}_1}(x_1 - y_1)^{H_1},$$

where $\tau_{\tilde{E}_1}(\cdot)$ is the radial part with respect to $\tilde{E}_1$.

Proof Throughout this proof, we fix $x = (x_1, \dots, x_m)$, $y = (y_1, \dots, y_m) \in [\frac{1}{2}, 1)^{d_1} \times \dots \times [\frac{1}{2}, 1)^{d_m}$. We will show that

$$\sigma(x, y) \geq Cr^{H_1} \quad (4.3)$$

for some constant $C > 0$ independent of x and y and $r = \tau_{\tilde{E}_1}(x_1 - y_1)$. Without loss of generality we will assume that $r > 0$, since for $r = 0$ (4.3) always holds. By definition, we have

$$\sigma^\alpha(x, y) = \int_{\mathbb{R}^d} \left| \prod_{j=1}^m (e^{i\langle x_j, \xi_j \rangle} - 1) - \prod_{j=1}^m (e^{i\langle y_j, \xi_j \rangle} - 1) \right|^\alpha \prod_{j=1}^m |\psi_j(\xi_j)|^{-\alpha H_j - q_j} d\xi. \quad (4.4)$$

Now, for every $j = 1, \dots, m$ we consider a so-called bump function $\delta_j \in C^\infty(\mathbb{R}^{d_j})$ with values in $[0, 1]$ such that $\delta_j(0) = 1$ and δ_j vanishes outside the open ball

$$B(K_j, 0) = \{z \in \mathbb{R}^{d_j} : \tau_{\tilde{E}_j}(z) < K_j\}$$

for

$$K_j = \min \left\{ 1, \frac{K_1^j}{K_2^j} \left(\sqrt{d_1} \frac{1}{2} \right)^{\frac{1}{a_1^j} - \frac{1}{a_{p_j}^j} + 2\varepsilon}, \frac{K_3^j}{K_4^j} \left(\sqrt{d_1} \frac{1}{2} \right)^{\frac{1}{a_{p_j}^j} - \frac{1}{a_1^j} - 2\varepsilon}, \frac{K_1^j}{K_4^j}, \frac{K_3^j}{K_2^j}, \right. \\ \left. K_1^j \left(\sqrt{d_1} \frac{1}{2} \right)^{\frac{1}{a_1^j} + \varepsilon}, K_3^j \left(\sqrt{d_1} \frac{1}{2} \right)^{\frac{1}{a_{p_j}^j} - \varepsilon} \right\},$$

where $\varepsilon > 0$ is some (sufficiently) small number and $K_1^j, \dots, K_4^j$ are the suitable constants derived from Lemma 2.1 corresponding to the matrix $\tilde{E}_j$. The choice of the constant $K_j > 0$ will be clear later in this proof. Let $\hat{\delta}_j$ be the Fourier transform of δ_j . It can be verified that $\hat{\delta}_j \in C^\infty(\mathbb{R}^{d_j})$ as well and that $\hat{\delta}_j(\lambda_j)$ decays rapidly as $\|\lambda_j\| \rightarrow \infty$. By the Fourier inversion formula, we have

$$\delta_j(s_j) = \frac{1}{(2\pi)^{d_j}} \int_{\mathbb{R}^{d_j}} e^{-i\langle s_j, \lambda_j \rangle} \hat{\delta}_j(\lambda_j) d\lambda_j \quad (4.5)$$

for all $s_j \in \mathbb{R}^{d_j}$. Let $\delta_1^r(s_1) = \frac{1}{r^{d_1}} \delta_1\left(\left(\frac{1}{r}\right)^{\tilde{E}_1} s_1\right)$. Then, by a change of variables in (4.5), for all $s_1 \in \mathbb{R}^{d_1}$ we obtain

$$\delta_1^r(s_1) = \frac{1}{(2\pi)^{d_1}} \int_{\mathbb{R}^{d_1}} e^{-i\langle s_1, \lambda_1 \rangle} \hat{\delta}_1(r^{\tilde{E}_1^T} \lambda_1) d\lambda_1. \quad (4.6)$$

Using Lemma 2.1 and the fact that $\tau_{\tilde{E}_1}(\cdot)$ is $\tilde{E}_1$ -homogeneous, it is straightforward to see that $\tau_{\tilde{E}_j}(x_j) \geq K_j$, $\tau_{\tilde{E}_1}\left(\left(\frac{1}{r}\right)^{\tilde{E}_1}(x_1 - y_1)\right) \geq K_1$ and $\tau_{\tilde{E}_1}\left(\left(\frac{1}{r}\right)^{\tilde{E}_1} x_1\right) \geq K_1$ for

$r = \tau_{\tilde{E}_1}(x_1 - y_1) > 0$. Therefore, we have $\delta_1^r(x_1) = 0$, $\delta_1^r(x_1 - y_1) = 0$ and $\delta_j(x_j) = 0$ for all $j = 2, \dots, m$. Hence, combining this with (4.5) and (4.6) it follows that

$$\begin{aligned}
 I &:= \int_{\mathbb{R}^d} \left(\prod_{j=1}^m (e^{i\langle x_j, \lambda_j \rangle} - 1) - \prod_{j=1}^m (e^{i\langle y_j, \lambda_j \rangle} - 1) \right) \\
 &\quad \cdot e^{-i\langle x_1, \lambda_1 \rangle} \hat{\delta}_1(r \tilde{E}_1^T \lambda_1) \prod_{j=2}^m \hat{\delta}_j(\lambda_j) d\lambda \\
 &= (2\pi)^d \left(\delta_1^r(0) - \delta_1^r(x_1) \right) \prod_{j=2}^m \left(\delta_j(0) - \delta_j(x_j) \right) \\
 &\quad - (2\pi)^d \left(\delta_1^r(x_1 - y_1) - \delta_1^r(x_1) \right) \prod_{j=2}^m \left(\delta_j(x_j - y_j) - \delta_j(x_j) \right) \\
 &= (2\pi)^d \frac{1}{r^{q_1}}.
 \end{aligned} \tag{4.7}$$

For every $\alpha \in (0, 2)$ we can choose $k \in \mathbb{N}$ such that $k\alpha \geq 1$ and let $\beta' > 1$ be the constant such that $\frac{1}{k\alpha} + \frac{1}{\beta'} = 1$. We first show that

$$\begin{aligned}
 &\left(\int_{\mathbb{R}^d} \left| \prod_{j=1}^m (e^{i\langle x_j, \lambda_j \rangle} - 1) - \prod_{j=1}^m (e^{i\langle y_j, \lambda_j \rangle} - 1) \right|^{k\alpha} \prod_{j=1}^m |\psi_j(\lambda_j)|^{-\alpha H_j - q_j} d\lambda \right)^{\frac{1}{k\alpha}} \\
 &\leq 2^{k\alpha(m+1)} \sigma(x, y)^{\frac{1}{k}}.
 \end{aligned} \tag{4.8}$$

For $\lambda = (\lambda_1, \dots, \lambda_m) \in \mathbb{R}^{d_1} \times \dots \times \mathbb{R}^{d_m}$, let

$$z(\lambda) = \prod_{j=1}^m (e^{i\langle x_j, \lambda_j \rangle} - 1) - \prod_{j=1}^m (e^{i\langle y_j, \lambda_j \rangle} - 1)$$

and note that, since $|e^{it} - 1|^2 = 2 - 2\cos t \leq 4$ for all $t \in \mathbb{R}$, it follows that

$$|z(\lambda)| \leq \prod_{j=1}^m |e^{i\langle x_j, \lambda_j \rangle} - 1| + \prod_{j=1}^m |e^{i\langle y_j, \lambda_j \rangle} - 1| \leq 2^{m+1}.$$

From this, we obtain

$$\begin{aligned}
 &\left(\int_{\mathbb{R}^d} |z(\lambda)|^{k\alpha} \prod_{j=1}^m |\psi_j(\lambda_j)|^{-\alpha H_j - q_j} d\lambda \right)^{\frac{1}{k\alpha}} \\
 &= \left(\int_{\{\lambda \in \mathbb{R}^d : |z(\lambda)| \leq 1\}} |z(\lambda)|^{k\alpha} \prod_{j=1}^m |\psi_j(\lambda_j)|^{-\alpha H_j - q_j} d\lambda \right)^{\frac{1}{k\alpha}}
 \end{aligned}$$

$$\begin{aligned}
& + \int_{\{\lambda \in \mathbb{R}^d : |z(\lambda)| > 1\}} |z(\lambda)|^{k\alpha} \prod_{j=1}^m |\psi_j(\lambda_j)|^{-\alpha H_j - q_j} d\lambda \Big)^{\frac{1}{k\alpha}} \\
& \leq \left(\int_{\{\lambda \in \mathbb{R}^d : |z(\lambda)| \leq 1\}} |z(\lambda)|^\alpha \prod_{j=1}^m |\psi_j(\lambda_j)|^{-\alpha H_j - q_j} d\lambda \right. \\
& \quad \left. + \int_{\{\lambda \in \mathbb{R}^d : |z(\lambda)| > 1\}} |z(\lambda)|^{k\alpha + \alpha} \prod_{j=1}^m |\psi_j(\lambda_j)|^{-\alpha H_j - q_j} d\lambda \right)^{\frac{1}{k\alpha}} \\
& \leq (2^{m+1})^{k\alpha} \left(\int_{\mathbb{R}^d} |z(\lambda)|^\alpha \prod_{j=1}^m |\psi_j(\lambda_j)|^{-\alpha H_j - q_j} d\lambda \right)^{\frac{1}{k\alpha}} = (2^{m+1})^{k\alpha} \sigma(x, y)^{\frac{1}{k}}.
\end{aligned}$$

Now, using (4.8), by Hölder's inequality and (4.4) we have

$$\begin{aligned}
I & \leq \left(\int_{\mathbb{R}^d} \left| \prod_{j=1}^m (e^{i\langle x_j, \lambda_j \rangle} - 1) - \prod_{j=1}^m (e^{i\langle y_j, \lambda_j \rangle} - 1) \right|^{k\alpha} \prod_{j=1}^m |\psi_j(\lambda_j)|^{-\alpha H_j - q_j} d\lambda \right)^{\frac{1}{k\alpha}} \\
& \quad \cdot \left(\int_{\mathbb{R}^d} \frac{1}{\left(\prod_{j=1}^m |\psi_j(\lambda_j)|^{-\alpha H_j - q_j} \right)^{\frac{\beta'}{k\alpha}}} |\hat{\delta}_1(r^{\tilde{E}_1^T} \lambda_1) \prod_{j=2}^m \hat{\delta}_j(\lambda_j)|^{\beta'} d\lambda \right)^{\frac{1}{\beta'}} \\
& \leq 2^{k\alpha(m+1)} \sigma(x, y)^{\frac{1}{k}} \cdot r^{-\frac{H_1}{k} - \frac{q_1}{k\alpha} - \frac{q_1}{\beta'}} \\
& \quad \cdot \left(\int_{\mathbb{R}^d} \frac{1}{\left(\prod_{j=1}^m |\psi_j(\lambda_j)|^{-H_j - \frac{q_j}{\alpha}} \right)^{\beta'}} \prod_{j=1}^m |\hat{\delta}_j(\lambda_j)|^{\beta'} d\lambda \right)^{\frac{1}{\beta'}} \\
& = \tilde{C} \cdot (\sigma(x, y) \cdot r^{-H_1 - kq_1})^{\frac{1}{k}}, \tag{4.9}
\end{aligned}$$

where $\tilde{C} > 0$ is a constant, which only depends on $H_1, \dots, H_m, q_1, \dots, q_m, k, \alpha, d$ and δ . It is clear that (4.3) follows from (4.7) and (4.9). This finishes the proof of the Theorem. $\square$

As an immediate consequence of Theorem 4.1, we obtain the following.

Corollary 4.3 *Let the assumptions of Theorem 3.1 hold. Then any continuous version of X_α admits $\min_{1 \leq j \leq m} \frac{H_j}{a_{p_j}}$ as the Hölder critical exponent.*

Remark 4.4 Let $\alpha = 2, d_j = \tilde{E}_j = 1$ for all $j = 1, \dots, m$ and consider the function $\psi(\xi_j) = |\xi_j|$ for all $\xi_j \in \mathbb{R}$. Clearly, ψ_j is a homogeneous function and satisfies $\psi_j(\xi_j) \neq 0$ for all $\xi_j \neq 0$. Thus, by Theorem 3.1, we can define

$$X_2(x) = \operatorname{Re} \int_{\mathbb{R}^d} \prod_{j=1}^d (e^{ix_j \xi_j} - 1) |\xi_j|^{-H_j - \frac{1}{2}} W_2(d\xi), \quad x = (x_1, \dots, x_d) \in \mathbb{R}^d,$$

for all $0 < H_j < 1$, $j = 1, \dots, d$ and the statement of Theorem 4.1 becomes

$$\dim_{\mathcal{H}} G_{X_2}([0, 1]^d) = \dim_{\mathcal{B}} G_{X_2}([0, 1]^d) = d + 1 - \min_{1 \leq j \leq d} H_j, \quad a.s.$$

Further, up to a multiplicative constant, the random field X_2 is a fractional Brownian sheet with Hurst indices $H_1, \dots, H_d$ (see [10]). Thus, Theorem 4.1 can be seen as a generalization of [2, Theorem 1.3]. Further, as noted above, for $m = 1$, $d = d_1$ and $E = E_1 = \tilde{E}_1$ the random field X_α given by (3.1) coincides with the random field in [4, Theorem 4.1] and the statement of Theorem 4.1 becomes

$$\dim_{\mathcal{H}} G_{X_\alpha}([0, 1]^d) = \dim_{\mathcal{B}} G_{X_\alpha}([0, 1]^d) = d + 1 - \frac{H_1}{a_{p_1}^1}, \quad a.s.$$

which is the statement of [4, Theorem 5.6] for $\alpha = 2$ and [5, Proposition 5.7] for $\alpha \in (0, 2)$. We finally remark that Theorem 4.1 can be proven similarly, if we replace $[0, 1]^d$ in (4.1) by any other compact cube of $\mathbb{R}^d$.

We finally remark that the Hausdorff dimension only depends on solely one index $\min_{1 \leq j \leq m} \frac{H_j}{a_{p_j}^1}$, whereas in higher space dimensions all indices are relevant and the corresponding results in more general dimensions can be found in [18]. See also [19, 20] for related results.

Acknowledgements The author would like to thank an anonymous referee and associate editor for some comments on earlier versions of the manuscript. A large part of this work has been written during the author's employment at Heinrich-Heine-Universität Düsseldorf. This work has been supported by Deutsche Forschungsgemeinschaft (DFG) under Grant KE1741/6-1.

Funding Open access funding provided by University of Klagenfurt.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

References

1. Anders, D., Hoffmann, A., Scheffler, H.P., Weinberg, K.: Application of operator-scaling anisotropic random fields to binary mixtures. *Phil. Mag.* **91**(29), 3766–3792 (2011)
2. Ayache, A.: Hausdorff dimension of the graph of the fractional Brownian sheet. *Rev. Mat. Iberoamericana* **20**(2), 395–412 (2004)
3. Ayache, A., Xiao, Y.: Asymptotic properties and Hausdorff dimensions of fractional Brownian sheets. *J. Fourier Anal. Appl.* **11**(4), 407–439 (2005)
4. Biermé, H., Meerschaert, M.M., Scheffler, H.P.: Operator scaling stable random fields. *Stoch. Proc. Appl.* **117**(3), 312–332 (2007)
5. Biermé, H., Lacaux, C.: Hölder regularity for operator scaling stable random fields. *Stoch. Proc. Appl.* **119**, 2222–2248 (2009)

6. Biermé, H., Lacaux, C.: Modulus of continuity of some conditionally sub-Gaussian fields, application to stable random fields. *Bernoulli* **21**, 1719–1759 (2015)
7. Bonami, A., Estrade, A.: Anisotropic analysis of some Gaussian models. *J. Fourier Anal. Appl.* **9**, 215–236 (2003)
8. Embrechts, P., Maehima, M.: Self-similar Processes. Princeton University Press, Princeton (2002)
9. Falconer, K.: Fractal Geometry: Mathematical Foundations and Applications. Wiley, Hoboken (1990)
10. Herbin, E.: From N parameter fractional Brownian motions to N parameter multifractional Brownian motions. *Rocky Mount. J. Math.* **36**, 1249–1284 (2006)
11. Hoffmann, A.: Operator Scaling Stable Random Sheets with Application to Binary Mixtures. Dissertation Universität Siegen (2011)
12. Kamont, A.: On the fractional anisotropic Wiener field. *Probab. Math. Stat.* **16**, 85–98 (1996)
13. Kolmogorov, A.N.: Wienersche Spiralen und einige andere interessante Kurven im Hilbertschen Raum. *C. R. Acad. Sci. URSS* **26**, 115–118 (1940)
14. Lévy Véhel, J.: Fractals in engineering: from theory to industrial applications. Springer, New York (1997)
15. Li, Y., Wang, W., Xiao, Y.: Exact moduli of continuity for operator scaling Gaussian random fields. *Bernoulli* **21**, 930–956 (2015)
16. Meerschaert, M.M., Scheffler, H.P.: Limit Distributions for Sums of Independent Random Vectors: Heavy Tails in Theory and Practice. Wiley, New York (2001)
17. Samorodnitsky, G., Taqqu, M.S.: Stable non-Gaussian random processes. Chapman and Hall, New York (1994)
18. Sönmez, E.: Hausdorff dimension results for operator-self-similar stable random fields. Dissertation Heinrich-Heine-Universität Düsseldorf (2017)
19. Sönmez, E.: The Hausdorff dimension of multivariate operator-self-similar Gaussian random fields. *Stoch. Process. Appl.* **128**, 426–444 (2018)
20. Sönmez, E.: Fractal behavior of multivariate operator-self-similar stable random fields. *Commun. Stoch. Anal.* **11**(2), 233–244 (2017)
21. Talagrand, M.: Hausdorff measure of trajectories of multiparameter fractional Brownian motion. *Ann. Probab.* **23**, 767–775 (1995)
22. Willinger, W., Paxson, V., Taqqu, M.S.: Self-similarity and heavy tails: structural modelling of network traffic, a practical guide to heavy tails. Birkhäuser, Boston (1998)
23. Wu, D., Xiao, Y.: Geometric properties of fractional Brownian sheets. *J. Fourier Anal. Appl.* **13**, 1–37 (2007)
24. Xiao, Y.: Properties of strong local nondeterminism and local times of stable random fields. *Random Fields Appl.* **6**, 279–310 (2011)
25. Xiao, Y.: Sample path properties of anisotropic Gaussian random fields. In: Khoshnevisan, D., Rassoul-Agha, F. (eds.) *A Minicourse on Stochastic Partial Differential Equations*, pp. 145–212. Springer, New York (2009)
26. Xiao, Y.: Strong local nondeterminism and sample path properties of Gaussian random fields, *Asymptot. Theory Probab. Stat. Appl.* 136–176 (2007)
27. Xiao, Y., Zhang, T.: Local times of fractional Brownian sheets. *Probab. Theory Relat. Fields* **124**, 204–226 (2002)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.