Functional Limit Theorems for the Fractional Ornstein–Uhlenbeck Process

Johann Gehringer[1] (ID) · **Xue-Mei Li[1]** (ID)

Abstract

We prove a functional limit theorem for vector-valued functionals of the fractional Ornstein–Uhlenbeck process, providing the foundation for the fluctuation theory of slow/fast systems driven by both long- and short-range-dependent noise. The limit process has both Gaussian and non-Gaussian components. The theorem holds for any L^2 functions, whereas for functions with stronger integrability properties the convergence is shown to hold in the Hölder topology, the rough topology for processes in $C^{\frac{1}{2}+}$. This leads to a 'rough creation' / 'rough homogenization' theorem, by which we mean the weak convergence of a family of random smooth curves to a non-Markovian random process with non-differentiable sample paths. In particular, we obtain effective dynamics for the second-order problem and for the kinetic fractional Brownian motion model.

Keywords Passive tracer · Fractional noise · Multi-scale · Mixed functional central and non-central limit theorems · Rough creation · Rough homogenization · Rough topology

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1 Introduction

The functional limit theorem we study here lays the foundation for the fluctuation problem for a slow/fast system with the fast variable given by a family of non-strong mixing stochastic processes, this will be discussed in [19], see [18] for the preliminary

✉ Johann Gehringer
johann.gehringer18@imperial.ac.uk

Xue-Mei Li
xue-mei.li@imperial.ac.uk

[1] Imperial College, London, UK

version. A pivot theorem for obtaining effective dynamics for the slow moving particles in a fast turbulent environment are scaling limit theorems for the convergence of the following functionals

$$X^\varepsilon := \left(X^{1,\varepsilon}, \ldots, X^{N,\varepsilon} \right), \qquad X^{k,\varepsilon} = \alpha_k(\varepsilon) \int_0^t G_k(y_s^\varepsilon) ds, \qquad (1.1)$$

with weak convergence in $\mathcal{C}^\alpha([0, T], \mathbf{R})$, where T is some finite fixed time horizon, $\alpha(\varepsilon)$ a suitable scaling and $G_k : \mathbf{R} \to \mathbf{R}$. If $y_t^\varepsilon = y_{\frac{t}{\varepsilon}}$ and y_t is a strong mixing process, $\alpha(\varepsilon) = \frac{1}{\sqrt{\varepsilon}}$ and the limit is a Markov process, for details see, e.g. the book [23] and the references therein. For stochastic processes whose auto-correlation function does not decay sufficiently fast at infinity there is no reason to have the $\sqrt{\varepsilon}$ scaling or to obtain a diffusive limit. Furthermore, the scaling limit and the limit function may depend on the individual functions G_k.

In this article, we take y_t^ε to be the stationary and rescaled fractional Ornstein–Uhlenbeck process with Hurst parameter H, which, for $H > \frac{1}{2}$, exhibits long-range dependence (LRD) and is not strong-mixing. Our interest for long range dependent / non-strong mixing noise comes the time series data of the river Nile. In a study of water flows of the Nile river, Hurst and his colleagues [20] observed long-range time dependence and found that the time dependence varies proportionally to t^H where $H \sim 0.73$, by contrast, Brownian motions and stable processes have independent increments. Fractional Brownian motions (fBM) were then proposed by Benoit Mandelbrot and John Van Ness [28] for modelling the Hurst phenomenon. A fBM is a continuous mean zero Gaussian process with stationary increments and covariance $\mathbf{E}(B_t - B_s)^2 = |t - s|^{2H}$, it is self-similar with similarity exponent H, and distinguished by the Gaussian property and stationary, but dependent increments. See, e.g. [26] and the reference therein for stochastic calculus for fBM's.

Self-similar processes appeared also in mathematically rigorous descriptions of critical phenomena and in renormalisation theory. In [38], Sinai constructed non-Gaussian self-similar fields; while Dobrushin [12] studied self-similar fields subordinated to self-similar Gaussian fields (multiple Wiener integrals). Those self-similar stochastic processes with stationary increments are a particular interesting class. When normalized to begin at 0, to have mean 0 and to have variance 1 at $t = 1$, they necessarily have the covariance $\frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H})$. Those of Gaussian variety are fBMs. Hermite processes are non-Gaussian self-similar processes with stationary increments and the above-mentioned covariance. They appeared as scaling limits of functionals of long-range dependent Gaussian processes, see [36]. Jona-Lasinio was also concerned with the construction of a systematic theory of limit distributions for sums of 'strongly dependent' random variables for which the classical central limit theorems do not hold, [22], see also the book [13].

Let us first consider the convergence of one single component, the scalar case. The scaling constant $\alpha_k(\varepsilon)$ depends on the function G_k, and is a reflection of the self-similarity exponents of the limiting process. If $\alpha_k(\varepsilon) = \frac{1}{\sqrt{\varepsilon}}$, the limit of $X^{k,\varepsilon}$ is a Wiener process and the functional central limit theorem is expected to hold. Let μ denote the centred and normalized Gaussian measure on $\mathbf{R}$ and m_k denote the Hermite

rank of a centred function $G_k \in L^2(\mu)$, which is the smallest nonzero term in its chaos expansion. Let $(H, m) \mapsto H^*(m)$ denote the function given by (1.3), which decreases with m. Then, the relevant scaling constants are given as below:

$$\alpha(\varepsilon, H^*(m_k)) = \begin{cases} \frac{1}{\sqrt{\varepsilon}}, & \text{if } H^*(m_k) < \frac{1}{2}, \\ \frac{1}{\sqrt{\varepsilon |\ln(\varepsilon)|}}, & \text{if } H^*(m_k) = \frac{1}{2}, \\ \varepsilon^{H^*(m)-1}, & \text{if } H^*(m_k) > \frac{1}{2}. \end{cases} \quad (1.2)$$

See Lemma 2.4 for a preliminary computation indicating the scales. In the past, the limit theorems for the non-Gaussian limits had been called non-central limit theorems, we use the terminology 'functional central limit theorems' for all cases.

The intuition for this comes from its counterpart for sequences. If Y_n is a mean zero, stationary, and strong mixing sequence, such that $\sigma_n^2 = \mathbf{E}(\sum_{i=1}^n Y_i)^2 \to \infty$, $\mathbf{E}(\sum_{i=1}^n Y_i)^4 = O(\sigma_n^4)$, then $\frac{1}{\sigma_n} \sum_{i=1}^n Y_i \longrightarrow N(0, 1)$. If Y_n is not strong mixing, this CLT may fail. Indeed, if X_n is a stationary mean zero variance 1 Gaussian sequence with auto-correlation $r(n) \sim cn^{-\gamma}$ for some $\gamma \in (0, 1)$ and $c \in \mathbf{R}$ (allowing for negative correlations), G a centred function with Hermite rank $m \geq 1$, and $A(n)$ a sequence such that

$$\lim_{n \to \infty} \text{var}\left(\frac{1}{A(n)} \sum_{k=1}^n G(X_k)\right) = 1.$$

Then, $z_n = \frac{1}{A(n)} \sum_{k=1}^{[nt]} G(X_n)$ is expected to converge in finite dimensional distributions. The scaling constant $A(n)$ is of the order $n^{1-\frac{1}{2}\gamma m}$ in the long-range-dependent case, of order $\sqrt{n}$ in the short-range-dependent case, and of order $\sqrt{n \ln n}$ for the borderline case, see [6,8]. By long-range dependence, we mean $\sum_{n=1}^{\infty} |r(n)| = +\infty$.

The limit process, $\lim_{n \to \infty} z_n$, is a Wiener process for fast decaying correlations, i.e. in case $\gamma \in (\frac{1}{m}, 1)$, [6]. In the borderline case, $\gamma = \frac{1}{m}$, the scaling limit is also a Wiener process. However, if $\gamma \in (0, \frac{1}{m})$ the correlations fail to decay sufficiently fast, the scaling limit is a Hermite process in the m-th chaos, [8,12]. The first convergence to a non-Gaussian similar process was shown in [36] where the aim was to construct a not strong mixing sequence of random variables, he achieved this by showing the sequence of random variables has a non-Gaussian scaling limit which is now known as the Rosenblatt process. In [8], vector valued combinations of short- and long-range-dependent sums were studied, however, the limit of each component is assumed to be moment determinate (which can only happen when they are in the L^2 chaos expansion of order less or equal to 2). This is due to a restriction in the asymptotic independence result in [33], which was extended in [29].

We return to the continuous functional limit theorems. For the scalar case, the continuous version CLT for $\gamma \in (\frac{1}{m}, 1)$ was obtained in [4], the borderline case $\gamma = \frac{1}{m}$ in [2]. These are shown for the convergence in finite dimensional distribution and for G to be a centred L^2 function. They also obtained uniform convergences in the continuous topology for a restrictive class of functions G (assuming sufficiently fast decay of the coefficients in the Wiener chaos expansion). This was extended in

[30] to vector valued X^ε, when each component of X^ε falls in the Brownian case, with convergence understood in the sense of finite dimensional distributions. The result in [30] was improved in [11], where the fast chaos decay restrictions on G_k, for $G_k \in L^p$ for $p > 2$, are removed with techniques from Malliavin calculus. In the continuous long-range-dependent case Taqqu, [40], obtained convergence in the continuous topology. These results, although fragmented (in some regimes these are only known for scalar valued processes or only at the level of sequences), provide a fairly good picture of what is going on.

There exists however no vector valued functional limit theorem with joint convergence, when the scaling limit of the components are mixed, in this article we provide a complete description for the joint convergence of $\{X^{k,\varepsilon}\}$ for $G_k \in L^2(\mu)$. We have a functional limit theorem for vector valued processes whose components may exhibit both short and long-range dependence. For G_k satisfying a stronger integrability condition, we can also show weak convergence in the $\mathcal{C}^\alpha([0, T], \mathbf{R}^d)$-topology and for each fixed time in L^2 for the low Hermite rank case, which already have interesting applications. Furthermore, they are the basis for the convergence in a suitable rough topology, which due to the change of the nature of the problem will appear in [18] where rough path theory is used to study slow/fast systems, leading to 'rough creation' / 'rough homogenization' in which the effective limit is not necessarily a Markov process.

Application. Consider the second-order equation on $\mathbf{R}$:

$$\dot{x}_t^\varepsilon = \varepsilon^{H-1} f(x_t^\varepsilon) y_t^\varepsilon, \quad x_0^\varepsilon = x_0$$

$$dy_t^\varepsilon = -\frac{1}{\varepsilon} y_t^\varepsilon dt + \frac{\sigma}{\varepsilon^H} dB_t^H, \quad y_0 \sim \varepsilon^{-H} \sigma \int_{-\infty}^0 e^{\frac{t-s}{\varepsilon}} dB_s^H.$$

Taking $\varepsilon \to 0$, does x_t^ε converge? In case $H = \frac{1}{2}$ and $f = 1$, this is essentially the Kramer-Smolouchowski limit (this is also called the kinetic Brownian motion model). For $H \neq \frac{1}{2}$ and for $f = 1$ this was shown in [1,7,43] to converge to a fBM, see also [14] for the case with a magnetic field. Given $H > \frac{1}{3}$ and $f \in C_b^3$ (for any H if $f = 1$), we can show x_t^ε converges to the solution of the equation $\dot{x}_t = f(x_t) \, dB_t^H$ with initial value x_0 where the integral in the differential equation is interpreted as a Riemann-Stieltjes integral. Furthermore, we obtain the following bound in $C^{\gamma'}$ where $0 < \gamma' < \gamma < H$:

$$\left\| |x_\cdot^\varepsilon - x_\cdot|_{C^{\gamma'}([0,T])} \right\|_{L^p} \lesssim T^\gamma \varepsilon^{H-\gamma},$$

This computation is straightforward, see Propositions 3.1 and 3.2 for detail.

With the functional limit theorem below, Theorem A, we can conclude also the convergence of solutions of the equations, for $h \in \mathcal{C}_b^2(\mathbf{R}^d, \mathbf{R}^d)$ and $g \in \mathcal{C}_b(\mathbf{R}, \mathbf{R})$,

$$\dot{x}_t^\varepsilon = \alpha(\varepsilon) f(x_t^\varepsilon) G(y_t^\varepsilon) + h(x_t^\varepsilon) g(y_t^\varepsilon).$$

We show that x_t^ε converges in $\mathcal{C}^\gamma([0, T], \mathbf{R})$ for $\gamma \in (0, H^*(m) \vee \frac{1}{2} - \frac{1}{p})$ either to the solution of the equation

$$d\bar{x}_t = cf(\bar{x}_t) \, dZ_t^{H^*(m),m} + \bar{g} \, h(\bar{x}_t),$$

where $Z_t^{H^*(m),m}$ is a Hermite process, or to the solution to the Stratonovich stochastic differential equation

$$d\bar{x}_t = cf(\bar{x}_t) \circ dW_t + \bar{g} \, h(\bar{x}_t)$$

where W_t is a standard Wiener process (given enough integrability on G). Here, c is a specific constant (c.f. Eq. (3.6)) depending on G arising from the homogenization procedure. For the above, we follow [11] and use Malliavin calculus to obtain suitable moment bounds on $\int_0^t G(y_s^\varepsilon)ds$. These results appeared in the previous version of the current paper [18]. Equations driven by fractional Brownian motions are also studied for the averaging regime, see [21] and [16]. A fluctuation theorem around the effective average was obtained [3].

Main Results We denote our underlying probability space by $(\Omega, \mathcal{F}, \mathbf{P})$. Let μ denote the standard Gaussian distribution and we choose σ such that the stationary scaled fOU process, to be defined below, satisfies $y_t^\varepsilon \sim \mu$. Let $\{H_m, m \geq 0\}$ be the orthogonal Hermite polynomials on $L^2(\mu)$, such that they have leading coefficient 1 and $L^2(\mu)$ norm $\sqrt{m!}$. Given $G \in L^2(\mu)$, then it posses an expansion of the form $G(x) = \sum_{k=0}^\infty c_k H_k(x)$, where $c_k = \frac{1}{k!}\langle G, H_k\rangle_{L^2(\mu)}$. A function G is centred if and only if $c_0 = 0$. The smallest m with $c_m \neq 0$ is called the Hermite rank of G. In case the correlations of y_t^ε do not decay sufficiently fast the path integral $\alpha(\varepsilon) \int_0^t G(y_s^\varepsilon)ds$ ought to be approximated by that of the first term of its Wiener chaos expansion. By orthogonality of the H_m's, it is sufficient to study the asymptotics of $\alpha(\varepsilon) \int_0^t H_m(y_s^\varepsilon)ds$ to deduce $\alpha(\varepsilon)$.

Although the solutions to the fOU equation converge exponentially fast to each other, their autocorrelation function decays only algebraically. The indicator for the behaviour of $\alpha(\varepsilon) \int_0^t H_m(y_s^\varepsilon)ds$ turns out to be

$$H^*(m) = m(H - 1) + 1, \tag{1.3}$$

and the self-similarity exponent of the limiting process is determined by $\alpha(\varepsilon, H^*(m))$. For large m, the limit will be a Wiener process, and otherwise the limit Z_t should have the scaling property: $\varepsilon^{H^*(m)} Z_{\frac{t}{\varepsilon}} \sim Z_t$. Indeed, Z_t are the self-similar Hermite processes. To state the functional limit theorem concisely, we make the following convention,

Convention 1.1 Given a collection of functions $(G_k \in L^2(\mu), k \leq N)$, we will label the high rank ones first, so the first n functions satisfy $H^*(m_k) \leq \frac{1}{2}$, where $n \geq 0$, and the remaining satisfy $H^*(m_k) > \frac{1}{2}$.

Theorem A Let y^ε be the stationary solution to the scaled fractional Ornstein–Uhlenbeck equation (2.4) with standard Gaussian distribution μ. Let $G_k : \mathbf{R} \to \mathbf{R}$ be

centred functions in $L^2(\mu)$ with Hermite ranks m_k. Write

$$G_k = \sum_{l=m_k}^{\infty} c_{k,l} H_l, \qquad \alpha_k(\varepsilon) = \alpha(H^*(m_k), m_k), \quad X^{k,\varepsilon} = \alpha_k(\varepsilon) \int_0^t G_k(y_s^\varepsilon) ds.$$

Set

$$X^{W,\varepsilon} = \left(X^{1,\varepsilon}, \ldots, X^{n,\varepsilon}\right), \qquad X^{Z,\varepsilon} = \left(X^{n+1,\varepsilon}, \ldots, X^{N,\varepsilon}\right).$$

Then, the following holds:

1. (a) *There exist stochastic processes $X^W = (X^1, \ldots, X^n)$ and $X^Z = (X^{n+1}, \ldots, X^N)$ such that on every finite interval $[0, T]$,*

$$(X^{W,\varepsilon}, X^{Z,\varepsilon}) \longrightarrow (X^W, X^Z),$$

in the sense of finite dimensional distributions. Furthermore, for any $t > 0$

$$\lim_{\varepsilon \to 0} \|X_t^{Z,\varepsilon} \to X_t^Z\|_{L^2(\Omega)} = 0.$$

   (b) *If furthermore each G_k satisfies Assumption 1.2 below, the convergence is weakly in $\mathcal{C}^\gamma([0, T], \mathbf{R}^N)$ for every $\gamma < \frac{1}{2} - \frac{1}{\min_{k \le n} p_k}$, if there is at least one component converging to a Wiener process. Otherwise they converge in $\mathcal{C}^\gamma([0, T], \mathbf{R}^N)$ for every $\gamma < \min_{k > n} H^*(m_k) - \frac{1}{p_k}$.*

2. *We now describe the limit $X = (X^W, X^Z)$.*
   (1) *$X^W \in \mathbf{R}^n$ and $X^Z \in \mathbf{R}^{N-n}$ are independent.*
   (2) *$X^W = U \hat{W}_t$ where $\hat{W}_t$ is a standard Wiener process and U is a square root of the matrix $(A^{i,j})$,*

$$A^{i,j} = \int_0^\infty \mathbf{E}\big(G_i(y_s) G_j(y_0)\big) ds = \sum_{q=m_i \vee m_j}^{\infty} c_{i,q} \, c_{j,q} \, (q!) \int_0^\infty \varrho(r)^q \, dr$$

   and $\varrho(r) = \mathbf{E}(y_r y_0)$. In other words, $\mathbf{E}\left(X_t^i X_s^j\right) = 2(t \wedge s) A^{i,j}$ for $i, j \le n$.
   (3) *Let $Z_t^{H^*(m_k), m_k}$ be the Hermite processes, represented by (2.1). Then,*

$$X^Z = (c_{n+1,m_{n+1}} Z_t^{n+1}, \ldots, c_{N,m_N} Z_t^N),$$

   where,

$$Z_t^k = \frac{m_k!}{K(H^*(m_k), m_k)} Z_t^{H^*(m_k), m_k}. \tag{1.4}$$

   We emphasize that the Wiener process W_t defining the Hermite processes is the same for every k (c.f. Eq. (2.1)), which is in addition independent of $\hat{W}_t$.

Assumption 1.2 *(Functional Limit $\mathcal{C}^\gamma$ assumptions) Let $G_k \in L^2(\mu)$ with Hermite rank $m_k \geq 1$.*

- *High rank case.   If $H^*(m_k) \leq \frac{1}{2}$, assume $G_k \in L^{p_k}(\mu)$ where $\frac{1}{2} - \frac{1}{p_k} > \frac{1}{3}$ (i.e. $p_k > 6$).*
- *Low rank case.   If $H^*(m_k) > \frac{1}{2}$, assume $G_k \in L^{p_k}(\mu)$ where $H^*(m_k) - \frac{1}{p_k} > \frac{1}{2}$.*

Remark 1.3 The case $H = \frac{1}{2}$ is classical, and is not of interest here. In this case the result is independent of the Hermite rank and the scaling is given by $\alpha(\varepsilon) = \frac{1}{\sqrt{\varepsilon}}$, due to the exponential decay of correlations.

An immediate application is the following rough homogenisation theorem for a toy model:

Theorem B *Let $H \in (\frac{1}{3}, 1) \setminus \{\frac{1}{2}\}$, $f \in \mathcal{C}_b^3(\mathbf{R}^d, \mathbf{R}^d)$, $h \in \mathcal{C}_b^2(\mathbf{R}^d; \mathbf{R}^d)$, $G \in \mathcal{C}(\mathbf{R}, \mathbf{R})$ satisfying Assumption 1.2 and $g \in \mathcal{C}_b(\mathbf{R}; \mathbf{R})$. Let $\alpha(\varepsilon) = \alpha(\varepsilon, H^*(m))$. Fix a finite time T and consider*

$$\dot{x}_t^\varepsilon = \alpha(\varepsilon) f(x_t^\varepsilon) G(y_t^\varepsilon) + h(x_t^\varepsilon) g(y_t^\varepsilon), \qquad x_0^\varepsilon = x_0. \tag{1.5}$$

1. *If $H^*(m) > \frac{1}{2}$, x_t^ε converges weakly in $\mathcal{C}^\gamma([0, T], \mathbf{R}^d)$ to the solution to the Young differential equation $d\bar{x}_t = c f(\bar{x}_t) \, dZ_t^{H^*(m),m} + \bar{g} h(\bar{x}_t) dt$ with initial value x_0 for $\gamma \in (0, H^*(m) - \frac{1}{p})$.*
2. *If $H^*(m) \leq \frac{1}{2}$, x_t^ε converges weakly in $\mathcal{C}^\gamma([0, T], \mathbf{R})$ to the solution of the Stratonovich stochastic differential equation $d\bar{x}_t = c f(\bar{x}_t) \circ dW_t + \bar{g} h(\bar{x}_t) dt$ with $\bar{x}_0 = x_0$, where $\gamma \in (0, \frac{1}{2} - \frac{1}{p})$.*

We also take the liberty to point out an intermediate result on the joint convergence of stochastic processes in finite L^2 chaos, for it maybe of service. The proof is a slight modification of results in [29,41].

Proposition 5.1 *Let $q_1 \leq q_2 \leq \cdots \leq q_n \leq p_1 \leq p_2 \leq \cdots \leq p_m$. Let $f_i^\varepsilon \in L^2(\mathbf{R}^{p_i})$, $g_i^\varepsilon \in L^2(\mathbf{R}^{q_i})$, $F^\varepsilon = \left(I_{p_1}(f_1^\varepsilon), \ldots, I_{p_m}(f_m^\varepsilon) \right)$ and $G^\varepsilon = \left(I_{q_1}(g_1^\varepsilon), \ldots, I_{q_n}(g_n^\varepsilon) \right)$, where I_q denotes the Wiener integral of order q. Suppose that for every i, j, and any $1 \leq r \leq q_i$:*

$$\| f_j^\varepsilon \otimes_r g_i^\varepsilon \| \to 0.$$

Then, $F^\varepsilon \to U$ and $G^\varepsilon \to V$ weakly imply that $(F^\varepsilon, G^\varepsilon) \to (U, V)$ jointly, where U and V are taken to be independent random variables.

2 Preliminaries

We take the Hermite polynomials of degree m to be

$$H_m(x) = (-1)^m e^{\frac{x^2}{2}} \frac{d^m}{dx^m} e^{\frac{x^2}{2}}.$$

Thus, $H_0(x) = 1$, $H_1(x) = x$. Let $\hat{H}$ be the inverse of $H^*(m) = m(H-1)+1$:

$$\hat{H}(m) = \frac{1}{m}(H-1)+1.$$

2.1 Hermite Processes

The rank 1 Hermite processes $Z^{H,1}$ are fractional BMs, the formulation (2.1) below is exactly the Mandelbrot-Van Ness representation for a fBM.

Definition 2.1 Let $m \in \mathbf{N}$ with $\hat{H}(m) > \frac{1}{2}$. The class of *Hermite processes* of rank m is given by the following mean-zero processes,

$$Z_t^{H,m} = \frac{K(H,m)}{m!} \int_{\mathbf{R}^m} \int_0^t \prod_{j=1}^m (s-\xi_j)_+^{-(\frac{1}{2}+\frac{1-H}{m})} \, ds \, dW(\xi_1) \ldots dW(\xi_m), \quad (2.1)$$

where the constant $K(H,m)$ is chosen so their variances are 1 at $t = 1$.

The integration in (2.1) is understood as a multiple Wiener-Itô integral (over the region $\mathbf{R}^n$ without the diagonal). Note $\hat{H}(1) = H$.

By the properties of Wiener integrals, two Hermite processes $Z^{H,m}$ and $Z^{H',m'}$, defined by the same Wiener process, are uncorrelated if $m \neq m'$. The Hermite processes have stationary increments and finite moments of all orders with covariance

$$\mathbf{E}(Z_t^{H,m} Z_s^{H,m}) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}). \quad (2.2)$$

Therefore, using Kolmogorov's theorem 2.6, one can show that the Hermite processes $Z_t^{H,m}$ have sample paths of Hölder regularity up to H. They are also self-similar with exponent H which means $\lambda^H Z_{\frac{\cdot}{\lambda}}^{H,m} \sim Z_{\cdot}^{H,m}$.

Remark 2.2 In some literature, see, e.g. [27] where further details on Hermite processes can also be found, the Hermite processes are defined with a different exponent as below:

$$\tilde{Z}_t^{H,m} = \frac{K(H,m)}{m!} \int_{\mathbf{R}^m} \int_0^t \prod_{j=1}^m (s-\xi_j)_+^{H-\frac{3}{2}} \, ds \, dW(\xi_1) \ldots dW(\xi_m).$$

The two notions are related by

$$Z_t^{H^*(m),m} = \tilde{Z}_t^{H,m}, \qquad Z_t^{H,m} = \tilde{Z}_t^{\hat{H}(m),m}. \quad (2.3)$$

We refer to [10,35,37] for detailed studies of fBM's which are used in this article.

2.2 Fractional Ornstein–Uhlenbeck Processes

Let us normalise the fractional Brownian motion, so that $B_0^H = 0$ and $\mathbf{E}(B_1^H)^2 = 1$. Disjoint increments of B_t^H have a covariance of the form:

$$\mathbf{E}(B_t - B_s)(B_u - B_v) = \frac{1}{2}\Big(|t - v|^{2H} + |s - u|^{2H} - |t - u|^{2H} - |s - v|^{2H}\Big).$$

We define the stationary fractional Ornstein–Uhlenbeck processes to be $y_t = \sigma \int_{-\infty}^t e^{-\sigma(t-s)}dB_s^H$, where B_t^H is a two-sided fractional BM, σ is chosen such that y_t is distributed as $\mu = N(0, 1)$. It is the solution of the following Langevin equation:

$$dy_t = -y_t dt + \sigma dB_t^H, \qquad y_0 = \sigma \int_{-\infty}^0 e^s dB_s^H.$$

We take y_t^ε, the fast or rescaled fOU, to be the stationary solution of

$$dy_t^\varepsilon = -\frac{1}{\varepsilon}y_t^\varepsilon dt + \frac{\sigma}{\varepsilon^H} dB_t^H. \tag{2.4}$$

Observe that y^ε and $y_{\frac{\cdot}{\varepsilon}}$ have the same distributions, $y_t^\varepsilon = \frac{\sigma}{\varepsilon^H} \int_{-\infty}^t e^{-\frac{1}{\varepsilon}(t-s)}dB_s^H$. In particular, both y_t and y_t^ε are Hölder continuous with Hölder exponents $\gamma \in (0, H)$. Let us denote their correlation functions by ϱ and ϱ^ε:

$$\varrho(s, t) := \mathbf{E}(y_s y_t), \qquad \varrho^\varepsilon(s, t) := \mathbf{E}(y_s^\varepsilon y_t^\varepsilon).$$

Let $\varrho(s) = \mathbf{E}(y_0 y_s)$ for $s \geq 0$ and extended to $\mathbf{R}$ by symmetry, then $\varrho(s, t) = \varrho(t - s)$ and similarly for ϱ^ε. For $H > \frac{1}{2}$, the set of functions for which Wiener integrals are defined include L^2 functions and so ϱ posses an analytical expression. Indeed, since

$$\mathbf{E}(B_t^H B_s^H) = H(2H - 1) \int_0^t \int_0^s |r_1 - r_2|^{2H-2}dr_1 dr_2,$$

we have

$$\frac{\partial^2}{\partial t \partial s}\mathbf{E}(B_t^H B_s^H) = H(2H - 1)|t - s|^{2H-2},$$

which is integrable, and therefore we may use the Wiener isometry to compute the covariances

$$\mathbf{E}(y_t y_s) = \sigma^2 H(2H - 1) \int_{-\infty}^t \int_{-\infty}^s e^{-(s+t-r_1-r_2)}|r_1 - r_2|^{2H-2}dr_1 dr_2.$$

For $u > 0$, we set

$$\varrho(u) = \sigma^2 H(2H - 1) \int_{-\infty}^u \int_{-\infty}^0 e^{-(u-r_1-r_2)}|r_1 - r_2|^{2H-2}dr_1 dr_2.$$

Using this, the following correlation decay was shown in [10].

Lemma 2.3 *Let $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$. Then, $\varrho(s) = 2\sigma^2 H(2H-1)s^{2H-2} + O(s^{2H-4})$ as $s \to \infty$. In particular, for any $t \neq s$,*

$$|\varrho(s, t)| \lesssim 1 \wedge |t - s|^{2H-2}. \tag{2.5}$$

Hence, $\int_0^\infty \varrho^m(s)ds$ is finite if and only if $H^*(m) < \frac{1}{2}$, or $H = \frac{1}{2}$ and $m \in \mathbf{N}$, as in the latter the usual OU process admits exponential decay of correlations.

Lemma 2.4 *Let $H \in (0, 1) \setminus \{\frac{1}{2}\}$, fix a finite time horizon T, then for $t \in [0, T]$ the following holds* uniformly *for $\varepsilon \in (0, \frac{1}{2}]$:*

$$
\left(\int_0^{\frac{t}{\varepsilon}} \int_0^{\frac{t}{\varepsilon}} |\varrho(u, r)|^m \, dr \, du \right)^{\frac{1}{2}}
$$

$$
\lesssim
\begin{cases}
\sqrt{\frac{t}{\varepsilon} \int_0^\infty |\varrho(s)|^m ds}, & \text{if } H^*(m) < \frac{1}{2}, \\
\sqrt{(\frac{t}{\varepsilon})| \ln (\frac{1}{\varepsilon})|}, & \text{if } H^*(m) = \frac{1}{2}, \\
(\frac{t}{\varepsilon})^{H^*(m)}, & \text{if } H^*(m) > \frac{1}{2}.
\end{cases}
\tag{2.6}
$$

$$
\left(\int_0^t \int_0^t |\varrho^\varepsilon(u, r)|^m \, dr \, du \right)^{\frac{1}{2}}
$$

$$
\lesssim
\begin{cases}
\sqrt{t\varepsilon \int_0^\infty |\varrho(s)|^m ds}, & \text{if } H^*(m) < \frac{1}{2}, \\
\sqrt{t\varepsilon| \ln (\frac{1}{\varepsilon})|}, & \text{if } H^*(m) = \frac{1}{2}, \\
(\frac{t}{\varepsilon})^{H^*(m)-1}, & \text{if } H^*(m) > \frac{1}{2}.
\end{cases}
\tag{2.7}
$$

Note, if $H = \frac{1}{2}$, for and any $m \in \mathbf{N}$, the bound is always $\sqrt{\frac{t}{\varepsilon} \int_0^\infty \varrho^m(s)ds}$. In particular,

$$t \int_0^t |\varrho^\varepsilon(s)|^m ds \lesssim \frac{t^{(2H^*(m)\vee 1)}}{\alpha(\varepsilon, H^*(m))^2}. \tag{2.8}$$

Proof We first observe that

$$\int_0^\infty \varrho^m(s)ds < \infty \iff H^*(m) < \frac{1}{2} \iff H < 1 - \frac{1}{2m}. \tag{2.9}$$

By a change of variables and using estimate (2.5) on the decay of the auto-correlation function,

$$
\int_0^{\frac{t}{\varepsilon}} \int_0^{\frac{t}{\varepsilon}} |\varrho(|u - r|)|^m dr du = 2\frac{t}{\varepsilon} \int_0^{\frac{t}{\varepsilon}} |\varrho(s)|^m ds
$$

$$
\lesssim
\begin{cases}
\frac{t}{\varepsilon} \int_0^\infty \varrho^m(s)ds, & \text{if } H^*(m) < \frac{1}{2}, \\
(\frac{t}{\varepsilon})^{2H^*(m)}, & \text{otherwise.}
\end{cases}
,
$$

For the case $H^*(m) = \frac{1}{2}$, we use

$$\int_0^{\frac{t}{\varepsilon}} |\varrho(s)|^m ds \leq \int_0^{\frac{T}{\varepsilon}} |\varrho(s)|^m ds \lesssim \int_0^{\frac{T}{\varepsilon}} \left(1 \wedge \frac{1}{s}\right) ds \lesssim \left|\ln\left(\frac{T}{\varepsilon}\right)\right| \lesssim \left|\ln\left(\frac{1}{\varepsilon}\right)\right|.$$

To complete the proof, we observe that by a simple change of variables,

$$\int_0^t \int_0^t |\varrho^\varepsilon(u, r)|^m \, dr \, du = \varepsilon^2 \int_0^{\frac{t}{\varepsilon}} \int_0^{\frac{t}{\varepsilon}} |\varrho(u, r)|^m \, dr \, du.$$

$\square$

Next, we recall the Garsia-Rodemich-Romsey-Kolmogorv inequality.

Lemma 2.5 *(Garsia-Rodemich-Romsey-Kolmogorov inequality) Let $T > 0$. Let $\theta :$ $[0, T] \to \mathbf{R}^d$. For any positive numbers γ, p, there exists a constant $C(\gamma, p)$ such that*

$$\sup_{s \neq t, s, t \in [0, T]} \frac{|\theta(t) - \theta(s)|}{|t - s|^\gamma} \leq C(\gamma, p) \left(\int_0^T \int_0^T \frac{|\theta_s - \theta_r|^p}{|s - r|^{\gamma p + 2}} ds dr\right)^{\frac{1}{p}}.$$

See [17, section A.2] in Appendix, as well as [39], for a proof. As a consequence of this inequality, one obtains the following theorem.

Theorem 2.6 *(Kolmogorov's Theorem) Let θ be a stochastic process. Suppose that for $s, t \in [0, T]$, $p > 1$ and $\delta > 0$,*

$$\mathbf{E}|\theta(t) - \theta(s)|^p \leq c_p |t - s|^{1+\delta},$$

where c_p is a constant. Then, for $\gamma < \frac{\delta}{p}$, $\theta \in \mathcal{C}^\gamma([0, T], \mathbf{R})$, and in particular

$$\||\theta|_{\mathcal{C}^\gamma([0,T])}\|_p \leq C(\gamma, p)(c_p)^{\frac{1}{p}} \left(\int_0^T \int_0^T |u - v|^{\delta - \gamma p - 1} du dv\right)^{\frac{1}{p}},$$

where right hand side is finite when $\gamma \in (0, \frac{\delta}{p})$.

Lemma 2.7 *For any $\gamma \in (0, H)$, $p > 1$, the following estimates hold:*

$$\sup_{s, t \geq 0} \frac{\|y_s - y_t\|_{L^p}}{1 \wedge |s - t|^H} \lesssim 1, \quad \mathbf{E} \sup_{s \neq t, s, t \in [0, T]} \left(\frac{|y_s - y_t|}{|t - s|^\gamma}\right)^p \lesssim T \, C(\gamma, p).$$

Proof We use the fractional Ornstein–Uhlenbeck equation

$$y_s - y_r = -\int_r^s y_u du + B_s^H - B_r^H,$$

to obtain $\mathbf{E}|y_s - y_r|^2 \lesssim (s-r)^2 \mathbf{E}|y_1|^2 + q|s-r|^{2H}$. Using the stationarity of y_t, one also has $\mathbf{E}|y_s - y_r|^2 \leq 2\mathbf{E}|y_1|^2 = 2$. Since for Gaussian random variables the L^2 norm controls the L^p norm we have

$$\|y_s - y_r\|_{L^p} \lesssim \begin{cases} 1, & \text{if } |s-r| \geq 1; \\ |s-r|^H, & \text{if } l|s-r| \leq 1. \end{cases}$$

Thus, by symmetry and a change of variables, $\int_0^T \int_0^T \frac{\mathbf{E}|y_s - y_r|^p}{|s-r|^{\gamma p+2}} ds\,dr \lesssim T$ and application of Kolmogorov's theorem 2.6 concludes the proof. □

3 Applications

3.1 The Second-Order Problem

If x is a stochastic process, we write $x_{s,t} = x_t - x_s$.

Proposition 3.1 *Let $H \in (0,1)$, $\gamma \in (0, H)$, $p > 1$ and fix a finite time T. Let $X_t^\varepsilon = \varepsilon^{H-1} \int_0^t y_s^\varepsilon ds$, then,*

$$\sup_{s,t \in [0,T]} \left\| X_{s,t}^\varepsilon - \sigma B_{s,t}^H \right\|_{L^p} \lesssim \varepsilon^H, \qquad \left\| \left| X^\varepsilon - \sigma B^H \right|_{\mathcal{C}^{\gamma'}([0,t],\mathbf{R})} \right\|_{L^p} \lesssim t^\gamma \varepsilon^{H-\gamma},$$

for any $\gamma' < \gamma < H$ and for $t \in [0,T]$.

Proof Set $v_t^\varepsilon = \varepsilon^{H-1} y_t^\varepsilon$, then, v_t^ε solves the following equation

$$dv_t^\varepsilon = -\frac{1}{\varepsilon} v_t^\varepsilon dt + \frac{\sigma}{\varepsilon} dB_t^H.$$

Using the equation for v_t^ε, we have

$$X_{s,t}^\varepsilon = \varepsilon^{H-1} \int_s^t y_r^\varepsilon dr = \int_s^t v_r^\varepsilon dr = \varepsilon(v_t^\varepsilon - v_s^\varepsilon) + \sigma B_{s,t}^H.$$

Therefore, for any $p > 1$,

$$\sup_{s,t \in [0,T]} \left\| X_{s,t}^\varepsilon - \sigma B_{s,t}^H \right\|_{L^p} = \sup_{s,t \in [0,T]} \left\| \varepsilon(v_t^\varepsilon - v_s^\varepsilon) \right\|_{L^p}$$

$$= \varepsilon^H \sup_{s,t \in [0,T]} \left\| y_t^\varepsilon - y_s^\varepsilon \right\|_{L^p} \lesssim \varepsilon^H.$$

In the last step, we used the stationarity of y_t^ε.

Thus, applying Kolmogorov's theorem 2.6 to $X^\varepsilon - \sigma B^H$, we see that the following holds for any $t \in [0, T]$, $p > 1$ and any $\gamma' < \gamma$

$$\||X^\varepsilon - \sigma B^H|_{\mathcal{C}^{\gamma'}([0,t],\mathbf{R})}\|_{L^p} \lesssim \varepsilon^H \left(\frac{t}{\varepsilon}\right)^\gamma,$$

hence, the claim follows. $\square$

As an application, we consider the following slow/fast system,

$$\begin{cases} \dot{x}_t^\varepsilon = \varepsilon^{H-1} f(x_t^\varepsilon) \, y_t^\varepsilon, \\ x_0^\varepsilon = x_0. \end{cases} \tag{3.1}$$

To describe its limit, we review the concept of Young integrals. If $f, g : [0, T] \to \mathbf{R}$ with $f \in \mathcal{C}^\alpha$ and $g \in \mathcal{C}^\beta$ such that $\alpha + \beta > 1$, then the Riemann-Stieljes integral makes sense, and

$$\int_0^t f_s dg_s = \lim_{|\mathcal{P}| \to 0} \sum_{[u,v] \subset \mathcal{P}} f_u(g_u - g_v) \in \mathcal{C}^\beta.$$

For details see [42]; this integral is called a Young integral. Since Young integrals have the regularity of its integrand, for $H > \frac{1}{2}$ the equation $\dot{x}_t = f(x_t) \, dB_t^H$ makes sense. In [25], it was shown that if $f \in \mathcal{C}_b^3$, then the equation has a unique global solution from each initial value. This type of equations are Young equation, the simplest type of rough differential equation. The notation $\mathcal{C}_b^3$ denotes the space of bounded functions such that their first three derivatives are bounded.

Proposition 3.2 *Let $H \in (\frac{1}{3}, 1)$ and $f \in \mathcal{C}_b^3(\mathbf{R}^d, \mathbf{R}^d)$. Then for any $\gamma \in (0, H)$, $\gamma' < \gamma$, x_t^ε converges in L^p in $\mathcal{C}^{\gamma'}([0, T], \mathbf{R}^d)$ to the solution of the rough differential equation:*

$$\dot{x}_t = \sigma f(x_t) \, dB_t^H, \quad x_0 = x_0. \tag{3.2}$$

Furthermore, for $t \in [0, T]$,

$$\left\||x^\varepsilon - x|_{\mathcal{C}^{\gamma'}([0,t],\mathbf{R})}\right\|_{L^p} \lesssim t^\gamma \varepsilon^{H-\gamma}.$$

Proof The idea is to consider Eq. (3.1) as a Young/rough differential equation. In case $H \in (\frac{1}{2}, 1)$, we can rewrite our equation as

$$\dot{x}_t^\varepsilon = f(x_t^\varepsilon) dX_t^\varepsilon,$$

where $X^\varepsilon = \varepsilon^{H-1} \int_0^t y_s^\varepsilon ds$ is as in the previous lemma. Young's continuity theorem, see Theorem 3.6, states that x^ε converges weakly provided X^ε converges weakly in a Hölder space of regularity greater than $\frac{1}{2}$.

For $H \in (\frac{1}{3}, \frac{1}{2})$, we need to rewrite our equation into a rough differential equation,

$$\dot{x}_t^\varepsilon = f(x_t^\varepsilon) d\mathbf{X}_t^\varepsilon,$$

where $\mathbf{X}^\varepsilon$ is given by X^ε enhanced with its canonical lift

$$\mathbb{X}_{s,t}^\varepsilon = \int_s^t (X_r^\varepsilon - X_s^\varepsilon) dX_r^\varepsilon.$$

As we restrict ourselves to one dimension we obtain, by symmetry, $\mathbb{X}_{s,t}^\varepsilon = \frac{1}{2}(X_{s,t}^\varepsilon)^2$, hence, $\mathbf{X}^\varepsilon$ converges to a fBm σB^H enhanced with $\mathbb{B}_{s,t}^H = \frac{1}{2}(\sigma B_{s,t}^H)^2$. As the solution map, Φ, to a RDE satisfies, see [15] or Theorem 3.6 below,

$$|\Phi(\mathbf{X}^\varepsilon) - \Phi(\mathbf{B}^H)|_{\mathcal{C}^{\gamma'}} \lesssim \varrho_\gamma(\mathbf{X}^\varepsilon, \mathbf{B}^H),$$

where ϱ_γ denotes the inhomogeneous rough path norm of regularity γ and $\mathbf{B}^H = (\sigma B^H, \frac{1}{2}(\sigma B_{s,t}^H)^2)$. Thus, the L^p convergence of the solutions follows from the L^p convergence of the drivers, hence, we can conclude the proof by Proposition 3.1. □

Remark 3.3 Krammer-Smoluchowski limits/Kinetic fBM's are studied in [1,7,43]. See also [14–16].

Remark 3.4

1. For $H < \frac{1}{2}$ and $m = 1$, Theorem A appears to contradict with Proposition 3.1; in the first we claim the limit is a Brownian motion, whereas in the second we claim that it is a fractional Brownian motion. Both results are correct and can be easily explained. It lies in the fact that $\int_{\mathbf{R}} \varrho(s) \, ds$ vanishes if $H < \frac{1}{2}$, and so the Brownian motion limit is degenerate. Since according to [10],

$$\varrho(s) = \sigma^2 \frac{\Gamma(2H+1)\sin(\pi H)}{2\pi} \int_{\mathbf{R}} e^{isx} \frac{|x|^{1-2H}}{1+x^2} dx, \qquad (3.3)$$

and by the decay estimate from (2.5), ϱ is integrable, $s(\lambda)$ is the value at zero of the inverse Fourier transform of $\varrho(s)$, which is up to a multiplicative constant $\frac{|\lambda|^{1-2H}}{1+\lambda^2}$. This is also the spectral density of y_t and has value 0 at 0. This means we have scaled too much and the correct scaling is to multiply the integral $\int_0^t y_s^\varepsilon ds$ by ε^{H-1} in which case we obtain a fBm as limit.

2. For $m > 1$ and $H < \frac{1}{2}$ the Wiener limit is not trivial. Indeed,

$$\int_{\mathbf{R}} \varrho(s)^m ds = C \int_{\mathbf{R}} \overbrace{\int_{\mathbf{R}} \cdots \int_{\mathbf{R}}}^{m} \prod_{k=1}^m e^{isx_k} \frac{|x_k|^{1-2H}}{1+|x_k|^2} dx_1 \ldots dx_m \, ds$$

$$= C \overbrace{\int_{\mathbf{R}} \cdots \int_{\mathbf{R}}}^{m} \frac{|x_2 + \cdots + x_m|^{1-2H}}{1+|x_2 + \cdots + x_m|^2} \prod_{k=2}^m \frac{|x_k|^{1-2H}}{1+|x_k|^2} \neq 0.$$

3.2 The 1d Fluctuation Problem

In this section, we give an application of Theorem A. Given a function $g \in L^1(\mu)$ we denote $\bar{g} = \int_{\mathbf{R}} g(y) \mu(dy)$.

Lemma 3.5 *The stationary Ornstein–Uhlenbeck process is ergodic. Thus, $\int_0^t g(y_s^\varepsilon) ds \to t\bar{g}$ in probability for every $g \in L^1(\mu)$.*

Proof A stationary Gaussian process is ergodic if its spectral measure has no atom, see [9,37]. The spectral measure F of a stationary Gaussian process is obtained from Fourier transforming its correlation function and $\varrho(\lambda) = \int_{\mathbf{R}} e^{i\lambda x} dF(x)$. According to [10]:

$$\varrho(s) = \frac{\Gamma(2H+1)\sin(\pi H)}{2\pi} \int_{\mathbf{R}} e^{isx} \frac{|x|^{1-2H}}{1+x^2} dx, \tag{3.4}$$

so the spectral measure is absolutely continuous with respect to the Lebesgue measure with spectral density, up to a nonzero constant, given by $s(x) = \frac{|x|^{1-2H}}{1+x^2}$. Since $\int_0^t g(y_t^\varepsilon) dt$ equals $\varepsilon \int_0^{\frac{t}{\varepsilon}} g(y_s) ds$ in law, the former converges in law to $t\bar{g}$ by Birkhoff's ergodic theorem. The claim now follows as weak convergence to a constant implies convergence in probability. □

In the following proof we will need the following theorem from Young/rough path theory for details we refer to [15,17,24,25]. We denote the space of rough paths of regularity β by $\mathscr{C}^\beta$.

Theorem 3.6 *Let $Y_0 \in \mathbf{R}^d$, $\beta \in (\frac{1}{3}, 1)$, $\gamma \in (\frac{1}{2}, 1)$ $f \in \mathcal{C}_b^3(\mathbf{R}^d, \mathbf{R}^d)$, $h \in \mathcal{C}_b^2(\mathbf{R}^d, \mathbf{R}^d)$, $\mathbf{X} \in \mathscr{C}^\beta([0, T], \mathbf{R})$ and $\mathbf{Z} \in \mathscr{C}^\gamma([0, T], \mathbf{R})$ such that $\beta + \gamma > 1$. Then, the differential equation*

$$Y_t = Y_0 + \int_0^t f(Y_s) d\mathbf{X}_s + \int_0^t h(Y_s) d\mathbf{Z}_s \tag{3.5}$$

has a unique solution which belongs to $\mathcal{C}^{\beta \wedge \gamma}$. Furthermore, the solution map $\Phi_{f,h} : \mathbf{R} \times \mathscr{C}^\beta([0, T], \mathbf{R}) \times \mathscr{C}^\gamma([0, T], \mathbf{R}) \to \mathcal{C}^{\beta \wedge \gamma}([0, T], \mathbf{R})$, where the first component is the initial condition and the second and third components the drivers, is continuous.

Given a centred function $G \in L^2(\mu)$, with chaos expansion $G = \sum_{k=m}^{\infty} c_k H_k$ and let $c \geq 0$ be given by

$$c^2 = \begin{cases} (\frac{c_m m!}{K(H,m)})^2, & H^*(m) > \frac{1}{2} \\ 2\sum_{k=m}^{\infty}(c_k)^2 k! \int_0^\infty \varrho^k(s) ds, & H^*(m) < \frac{1}{2} \\ 2m!(c_m)^2, & H^*(m) = \frac{1}{2}. \end{cases} \tag{3.6}$$

Theorem 3.7 *Let $H \in (\frac{1}{3}, 1) \setminus \{\frac{1}{2}\}$, $f \in \mathcal{C}_b^3(\mathbf{R}^d, \mathbf{R}^d)$, $h \in \mathcal{C}_b^2(\mathbf{R}^d; \mathbf{R}^d)$, $G \in \mathcal{C}(\mathbf{R}, \mathbf{R})$ satisfying Assumption 1.2 and $g \in \mathcal{C}_b(\mathbf{R}; \mathbf{R})$. Let $\alpha(\varepsilon) = \alpha(\varepsilon, H^*(m))$. Fix a finite time T and consider*

$$\dot{x}_t^\varepsilon = \alpha(\varepsilon) f(x_t^\varepsilon) G(y_t^\varepsilon) + h(x_t^\varepsilon) g(y_t^\varepsilon), \qquad x_0^\varepsilon = x_0. \tag{3.7}$$

1. If $H^*(m) > \frac{1}{2}$, x_t^ε converges weakly in $\mathcal{C}^\gamma([0, T], \mathbf{R}^d)$ to the solution to the Young differential equation $d\bar{x}_t = cf(\bar{x}_t)\,dZ_t^{H^*(m),m} + \bar{g}h(\bar{x}_t)dt$ with initial value x_0 for $\gamma \in (0, H^*(m) - \frac{1}{p})$.

2. If $H^*(m) \leq \frac{1}{2}$, x_t^ε converges weakly in $\mathcal{C}^\gamma([0, T], \mathbf{R})$ to the solution of the Stratonovich stochastic differential equation $d\bar{x}_t = cf(\bar{x}_t) \circ dW_t + \bar{g}h(\bar{x}_t)dt$ with $\bar{x}_0 = x_0$, where $\gamma \in (0, \frac{1}{2} - \frac{1}{p})$.

Proof As in Proposition 3.2 we can rewrite our equations as Young/rough differential equations and therefore reduce our analysis to the drivers $\left(\alpha(\varepsilon) \int_0^t G(y_s^\varepsilon)ds, \int_0^t g(y_s^\varepsilon)ds\right)$. By Theorem A, $\alpha(\varepsilon) \int_0^t G(y_s^\varepsilon)ds$ converges in finite dimensional distributions either to a Wiener or a Hermite process. By Lemma 3.5, $\int_0^t g(y_s^\varepsilon)ds$ converges in probability to the deterministic path $t\bar{g}$. Hence, $\left(\alpha(\varepsilon) \int_0^t G(y_s^\varepsilon)ds, \int_0^t g(y_s^\varepsilon)ds\right)$ converges jointly in finite dimensional distributions. Furthermore, $\|\int_0^t g(y_s^\varepsilon)ds\|_\infty \leq t\|g\|_\infty$, this combined with the moment bounds obtained in Theorem A enables us to apply Theorem 3.6 to conclude the proof. $\qquad\square$

Remark 3.8 The constant c could be 0, for further details see Remark 3.4.

4 Proof of Theorem A

We first establish the $L^2(\Omega)$ convergence of $X_t^\varepsilon = \alpha(\varepsilon) \int_0^t G(y_s^\varepsilon)ds$, where $G = \sum_{k=m}^\infty c_k H_k$ has low Hermite rank, followed by a reduction theorem. We then prove moment bounds and conclude the proof of Theorem A.

4.1 Preliminary Lemmas

The basic scalar valued functional limit theorem, for low rank Hermite functions, was proved in [40] for $\varepsilon^{H^*(m)} \int_0^{\frac{t}{\varepsilon}} G(X_s)ds$ with $X_t = \int_\mathbf{R} p(t - \xi)dW_\xi$ a moving average, where in order to prove convergence one uses the self-similarity of a Wiener process leading to weak convergence as this equivalence of course is only in law. Nevertheless, in our case we can choose a properly scaled fast variable and write, $y_t^\varepsilon = \int_\mathbf{R} \hat{g}(\frac{t-\xi}{\varepsilon})dW_\xi$ for a function $\hat{g}$, and thus avoid using self-similarity. The key idea is to write a Wiener integral representation beginning with

$$y_t^\varepsilon = \varepsilon^{-H}\sigma \int_{-\infty}^t e^{-\frac{t-r}{\varepsilon}}\,dB_r^H = \int_\mathbf{R} h_\varepsilon(t, s)dW_s, \quad \text{where,}$$

$$h_\varepsilon(t, s) = \varepsilon^{-\frac{1}{2}}\frac{\sigma}{c_1(H)}e^{-\frac{t-s}{\varepsilon}}\int_0^{\frac{t-s}{\varepsilon}} e^v\,v_+^{H-\frac{3}{2}}\,dv,$$

(4.1)

and $c_1(H) = \sqrt{\int_{-\infty}^{0} \left((1-s)^{H-\frac{1}{2}} - (-s)^{H-\frac{1}{2}} \right)^2 ds + \frac{1}{2H}}$. This can be obtained by applying the integral representation for fBM's:

$$B_t^H = \int_{-\infty}^{\infty} g(t,s)dW_s, \qquad \text{where} \quad g(t,s) = \frac{1}{c_1(H)} \int_0^t (r-s)_+^{H-\frac{3}{2}} dr, \quad (4.2)$$

and by repeated applications of integration by parts (to the Young integrals):

$$\sigma \int_{-\infty}^{t} e^{-\frac{t-s}{\varepsilon}} dB_s^H = \sigma B_t^H - \frac{\sigma}{\varepsilon} \int_{-\infty}^{t} e^{-\frac{t-s}{\varepsilon}} B_s^H ds$$

$$= \sigma B_t^H - \frac{\sigma}{\varepsilon} \int_{-\infty}^{t} e^{-\frac{t-s}{\varepsilon}} \left(\int_{\mathbf{R}} g(s,r)dW_r \right) ds = \sigma \int_{\mathbf{R}} \int_{-\infty}^{t} e^{-\frac{t-s}{\varepsilon}} \partial_s g(s,r) ds dW_r$$

$$= \frac{\sigma}{c_1(H)} \int_{\mathbf{R}} \int_{-\infty}^{t} e^{-\frac{t-s}{\varepsilon}} (s-r)_+^{H-\frac{3}{2}} ds dW_r.$$

One may also use the following, see [34], taking $f \in L^1 \cap L^2$:

$$\int_{\mathbf{R}} f(u)dB_u^H = \frac{1}{c_1(H)} \int_{\mathbf{R}} \int_{\mathbf{R}} f(u)(u-s)_+^{H-\frac{3}{2}} du\, dW_s.$$

Lemma 4.1 *Let λ denote the Lebesgue measure, then as $\varepsilon \to 0$, $\varepsilon^{H^*(m)-1} \int_0^t H_m(y_s^\varepsilon) ds$ converges to $\frac{m!}{K(H,m)} Z_t^{H^*(m),m}$ in $L^2(\Omega)$. Equivalently,*

$$\left\| \int_0^t \prod_{i=1}^m h_\varepsilon(s,u_i) ds - \int_0^t \prod_{i=1}^m (s-u_i)_+^{H-\frac{3}{2}} ds \right\|_{L^2(\mathbf{R}^m,\lambda)} \to 0. \qquad (4.3)$$

Proof This can be shown by applying [40, Theorem 4.7], where weak convergence is obtained. With a small modification and using [40, Lemma 4.5, Lemma 4.6] directly we obtain the $L^2(\Omega)$ convergence:

$$\mathbf{E}\left(\int_{\mathbf{R}^m} \int_0^t \prod_{i=1}^m p\left(\frac{s-\xi}{\varepsilon} \right) \varepsilon^{H-\frac{3}{2}} ds dW_\xi - \frac{Z^{H^*(m),m}}{K(H,m)} \right)^2 \to 0,$$

using the Wiener integral representation of the Hermite processes this is equivalent, by a multiple Wiener-Itô isometry, to

$$\int_{\mathbf{R}^m} \left(\int_0^t \prod_{i=1}^m p\left(\frac{s-\xi_i}{\varepsilon} \right) \varepsilon^{H-\frac{3}{2}} ds - \int_0^t \prod_{i=1}^m (s-\xi_i)_+^{H-\frac{3}{2}} ds \right)^2 d\xi_1 \ldots d\xi_m \to 0.$$

$$(4.4)$$

Examining Taqqu's proof, we note that in fact the L^2 convergence of (4.4) is obtained under the following conditions.

1 $\int_{\mathbf{R}} p(s)^2 ds < \infty$.

2 $|p(s)| \leq C s^{H-\frac{3}{2}} L(u)$ for almost all $s > 0$.

3 $p(s) \sim s^{H-\frac{3}{2}} L(s)$ as $s \to \infty$.

4 There exists a constant γ such that $0 < \gamma < (1-H) \wedge (H - (1 - \frac{1}{2m}))$ such that $\int_{-\infty}^{0} |p(s)p(xy+s)|ds = o(x^{2H-2}L^2(x))y^{2H-2-2\gamma}$ as $x \to \infty$ uniformly in $y \in (0, t]$.

where L denotes a slowly varying function (for every $\lambda > 0$ $\lim_{x \to \infty} \frac{L(\lambda x)}{L(x)}) = 1$). Set

$$\hat{g}(s) = \frac{\sigma}{c_1(H)} e^{-s} \int_0^s e^u u_+^{H-\frac{3}{2}} du, \qquad (4.5)$$

then,

$$y_t^\varepsilon = \varepsilon^{-\frac{1}{2}} \int_{-\infty}^{t} \hat{g}\left(\frac{t-s}{\varepsilon}\right) dW_s.$$

We are now in Taqqu's framework and it is only left the check $\hat{g}$ defined by (4.5) satisfies these conditions. To increase readability, we suppress the constant $\frac{\sigma}{c_1(H)}$ in the following computations. For $s < 1$,

$$e^{-s} \int_0^s e^u u^{H-\frac{3}{2}} du \leq \int_0^s u^{H-\frac{3}{2}} du \lesssim s^{H-\frac{1}{2}}.$$

We calculate for $s > 1$ via integration by parts

$$e^{-s} \int_0^s e^u u^{H-\frac{3}{2}} du \leq e^{-s} \int_0^1 e^u u^{H-\frac{3}{2}} du + e^{-s} \int_1^s e^u u^{H-\frac{3}{2}} du$$

$$\lesssim e^{-s} + s^{H-\frac{3}{2}} + e^{-s} \int_1^s e^u u^{H-\frac{5}{2}} du \lesssim s^{H-\frac{3}{2}}.$$

This of course implies that $\hat{g}$ is $L^2(\lambda)$ integrable. Finally, observe that $\int_{-\infty}^{0} |\hat{g}(s)\hat{g}(xy+s)|ds = 0$ as $\hat{g}(s) = 0$ for $s < 0$. With these we apply [40, Theorem 4.7] to conclude the L^2 convergence of the kernels. $\qquad \square$

Lemma 4.2 *Let $G \in L^2(\mu)$ be a centred function with Hermite rank m satisfying $H^*(m) > \frac{1}{2}$. Let $H \in (\frac{1}{2}, 1)$. Then, the following statements hold for the stationary scaled fOU process y_s^ε. Fix $t > 0$, then,*

$$\left\| \varepsilon^{H^*(m)-1} \int_0^t G(y_s^\varepsilon) ds - \frac{c_m m!}{K(H^*(m), m)} Z_t^{H^*(m),m} \right\|_{L^2(\Omega)} \to 0.$$

Proof For $G = H_m$, the claim has already been shown in Lemma 4.1. To conclude the claim in the case of a general G, we compute,

$$
\left\| \varepsilon^{H^*(m)-1} \int_0^t (G - c_m H_m)(y_s^\varepsilon) ds \right\|_{L^2(\Omega)}^2
$$
$$
= \varepsilon^{2H^*(m)-2} \sum_{k=m+1}^\infty c_k^2 k! o(\varepsilon^{2H^*(m)-2}) \to 0
\tag{4.6}
$$

as $\sum_{k=m+1}^\infty c_k^2 \sqrt{k!} < \infty$ as $G \in L^2(\mu)$. This finishes the proof. $\qquad\square$

The fact that only the first term in the chaos expansion gives a contributions is in the literature often called a reduction lemma. In the high Hermite rank case however it is not possible to restrict one's analysis to a pure Hermite polynomial, but as the next lemma shows finite linear combinations are indeed sufficient. To make the application later on easier, we directly prove it in the multi-dimensional case.

Lemma 4.3 *(Reduction Lemma) Fix $H \in (0, 1) \setminus \{\frac{1}{2}\}$. For $M \in \mathbf{N}$, define the truncated functions:*

$$
G_{k,M} = \sum_{j=m_k}^M c_{k,j} H_j, \qquad X_M^{k,\varepsilon}(t) = \alpha_k(\varepsilon) \int_0^t G_{k,M}(y_s^\varepsilon) ds.
$$

If for every $M \in \mathbf{N}$, $(X_M^{1,\varepsilon}, \ldots, X_M^{N,\varepsilon}) \overset{(\varepsilon \to 0)}{\longrightarrow} (X_M^1, \ldots, X_M^N)$ in finite dimensional distributions, then,

$$
\left(X^{1,\varepsilon}, \ldots, X^{N,\varepsilon} \right) \overset{(\varepsilon \to 0)}{\longrightarrow} (X^1, \ldots, X^N)
$$

in finite dimensional distributions.

Proof Firstly,

$$
X^{k,\varepsilon}(t) - X_M^{k,\varepsilon}(t) = \alpha_k(\varepsilon) \int_0^t \left(G_k(y_s^\varepsilon) - G_{k,M}(y_s^\varepsilon) \right) ds
$$
$$
= \alpha_k(\varepsilon) \int_0^t \sum_{j=M+1}^\infty c_{k,j} H_j(y_s^\varepsilon) ds.
$$

Using properties of the Hermite polynomials, we obtain

$$\mathbf{E}\left(\alpha_k(\varepsilon)\int_0^t \sum_{j=M+1}^{\infty} c_{k,j}H_j(y_s^\varepsilon)ds\right)^2$$

$$= \alpha_k(\varepsilon)^2 \int_0^t \int_0^t \sum_{j=M+1}^{\infty} (c_{k,j})^2 \mathbf{E}\big(H_j(y_s^\varepsilon)H_j(y_r^\varepsilon)\big)drds$$

$$= \alpha_k(\varepsilon)^2 \sum_{j=M+1}^{\infty} (c_{k,j})^2 j! \int_0^t \int_0^t \varrho^\varepsilon(|s-r|)^j drds \lesssim \sum_{j=M+1}^{\infty} (c_{k,j})^2 j!$$

As $\sum_{j=m}^{\infty}(c_{k,j})^2 j! < \infty$ we obtain $\sum_{j=M+1}^{\infty}(c_{k,j})^2 j! \to 0$ as $M \to \infty$. Thus,

$$\lim_{M\to\infty}\lim_{\varepsilon\to 0}\mathbf{E}\left(\alpha_k(\varepsilon)\int_0^t G_k(y_s^\varepsilon)ds - \alpha_k(\varepsilon)\int_0^t G_{k,M}(y_s^\varepsilon)ds\right)^2 \to 0, \qquad (4.7)$$

Let $\{t_{\gamma_{k,l}}, k \le N, l \le A\}$ be a sequence of positive numbers. Now, by the triangle inequality,

$$\lim_{M\to\infty}\lim_{\varepsilon\to 0}\left\|\sum_{k,l}\gamma_{k,l}\left(X^{k,\varepsilon}(t_l) - X_M^{k,\varepsilon}(t_l)\right)\right\|_{L^2(\Omega)} \to 0.$$

With Theorem 3.2 in [5], this proves the claim. □

4.2 Moment Bounds

We will use some results from Malliavin Calculus. Let x_s be a stationary Gaussian process with $\beta(s) = \mathbf{E}(x_s x_0)$, such that $\beta(0) = 1$. As a real separable Hilbert space we use $\mathscr{H} = L^2(\mathbf{R}_+, \nu)$ where for a Borel-set A we have $\nu(A) = \int_{\mathbf{R}_+} \mathbf{1}_A(s)d\beta_s$. Let $\mathscr{H}^{\otimes q}$ denote the q-th tensor product of $\mathscr{H}$. For $h \in \mathscr{H}$, we may define the Wiener integrals $W(h) = \int_0^\infty h_s dx_s$ by $W([a,b]) = x(b) - x(a)$ (where $a, b \ge 0$), linearity and the Wiener isometry ($\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]}\rangle = \beta(t-s)$). Iterated Wiener integrals are defined similarly and by its values on indictor functions: $I_m(\mathbf{1}_{A_1\times\cdots\times A_m}) = \prod_{i=1}^m W(A_i)$ where A_i are pairwise disjoint Borel subsets of $\mathbf{R}_+$. If $\mathcal{F}$ denotes the σ-field generated by x, then any $\mathcal{F}$-measurable $L^2(\Omega)$ function F has the chaos expansion: $F = \mathbf{E}F + \sum_{m=1}^{\infty} I_m(f_m)$ where $f_m \in L^2(\mathbf{R}_+^m)$. This is due to the fact that $L^2(\Omega) = \bigoplus_{m=0}^{\infty} \mathscr{H}_m$ where $\mathscr{H}_m$ is the closed linear space generated by $\{H_m(W(h)) : \|h\|_{L^2} = 1\}$, H_m are the m-th Hermite polynomials, and $\mathscr{H}_m = I_m(L^2_{\text{sym}}(\mathbf{R}_+^m))$. The last fact is due to $H_m(W(h)) = I_m(h^{\otimes^m})$. In the following $\mathbb{D}^{k,p}(\mathscr{H}^{\otimes m})$ denotes the closure of Malliavin smooth random variables under the following norm $\|u\|_{\mathbb{D}^{k,p}(\mathscr{H}^{\otimes m})} = \left(\sum_{j=0}^k \mathbf{E}\big(\|D^j u\|_{\mathscr{H}^{\otimes m}}^p\big)\right)^{\frac{1}{p}}$.

Lemma 4.4 *(Meyer's inequality) [32] Let δ denote the divergence operator. Then for $u \in \mathbb{D}^{k,p}(\mathscr{H}^{\otimes m})$,*

$$\|\delta^m(u)\|_{L^p(\Omega)} \lesssim \sum_{k=0}^m \|u\|_{\mathbb{D}^{k,p}(\mathscr{H}^{\otimes m})}. \tag{4.8}$$

Lemma 4.5 *[11] If $G : \mathbf{R} \to \mathbf{R}$ is a function of Hermite rank m, then G has the following multiple Wiener-Itô-integral representation:*

$$G(x_s) = \delta^m \Big(G_m(x_s) \mathbf{1}_{[0,s]}^{\otimes m} \Big), \tag{4.9}$$

where G_m has the following properties:

(1) $\|G_m\|_{L^p(\mu)} \lesssim \|G\|_{L^p(\mu)}$,
(2) $G_m(x_1)$ is m times Malliavin differentiable and its kth derivative, denoted by $G_m^{(k)}(x_1) \mathbf{1}_{[0,1]}^{\otimes k}$, satisfies $\|G_m^{(k)}\|_{L^p(\mu)} \lesssim \|G\|_{L^p(\mu)}$.

In the lemma below we estimate the moments of $\int_0^t G(x_{\frac{r}{\varepsilon}}) dr$, where we need the multiple Wiener-Itô-integral representation above to transfer the correlation function to L^2 norms of indicator functions. We use an idea from [11] for the estimates below.

Lemma 4.6 *Let $x_t = W([0,t])$ be a stationary Gaussian process with correlation $\beta(t) = \big|\mathbf{E}(x_t x_0)\big|$, stationary distribution μ and $\mathscr{H}$ the L^2 space over $\mathbf{R}_+$ with measure $\beta(r) dr$. If G is a function of Hermite rank m and $G \in L^p(\mu)$, for $p > 2$, then,*

$$\left\| \frac{1}{\varepsilon} \int_0^t G(x_{\frac{r}{\varepsilon}}) dr \right\|_{L^p(\Omega)} \lesssim \|G\|_{L^p(\mu)} \left(\int_0^{\frac{t}{\varepsilon}} \int_0^{\frac{t}{\varepsilon}} \beta(|u-r|)^m dr du \right)^{\frac{1}{2}}. \tag{4.10}$$

For the stationary scaled fractional OU process y_t^ε, we have

$$\left\| \frac{1}{\varepsilon} \int_0^t G(y_r^\varepsilon) dr \right\|_{L^p(\Omega)} \lesssim \begin{cases} \|G\|_{L^p(\mu)} \sqrt{\frac{t}{\varepsilon} \int_0^\infty \big|\varrho^m(s)\big| ds}, & \text{if } H^*(m) < \frac{1}{2}, \\ \|G\|_{L^p(\mu)} \sqrt{\frac{t}{\varepsilon} \ln|\frac{1}{\varepsilon}|}, & \text{if } H^*(m) = \frac{1}{2}, \\ \|G\|_{L^p(\mu)} \left(\frac{t}{\varepsilon}\right)^{H^*(m)}, & \text{otherwise.} \end{cases} \tag{4.11}$$

in particular,

$$\left\| \int_0^t G(y_r^\varepsilon) dr \right\|_{L^p(\Omega)} \lesssim \frac{\|G\|_{L^p(\mu)} t^{H^*(m) \vee \frac{1}{2}}}{\alpha(\varepsilon, H^*(m))}. \tag{4.12}$$

Proof We first use Lemma 4.5 and then apply Meyer's inequality from Lemma 4.4 to obtain

$$\left\| \frac{1}{\varepsilon} \int_0^t G(x_{\frac{r}{\varepsilon}}) dr \right\|_{L^p(\Omega)} = \left\| \int_0^{\frac{t}{\varepsilon}} G(x_r) dr \right\|_{L^p(\Omega)}$$

$$= \left\| \int_0^{\frac{t}{\varepsilon}} \delta^m \left(G_m(x_r) \mathbf{1}_{[0,r]}^{\otimes m} \right) dr \right\|_{L^p(\Omega)}$$

$$\lesssim \sum_{k=0}^m \left\| \int_0^{\frac{t}{\varepsilon}} D^k \left(G_m(x_r) \mathbf{1}_{[0,r]}^{\otimes m} \right) dr \right\|_{L^p(\Omega, \mathscr{H}^{\otimes m+k})}$$

$$= \sum_{k=0}^m \left\| \int_0^{\frac{t}{\varepsilon}} G_m^{(k)}(x_r) \mathbf{1}_{[0,r]}^{\otimes m+k} dr \right\|_{L^p(\Omega, \mathscr{H}^{\otimes m+k})}.$$

Here for G_m is as given in Lemma 4.5 and $G_m^{(k)}$ denotes its k-th order Malliavin derivative. We estimate the individual terms using the linearity of the inner product and the isometry $\langle \mathbf{1}_{[0,r]}, \mathbf{1}_{[0,s]} \rangle_{\mathscr{H}} = \mathbf{E}(x_r x_s) = \beta(r-s)$,

$$\left(\left\| \int_0^{\frac{t}{\varepsilon}} G_m^{(k)}(x_r) \mathbf{1}_{[0,r]}^{\otimes m+k} dr \right\|_{\mathscr{H}^{\otimes m+k}} \right)^2$$

$$= \left\langle \int_0^{\frac{t}{\varepsilon}} G_m^{(k)}(x_r) \mathbf{1}_{[0,r]}^{\otimes m+k} dr, \int_0^{\frac{t}{\varepsilon}} G_m^{(k)}(x_u) \mathbf{1}_{[0,r]}^{\otimes m+k} du \right\rangle_{\mathscr{H}^{\otimes m+k}}$$

$$= \int_0^{\frac{t}{\varepsilon}} \int_0^{\frac{t}{\varepsilon}} G_m^{(k)}(x_r) G_m^{(k)}(x_u) \langle \mathbf{1}_{[0,r]}^{\otimes m+k}, \mathbf{1}_{[0,u]}^{\otimes m+k} \rangle_{\mathscr{H}^{\otimes m+k}} dr du$$

$$= \int_0^{\frac{t}{\varepsilon}} \int_0^{\frac{t}{\varepsilon}} G_m^{(k)}(x_r) G_m^{(k)}(x_u) \left(\beta(r-u) \right)^{m+k} dr du.$$

Using Minkowski's inequality, we obtain

$$\sum_{k=0}^m \left\| \int_0^{\frac{t}{\varepsilon}} G_m^{(k)}(x_r) \mathbf{1}_{[0,r]}^{\otimes m+k} dr \right\|_{L^p(\Omega, \mathscr{H}^{\otimes m+k})}$$

$$\leq \sum_{k=0}^m \left(\left\| \int_0^{\frac{t}{\varepsilon}} \int_0^{\frac{t}{\varepsilon}} G_m^{(k)}(x_r) G_m^{(k)}(x_u) \beta(r-u)^{m+k} dr du \right\|_{L^{\frac{p}{2}}(\Omega)} \right)^{\frac{1}{2}}$$

$$\leq \sum_{k=0}^m \left(\int_0^{\frac{t}{\varepsilon}} \int_0^{\frac{t}{\varepsilon}} \left\| G_m^{(k)}(x_r) G_m^{(k)}(x_u) \right\|_{L^{\frac{p}{2}}(\Omega)} \beta(r-u)^{m+k} dr du \right)^{\frac{1}{2}}.$$

We then estimate $\mathbf{E}|G_m^{(k)}(x_r)G_m^{(k)}(x_u)|^{\frac{p}{2}}$ by Hölder's inequality and use the fact that x_t is stationary. Since the right hand side is then controlled by

$$
\begin{aligned}
RHS &\leq \sum_{k=0}^{m} \|G_m^{(k)}\|_{L^p(\mu)} \left(\int_0^{\frac{t}{\varepsilon}} \int_0^{\frac{t}{\varepsilon}} |\beta(|u-r|)|^{m+k} dr du \right)^{\frac{1}{2}} \\
&\lesssim \|G\|_{L^p(\mu)} \left(\int_0^{\frac{t}{\varepsilon}} \int_0^{\frac{t}{\varepsilon}} |\beta(|u-r|)|^m dr du \right)^{\frac{1}{2}},
\end{aligned}
$$

concluding (4.10). We finally apply Lemma 2.4 to conclude (4.11). $\square$

Now we are ready to prove our main theorem.

4.3 Concluding the Proof

Proof *Step 1, CLT in the pure Wiener case*
We first deal with the high Hermite rank components. For $k \leq n$ we define the truncated functions $G_{k,M} = \sum_{j=m_k}^{M} c_{k,j} H_j$ and set

$$
X_M^{k,\varepsilon} = \alpha_k(\varepsilon) \int_0^t G_{k,M}(y_s^\varepsilon) ds.
$$

Then, by the reduction Lemma 4.3 above, it is sufficient to show the convergence of $(X_M^{1,\varepsilon}, \ldots, X_M^{n,\varepsilon})$ for every M. By [4] and [2] each component alone converges to a Wiener process. Hence, as each $X_M^{k,\varepsilon}$ belongs to a finite chaos we can make use of the normal approximation theorem from [32, Theorem 6.2.3]: if each component of a family of mean zero vector valued stochastic processes, with components of the form $I_{q_i}(f_{i,n})$, where $f_{i,n}$ are symmetric L^2 functions in q_i variables, converges in law to a Gaussian process, then they converge jointly in law to a vector valued Gaussian process, provided that their covariance functions converge. Furthermore, the covariance functions of the limit distribution are $\lim_{\varepsilon \to 0} \mathbf{E}[X^{i,\varepsilon}(t)X^{j,\varepsilon}(s)]$. Let $m = \min(m_i, m_j)$ we use

$$
\mathbf{E}(H_k(y_t^\varepsilon)H_l(y_s^\varepsilon)) = \delta_{k,l}\left(\mathbf{E}(y_s^\varepsilon y_t^\varepsilon)\right)^k
$$

to obtain, for $s \leq t$,

$$\mathbf{E}\left[\alpha_i(\varepsilon)\alpha_j(\varepsilon)\int_0^t G_{i,M}(y_u^\varepsilon)du \int_0^s G_{j,M}(y_r^\varepsilon)dr\right]$$

$$= \sum_{k=m}^M \alpha_i(\varepsilon)\alpha_j(\varepsilon)c_{i,k}c_{j,k}(k!)^2 \int_0^t \int_0^s (\mathbf{E}(y_r^\varepsilon y_u^\varepsilon))^k drdu$$

$$= \sum_{k=m}^M \alpha_i(\varepsilon)\alpha_j(\varepsilon)c_{i,k}c_{j,k}(k!)^2 \left(\int_0^s \int_0^s \varrho^\varepsilon(u-r)^k drdu\right.$$

$$+ \left.\int_s^t \int_0^s \varrho^\varepsilon(u-r)^k drdu\right).$$

By Lemma 2.4, we obtain, for $\varepsilon \to 0$,

$$\alpha_i(\varepsilon)\alpha_j(\varepsilon)\int_s^t \int_0^s \varrho^\varepsilon(u-r)^k drdu \to 0.$$

Hence,

$$\lim_{\varepsilon \to 0} RHS = 2\sum_{k=m}^M c_{i,k}c_{j,k}(k!)^2 \lim_{\varepsilon \to 0}\left(\varepsilon\alpha_i(\varepsilon)\alpha_j(\varepsilon)s \int_0^{\frac{s}{\varepsilon}} (\varrho(v))^k dv\right)$$

$$= 2(s \wedge t)\sum_{k=m}^M c_{i,k}c_{j,k}(k!)^2 \int_0^\infty \varrho(u)^k du$$

$$= 2(s \wedge t)\int_0^\infty \mathbf{E}(G_{i,M}(y_s)G_{j,M}(y_0))ds,$$

proving the finite chaos case. We now prove that the correlations of the limit converge as $M \to \infty$. Indeed,

$$\lim_{M \to \infty} 2(s \wedge t)\sum_{k=m}^M c_{i,k}c_{j,k}(k!)^2 \int_0^\infty \varrho(u)^k du$$

$$= 2(s \wedge t)\sum_{k=m}^\infty c_{i,k}c_{j,k}(k!)^2 \int_0^\infty \varrho(u)^k du$$

$$= 2(s \wedge t)\int_0^\infty \mathbf{E}(G_i(y_s)G_j(y_0))ds.$$

As $G_{i,M} \to G_i$ in $L^2(\mu)$, and similarly for j, this proves the joint convergence of the high Hermite rank components.

Step 2, CLT in the pure Hermite case

In this step, we focus on the vector component whose entries satisfy $H^*(m_k) > \frac{1}{2}$. Recall, this implies $H > \frac{1}{2}$. By Lemma 4.2, evaluations of each component of

$(X^{n+1}, \ldots, X^N)$ converge in $L^2(\Omega)$. Hence, they converge as well jointly in $L^2(\Omega)$. Now, choose finitely many points $t_{k,j} \in [0,T]$ and constants $a_{k,j} \in \mathbf{R}$, then, $\sum_{k,j} a_{k,j} X_{t_{k,j}}^{k,\varepsilon}$ converges in $L^2(\Omega)$ to $\sum_{k,j} a_{k,j} X_{t_{k,j}}^k$ and thus we may conclude joint convergence in finite dimensional distributions by an application of the Cramer-Wold theorem.

Step 3, Joint convergence

We have already shown that $X^{W,\varepsilon} \to X^W$ and $X^{Z,\varepsilon} \to X^Z$ in finite dimensional distributions, it is only left to prove their joint convergence. By Lemma 4.3 and Eq. (4.6), we may again reduce the problem to

$$G_i = \sum_{k=m_i}^M c_{i,k} H_k, \quad G_j = c_{j,m_j} H_{m_j}, \qquad 1 \le i \le n, \ j > n.$$

Now, we can rewrite $H_m(y_s^\varepsilon) = I_m(f_s^{m,\varepsilon})$, where I_m denotes a m-fold Wiener-Itô integral and a function $f_s^{m,\varepsilon} \in L^2(\mathbf{R}^m, \mu)$. Now, for $1 \le i \le n$ we obtain,

$$\alpha_i(\varepsilon) \int_0^t G_i(y_s^\varepsilon)ds = \alpha_i(\varepsilon) \int_0^t \sum_{k=m_i}^M c_{i,k} H_k(y_s^\varepsilon)ds$$

$$= \alpha_i(\varepsilon) \int_0^t \sum_{k=m_i}^M c_{i,k} I_k(f_s^{k,\varepsilon})ds = \sum_{k=m_i}^M c_{i,k} I_k(\hat{f}_t^{k,\varepsilon}),$$

where

$$\hat{f}_t^{k,\varepsilon} = \int_0^t f_s^{k,\varepsilon}ds.$$

Similarly for $j > n$,

$$\int_0^t G_j(y_s^\varepsilon)ds = \int_0^t c_{j,m_j} H_{m_j}(y_s^\varepsilon)ds = c_{j,m_j} I_{m_j}(\hat{f}_t^{m_j,\varepsilon}).$$

Hence, we only need to show that the collection of stochastic processes of the form $I_{m_k}(\hat{f}_t^{m_k,\varepsilon})$ converges jointly in finite dimensional distribution. It is thus sufficient to show that for every finite collection of times, $t_1, \ldots, t_Q \in [0,T]$, the vector, $\left\{ I_k(\hat{f}_{t_l}^{k,\varepsilon}), k = m, \ldots, M, l = 1, \ldots, Q \right\}$ converges jointly, where $m = \min_{k=1,\ldots,N} m_k$. Let n_0 denote the smallest natural number such that $H^*(n_0) < \frac{1}{2}$. For $k > n_0$, the collection $I_k(\hat{f}_{t_l}^{k,\varepsilon})$ converges to a normal distribution, hence, by the moment bounds in Lemma 4.6 $\|(k!) I_k(\hat{f}_{t_l}^{k,\varepsilon})\|_{H^{\otimes k}} = \sqrt{\mathbf{E}\left(I_k(\hat{f}_{t_l}^{k,\varepsilon})^2 \right)}$ converges to a constant. The convergence of such a sequence is equivalent to the convergence of the contractions, as defined in Eq. (5.1), of their kernels. Indeed, by a generalised fourth

moment theorem [31, Theorem 1], we have

$$\| \hat{f}_{t_l}^{k,\varepsilon} \otimes_r \hat{f}_{t_l}^{k,\varepsilon} \|_{\mathscr{H}^{2k-2r}} \to 0, \qquad r = 1, \ldots, k-1.$$

By Cauchy-Schwarz, we obtain for $r = 1, \ldots, k_1$,

$$\left\| \hat{f}_{t_{l_1}}^{k_1,\varepsilon} \otimes_r \hat{f}_{t_{l_2}}^{k_2,\varepsilon} \right\|_{\mathscr{H}^{k_1+k_2-2r}} \le \left\| \hat{f}_{t_{l_1}}^{k_1,\varepsilon} \otimes_r \hat{f}_{t_{l_1}}^{k_1,\varepsilon} \right\|_{\mathscr{H}^{p-r}} \left\| \hat{f}_{t_{l_2}}^{k_2,\varepsilon} \otimes_r \hat{f}_{t_{l_2}}^{k_2,\varepsilon} \right\|_{\mathscr{H}^{q-r}} \to 0,$$

for all $t_{l_1}, t_{l_2} \in [0, T]$, $1 \le k_1 \le n_0 < k_2 \le M$. We can now apply an asymptotic independence result, Proposition 5.1 in Appendix, to conclude the joint convergence in finite dimensional distributions of X^ε to (X^W, X^Z). Furthermore, X^W is independent of X^Z.

The correlations between X_t^i and $X_{t'}^j$, where $i, j > n$, are 0 if $m_i \ne m_j$, otherwise given by the L^2 norm of their integrands, which follows from the Wiener-Itô isometry and are given by

$$c_{i,m_i} c_{j,m_j} \int_0^t \int_0^{t'} \int_{\mathbf{R}^{m_i}} \prod_{i=1}^{m_i} (s-\xi_i)_+^{\hat{H}(m)-\frac{3}{2}} \prod_{i=1}^{m_i} (r-\xi_i)_+^{\hat{H}(m)-\frac{3}{2}} d\xi_1 \ldots \xi_{m_i} dr ds.$$

Step 4, Convergence in the Hölder norms.
We first choose $\gamma' \in \left(\gamma, (H^*(m_k) \wedge \frac{1}{2}) - \frac{1}{p_k} \right)$. Then, using the Markov-Chebyshev inequality, we obtain that, as M tends to ∞,

$$\mathbf{P}\left(|X^{k,\varepsilon}|_{\mathcal{C}^{\gamma'}([0,T],\mathbf{R})} > M \right) \le \frac{\|\theta|_{\mathcal{C}^{\gamma'}([0,T],\mathbf{R})}\|_{p_k}}{M^{p_k}}$$
$$\to 0,$$

since $\|\theta|_{\mathcal{C}^{\gamma'}}\|_{p_k} < \infty$ by Lemma 4.6 and an application of Kolmogorov's theorem 2.6. Furthermore, since $\mathcal{C}^{\gamma'}([0,T];\mathbf{R})$ is compactly embedded in $\mathcal{C}^\gamma([0, T]; \mathbf{R})$ for $1 > \gamma' > \gamma$, by an type Arzela-Ascoli argument, the sets $\{X_t^{k,\varepsilon} : |X^{k,\varepsilon}|_{\mathcal{C}^{\gamma'}([0,T],\mathbf{R})} \le M\}$ are sequentially compact in $\mathcal{C}^\gamma([0, T], \mathbf{R})$. Hence, $\{X_t^{k,\varepsilon}\}$ is tight in $\mathcal{C}^\gamma([0, T], \mathbf{R})$. See, e.g. [17]. As tightness in each component implies joint tightness, we obtain that X^ε is tight in $\mathcal{C}^\gamma([0, T], \mathbf{R}^N)$, where $\gamma \in (0, \frac{1}{2} - \frac{1}{\min_{k \le n} p_k})$ in case $0 < n$ and $\gamma \in (0, \min_{k > n} H^*(m_k) - \frac{1}{p_k})$ otherwise. By the above discussion, any limit of a converging subsequence has the same finite dimensional distributions, hence, every subsequence converges to the same limit. This concludes the proof for the convergence in the Hölder norm.   □

5 Appendix: Joint Convergence by Asymptotic Independence

For the proof in the previous section, we need the following which modifies results from [33] and [29]. Let $I_p(f)$ denote the pth iterated Itô-Wiener integral of a symmetric

function f of p variables,

$$I_p(f) = p! \int_{-\infty}^{\infty} \int_{-\infty}^{s_{p-1}} \cdots \int_{-\infty}^{s_2} f(s_1, \ldots, s_p) dW_{s_1} dW_{s_2} \ldots dW_{s_p}.$$

If $f \in L^2(\mathbf{R}^p)$ and $g \in L^2(\mathbf{R}^q)$ are symmetric functions and $p, q \geq 1$, their r^{th}-contraction is given by

$$f \otimes_r g = \int_{\mathbf{R}^r} f(x_1, \ldots, x_{p-r}, s_1, \ldots, s_r) g(y_1, \ldots, y_{q-r}, s_1, \ldots, s_r) ds_1 \ldots ds_r,$$

(5.1)

where $r \leq p \wedge q$. If $f \otimes_1 g = \int_{\mathbf{R}} f(x_1, \ldots, x_{p-1}, s) g(y_1, \ldots, y_{q-1}, s) ds$ vanishes, so do all higher order contractions.

Proposition 5.1 *Let $q_1 \leq q_2 \leq \cdots \leq q_n \leq p_1 \leq p_2 \leq \cdots \leq p_m$. Let $f_i^\varepsilon \in L^2(\mathbf{R}^{p_i})$, $g_i^\varepsilon \in L^2(\mathbf{R}^{q_i})$, $F^\varepsilon = \left(I_{p_1}(f_1^\varepsilon), \ldots, I_{p_m}(f_m^\varepsilon) \right)$ and $G^\varepsilon = \left(I_{q_1}(g_1^\varepsilon), \ldots, I_{q_n}(g_n^\varepsilon) \right)$. Suppose that for every i, j, and any $1 \leq r \leq q_i$:*

$$\| f_j^\varepsilon \otimes_r g_i^\varepsilon \| \to 0.$$

Then $F^\varepsilon \to U$ and $G^\varepsilon \to V$ weakly imply that $(F^\varepsilon, G^\varepsilon) \to (U, V)$ jointly, where U and V are taken to be independent random variables.

These results benefit from the insights of Üstünel-Zakai [41] on the independence of two iterated integrals $I_p(f)$ and $I_q(g)$. They are independent if and only if the 1-contraction between f and g vanishes almost surely with respect to the Lebesgue measure. An asymptotic independence result follows as below,

Lemma 5.2 *[29, Thm. 3.1] Let $F^\varepsilon = I_p(f^\varepsilon)$ and $G^\varepsilon = I_q(g^\varepsilon)$, where $f^\varepsilon \in L^2(\mathbf{R}^p)$ and $g^\varepsilon \in L^2(\mathbf{R}^q)$. Then,*

$$\mathrm{Cov}\left(\left(F^\varepsilon \right)^2, \left(G^\varepsilon \right)^2 \right) \to 0$$

is equivalent to $\| f^\varepsilon \otimes_r g^\varepsilon \| \to 0$, for $1 \leq r \leq p \wedge q$.

It is also known that if two integrals $I_p(f)$ and $I_q(g)$ are independent, then their Malliavin derivatives are orthogonal, see [41]. This explains why Malliavin calculus comes into prominent play, which has been developed to its perfection in [29, Lemma 3.2]. Given a smooth test function φ we define,

$$\| \varphi \|_q = \| \varphi \|_\infty + \sum_{|k|=1}^{q} \left\| \frac{\partial^k}{\partial^k x} \right\|_\infty,$$

where the sum runs over multi-indices $k = (k_1, \ldots, k_m)$. Let $L = -\delta D$ and throughout this section $f_i : \mathbf{R}^{p_i} \to \mathbf{R}$ and $g : \mathbf{R}^q \to \mathbf{R}$ denote symmetric functions.

Lemma 5.3 *[29] Let $q \leq p_i$, $g \in L^2(\mathbf{R}^q)$, $G = I_q(g)$, $f_i \in L^2(\mathbf{R}^{p_i})$, and $F_i = I_{p_i}(f_i)$ with $\mathbf{E}(F_i^2) = 1$. Set $F = (F_1, \ldots, F_m)$ and let θ be a smooth test function. Then,*

$$\mathbf{E}\left|\left\langle (I-L)^{-1}\theta(F)DF_j, DG \right\rangle_{\mathcal{H}}\right| \leq c\,\|\theta\|_q \operatorname{Cov}(F_j^2, G^2),$$

where c is a constant depending on $\|F\|_{L^2}$, $\|G\|_{L^2}$, and q, m, $p_1, \ldots, p_m$.

The final piece of the puzzle is the observation that the defect in being independent is quantitatively controlled by the covariance of the squares of the relative components. The following is from [29], our only modification is to take G to be vector valued. Let $g_i : \mathbf{R}^{q_i} \to \mathbf{R}$ be symmetric functions.

Lemma 5.4 *Given $F = \left(I_{p_1}(f_1), \ldots I_{p_m}(f_m)\right)$ and $G = \left(I_{q_1}(g_1), \ldots, I_{q_n}(g_n)\right)$ such that $p_k \geq q_l$ for every pair of k, l. Then, for all test functions φ and ψ, the following holds for some constant c, depending on $\|F\|_{L^2}$, $\|G\|_{L^2}$, and m, n, $p_1, \ldots, p_m$, $q_1, \ldots, q_n$,*

$$\mathbf{E}(\varphi(F)\psi(G)) - \mathbf{E}(\varphi(F))\mathbf{E}(\psi(G)) \leq c\|D\psi\|_\infty \|\varphi\|_{q_n} \sum_{i=1}^{m}\sum_{j=1}^{n} \operatorname{Cov}(F_i^2, G_j^2)$$

Proof Define $L^{-1}(\sum_{k=0}^{\infty} I_k(h_m)) = \sum_{k=1}^{\infty} \frac{1}{k} I_k(h_m) \in \mathbb{D}^{2,2}$. The key equality is $-DL^{-1} = (I-L)^{-1}D$. As in [29],

$$\varphi(F) - \mathbf{E}(\varphi(F)) = LL^{-1}\varphi(F) = \sum_{j=1}^{m} \delta((I-L)^{-1}\partial_j\varphi(F)DF_j).$$

Multiplying both sides by $\psi(G)$, taking expectations and using integration by parts we obtain

$$\begin{aligned}
&\mathbf{E}(\varphi(F)\psi(G)) - \mathbf{E}(\varphi(F))\mathbf{E}(\psi(G)) \\
&= \sum_{j=1}^{m}\sum_{i=1}^{n} \mathbf{E}\left(\langle (I-L)^{-1}\partial_j\varphi(F)DF_j, DG_i\rangle_{\mathcal{H}}\,\partial_i\psi(G)\right) \\
&\leq \|D\psi\|_\infty \sum_{j=1}^{m}\sum_{i=1}^{n} \left|\mathbf{E}\left(\langle (I-L)^{-1}\partial_j\varphi(F)DF_j, DG_i\rangle_{\mathcal{H}}\right)\right|.
\end{aligned}$$

To conclude, apply to each summand Lemma 5.3 with $\theta = \partial_j\varphi$ and $G = G_i$. □

Lemma 5.5 *Let $F^\varepsilon = \left(I_{p_1}(f_1^\varepsilon), \ldots I_{p_m}(f_m^\varepsilon)\right)$ and $G^\varepsilon = \left(I_{q_1}(g_1^\varepsilon), \ldots, I_{q_n}(g_n^\varepsilon)\right)$ with $q_1 \leq q_2, \ldots, q_n \leq p_1 \leq p_2 \leq \cdots \leq p_m$. Then for every $i \leq m$, $j \leq n$,*

$$\|f_j^\varepsilon \otimes_r g_i^\varepsilon\| \to 0, \quad 1 \leq r \leq p_j \wedge q_i$$

implies that for any smooth test functions φ and ψ,

$$\mathbf{E}\big(\psi(F^\varepsilon)\psi(G^\varepsilon)\big) - \mathbf{E}\big(\psi(F^\varepsilon)\big)\mathbf{E}\big(\psi(G^\varepsilon)\big) \to 0.$$

Proof Just combine Lemmas 5.4 and 5.2. □

Finally, we finish the proof of Proposition 5.1.

Proof Since $(F^\varepsilon, G^\varepsilon)$ is bounded in $L^2(\Omega)$ it is tight. Now choose a weakly converging subsequence (F^n, G^n) with limit denoted by (X, Y). Let φ and ψ be smooth test functions, then by Lemma 5.5 and the bounds on φ, ψ, we pass to the limit under the expectation sign and obtain

$$\mathbf{E}(\varphi(X)\psi(Y)) = \mathbf{E}(\varphi(X))\mathbf{E}(\psi(Y)).$$

Thus, every limit measure is the product measure determined by U and V, hence, $(F^\varepsilon, G^\varepsilon)$ converges as claimed. □

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