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Central Limit Theorem for Linear Eigenvalue Statistics for a Tensor Product Version of Sample Covariance Matrices

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Abstract For $k, m, n \in \mathbb{N}$, we consider $n^k \times n^k$ random matrices of the form

$$\mathcal{M}_{n,m,k}(\mathbf{y}) = \sum_{\alpha=1}^{m} \tau_\alpha Y_\alpha Y_\alpha^T, \quad Y_\alpha = \mathbf{y}_\alpha^{(1)} \otimes \cdots \otimes \mathbf{y}_\alpha^{(k)},$$

where $\tau_\alpha, \alpha \in [m]$, are real numbers and $\mathbf{y}_\alpha^{(j)}, \alpha \in [m], j \in [k]$, are i.i.d. copies of a normalized isotropic random vector $\mathbf{y} \in \mathbb{R}^n$. For every fixed $k \geq 1$, if the Normalized Counting Measures of $\{\tau_\alpha\}_\alpha$ converge weakly as $m, n \to \infty$, $m/n^k \to c \in [0, \infty)$ and $\mathbf{y}$ is a good vector in the sense of Definition 1.1, then the Normalized Counting Measures of eigenvalues of $\mathcal{M}_{n,m,k}(\mathbf{y})$ converge weakly in probability to a nonrandom limit found in Marchenko and Pastur (Math USSR Sb 1:457–483, 1967). For $k = 2$, we define a subclass of good vectors $\mathbf{y}$ for which the centered linear eigenvalue statistics $n^{-1/2} \operatorname{Tr} \varphi(\mathcal{M}_{n,m,2}(\mathbf{y}))^\circ$ converge in distribution to a Gaussian random variable, i.e., the Central Limit Theorem is valid.

Keywords Random matrices · Sample covariance matrices · Central Limit Theorem · Linear eigenvalue statistics

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1 Introduction: Problem and Main Result

For every $k \in \mathbb{N}$, consider random vectors of the form

$$Y = \mathbf{y}^{(1)} \otimes \cdots \otimes \mathbf{y}^{(k)} \in (\mathbb{R}^n)^{\otimes k}, \tag{1.1}$$

where $\mathbf{y}^{(1)}, \ldots, \mathbf{y}^{(k)}$ are i.i.d. copies of *a normalized isotropic* random vector $\mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$,

$$\mathbf{E}\{y_j\} = 0, \quad \mathbf{E}\{y_i y_j\} = \delta_{ij} n^{-1}, \quad i, j \in [n], \tag{1.2}$$

$[n] = \{1, \ldots, n\}$. The components of Y have the form

$$Y_{\mathbf{j}} = y_{j_1}^{(1)} \times \ldots \times y_{j_k}^{(k)},$$

where we use the notation $\mathbf{j}$ for k-multiindex:

$$\mathbf{j} = \{j_1, \ldots, j_k\}, \quad j_1, \ldots, j_k \in [n].$$

For every $m \in \mathbb{N}$, let $\{Y_\alpha\}_{\alpha=1}^m$ be i.i.d. copies of Y, and let $\{\tau_\alpha\}_{\alpha=1}^m$ be a collection of real numbers. Consider an $n^k \times n^k$ real symmetric random matrix corresponding to a normalized isotropic random vector $\mathbf{y}$,

$$\mathcal{M}_n = \mathcal{M}_{n,m,k} = \mathcal{M}_{n,m,k}(\mathbf{y}) = \sum_{\alpha=1}^m \tau_\alpha Y_\alpha Y_\alpha^T. \tag{1.3}$$

We suppose that

$$m \to \infty \quad \text{and} \quad m/n^k \to c \in (0, \infty) \quad \text{as} \quad n \to \infty. \tag{1.4}$$

Note that $\mathcal{M}_{n,m,k}$ can be also written in the form

$$\mathcal{M}_{n,m,k} = B_{n,m,k} T_m B_{n,m,k}^T, \tag{1.5}$$

where

$$B_{n,m,k} = (Y_1 \ Y_2 \ \ldots \ Y_m), \quad T_m = \{\tau_\alpha \delta_{\alpha\beta}\}_{\alpha,\beta=1}^m.$$

Such matrices with $T_m \geq 0$ (not necessarily diagonal) are known as sample covariance matrices. The asymptotic behavior of their spectral statistics is well studied when all entries of Y_α are independent. Much less is known in the case when columns Y_α have dependence in their structure.

The model constructed in (1.3) appeared in *the quantum information theory* and was introduced to random matrix theory by Hastings (see [3,14,15]). In [3], it was studied as a quantum analog of the classical probability problem on the allocation of p balls among q boxes (a quantum model of data hiding and correlation locking scheme). In particular, by combinatorial analysis of moments of $n^{-k} \operatorname{Tr} \mathcal{M}_n^p, p \in \mathbb{N}$, it was proved

that for the special cases of random vectors $\mathbf{y}$ uniformly distributed on the unit sphere in $\mathbb{C}^n$ or having Gaussian components, the expectations of the Normalized Counting Measures of eigenvalues of the corresponding matrices converge to the Marchenko–Pastur law [17]. The main goal of the present paper is to extend this result of [3] to a wider class of matrices $M_{n,m,k}(\mathbf{y})$ and also to prove the Central Limit Theorem for linear eigenvalue statistics in the case $k = 2$.

Let $\{\lambda_l^{(n)}\}_{l=1}^{n^k}$ be the eigenvalues of $\mathcal{M}_n$ counting their multiplicity, and introduce their Normalized Counting Measure (NCM) N_n, setting for every $\Delta \subset \mathbb{R}$

$$N_n(\Delta) = \mathrm{Card}\{l \in [n^k] : \lambda_l^{(n)} \in \Delta\}/n^k.$$

Likewise, define the NCM σ_m of $\{\tau_\alpha\}_{\alpha=1}^m$,

$$\sigma_m(\Delta) = \mathrm{Card}\{\alpha \in [m] : \tau_\alpha \in \Delta\}/m. \tag{1.6}$$

We assume that the sequence $\{\sigma_m\}_{m=1}^\infty$ converges weakly:

$$\lim_{m\to\infty} \sigma_m = \sigma, \ \sigma(\mathbb{R}) = 1. \tag{1.7}$$

In the case $k = 1$, there are a number of papers devoted to the convergence of the NCMs of the eigenvalues of $\mathcal{M}_{n,m,1}$ and related matrices (see [1,6,12,17,20,27] and references therein). In particular, in [20] the convergence of NCMs of eigenvalues of $\mathcal{M}_{n,m,1}$ was proved in the case when corresponding vectors $\{Y_\alpha\}_\alpha$ are "good vectors" in the sense of the following definition.

Definition 1.1 We say that a normalized isotropic vector $\mathbf{y} \in \mathbb{R}^n$ is good, if for every $n \times n$ complex matrix H_n which does not depend on $\mathbf{y}$, we have

$$\mathbf{Var}\{(H_n\mathbf{y}, \mathbf{y})\} \leq ||H_n||^2 \delta_n, \quad \delta_n = o(1), \ n \to \infty, \tag{1.8}$$

where $||H_n||$ is the Euclidean operator norm of H_n.

Following the scheme of the proof proposed in [20], we show that despite the fact that the number of independent parameters, $kmn = O(n^{k+1})$ for $k \geq 2$, is much less than the number of matrix entries, n^{2k}, the limiting distribution of eigenvalues still obeys the Marchenko–Pastur law. We have:

Theorem 1.2 Fix $k \geq 1$. Let n and m be positive integers satisfying (1.4), let $\{\tau_\alpha\}_\alpha$ be real numbers satisfying (1.7), and let $\mathbf{y}$ be a good vector in the sense of Definition 1.1. Then there exists a nonrandom measure N of total mass 1 such that the NCMs N_n of the eigenvalues of $\mathcal{M}_n$ (1.3) converge weakly in probability to N as $n \to \infty$. The Stieltjes transform f of N,

$$f(z) = \int \frac{N(d\lambda)}{\lambda - z}, \quad \Im z \neq 0, \tag{1.9}$$

is the unique solution of the functional equation

$$zf(z) = c - 1 - c \int (1 + \tau f(z))^{-1} \sigma(d\tau) \tag{1.10}$$

in the class of analytic in $\mathbb{C} \setminus \mathbb{R}$ functions such that $\Im f(z)\Im z \geq 0$, $\Im z \neq 0$.

We use the notation $\int$ for the integrals over $\mathbb{R}$. Note that in [26] there was proved an analog of this statement for a deformed version of $M_{n,m,2}$.

It follows from Theorem 1.2 that if

$$\mathcal{N}_n[\varphi] = \sum_{j=1}^{n^k} \varphi(\lambda_j^{(n)}) \tag{1.11}$$

is the *linear eigenvalue statistic* of $\mathcal{M}_n$ corresponding to a bounded continuous *test function* $\varphi : \mathbb{R} \to \mathbb{C}$, then we have in probability

$$\lim_{n \to \infty} n^{-k} \mathcal{N}_n[\varphi] = \int \varphi(\lambda) dN(\lambda). \tag{1.12}$$

This can be viewed as an analog of the Law of Large Numbers in probability theory for (1.11). Since the limit is nonrandom, the next natural step is to investigate the fluctuations of $\mathcal{N}_n[\varphi]$. This corresponds to the question of validity of the Central Limit Theorem (CLT). The main goal of this paper is to prove the CLT for the linear eigenvalue statistics of the tensor version of the sample covariance matrix $\mathcal{M}_{n,m,2}$ defined in (1.3).

There are a considerable number of papers on the CLT for linear eigenvalue statistics of sample covariance matrices $\mathcal{M}_{n,m,1}$ (1.5), where all entries of the matrix $B_{n,m,1}$ are independent (see [4,7–9,11,16,18,19,21,25] and references therein). Less is known in the case where the components of vector **y** are dependent. In [13], the CLT was proved for linear statistics of eigenvalues of $\mathcal{M}_{n,m,1}$, corresponding to some special class of isotropic vectors defined below.

Definition 1.3 The distribution of a random vector $\mathbf{y} \in \mathbb{R}^n$ is called *unconditional* if its components $\{y_j\}_{j=1}^n$ have the same joint distribution as $\{\pm y_j\}_{j=1}^n$ for any choice of signs.

Definition 1.4 We say that normalized isotropic vectors $\mathbf{y} \in \mathbb{R}^n, n \in \mathbb{N}$, are *very good* if they have unconditional distributions, their mixed moments up to the fourth order do not depend on i, j, n, there exist n-independent $a, b \in \mathbb{R}$ such that as $n \to \infty$,

$$a_{2,2} := \mathbf{E}\{y_i^2 y_j^2\} = n^{-2} + an^{-3} + O(n^{-4}), \quad i \neq j, \tag{1.13}$$

$$\kappa_4 := \mathbf{E}\{y_j^4\} - 3a_{2,2} = bn^{-2} + O(n^{-3}), \tag{1.14}$$

and for every $n \times n$ complex matrix H_n which does not depend on **y**,

$$\mathbf{E}\{|(H_n\mathbf{y}, \mathbf{y})^\circ|^4\} \leq C||H_n||^4 n^{-2}. \tag{1.15}$$

Here and in what follows we use the notation $\xi^\circ = \xi - \mathbf{E}\{\xi\}$.

An important step in proving the CLT for linear eigenvalue statistics is the asymptotic analysis of their variances $\mathbf{Var}\{\mathcal{N}_n[\varphi]\} := \mathbf{E}\{|\mathcal{N}_n^\circ[\varphi]|^2\}$, in particular, the proof of the bound

$$\mathbf{Var}\{\mathcal{N}_n[\varphi]\} \le C_n \|\varphi\|_{\mathcal{H}}^2, \tag{1.16}$$

where $\|\dots\|_{\mathcal{H}}$ is a functional norm and C_n depends only on n. This bound determines the normalization factor in front of $\mathcal{N}_n^\circ[\varphi]$ and the class $\mathcal{H}$ of the test functions for which the CLT, if any, is valid. It appears that for many random matrices normalized so that there exists a limit of their NCMs, in particular for sample covariance matrices $\mathcal{M}_{n,m,1}$, the variance of the linear eigenvalue statistic corresponding to a smooth enough test function does not grow with n, and the CLT is valid for $\mathcal{N}_n^\circ[\varphi]$ itself without any n-dependent normalization factor in front. Consider the test functions $\varphi : \mathbb{R} \to \mathbb{R}$ from the Sobolev space $\mathcal{H}_s$, possessing the norm

$$\|\varphi\|_s^2 = \int (1 + |t|)^{2s} |\widehat{\varphi}(t)|^2 dt, \quad \widehat{\varphi}(t) = \int e^{it\theta} \varphi(\theta) d\theta. \tag{1.17}$$

The following statement was proved in [13] (see Theorem 1.8 and Remark 1.11):

Theorem 1.5 *Let m and n be positive integers satisfying* (1.4) *with $k = 1$, let $\{\tau_\alpha\}_{\alpha=1}^m$ be a collection of real numbers satisfying* (1.7) *and*

$$\sup_m \int \tau^4 d\sigma_m(\tau) < \infty, \tag{1.18}$$

and let $\mathbf{y}$ be a very good vector in the sense of Definition 1.4. *Consider matrix $\mathcal{M}_{n,m,1}(\mathbf{y})$* (1.3) *and the linear statistic of its eigenvalues $\mathcal{N}_n[\varphi]$* (1.11) *corresponding to a test function $\varphi \in \mathcal{H}_s$, $s > 2$. Then $\{\mathcal{N}_n^\circ[\varphi]\}_n$ converges in distribution to a Gaussian random variable with zero mean and the variance $V[\varphi] = \lim_{\eta \downarrow 0} V_\eta[\varphi]$, where*

$$V_\eta[\varphi] = \frac{1}{2\pi^2} \int \int \Re\big[L(z_1, z_2) - L(z_1, \overline{z_2})\big](\varphi(\lambda_1) - \varphi(\lambda_2))^2 d\lambda_1 d\lambda_2$$

$$+ \frac{(a+b)c}{\pi^2} \int \tau^2 \bigg(\Im \int \frac{f'(z_1)}{(1 + \tau f(z_1))^2} \varphi(\lambda_1) d\lambda_1 \bigg)^2 d\sigma(\tau),$$

$$L(z_1, z_2) = \frac{\partial^2}{\partial z_1 \partial z_2} \log \frac{\Delta f}{\Delta z},$$

$z_{1,2} = \lambda_{1,2} + i\eta$, $\Delta f = f(z_1) - f(z_2)$, $\Delta z = z_1 - z_2$, *and f given by* (1.10).

Here we prove an analog of Theorem 1.5 in the case $k = 2$. We start with establishing a version of (1.16) in general case $k \ge 1$:

Lemma 1.6 *Let $\{\tau_\alpha\}_\alpha$ be a collection of real numbers satisfying* (1.7) *and* (1.18), *and let $\mathbf{y}$ be a normalized isotropic vector having an unconditional distribution, such that*

$$a_{2,2} = n^{-2} + O(n^{-3}), \quad \kappa_4 = O(n^{-2}). \tag{1.19}$$

Consider the corresponding matrix $\mathcal{M}_n$ (1.3) *and a linear statistic of its eigenvalues $\mathcal{N}_n[\varphi]$. Then for every $\varphi \in \mathcal{H}_s$, $s > 5/2$, and for all sufficiently large m and n, we have*

$$\mathbf{Var}\{\mathcal{N}_n[\varphi]\} \leq Cn^{k-1}||\varphi||_s^2, \tag{1.20}$$

where C does not depend on n and φ.

It follows from Lemma 1.6 that in order to prove the CLT (if any) for linear eigenvalue statistics of $\mathcal{M}_n$, one needs to normalize them by $n^{-(k-1)/2}$. To formulate our main result we need more definitions.

Definition 1.7 We say that the distribution of a random vector $\mathbf{y} \in \mathbb{R}^n$ is *permutationally invariant* (or *exchangeable*) if it is invariant with respect to the permutations of entries of $\mathbf{y}$.

Definition 1.8 We say that normalized isotropic vectors $\mathbf{y} \in \mathbb{R}^n$, $n \in \mathbb{N}$, are the *CLT-vectors* if they have unconditional permutationally invariant distributions and satisfy the following conditions:

 (i) their fourth moments satisfy (1.13)–(1.14),
 (ii) their sixth moments satisfy conditions

$$a_{2,2,2} := \mathbf{E}\{y_i^2 y_j^2 y_k^2\} = n^{-3} + O(n^{-4}),$$
$$a_{2,4} := \mathbf{E}\{y_i^2 y_j^4\} = O(n^{-3}), \quad a_6 := \mathbf{E}\{y_i^6\} = O(n^{-3}), \tag{1.21}$$

(iii) for every $n \times n$ matrix H_n which does not depend on $\mathbf{y}$,

$$\mathbf{E}\{|(H_n\mathbf{y}, \mathbf{y})^\circ|^6\} \leq C||H_n||^6 n^{-3}. \tag{1.22}$$

It can be shown that a vector of the form $\mathbf{y} = \mathbf{x}/n^{1/2}$, where $\mathbf{x}$ has i.i.d. components with even distribution and bounded twelfth moment is a CLT-vector as well as a vector uniformly distributed on the unit ball in $\mathbb{R}^n$ or a properly normalized vector uniformly distributed on the unit ball $B_p^n = \{\mathbf{x} \in \mathbb{R}^n : \sum_{j=1}^n |x_j|^p \leq 1\}$ in l_p^n (see [13], Section 2 for $k = 1$).

The main result of the present paper is:

Theorem 1.9 *Let m and n be positive integers satisfying* (1.4) *with $k = 2$, and let $\{\tau_\alpha\}_{\alpha=1}^m$ be a set of real numbers uniformly bounded in α and m and satisfying* (1.7). *Consider matrices $\mathcal{M}_{n,m,2}(\mathbf{y})$* (1.3) *corresponding to CLT-vectors $\mathbf{y} \in \mathbb{R}^n$. If $\mathcal{N}_n[\varphi]$ are the linear statistics of their eigenvalues* (1.11) *corresponding to a test function $\varphi \in \mathcal{H}_s$, $s > 5/2$, then $\{n^{-1/2}\mathcal{N}_n^\circ[\varphi]\}_n$ converges in distribution to a Gaussian random variable with zero mean and the variance $V[\varphi] = \lim_{\eta\downarrow0} V_\eta[\varphi]$, where*

$$V_\eta[\varphi] = \frac{2(a+b+2)c}{\pi^2} \int \tau^2 \left(\Im \int \frac{f'(\lambda + i\eta)}{(1 + \tau f(\lambda + i\eta))^2} \varphi(\lambda)d\lambda\right)^2 d\sigma(\tau) \tag{1.23}$$

and f is given by (1.10).

Remark 1.10 (i) In particular, if $\tau_1 = \cdots = \tau_m = 1$, then

$$V[\varphi] = \frac{(a+b+2)}{2c\pi^2}\left(\int_{a_-}^{a_+}\varphi(\mu)\frac{\mu - a_m}{\sqrt{(a_+ - \mu)(\mu - a_-)}}d\mu\right)^2,$$

where $a_{\pm} = (1 \pm \sqrt{c})^2$ and $a_m = 1 + c$.

(ii) We can replace the condition of the uniform boundedness of τ_α with the condition of uniform boundedness of eighth moments of the Normalized Counting Measures σ_n, or take $\{\tau_\alpha\}_\alpha$ being real random variables independent of **y** with common probability law σ having finite eighth moment. In general, it is clear from (1.23) that it should be enough to have second moments of σ_n being uniformly bounded in n.

(iii) If in (1.23) $a + b + 2 = 0$, then to prove the CLT one needs to renormalize linear eigenvalue statistics. In particular, it can be shown that if **y** in the definition of $\mathcal{M}_{n,m,k}(\mathbf{y})$ is uniformly distributed on the unit sphere in $\mathbb{R}^n$, then $a + b + 2 = 0$ and under additional assumption $m/n = c + O(n^{-1})$ the variance of the linear eigenvalue statistic corresponding to a smooth enough test function is of the order $O(n^{k-2})$ (cf 1.20).

The paper is organized as follows. Section 3 contains some known facts and auxiliary results. In Sect. 4, we prove Theorem 1.2 on the convergence of the NCMs of eigenvalues of $\mathcal{M}_{n,m,k}$. Sections 5 and 7 present some asymptotic properties of bilinear forms (HY, Y), where Y is given by (1.1) and H does not depend on Y. In Sect. 6, we prove Lemma 1.6. In Sect. 8, the limit expression for the covariance of the resolvent traces is found. Section 9 contains the proof of the main result, Theorem 1.9.

2 Notations

Let I be the $n^k \times n^k$ identity matrix. For $z \in \mathbb{C}$, $\Im z \neq 0$, let $G(z) = (\mathcal{M}_n - zI)^{-1}$ be the resolvent of $\mathcal{M}_n$, and

$$\gamma_n(z) = \text{Tr}\, G(z) = \sum_{\mathbf{j}} G_{\mathbf{jj}}(z),$$

$$g_n(z) = n^{-k}\gamma_n(z), \quad f_n(z) = \mathbf{E}\{g_n(z)\}.$$

Here and in what follows

$$\sum_{\mathbf{j}} = \sum_{j_1,\dots,j_k}, \quad \sum_j = \sum_{j=1}^{n}, \quad \text{and} \quad \sum_\alpha = \sum_{\alpha=1}^{m},$$

so that for the nonbold Latin and Greek indices the summations are from 1 to n and from 1 to m, respectively. For $\alpha \in [m]$, let

$$\mathcal{M}_n^\alpha = \mathcal{M}_n\big|_{\tau_\alpha=0} = \mathcal{M}_n - \tau_\alpha Y_\alpha Y_\alpha^T, \quad G^\alpha(z) = (\mathcal{M}_n^\alpha - zI)^{-1},$$
$$\gamma_n^\alpha = \mathrm{Tr}\, G^\alpha, \quad g_n^\alpha = n^{-k}\gamma_n^\alpha, \quad f_n^\alpha = \mathbf{E}\{g_n^\alpha\}. \tag{2.1}$$

Thus the upper index α indicates that the corresponding function does not depend on Y_α. We use the notations $\mathbf{E}_\alpha\{\ldots\}$ and $(\ldots)_\alpha^\circ$ for the averaging and the centering with respect to Y_α, so that $(\xi)_\alpha^\circ = \xi - \mathbf{E}_\alpha\{\xi\}$.

In what follows we also need functions (see (4.5) below)

$$A_\alpha = A_\alpha(z) := 1 + \tau_\alpha(G^\alpha Y_\alpha, Y_\alpha) \quad \text{and} \quad B_\alpha = B_\alpha(z) := \tau_\alpha((G^\alpha)^2 Y_\alpha, Y_\alpha).$$

Writing $O(n^{-p})$ or $o(n^{-p})$ we suppose that $n \to \infty$ and that the coefficients in the corresponding relations are uniformly bounded in $\{\tau_\alpha\}_\alpha$, $n \in \mathbb{N}$, and $z \in K$. We use the notation K for any compact set in $\mathbb{C} \setminus \mathbb{R}$.

Given matrix H, $||H||$ and $||H||_{HS}$ are the Euclidean operator norm and the Hilbert-Schmidt norm, respectively. We use C for any absolute constant which can vary from place to place.

3 Some Facts and Auxiliary Results

We need the following bound for the martingales moments, obtained in [10]:

Proposition 3.1 *Let $\{S_m\}_{m\geq 1}$ be a martingale, i.e., $\forall m$, $\mathbf{E}\{S_{m+1} \mid S_1, \ldots, S_m\} = S_m$ and $\mathbf{E}\{|S_m|\} < \infty$. Let $S_0 = 0$. Then for every $v \geq 2$, there exists an absolute constant C_v such that for all $m = 1, 2 \ldots$*

$$\mathbf{E}\{|S_m|^v\} \leq C_v m^{v/2-1} \sum_{j=1}^m \mathbf{E}\{|S_j - S_{j-1}|^v\}. \tag{3.1}$$

Lemma 3.2 *Let $\{\xi_\alpha\}_\alpha$ be independent random variables assuming values in $\mathbb{R}^{n_\alpha}$ and having probability laws P_α, $\alpha \in [m]$, and let $\Phi : \mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_m} \to \mathbb{C}$ be a Borel measurable function. Then for every $v \geq 2$, there exists an absolute constant C_v such that for all $m = 1, 2 \ldots$*

$$\mathbf{E}\{|\Phi - \mathbf{E}\{\Phi\}|^v\} \leq C_v m^{v/2-1} \sum_{\alpha=1}^m \mathbf{E}\{|(\Phi)_\alpha^\circ|^v\}, \tag{3.2}$$

where $(\Phi)_\alpha^\circ = \Phi - \mathbf{E}_\alpha\{\Phi\}$, and $\mathbf{E}_\alpha$ is the averaging with respect to ξ_α.

Proof This simple statement is hidden in the proof of Proposition 1 in [25]. We give its proof for the sake of completeness. For $\alpha \in [m]$, denote $\mathbf{E}_{\geq\alpha} = \mathbf{E}_\alpha \ldots \mathbf{E}_m$. Applying Proposition 3.1 with $S_0 = 0$, $S_\alpha = \mathbf{E}_{\geq\alpha+1}\{\Phi\} - \mathbf{E}\{\Phi\}$, $S_m = \Phi - \mathbf{E}\{\Phi\}$, we get

$$\mathbf{E}\{|\Phi - \mathbf{E}\{\Phi\}|^{\nu}\} \le C_{\nu} m^{\nu/2-1} \sum_{\alpha=1}^{m} \mathbf{E}\{|\mathbf{E}_{\ge\alpha+1}\{\Phi\} - \mathbf{E}_{\ge\alpha}\{\Phi\}|^{\nu}\}.$$

By the Hölder inequality

$$|\mathbf{E}_{\ge\alpha+1}\{\Phi\} - \mathbf{E}_{\ge\alpha}\{\Phi\}|^{\nu} = |\mathbf{E}_{\ge\alpha+1}\{(\Phi)_{\alpha}^{\circ}\}|^{\nu} \le \mathbf{E}_{\ge\alpha+1}\{|(\Phi)_{\alpha}^{\circ}|^{\nu}\},$$

which implies (3.2). □

Lemma 3.3 *Fix $\ell \ge 2$ and $k \ge 2$. Let $\mathbf{y} \in \mathbb{R}^n$ be a normalized isotropic random vector* (1.2) *such that for every $n \times n$ complex matrix H which does not depend on $\mathbf{y}$, we have*

$$\mathbf{E}\{|(H\mathbf{y}, \mathbf{y})^{\circ}|^{\ell}\} \le ||H||^{\ell} \delta_n, \quad \delta_n = o(1), \ n \to \infty. \tag{3.3}$$

Then there exists an absolute constant C_{ℓ} such that for every $n^k \times n^k$ complex matrix $\mathcal{H}$ which does not depend on $\mathbf{y}$, we have

$$\mathbf{E}\{|(\mathcal{H}Y, Y)^{\circ}|^{\ell}\} \le C_{\ell} k^{\ell/2} ||\mathcal{H}||^{\ell} \delta_n, \tag{3.4}$$

where $Y = \mathbf{y}^{(1)} \otimes \ldots \otimes \mathbf{y}^{(k)}$, and $\mathbf{y}^{(j)}$, $j \in [k]$, are i.i.d. copies of $\mathbf{y}$.

Proof It follows from (3.2) that

$$\mathbf{E}\{|(\mathcal{H}Y, Y)^{\circ}|^{\ell}\} \le C_{\ell} k^{\ell/2-1} \sum_{j=1}^{k} \mathbf{E}\{|(\mathcal{H}Y, Y)_j^{\circ}|^{\ell}\}, \tag{3.5}$$

where $\xi_j^{\circ} = \xi - \mathbf{E}_j\{\xi\}$ and $\mathbf{E}_j$ is the averaging w.r.t. $y^{(j)}$. We have

$$(\mathcal{H}Y, Y) = \sum_{\mathbf{p}, \mathbf{q}} \mathcal{H}_{\mathbf{p}, \mathbf{q}} Y_{\mathbf{p}} Y_{\mathbf{q}} = (H^{(j)} \mathbf{y}^{(j)}, \mathbf{y}^{(j)}),$$

where $H^{(j)}$ is an $n \times n$ matrix with the entries

$$(H^{(j)})_{st} = \sum_{\mathbf{p}, \mathbf{q}} \mathcal{H}_{\mathbf{p}, \mathbf{q}} \delta_{p_j s} \delta_{q_j t} \ y_{p_1}^{(1)} \ldots y_{p_{j-1}}^{(j-1)} y_{p_{j+1}}^{(j+1)} \ldots y_{p_k}^{(k)} \ y_{q_1}^{(1)} \ldots y_{q_{j-1}}^{(j-1)} y_{q_{j+1}}^{(j+1)} \ldots y_{q_k}^{(k)}.$$

This and (3.3) yield

$$\mathbf{E}_j \left\{ |(\mathcal{H}Y, Y)_j^{\circ}|^{\ell} \right\} = \mathbf{E}_j\{|(H^{(j)} \mathbf{y}^{(j)}, \mathbf{y}^{(j)})^{\circ}|^{\ell}\} \le ||H^{(j)}||^{\ell} \delta_n.$$

We have

$$||H^{(j)}|| \le ||\mathcal{H}|| \prod_{i \neq j} ||\mathbf{y}^{(i)}||^2.$$

For $i \in [k]$, since by (1.2) $\mathbf{E}\{||y^{(i)}||\} = 1$, we have by (3.3) $\mathbf{E}\{||y^{(i)}||^{2\ell}\} \le C$.

Hence

$$\mathbf{E}_j\{|(\mathcal{H}Y, Y)_j^\circ|^\ell\} \leq ||\mathcal{H}||^\ell \prod_{i \neq j} \mathbf{E}\{||\mathbf{y}^{(i)}||^{2\ell}\}\delta_n \leq C||\mathcal{H}||^\ell \delta_n.$$

This and (3.5) lead to (3.4), which completes the proof of the lemma. □

The following statement was proved in [20].

Proposition 3.4 *Let N_n be the NCM of the eigenvalues of $M_n = \sum_\alpha \tau_\alpha Y_\alpha Y_\alpha^T$, where $\{Y_\alpha\}_{\alpha=1}^m \in \mathbb{R}^p$ are i.i.d. random vectors and $\{\tau_\alpha\}_{\alpha=1}^m$ are real numbers. Then*

$$\mathbf{Var}\{N_n(\Delta)\} \leq 4m/p^2, \quad \forall \Delta \subset \mathbb{R}, \tag{3.6}$$

$$\mathbf{Var}\{g_n(z)\} \leq 4m/(p|\Im z|)^2, \quad \forall z \in \mathbb{C} \setminus \mathbb{R}. \tag{3.7}$$

Also, we will need the following simple claim:

Claim 3.5 *If h_1, h_2 are bounded random variables, then*

$$\mathbf{Var}\{h_1 h_2\} \leq C\big(\mathbf{Var}\{h_1\} + \mathbf{Var}\{h_2\}\big). \tag{3.8}$$

4 Proof of Theorem 1.2

Theorem 1.2 essentially follows from Theorem 3.3 of [20] and Lemma 3.3; here we give a proof for the sake of completeness. In view of (3.6) with $p = n^k$, it suffices to prove that the expectations $\overline{N}_n = \mathbf{E}\{N_n\}$ of the NCMs of the eigenvalues of $\mathcal{M}_n$ converge weakly to N. Due to the one-to-one correspondence between nonnegative measures and their Stieltjes transforms (see, e.g., [2]), it is enough to show that the Stieltjes transforms of $\overline{N}_n$,

$$f_n(z) = \int \frac{\overline{N}_n(d\lambda)}{\lambda - z},$$

converge to the solution f of (1.10) uniformly on every compact set $K \subset \mathbb{C} \setminus \mathbb{R}$, and that

$$\lim_{\eta \to \infty} \eta|f(i\eta)| = 1. \tag{4.1}$$

In [20], it is proved that the solution of (1.10) satisfies (4.1), so it is enough to show that

$$f_n(z) \underset{n \to \infty}{\rightrightarrows} f(z), \quad z \in K, \tag{4.2}$$

where we use the double arrow notation for the uniform convergence. Assume first that all τ_α are bounded:

$$\forall m \ \forall \alpha \in [m] \ \ |\tau_\alpha| \leq L. \tag{4.3}$$

Since $\mathcal{M}_n - \mathcal{M}_n^\alpha = \tau_\alpha Y_\alpha Y_\alpha^T$, the rank one perturbation formula

$$G - G^\alpha = -\frac{\tau_\alpha G^\alpha Y_\alpha Y_\alpha^T G^\alpha}{1 + \tau_\alpha (G^\alpha Y_\alpha, Y_\alpha)} \tag{4.4}$$

implies that

$$\gamma_n - \gamma_n^\alpha = -\frac{\tau_\alpha((G^\alpha)^2 Y_\alpha, Y_\alpha)}{1 + \tau_\alpha(G^\alpha Y_\alpha, Y_\alpha)} = -\frac{B_\alpha}{A_\alpha}. \tag{4.5}$$

It follows from the spectral theorem for the real symmetric matrices that there exists a nonnegative measure m^α such that

$$(G^\alpha Y_\alpha, Y_\alpha) = \int \frac{m^\alpha(d\lambda)}{\lambda - z}, \quad ((G^\alpha)^2 Y_\alpha, Y_\alpha) = \int \frac{m^\alpha(d\lambda)}{(\lambda - z)^2}. \tag{4.6}$$

This yields

$$|A_\alpha| \geq |\Im A_\alpha| = |\tau_\alpha||\Im z| \int \frac{m^\alpha(d\lambda)}{|\lambda - z|^2}, \quad |B_\alpha| \leq |\tau_\alpha| \int \frac{m^\alpha(d\lambda)}{|\lambda - z|^2},$$

implying that

$$|B_\alpha/A_\alpha| \leq 1/|\Im z|. \tag{4.7}$$

It also follows from (4.4) that $A_\alpha^{-1} = 1 - \tau_\alpha(G Y_\alpha, Y_\alpha)$. Hence,

$$|A_\alpha^{-1}| \leq 1 + |\tau_\alpha| \cdot ||Y_\alpha||^2/|\Im z|, \tag{4.8}$$

where we use $||G|| \leq |\Im z|^{-1}$. Let us show that

$$|\mathbf{E}_\alpha\{A_\alpha\}|^{-1}, \; |\mathbf{E}\{A_\alpha\}|^{-1} \leq 4(1 + |\tau_\alpha|/|\Im z|). \tag{4.9}$$

It follows from (1.2) that

$$\mathbf{E}_\alpha\{A_\alpha\} = 1 + \tau_\alpha g_n^\alpha(z), \quad \mathbf{E}\{A_\alpha\} = 1 + \tau_\alpha f_n^\alpha(z). \tag{4.10}$$

Consider $\mathbf{E}_\alpha\{A_\alpha\}$. By the spectral theorem for the real symmetric matrices,

$$\mathbf{E}_\alpha\{A_\alpha\} = 1 + \tau_\alpha n^{-k} \int \frac{\mathcal{N}_n^\alpha(d\lambda)}{\lambda - z},$$

where $\mathcal{N}_n^\alpha$ is the counting measure of the eigenvalues of $\mathcal{M}_n^\alpha$. For every $\eta \in \mathbb{R} \setminus \{0\}$, consider

$$E_\eta = \left\{ z = \mu + i\eta : \left| n^{-k} \int \frac{\mathcal{N}_n^\alpha(d\lambda)}{\lambda - z} \right| \leq \frac{1}{2|\tau_\alpha|} \right\}.$$

Clearly, for $z \in E_\eta$, $|\mathbf{E}_\alpha\{A_\alpha\}| \geq 1/2$. If $z = \mu + i\eta \notin E_\eta$, then

$$\frac{1}{2|\tau_\alpha|} < \left| n^{-k} \int \frac{\mathcal{N}_n^\alpha(d\lambda)}{\lambda - z} \right| \leq \left(n^{-k} \int \frac{\mathcal{N}_n^\alpha(d\lambda)}{|\lambda - z|^2} \right)^{1/2},$$

so that

$$|\mathbf{E}_\alpha\{A_\alpha\}| \geq |\Im \mathbf{E}_\alpha\{A_\alpha\}| = |\tau_\alpha||\eta|n^{-k} \int \frac{\mathcal{N}_n^\alpha(d\lambda)}{|\lambda - z|^2} \geq \frac{|\eta|}{4|\tau_\alpha|}.$$

This leads to (4.9) for $\mathbf{E}_\alpha\{A_\alpha\}$. Replacing in our argument $\mathcal{N}_n^\alpha$ with $\overline{\mathcal{N}}_n^\alpha$, we get (4.9) for $\mathbf{E}\{A_\alpha\}$.

It follows from the resolvent identity and (4.4) that

$$z g_n(z) = -1 + n^{-k} \operatorname{Tr} \mathcal{M}_n G = \left(-1 + mn^{-k}\right) - n^{-k} \sum_\alpha A_\alpha^{-1}. \qquad (4.11)$$

This and the identity

$$\frac{1}{A_\alpha} = \frac{1}{\mathbf{E}\{A_\alpha\}} - \frac{A_\alpha^\circ}{A_\alpha \mathbf{E}\{A_\alpha\}} \qquad (4.12)$$

lead to

$$z f_n(z) = \left(-1 + n^{-k}\right) - n^{-k} \sum_\alpha \mathbf{E}\{A_\alpha\}^{-1} + r_n(z),$$

$$r_n(z) = n^{-k} \sum_\alpha \frac{1}{\mathbf{E}\{A_\alpha\}} \mathbf{E}\left\{\frac{A_\alpha^\circ}{A_\alpha}\right\}.$$

It follows from the Schwarz inequality that

$$|\mathbf{E}\{A_\alpha^\circ A_\alpha^{-1}\}| \leq \mathbf{E}\{|A_\alpha^\circ|^2\}^{1/2} \mathbf{E}\{|A_\alpha^{-2}|\}^{1/2}.$$

Note that since $\mathbf{E}\{\|Y_\alpha\| = 1\}$, we have by (1.8) $\mathbf{E}\{\|Y_\alpha\|^4\} \leq C$. This and (4.8) imply that $\mathbf{E}\{|A_\alpha^{-2}|\}$ is uniformly bounded in $|\tau_\alpha| \leq L$ and $z \in K$. We also have

$$A_\alpha^\circ = (A_\alpha)_\alpha^\circ + \tau_\alpha(g_n^\alpha)^\circ = \tau_\alpha\big[(G^\alpha Y_\alpha, Y_\alpha)_\alpha^\circ + (g_n^\alpha)^\circ\big], \qquad (4.13)$$

hence

$$\mathbf{E}\{|A_\alpha^\circ|^2\} = \tau_\alpha^2\Big(\mathbf{E}\{\mathbf{E}_\alpha\{|(G^\alpha Y_\alpha, Y_\alpha)_\alpha^\circ|^2\}\} + \mathbf{E}\{|(g_n^\alpha)^\circ|^2\}\Big).$$

By (1.4) and (3.7) with $p = n^k$, $\mathbf{Var}\{g_n^\alpha\} \leq Cn^{-k}|\Im z|^{-2}$. It follows from (1.8) and Lemma 3.3 with $\mathcal{H} = G^\alpha$ and $\ell = 2$ that

$$\mathbf{E}_\alpha\{|(G^\alpha Y_\alpha, Y_\alpha)_\alpha^\circ|^2\} \leq C_2 k\|G^\alpha\|^2 \delta_n \leq C_2 k|\Im z|^{-2}\delta_n.$$

Thus, $\mathbf{E}\{|A_\alpha^\circ|^2\} \leq CL^2|\Im z|^{-2}(k\delta_n + n^{-k})$. This and (4.9) yield

$$|r_n| \leq C(k\delta_n + n^{-k})^{1/2}. \qquad (4.14)$$

uniformly in $|\tau_\alpha| \leq L$ and $z \in K$. Hence

$$z f_n(z) = (-1 + mn^{-k}) - n^{-k} \sum_\alpha (1 + \tau_\alpha f_n^\alpha(z))^{-1} + o(1). \qquad (4.15)$$

It follows from (4.5) and (4.7) that

$$|f_n(z) - f_n^\alpha(z)| \leq n^{-k}|\Im z|^{-1}. \tag{4.16}$$

This and (4.9) imply that $|1 + \tau_\alpha f_n(z)|^{-1}$ is uniformly bounded in $|\tau_\alpha| \leq L$ and $z \in K$. Hence, in (4.15) we can replace f_n^α with f_n (the corresponding error term is of the order $O(n^{-k})$) and pass to the limit as $n \to \infty$. Taking into account (1.7) we get that the limit of every convergent subsequence of $\{f_n(z)\}_n$ satisfies (1.10). This finishes the proof of the theorem under assumption (4.3).

Consider now the general case and take any sequence $\{\sigma_n\} = \{\sigma_{m(n)}\}$ satisfying (1.7). For any $L > 0$, introduce the truncated random variables

$$\tau_\alpha^L = \begin{cases} \tau_\alpha, & |\tau_\alpha| < L, \\ 0, & \text{otherwise.} \end{cases}$$

Denote $\mathcal{M}_n^L = \sum_{\alpha=1}^m \tau_\alpha^L Y_\alpha Y_\alpha^T$. Then

$$\text{rank}(\mathcal{M}_n - \mathcal{M}_n^L) \leq \text{Card}\{\alpha \in [m] : |\tau_\alpha| \geq L\}.$$

Take any sequence $\{L_i\}_i$ which does not contain atoms of σ and tends to infinity as $i \to \infty$. If $N_n^{L_i}$ is the NCM of the eigenvalues of $\mathcal{M}_n^{L_i}$ and $\overline{N}_n^{L_i}$ is its expectation, then the mini-max principle implies that for any interval $\Delta \subset \mathbb{R}$:

$$|\overline{N}_n(\Delta) - \overline{N}_n^{L_i}(\Delta)| \leq \int_{|\tau| \geq L_i} \sigma_n(d\tau).$$

We have

$$\int_{|\tau| \geq L_i} \sigma_n(d\tau) = \int_{|\tau| \geq L_i} (\sigma_n - \sigma)(d\tau) + \int_{|\tau| \geq L_i} \sigma(d\tau),$$

where by (1.7) the first term on the r.h.s. tends to zero as $n \to \infty$. Hence,

$$\lim_{L_i \to \infty} \lim_{n \to \infty} \int_{|\tau| \geq L_i} \sigma_n(d\tau) = 0.$$

Thus if f and f^{L_i} are the Stieltjes transforms of $\overline{N}$ and $\lim_{n \to \infty} \overline{N}_n^{L_i}$, then

$$f(z) = \lim_{i \to \infty} f^{L_i}(z)$$

uniformly on K. It follows from the first part of the proof that

$$zf^{L_i}(z) = -1 - c_{L_i} f^{L_i}(z) \int_{-L_i}^{L_i} \tau(1 + \tau f^{L_i}(z))^{-1} \sigma(d\tau), \tag{4.17}$$

where $c_{L_i} = c\sigma[-L_i, L_i] \to c$ as $L_i \to \infty$. Since $N(\mathbb{R}) = 1$, there exists $C > 0$, such that

$$\min_{z \in K} |\Im f(z)| = C > 0.$$

Hence we have for all sufficiently big L_i:

$$\min_{z \in K} |\Im f^{L_i}(z)| = C/2 > 0.$$

Thus $|\tau/(1 + \tau f^{L_i}(z))| \leq |\Im f^{L_i}(z)|^{-1} \leq 2/C < \infty$, $z \in K$. This allows us to pass to the limit $L_i \to \infty$ in (4.17) and to obtain (1.10) for f, which completes the proof of the theorem. $\qquad\square$

Remark 4.1 It follows from the proof that in the model we can take k depending on n such that

$$k \to \infty \quad \text{and} \quad k\delta_n \to 0$$

as $n \to \infty$, and the theorem remains valid (see 4.14).

5 Variance of Bilinear Forms

Lemma 5.1 *Let Y be defined in (1.1–1.2), where $\mathbf{y}$ has an unconditional distribution and satisfies (1.19). Then for every symmetric $n^k \times n^k$ matrix H which does not depend on $\mathbf{y}$ and whose operator norm is uniformly bounded in n, there is an absolute constant C such that*

$$n\mathbf{Var}\{(HY, Y)\} \leq Cn^{-k}||H||_{HS}^2 \leq C||H||^2. \tag{5.1}$$

If additionally $\mathbf{y}$ satisfies (1.13–1.14), then we have

$$\begin{aligned}
n\mathbf{Var}\{(HY, Y)\} = {}& ka|n^{-k} \operatorname{Tr} H|^2 \\
& + n^{-2k+1} \sum_{i=1}^{k} \sum_{\mathbf{j}, \mathbf{p}} \left[2H_{\mathbf{j}, \mathbf{j}(p_i)} \overline{H}_{\mathbf{p}, \mathbf{p}(j_i)} + b H_{\mathbf{j}, \mathbf{j}} \overline{H}_{\mathbf{p}, \mathbf{p}} \delta_{p_i j_i} \right] \\
& + O(n^{-1}),
\end{aligned} \tag{5.2}$$

where $\mathbf{j}(p_i) = \{j_1, \ldots, j_{i-1}, p_i, j_{i+1}, \ldots, j_k\}$.

Proof Since $\mathbf{y}$ has an unconditional distribution, we have

$$\mathbf{E}\{y_j y_s y_p y_q\} = a_{2,2}(\delta_{js}\delta_{pq} + \delta_{jp}\delta_{sq} + \delta_{jq}\delta_{sp}) + \kappa_4 \delta_{js}\delta_{jp}\delta_{jq}. \tag{5.3}$$

Hence,

$$\mathbf{E}\{|(HY, Y)|^2\} = \sum_{\mathbf{j,s,p,q}} H_{\mathbf{j,s}} \overline{H}_{\mathbf{p,q}} \prod_{i=1}^{k} \left[a_{2,2} \delta_{j_i s_i} \delta_{p_i q_i} + w_i \right],$$

where

$$w_i = w_i(\mathbf{j, s, p, q}) = a_{2,2} (\delta_{j_i p_i} \delta_{s_i q_i} + \delta_{j_i q_i} \delta_{s_i p_i}) + \kappa_4 \delta_{j_i s_i} \delta_{j_i p_i} \delta_{j_i q_i}.$$

For $W \subset [k]$, $W^c = [k] \setminus W$, denote

$$\Lambda(W, \mathbf{j, s, p, q}) = \prod_{i \in W^c} (a_{2,2} \delta_{j_i s_i} \delta_{p_i q_i}) \prod_{\ell \in W} w_\ell.$$

For every fixed $W, \mathbf{j, s}$, we have

$$\sum_{\mathbf{p,q}} \Lambda(W, \mathbf{j, s, p, q}) = O(n^{-k-|W|}). \tag{5.4}$$

Indeed, the number of pairs for which $\Lambda(W, \mathbf{j, s, p, q}) \neq 0$ does not exceed $2^{|W|} n^{k-|W|}$ (the number of choices of indices $p_i = q_i$ for $i \notin W$ equals to $n^{k-|W|}$; all other indices p_ℓ, q_ℓ ($\ell \in W$) must satisfy $\{p_\ell, q_\ell\} = \{j_\ell, s_\ell\}$ and, therefore, can be chosen in at most two ways each). Since $a_{2,2}, w_i = O(n^{-2})$, (5.4) follows.

For every fixed W,

$$\sum_{\mathbf{j,s,p,q}} |H_{\mathbf{j,s}}||H_{\mathbf{p,q}}|\Lambda(W, \mathbf{j, s, p, q}) \leq \sum_{\mathbf{j,s,p,q}} \left(|H_{\mathbf{j,s}}|^2 + |H_{\mathbf{p,q}}|^2 \right) \Lambda(W, \mathbf{j, s, p, q})/2$$

$$= O(n^{-k-|W|})||H||_{HS}^2. \tag{5.5}$$

Since by (1.2) $\mathbf{E}\{(HY, Y)\} = n^{-k} \operatorname{Tr} H$, we have

$$\mathbf{Var}\{(HY, Y)\} = \sum_{r=0}^{k} \sum_{|W|=r} \sum_{\mathbf{j,s,p,q}} H_{\mathbf{j,s}} \overline{H}_{\mathbf{p,q}} \Lambda(W, \mathbf{j, s, p, q}) - n^{-2k} |\operatorname{Tr} H|^2. \tag{5.6}$$

By (1.19), the term corresponding to $W = \emptyset$, $W^c = [k]$, has the form

$$T_0 := \sum_{\mathbf{j,s,p,q}} H_{\mathbf{j,s}} \overline{H}_{\mathbf{p,q}} \prod_{i=1}^{k} (a_{2,2} \delta_{j_i s_i} \delta_{p_i q_i}) = a_{2,2}^k |\operatorname{Tr} H|^2.$$

This and (1.19) imply that

$$n|T_0 - n^{-2k}|\operatorname{Tr} H|^2| \leq C n^{-k} ||H||_{HS}^2,$$

and by (1.13),

$$n(T_0 - n^{-2k}|\operatorname{Tr} H|^2) = kan^{-2k}|\operatorname{Tr} H|^2 + O(n^{-1}).\tag{5.7}$$

The term corresponding to $\sum_{|W|=1}$ (i.e., $W = \{1\}, \ldots, W = \{k\}$), has the form

$$\begin{aligned}
T_1 &:= \sum_{i=1}^{k} \sum_{\mathbf{j},\mathbf{s},\mathbf{p},\mathbf{q}} H_{\mathbf{j},\mathbf{s}} \overline{H}_{\mathbf{p},\mathbf{q}}\, w_i(\mathbf{j},\mathbf{s},\mathbf{p},\mathbf{q}) \prod_{\ell \neq i} a_{2,2} \delta_{j_\ell s_\ell} \delta_{p_\ell q_\ell} \\
&= \sum_{i=1}^{k} \sum_{\mathbf{j},\mathbf{p}} \left[a_{2,2}^{k} H_{\mathbf{j},\mathbf{j}(p_i)} \overline{H}_{\mathbf{p},\mathbf{p}(j_i)} + a_{2,2}^{k-1} \kappa_4 H_{\mathbf{j},\mathbf{j}} \overline{H}_{\mathbf{p},\mathbf{p}} \delta_{p_i j_i} \right],
\end{aligned}$$

and by (1.13)

$$nT_1 = n^{-2k+1} \sum_{i=1}^{k} \sum_{\mathbf{j},\mathbf{p}} \left[2H_{\mathbf{j},\mathbf{j}(p_i)} \overline{H}_{\mathbf{p},\mathbf{p}(j_i)} + b H_{\mathbf{j},\mathbf{j}} \overline{H}_{\mathbf{p},\mathbf{p}} \delta_{p_i j_i} \right] + O(n^{-1}).\tag{5.8}$$

Also it follows from (5.5) that the terms corresponding to $W : |W| \geq 2$ are less than $Cn^{-k-2}\|H\|_{HS}^2$. Summarizing (5.6–5.8), we get (5.1) and (5.2) and complete the proof of the lemma. $\qquad\square$

6 Proof of Lemma 1.6

Lemma 6.1 *Let $\{\tau_\alpha\}_\alpha$ be a collection of real numbers satisfying (1.7), (1.18), and let $\mathbf{y}$ be a normalized isotropic vector having an unconditional distribution and satisfying (1.19). Consider the corresponding matrix $\mathcal{M}_n$ (1.3) and the trace of its resolvent $\gamma_n(z) = \operatorname{Tr}(\mathcal{M}_n - zI)^{-1}$. We have*

$$\mathbf{Var}\{\gamma_n(z)\} \leq Cn^{k-1}|\Im z|^{-6}.\tag{6.1}$$

If additionally $\mathbf{y}$ satisfies (1.15) and τ_α are uniformly bounded in α and m, then

$$\mathbf{E}\{|\gamma_n^\circ(z)|^4\} \leq Cn^{2k-2}|\Im z|^{-12}.\tag{6.2}$$

Proof The proof follows the scheme proposed in [25] (see also Lemma 3.2 of [13]). For $q = 1, 2$, by (3.2) we have

$$\mathbf{E}\{|\gamma_n^\circ|^{2q}\} \leq Cm^{q-1} \sum_{\alpha} \mathbf{E}\{|(\gamma_n)_\alpha^\circ|^{2q}\}.\tag{6.3}$$

Applying (4.5), (4.7), and (4.9) we get

$$
\begin{aligned}
\mathbf{E}\{|(\gamma_n)^\circ_\alpha|^{2q}\} &= \mathbf{E}\{|\gamma_n - \gamma_n^\alpha - \mathbf{E}_\alpha\{\gamma_n - \gamma_n^\alpha\}|^{2q}\} \\
&\le C\mathbf{E}\Big\{\Big|\frac{B_\alpha}{A_\alpha} - \frac{\mathbf{E}_\alpha\{B_\alpha\}}{\mathbf{E}_\alpha\{A_\alpha\}}\Big|^{2q}\Big\} = C\mathbf{E}\Big\{\Big|\frac{(B_\alpha)^\circ_\alpha}{\mathbf{E}_\alpha\{A_\alpha\}} - \frac{B_\alpha}{A_\alpha}\cdot\frac{(A_\alpha)^\circ_\alpha}{\mathbf{E}_\alpha\{A_\alpha\}}\Big|^{2q}\Big\} \\
&\le C(1 + |\tau_\alpha|/|\Im z|)^{2q}\mathbf{E}\big\{\mathbf{E}_\alpha\{|(B_\alpha)^\circ_\alpha|^{2q}\} + \mathbf{E}_\alpha\{|(A_\alpha)^\circ_\alpha|^{2q}\}/|\Im z|^{2q}\big\}.
\end{aligned}
\tag{6.4}
$$

Here by (5.1)

$$
n\tau_\alpha^{-2}\mathbf{E}_\alpha\{|(A_\alpha)^\circ_\alpha|^2\} = n\mathbf{E}_\alpha\{|(G^\alpha Y_\alpha, Y_\alpha)^\circ_\alpha|^2\} \le Cn^{-k}\|G^\alpha\|_{HS}^2 \le C|\Im z|^{-2} \tag{6.5}
$$

and

$$
n\tau_\alpha^{-2}\mathbf{E}_\alpha\{|(B_\alpha)^\circ_\alpha|^2\} \le Cn^{-k}\|(G^\alpha)^2\|_{HS}^2 \le |\Im z|^{-4}. \tag{6.6}
$$

This and (6.3–6.4) lead to (6.1). Also it follows from (1.15) and Lemma 3.3 that

$$
\mathbf{E}_\alpha\{|(B_\alpha)^\circ_\alpha|^4\}, \quad \mathbf{E}_\alpha\{|(A_\alpha)^\circ_\alpha|^4\}/|\Im z|^4 \le C\tau_\alpha^4|\Im z|^{-8}n^{-2},
$$

which leads to (6.2). $\square$

Proof of Lemma 1.6 The proof of (1.20) is based on the following inequality obtained in [25]: for $\varphi \in \mathcal{H}_s$ (see 1.17),

$$
\mathbf{Var}\{\mathcal{N}_n[\varphi]\} \le C_s\|\varphi\|_s^2 \int_0^\infty \mathrm{d}\eta\, e^{-\eta}\eta^{2s-1} \int \mathbf{Var}\{\gamma_n(\mu + i\eta)\}\mathrm{d}\mu.
$$

Let $z = \mu + i\eta$, $\eta > 0$. It follows from (6.3) – (6.6) that

$$
\begin{aligned}
\mathbf{Var}\{\gamma_n\} &\le \sum_\alpha \mathbf{E}\{|(\gamma_n)^\circ_\alpha|^2\} \\
&\le Cn^{-k-1}\sum_\alpha \tau_\alpha^2(1 + \eta^{-2}\tau_\alpha^2)\mathbf{E}\{\|(G^\alpha)^2\|_{HS}^2 + \eta^{-2}\|G^\alpha\|_{HS}^2\}.
\end{aligned}
$$

By the spectral theorem for the real symmetric matrices,

$$
\mathbf{E}\{\|G^\alpha\|_{HS}^2\} = \int \frac{\overline{\mathcal{N}_n^\alpha}(\mathrm{d}\lambda)}{|\lambda - z|^2}, \quad \mathbf{E}\{\|(G^\alpha)^2\|_{HS}^2\} = \int \frac{\overline{\mathcal{N}_n^\alpha}(\mathrm{d}\lambda)}{|\lambda - z|^4},
$$

where $\overline{\mathcal{N}_n^\alpha}$ is the expectation of the counting measure of the eigenvalues of $\mathcal{M}_n^\alpha$. We have

$$
n^{-k}\int\int \frac{\overline{\mathcal{N}_n^\alpha}(\mathrm{d}\lambda)}{|\lambda - z|^2}\mathrm{d}\mu \le C\eta^{-1}, \quad n^{-k}\int\int \frac{\overline{\mathcal{N}_n^\alpha}(\mathrm{d}\lambda)}{|\lambda - z|^4}\mathrm{d}\mu \le C\eta^{-3}.
$$

Summarizing, we get

$$\mathbf{Var}\{\mathcal{N}_n[\varphi]\} \leq Cn^{k-1}||\varphi||_s^2 \int_0^\infty \mathrm{d}\eta e^{-\eta}\eta^{2s-6} \leq Cn^{k-1}||\varphi||_s^2$$

provided that $s > 5/2$. This finishes the proof of Lemma 1.6. $\square$

7 Case $k = 2$: Some Preliminary Results

From now on we fix $k = 2$ and consider matrices $\mathcal{M}_n = \mathcal{M}_{n,m,2}$. For every $\mathbf{j} = \{j_1, j_2\} = j_1 j_2$,

$$\sum_{\mathbf{j}} = \sum_{j_1, j_2}, \quad \sum_j = \sum_{j=1}^n.$$

In this section we establish some asymptotic properties of A_α, $(G^\alpha Y_\alpha, Y_\alpha)$, and their central moments. We start with

Lemma 7.1 *Under conditions of Theorem 1.9,*

$$\begin{aligned}
\mathbf{E}_\alpha\{|(A_\alpha)_\alpha^\circ|^p\} &\leq C(\tau_\alpha/|\Im z|)^p n^{-p/2}, \\
\mathbf{E}_\alpha\{|(B_\alpha)_\alpha^\circ|^p\} &\leq C(\tau_\alpha/|\Im z|^2)^p n^{-p/2},
\end{aligned} \tag{7.1}$$

and

$$\mathbf{E}\{|A_\alpha^\circ|^p\}, \ \mathbf{E}\{|B_\alpha^\circ|^p\} = O(n^{-p/2}), \quad 2 \leq p \leq 6. \tag{7.2}$$

Proof Since $(A_\alpha)_\alpha^\circ = \tau_\alpha(G^\alpha Y_\alpha, Y_\alpha)_\alpha^\circ$, Lemma 3.3 and (1.22) imply that

$$\mathbf{E}_\alpha\{|(A_\alpha)_\alpha^\circ|^6\} \leq C(\tau_\alpha/|\Im z|)^6 n^{-3},$$

and by the Hölder inequality we get the first estimate in (7.1). Analogously one can get the second estimate in (7.1). Also we have by (6.1)

$$\mathbf{E}\{|(g_n^\alpha)^\circ|^p\} \leq |\Im z|^{2-p}\mathbf{E}\{|(g_n^\alpha)^\circ|^2\} = O(n^{-3}), \quad p \geq 2,$$

which together with (4.13) and (7.1) leads to (7.2). $\square$

Let

$$H = H(z) = G^\alpha(z).$$

It follows from (5.2) with $k = 2$ that

$$
\begin{aligned}
n\mathbf{Var}\{(HY, Y)\} =&2a|n^{-2}\operatorname{Tr} H|^2 \\
&+ 2n^{-3}\sum_{\mathbf{j},\mathbf{p}}\left[H_{\mathbf{j},\, j_1 p_2}\overline{H}_{\mathbf{p},\, p_1 j_2} + H_{\mathbf{j},\, p_1 j_2}\overline{H}_{\mathbf{p},\, j_1 p_2}\right] \\
&+ bn^{-3}\sum_{\mathbf{j},\mathbf{p}} H_{\mathbf{j},\mathbf{j}}\overline{H}_{\mathbf{p},\mathbf{p}}(\delta_{p_1 j_1} + \delta_{p_2 j_2}) + O(n^{-1}). \quad (7.3)
\end{aligned}
$$

Consider an $n \times n$ matrix of the form

$$
\mathcal{G} = \{\mathcal{G}_{s,p}\}_{s,p=1}^n, \quad \mathcal{G}_{s,p} = \sum_j H_{js,\, jp}.
$$

Since $\mathcal{G} = \sum_j \mathcal{G}^{(j)}$, where for every j, $\mathcal{G}^{(j)} = \{H_{js,\, jp}\}_{s,p}$ is a block of G^α, we have

$$
||\mathcal{G}|| \le \sum_j ||\mathcal{G}^{(j)}|| \le n||G^\alpha|| \le n/|\Im z|. \quad (7.4)
$$

We define functions

$$
\begin{aligned}
g_n^{(1)}(z_1, z_2) :=&n^{-3}\sum_{\mathbf{j},\mathbf{p}} H_{\mathbf{j},\, j_1 p_2}(z_1)H_{\mathbf{p},\, p_1 j_2}(z_2) = n^{-3}\operatorname{Tr}\mathcal{G}(z_1)\mathcal{G}(z_2), \\
g_n^{(2)}(z_1, z_2) :=&n^{-3}\sum_{i,s,j} H_{is,\, is}(z_1)H_{js,\, js}(z_2) = n^{-3}\sum_s \mathcal{G}_{ss}(z_1)\mathcal{G}_{ss}(z_2).
\end{aligned}
$$

Similarly, we introduce the matrix

$$
\widetilde{\mathcal{G}} = \{\widetilde{\mathcal{G}}_{i,j}\}_{i,j=1}^n, \quad \widetilde{\mathcal{G}}_{i,j} = \sum_s H_{is,\, js}
$$

and define functions

$$
\widetilde{g}_n^{(1)}(z_1, z_2) = n^{-3}\operatorname{Tr}\widetilde{\mathcal{G}}(z_1)\widetilde{\mathcal{G}}(z_2), \quad \widetilde{g}_n^{(2)}(z_1, z_2) = n^{-3}\sum_i \widetilde{\mathcal{G}}_{ii}(z_1)\widetilde{\mathcal{G}}_{ii}(z_2). \quad (7.5)
$$

It follows from (7.3) that

$$
\begin{aligned}
n\mathbf{E}_\alpha\left\{((H(z)Y_\alpha, Y_\alpha)_\alpha^\circ)^2\right\} =&2a(g_n^\alpha(z))^2 + 2(g_n^{(1)}(z, z) + \widetilde{g}_n^{(1)}(z, z)) \\
&+ b(g_n^{(2)}(z, z) + \widetilde{g}_n^{(2)}(z, z)) + O(n^{-1}). \quad (7.6)
\end{aligned}
$$

We have:

Lemma 7.2 *Under conditions of Theorem* 1.9, *we have for* $i = 1, 2$:

$$\mathbf{Var}\{g_n^{(i)}\}, \ \mathbf{Var}\{\widetilde{g}_n^{(i)}\} = O(n^{-2}), \tag{7.7}$$

$$\lim_{n \to \infty} \mathbf{E}\{g_n^{(i)}(z_1, z_2)\} = \lim_{n \to \infty} \mathbf{E}\{\widetilde{g}_n^{(i)}(z_1, z_2)\} = f(z_1)f(z_2), \tag{7.8}$$

where f is the solution of (1.10).

Proof We prove the lemma for $g_n^{(1)}$; the cases of $\widetilde{g}_n^{(2)}$, $g_n^{(2)}$, and $\widetilde{g}_n^{(2)}$ can be treated similarly. Without loss of generality we can assume that in the definitions of $\mathcal{G}$ and $g_n^{(1)}$, $H = G$. It follows from (3.2) that

$$\mathbf{Var}\{g_n^{(1)}\} \le \sum_\alpha \mathbf{E}\{|(g_n^{(1)})_\alpha^\circ|^2\}.$$

We have

$$
\begin{aligned}
g_n^{(1)} - g_n^{(1)\alpha} &= n^{-3} \operatorname{Tr}(\mathcal{G}(z_1) - \mathcal{G}^\alpha(z_1))\mathcal{G}(z_2) \\
&\quad + n^{-3} \operatorname{Tr} \mathcal{G}^\alpha(z_1)(\mathcal{G}(z_2) - \mathcal{G}^\alpha(z_2)) =: S_n^{(1)} + S_n^{(2)}.
\end{aligned}
$$

Hence

$$(g_n^{(1)})_\alpha^\circ = g_n^{(1)} - g_n^{(1)\alpha} - \mathbf{E}_\alpha\{g_n^{(1)} - g_n^{(1)\alpha}\} = (S_n^{(1)})_\alpha^\circ + (S_n^{(2)})_\alpha^\circ,$$

and to get (7.7), it is enough to show that

$$\mathbf{E}\{|S_n^{(j)}|^2\} = O(n^{-4}), \quad j = 1, 2. \tag{7.9}$$

Consider $S_n^{(1)}$. It follows from (4.4) that

$$S_n^{(1)} = A_\alpha^{-1} n^{-3} \sum_{s,p} \sum_j \mathcal{G}_{s,p}(H^\alpha Y_\alpha)_{js}(H^\alpha Y_\alpha)_{jp}. \tag{7.10}$$

Since for $x, \xi \in \mathbb{R}^n$ and an $n \times n$ matrix D

$$\left| \sum_{i,j} D_{ij} x_i \xi_j \right| \le ||D|| \cdot ||x|| \cdot ||\xi||, \tag{7.11}$$

taking into account $||H|| \le 1/|\Im z|$, (4.8), and (7.4) we get

$$
\begin{aligned}
|S_n^{(1)}| &\le n^{-3}(1 + |\tau_\alpha| \cdot |\Im z|^{-1} ||Y_\alpha||^2) \cdot ||\mathcal{G}|| \cdot ||H^\alpha Y_\alpha||^2 \\
&\le n^{-2}(1 + |\tau_\alpha| \cdot |\Im z|^{-1} ||Y_\alpha||^2) |\Im z|^{-3} ||Y_\alpha||^2.
\end{aligned} \tag{7.12}
$$

This and following from (1.2) and (1.22) bound

$$\mathbf{E}\{||Y_\alpha||^p\} \le C, \quad p \le 12 \tag{7.13}$$

imply (7.9) for $j = 1$. The case $j = 2$ can be treated similarly. So we get (7.7) for $g_n^{(1)}$.

Let us prove (7.8) for $g_n^{(1)}$. Let $f_n^{(1)} = \mathbf{E}\{g_n^{(1)}\}$. For a convergent subsequence $\{f_{n_i}^{(1)}\}$, put $f^{(1)} := \lim_{n_i \to \infty} f_{n_i}^{(1)}$. It follows from (4.4) that

$$(Y_\alpha Y_\alpha^T H)_{\mathbf{j},\,\mathbf{q}} = A_\alpha^{-1} Y_{\alpha\mathbf{j}}(H^\alpha Y_\alpha)_{\mathbf{q}}.$$

This and the resolvent identity yield

$$H_{\mathbf{j},\,\mathbf{q}}(z_1) = -z_1^{-1}\delta_{\mathbf{j},\,\mathbf{q}} + z_1^{-1}\sum_\alpha \tau_\alpha A_\alpha^{-1}(z_1) Y_{\alpha\mathbf{j}}(H^\alpha(z_1)Y_\alpha)_{\mathbf{q}}.$$

Hence,

$$z_1 f_n^{(1)}(z_1, z_2) = -f_n(z_2) + n^{-3}\sum_{\mathbf{j},\mathbf{p}}\sum_\alpha \tau_\alpha \mathbf{E}\left\{\frac{Y_{\alpha\mathbf{j}}(H^\alpha(z_1)Y_\alpha)_{j_1 p_2}}{A_\alpha(z_1)} H_{\mathbf{p},\,p_1 j_2}^\alpha(z_2)\right\}$$

$$- n^{-3}\sum_{\mathbf{j},\mathbf{p}}\sum_\alpha \tau_\alpha^2 \mathbf{E}\left\{\frac{Y_{\alpha\mathbf{j}}(H^\alpha(z_1)Y_\alpha)_{j_1 p_2}}{A_\alpha(z_1)} \cdot \frac{(H^\alpha(z_2)Y_\alpha)_{\mathbf{p}}(H^\alpha(z_2)Y_\alpha)_{p_1 j_2}}{A_\alpha(z_2)}\right\}$$

$$= -f_n(z_2) + T_n^{(1)} + T_n^{(2)}.$$

By the Hölder inequality, (4.8), and (7.13)

$$|T_n^{(2)}| \leq n^{-3}\sum_\alpha \tau_\alpha^2 \mathbf{E}\left\{\frac{||Y_\alpha|| \cdot ||H^\alpha(z_1)Y_\alpha||}{|A_\alpha(z_1)|} \cdot \frac{||H^\alpha(z_2)Y_\alpha|| \cdot ||H^\alpha(z_2)Y_\alpha||}{|A_\alpha(z_2)|}\right\}$$

$$\leq Cn^{-3}\sum_\alpha \tau_\alpha^2 \mathbf{E}\{||Y_\alpha||^4|A_\alpha(z_1)|^{-1}|A_\alpha(z_2)|^{-1}\} = O(n^{-1}).$$

It follows from (1.2) that

$$\mathbf{E}_\alpha\{Y_{\alpha\mathbf{j}}(H^\alpha Y_\alpha)_{j_1 p_2}\} = n^{-2}H_{\mathbf{j},\,j_1 p_2}^\alpha.$$

This and (4.12) yield

$$T_n^{(1)} = n^{-5}\sum_{\mathbf{j},\mathbf{p}}\sum_\alpha \tau_\alpha \frac{\mathbf{E}\{H_{\mathbf{j},\,j_1 p_2}^\alpha H_{\mathbf{p},\,p_1 j_2}^\alpha(z_2)\}}{1 + \tau_\alpha f_n^\alpha(z_1)} + r_n,$$

$$r_n = n^{-3}\sum_{\mathbf{j},\mathbf{p}}\sum_\alpha \frac{\tau_\alpha}{\mathbf{E}\{A_\alpha(z_1)\}}\mathbf{E}\left\{A_\alpha^\circ(z_1)\frac{Y_{\alpha\mathbf{j}}(H^\alpha(z_1)Y_\alpha)_{j_1 p_2}}{A_\alpha(z_1)} H_{\mathbf{p},\,p_1 j_2}^\alpha(z_2)\right\}.$$

Treating r_n we note that

$$n^{-1}\sum_{\mathbf{j},p_2}\left|Y_{\alpha\mathbf{j}}(H^\alpha Y_\alpha)_{j_1 p_2}\mathcal{G}_{p_2,\,j_2}^\alpha\right| \leq n^{-1}||\mathcal{G}^\alpha|| \cdot ||Y_\alpha|| \cdot ||H^\alpha Y_\alpha|| \leq C||Y_\alpha||^2.$$

Hence, by the Schwarz inequality, (4.8), (4.9), (7.2), and (7.13)

$$|r_n| \leq Cn^{-2} \sum_\alpha \mathbf{E}\{|A_\alpha^\circ| \cdot |A_\alpha|^{-1} ||Y_\alpha||^2\}$$

$$\leq Cn^{-2} \sum_\alpha \mathbf{E}\{|A_\alpha^\circ|^2\}^{1/2} \mathbf{E}\{|A_\alpha|^{-2} ||Y_\alpha||^4\}^{1/2} = O(n^{-1/2}).$$

Also one can replace f_n^α and H^α with f_n and G (the error term is of the order $O(n^{-1})$). Hence,

$$z_1 f_n^{(1)}(z_1, z_2) = -f_n(z_2) + f_n^{(1)}(z_1, z_2) n^{-2} \sum_\alpha \frac{\tau_\alpha}{1 + \tau_\alpha f_n(z_1)} + o(1).$$

This, (1.4), (1.7), and (1.10) lead to

$$f^{(1)}(z_1, z_2) = f(z_2) \left(c \int \frac{\tau \, d\sigma(\tau)}{1 + \tau f(z_1)} - z_1 \right)^{-1} = f(z_1) f(z_2)$$

and finishes the proof of the lemma. $\qquad\qquad\square$

It follows from Lemmas 5.1 and 7.2 that under conditions of Theorem 1.9

$$\lim_{n\to\infty} n\tau_\alpha^{-2} \mathbf{E}\{A_\alpha^\circ(z_1) A_\alpha(z_2)\} = 2(a + b + 2) f(z_1) f(z_2), \qquad (7.14)$$

where f is the solution of (1.10).

Lemma 7.3 *Under conditions of Theorem* 1.9

$$\mathbf{Var}\{\mathbf{E}_\alpha\{(A_\alpha^\circ)^p\}\} = O(n^{-4}), \quad p = 2, 3. \qquad (7.15)$$

Proof Since $\tau_\alpha, \alpha \in [m]$, are uniformly bounded in α and n, then to get the desired bounds it is enough to consider the case $\tau_\alpha = 1, \alpha \in [m]$. By (4.13), we have

$$\mathbf{E}_\alpha\{(A_\alpha^\circ)^2\} = \mathbf{E}_\alpha\{(HY_\alpha, Y_\alpha)_\alpha^{\circ 2}\} + (g_n^\alpha)^{\circ 2},$$
$$\mathbf{E}_\alpha\{(A_\alpha^\circ)^3\} = \mathbf{E}_\alpha\{(HY_\alpha, Y_\alpha)_\alpha^{\circ 3}\} + 3\mathbf{E}_\alpha\{(HY_\alpha, Y_\alpha)_\alpha^{\circ 2}\} g_n^{\alpha\circ} + (g_n^\alpha)^{\circ 3},$$

where by (6.2) $\mathbf{E}\{|(g_n^\alpha)^\circ|^{2p}\} = O(n^{-6}), p = 2, 3$, and by (7.1) and (6.1)

$$\mathbf{E}\{|\mathbf{E}_\alpha\{(HY_\alpha, Y_\alpha)_\alpha^{\circ 2}\}(g_n^\alpha)^\circ|^2\} = O(n^{-2})\mathbf{E}\{|(g_n^\alpha)^\circ|^2\} = O(n^{-5}).$$

Hence,

$$\mathbf{Var}\{\mathbf{E}_\alpha\{(A_\alpha^\circ)^p\}\} \leq 2\mathbf{Var}\{\mathbf{E}_\alpha\{(HY_\alpha, Y_\alpha)_\alpha^{\circ p}\}\} + O(n^{-4}), \quad p = 2, 3.$$

It also follows from (7.6) and Lemmas 6.1 and 7.2 that

$$\mathbf{Var}\{\mathbf{E}_\alpha\{(HY_\alpha, Y_\alpha)_\alpha^{\circ 2}\}\} = O(n^{-4}), \tag{7.16}$$

which leads to (7.15) for $p = 2$. To get (7.15) for $p = 3$, it is enough to show that

$$\mathbf{Var}\{\mathbf{E}_\alpha\{(HY_\alpha, Y_\alpha)_\alpha^{\circ 3}\}\} = O(n^{-4}). \tag{7.17}$$

We have

$$
\begin{aligned}
\mathbf{E}_\alpha\{(HY_\alpha, Y_\alpha)_\alpha^{\circ 3}\} =& \mathbf{E}_\alpha\{(HY_\alpha, Y_\alpha)^3\} - \mathbf{E}_\alpha\{(HY_\alpha, Y_\alpha)\}^3 \\
& - 3\mathbf{E}_\alpha\{(HY_\alpha, Y_\alpha)\} \cdot \mathbf{E}_\alpha\{(HY_\alpha, Y_\alpha)_\alpha^{\circ 2}\} \\
=& \mathbf{E}_\alpha\{(HY_\alpha, Y_\alpha)^3\} - g_n^{\alpha 3} - 3g_n^\alpha \cdot \mathbf{E}_\alpha\{(HY_\alpha, Y_\alpha)_\alpha^{\circ 2}\}.
\end{aligned}
$$

It follows from (6.1), (7.16), and (3.8) with $h_1 = g_n^\alpha$, $h_2 = n\mathbf{E}_\alpha\{(HY_\alpha, Y_\alpha)_\alpha^{\circ 2}\}$ that

$$\mathbf{Var}\{g_n^\alpha \cdot n\mathbf{E}_\alpha\{(HY_\alpha, Y_\alpha)_\alpha^{\circ 2}\}\} \leq C\big(\mathbf{Var}\{g_n^\alpha\} + \mathbf{Var}\{n\mathbf{E}_\alpha\{(HY_\alpha, Y_\alpha)_\alpha^{\circ 2}\}\}\big) = O(n^{-2}).$$

Hence,

$$\mathbf{Var}\{\mathbf{E}_\alpha\{(HY_\alpha, Y_\alpha)_\alpha^{\circ 3}\}\} \leq 2\mathbf{Var}\Big\{\mathbf{E}_\alpha\{(HY_\alpha, Y_\alpha)^3\} - g_n^{\alpha 3}\Big\} + O(n^{-4}),$$

and to get (7.17) for $p = 3$ it is enough to show that

$$\mathbf{Var}\Big\{\mathbf{E}_\alpha\{(HY_\alpha, Y_\alpha)^3\} - g_n^{\alpha 3}\Big\} = O(n^{-4}). \tag{7.18}$$

We have

$$\mathbf{E}_\alpha\{(HY_\alpha, Y_\alpha)^3\} = \sum_{\mathbf{i,j,p,q,s,t}} H_{\mathbf{i,j}} H_{\mathbf{p,q}} H_{\mathbf{s,t}} \Lambda(\mathbf{i, j, p, q, s, t}), \tag{7.19}$$

where

$$\Lambda(\mathbf{i, j, p, q, s, t}) = \prod_{k=1}^2 \mathbf{E}_\alpha\{(y_\alpha^{(k)})_{i_k}(y_\alpha^{(k)})_{j_k}(y_\alpha^{(k)})_{p_k}(y_\alpha^{(k)})_{q_k}(y_\alpha^{(k)})_{s_k}(y_\alpha^{(k)})_{t_k}\}$$

and by (1.21)

$$\Lambda(\mathbf{i, j, p, q, s, t}) = O(n^{-6}). \tag{7.20}$$

Also, due to the unconditionality of the distribution, Λ contains only even moments. Thus in the index pairs $\mathbf{i, j, p, q, s, t} \in [n]^2$, every index (both on the first positions and on the second positions) is repeated an even number of times. Hence, there are at most 6 independent indices: ≤ 3 on the first positions (call them i, j, k) and ≤ 3 on the second

positions (call them u, v, w). For every fixed set of independent indices, consider maps Φ from this set to the sets of index pairs $\{\mathbf{i}, \mathbf{j}, \mathbf{p}, \mathbf{q}, \mathbf{s}, \mathbf{t}\}$. We call such maps *the index schemes*. Let $|\Phi|$ be the cardinality of the corresponding set of independent indices. For example,

$$\Phi \; : \; \{i, j \,; u, v, w\} \rightarrow \{(i, u), (i, v);\ (i, w), (i, u);\ (j, w), (j, v)\}, \quad |\Phi| = 5,$$

is an index scheme with 5 independent indices (i, j on the first positions and u, v, w on the second positions). The inclusion–exclusion principle allows to split the expression (7.19) into the sums over fixed sets of independent indices of cardinalities from 2 to 6 with the fixed coefficients depending on $a_{2,2,2}$, $a_{2,4}$, and a_6 in front of every such sum. We have

$$\mathbf{E}_\alpha\{(HY_\alpha, Y_\alpha)^3\} = \sum_{\ell=2}^{6} S_\ell, \quad S_\ell = \sum_{\Phi : |\Phi|=\ell} \sum H_{\mathbf{i},\mathbf{j}} H_{\mathbf{p},\mathbf{q}} H_{\mathbf{s},\mathbf{t}} \Lambda'(\Phi), \qquad (7.21)$$

where the last sum is taken over the set of independent indices of cardinality ℓ, Φ is an index scheme constructing pairs $\{\mathbf{i}, \mathbf{j}, \mathbf{p}, \mathbf{q}, \mathbf{s}, \mathbf{t}\}$ from this set, and $\Lambda'(\Phi)$ is a certain expression, depending on Φ, $a_{2,2,2}$, $a_{2,4}$, and a_6. For example,

$$S_2 = F(a_{2,2,2}, a_{2,4}, a_6) \sum_{i,u} (H_{iu, iu})^3,$$

where $F(a_{2,2,2}, a_{2,4}, a_6)$ can be found by using the inclusion–exclusion formulas. As to $\Lambda'(\Phi)$ in (7.21), the only thing we need to know is that

$$\Lambda'(\Phi) = O(n^{-6}), \qquad (7.22)$$

and that in the particular case of

$$\Phi_{\mathrm{Tr}} \; : \; \{i, j, k \,; u, v, w\} \rightarrow \{(i, u), (i, u);\ (j, v), (j, v);\ (k, w), (k, w)\},$$

we have by (1.21)

$$\Lambda'(\Phi) = a_{2,2,2}^2 = n^{-6} + O(n^{-7}),$$

and the corresponding term in S_6 has the form $a_{2,2,2}^2 (\operatorname{Tr} H)^3$.

Note that by (7.20), S_2 is of the order $O(n^{-4})$. By the same reason

$$\left| \sum_{\ell=2}^{4} S_\ell \right| = O(n^{-2})$$

so that

$$\mathbf{Var}\left\{\left\|\sum_{\ell=2}^{4} S_\ell\right\|\right\} = O(n^{-4}).$$

Hence to get (7.18) it suffices to consider terms with 5 and 6 independent indices and show that

$$\mathbf{Var}\{S_5\}, \ \ \mathbf{Var}\{S_6 - g_n^{\alpha 3}\} = O(n^{-4}). \tag{7.23}$$

Consider S_5. In this case we have exactly 5 independent indices. By the symmetry we can suppose that there are two first independent indices, i, j, and three second independent indices, u, v, w, and that we have i on four places and j on two places. Thus, S_5 is equal to the sum of terms of the form

$$S_5' = O(n^{-6}) \sum_{i,j,u,v,w} H_{i\cdot,\,i\cdot} H_{i\cdot,\,j\cdot} H_{i\cdot,\,j\cdot} \quad \text{or}$$

$$S_5'' = O(n^{-6}) \sum_{i,j,u,v,w} H_{i\cdot,\,i\cdot} H_{i\cdot,\,i\cdot} H_{j\cdot,\,j\cdot}.$$

Here we suppose that there are some fixed indices on the dot places, which are different from explicitly mentioned ones. Note that S_5' has a single "external" pairing with respect to j. While estimating the terms, our argument is essentially based on the simple relations

$$\sum_{j,v} |H_{iu,\,jv}|^2 = O(1), \quad |H_{iu,\,jv}| = O(1), \quad \|H\| = O(1), \tag{7.24}$$

and on the observation that the more the mixing of matrix entries we have the lower order of sums we get. Let $V \subset \mathbb{R}^n$ be the set of vectors of the form

$$\xi = \{\xi_j\}_{j=1}^n = \{H_{\cdot\cdot,\,j\cdot}\}_{j=1}^n \quad \text{or} \quad \xi = \{H_{\cdot\cdot,\,\cdot u}\}_{u=1}^n,$$

and let W be the set of $n \times n$ matrices of the form

$$D = \{H_{i\cdot,\,j\cdot}\}_{i,j=1}^n, \quad \text{or} \quad D = \{H_{i\cdot,\,\cdot u}\}_{i,u=1}^n, \quad \text{or} \quad D = \{H_{\cdot u,\,\cdot v}\}_{u,v=1}^n.$$

It follows from (7.24) that

$$\forall \xi \in V \ \ \|\xi\| = O(1) \quad \text{and} \quad \forall D \in W \ \ \|D\| = O(1).$$

Hence,

$$\sum_j |H_{\cdot\cdot,\,j}.H_{\cdot\cdot,\,j}.| = O(1), \quad \sum_u |H_{\cdot\cdot,\,\cdot u}H_{\cdot\cdot,\,\cdot u}| = O(1), \tag{7.25}$$

$$\sum_{i,j} H_{i\cdot,\,j}.H_{i\cdot,\,\cdot\cdot}H_{\cdot\cdot,\,j}. = O(1), \quad \text{and} \quad \sum_{i,u} H_{i\cdot,\,\cdot u}H_{i\cdot,\,\cdot\cdot}H_{\cdot\cdot,\,\cdot u} = O(1). \tag{7.26}$$

In particular, by (7.24) and (7.25), we have for S_5'

$$|S_5'| \leq O(n^{-6}) \sum_{i,u,v,w} \sum_j |H_{i\cdot,\,j}.H_{i\cdot,\,j}.| = O(n^{-2}),$$

so that $\mathbf{Var}\{S_5'\} = O(n^{-4})$. Consider S_5''. Note that if in S_5'' we have a single "external" pairing with respect to at least one index on the second positions, then similar to S_5', the variance of this term is of the order $O(n^{-4})$. So we are left with the terms of the form

$$S_5''' = O(n^{-6}) \sum_{i,j,u,v,w} H_{iu,\,iu}H_{iv,\,iv}H_{jw,\,jw}.$$

It follows from (7.5) that

$$S_5''' = O(n^{-1}) \cdot g_n^\alpha(z) \cdot \widetilde{g}_n^{(2)}(z,z).$$

Now (3.8), (6.1), and (7.7) imply that

$$\mathbf{Var}\{S_5'''\} \leq Cn^{-2}(\mathbf{Var}\{g_n^\alpha\} + \mathbf{Var}\{\widetilde{g}_n^{(2)}\}) = O(n^{-4}).$$

Summarizing we get $\mathbf{Var}\{S_5\} = O(n^{-4})$.

Consider S_6 and show that $\mathbf{Var}\{S_6 - g_n^{\alpha 3}\} = O(n^{-4})$. In this case we have 6 independent indices, $i,\,j,\,k$ for the first positions and $u,\,v,\,w$ for the second positions. Suppose that we have two single external pairing with respect to two different first indices and consider terms of the form

$$S_6' = O(n^{-6}) \sum_{i,j,k,u,v,w} H_{i\cdot,\,j}.H_{i\cdot,\,\cdot\cdot}H_{\cdot\cdot,\,j}.,$$

$$S_6'' = O(n^{-6}) \sum_{i,j,k,u,v,w} H_{i\cdot,\,j}.H_{i\cdot,\,j}.H_{\cdot\cdot,\,\cdot\cdot}.$$

It follows from (7.26) that $S_6' = O(n^{-2})$; hence $\mathbf{Var}\{S_6'\} = O(n^{-4})$. Consider S_6''

$$S_6'' = O(n^{-6}) \sum_{i,j,k,u,v,w} H_{i\cdot,\,j}.H_{i\cdot,\,j}.H_{k\cdot,\,k}.. \tag{7.27}$$

If the second indices in $H_{k\cdot,\,k}$ are not equal, then we get the expression of the form

$$S_6''' = O(n^{-6}) \sum_{i,j,k,u,v,w} H_{i\cdot,\,ju} H_{i\cdot,\,j} H_{k\cdot,\,ku}.$$

It follows from (7.26) that $S_6''' = O(n^{-2})$; hence $\mathbf{Var}\{S_6'''\} = O(n^{-4})$. If the second indices in $H_{k\cdot,\,k}$ in (7.27) are equal, then we get the expressions of three types:

$$O(n^{-6}) \sum_{i,j,k,u,v,w} H_{iu,\,jv} H_{iu,\,jv} H_{kw,\,kw} = g_n^\alpha n^{-4} \sum_{i,j,u,v} (H_{iu,\,jv})^2 = O(n^{-2}),$$

$$O(n^{-6}) \sum_{i,j,k,u,v,w} H_{iu,\,jv} H_{iv,\,ju} H_{kw,\,kw} = g_n^\alpha n^{-4} \sum_{i,j,u,v} H_{iu,\,jv} H_{iv,\,ju} = O(n^{-2}),$$

$$O(n^{-6}) \sum_{i,j,k,u,v,w} H_{iu,\,ju} H_{iv,\,jv} H_{kw,\,kw} = O(n^{-1}) g_n^{(1)}(z,z) g_n^\alpha(z),$$

where we used (7.24) to estimate the first two expressions, so that their variances are of the order $O(n^{-4})$. It also follows from (3.8), (6.1), and (7.7) that the variance of the third expression is of the order $O(n^{-4})$. Hence, $\mathbf{Var}\{S_6'''\} = O(n^{-4})$. It remains to consider the term without external pairing, which corresponds to

$$(a_{2,2,2})^2 \sum_{i,j,k,u,v,w} H_{iu,\,iu} H_{jv,\,jv} H_{kw,\,kw} = (a_{2,2,2})^2 \gamma_n^3$$

(see (7.22)). Summarizing we get

$$\begin{aligned} \mathbf{Var}\{S_6 - g_n^{\alpha 3}\} &\le 2\mathbf{Var}\{((a_{2,2,2})^2 - n^{-6})\gamma_n^{\alpha 3}\} + O(n^{-4}) \\ &= O(n^{-2})\mathbf{Var}\{g_n^{\alpha 3}\} + O(n^{-4}) = O(n^{-4}), \end{aligned}$$

where we used (1.21) and (6.1). This leads to (7.23) and completes the proof of the lemma. $\qquad\square$

8 Covariance of the Resolvent Traces

Lemma 8.1 *Suppose that the conditions of Theorem* 1.9 *are fulfilled. Let*

$$C_n(z_1, z_2) := n^{-1} \mathbf{Cov}\{\gamma_n(z_1),\ \gamma_n(z_2)\} = n^{-1}\mathbf{E}\{\gamma_n(z_1)\gamma_n^\circ(z_2)\}.$$

Then $\{C_n(z_1, z_2)\}_n$ *converges uniformly in* $z_{1,2} \in K$ *to*

$$C(z_1, z_2) = 2(a + b + 2)c \int \frac{f'(z_1)}{(1 + \tau f(z_1))^2} \frac{f'(z_2)}{(1 + \tau f(z_2))^2} \tau^2 d\sigma(\tau). \qquad (8.1)$$

Proof For a convergent subsequence $\{C_{n_i}\}$, denote

$$C(z_1, z_2) := \lim_{n_i \to \infty} C_{n_i}(z_1, z_2).$$

We will show that for every converging subsequence, its limit satisfies (8.1). Applying the resolvent identity, we get (see (4.11))

$$C_n(z_1, z_2) = -\frac{1}{nz_1} \sum_\alpha \mathbf{E}\{A_\alpha^{-1}(z_1)\gamma_n^\circ(z_2)\} = -\frac{1}{nz_1} \sum_\alpha \mathbf{E}\{A_\alpha^{-1}(z_1)\gamma_n^{\alpha\circ}(z_2)\}$$

$$-\frac{1}{nz_1} \sum_\alpha \mathbf{E}\{A_\alpha^{-1}(z_1)(\gamma_n - \gamma_n^\alpha)^\circ(z_2)\} =: T_n^{(1)} + T_n^{(2)}. \qquad (8.2)$$

Consider $T_n^{(1)}$. Iterating (4.12) four times, we get

$$T_n^{(1)} = \frac{1}{nz_1} \sum_\alpha \left[\frac{\mathbf{E}\{A_\alpha(z_1)\gamma_n^{\alpha\circ}(z_2)\}}{\mathbf{E}\{A_\alpha(z_1)\}^2} - \frac{\mathbf{E}\{A_\alpha^{\circ 2}(z_1)\gamma_n^{\alpha\circ}(z_2)\}}{\mathbf{E}\{A_\alpha(z_1)\}^3} + \frac{\mathbf{E}\{A_\alpha^{\circ 3}(z_1)\gamma_n^{\alpha\circ}(z_2)\}}{\mathbf{E}\{A_\alpha(z_1)\}^4} \right.$$

$$\left. -\frac{\mathbf{E}\{A_\alpha^{-1}(z_1)A_\alpha^{\circ 4}(z_1)\gamma_n^{\alpha\circ}(z_2)\}}{\mathbf{E}\{A_\alpha(z_1)\}^4} \right] =: S_n^{(1)} + S_n^{(2)} + S_n^{(3)} + S_n^{(4)}.$$

It follows from (4.9), (6.1), and (7.15) that $S_n^{(i)} = O(n^{-1/2})$, $i = 2, 3$. Also, by (4.8) we have

$$\mathbf{E}\{|A_\alpha^{-1} A_\alpha^{\circ 4} \gamma_n^{\alpha\circ}|\} \leq \mathbf{E}\{(1 + |\tau_\alpha|\|Y_\alpha\|^2/|\Im z|)|A_\alpha^{\circ 4}\gamma_n^{\alpha\circ}|\},$$

where by the Schwarz inequality, (6.2), (7.1), and (7.13)

$$\mathbf{E}\{\|Y_\alpha\|^2 |A_\alpha^{\circ 4}\gamma_n^{\alpha\circ}|\} \leq \mathbf{E}\{|A_\alpha^\circ|^6\}^{1/2}\mathbf{E}\{|A_\alpha^\circ|^4\}^{1/4}\mathbf{E}\{\mathbf{E}_\alpha\{\|Y_\alpha\|^8\}|\gamma_n^{\alpha\circ}|^4\}^{1/4} = O(n^{-3/2}).$$

Hence $S_n^{(4)} = O(n^{-1/2})$, and we are left with $S_n^{(1)}$. We have

$$\mathbf{E}\{A_\alpha(z_1)\gamma_n^{\alpha\circ}(z_2)\} = \mathbf{E}\{\mathbf{E}_\alpha\{A_\alpha(z_1)\}\gamma_n^{\alpha\circ}(z_2)\} = n^{-2}\tau_\alpha \mathbf{E}\{\gamma_n^{\alpha\circ}(z_1)\gamma_n^{\alpha\circ}(z_2)\}.$$

It follows from (4.5) and (4.7) that $|\gamma_n(z) - \gamma_n^\alpha(z)| \leq 1/|\Im z|$. This and (6.1) yield

$$|\mathbf{E}\{(\gamma_n^\alpha - \gamma_n)^\circ(z_1)\gamma_n^{\alpha\circ}(z_2)\} + \mathbf{E}\{(\gamma_n^\circ(z_1)(\gamma_n^\alpha - \gamma_n)^\circ(z_2)\}|$$

$$\leq \mathbf{E}\{|\gamma_n^{\alpha\circ}(z_2)|\}/|z_1| + \mathbf{E}\{|\gamma_n^\circ(z_1)|\}/|z_2| = O(n^{1/2}).$$

Hence,

$$\mathbf{E}\{A_\alpha(z_1)\gamma_n^{\alpha\circ}(z_2)\} = n^{-1}\tau_\alpha C_n(z_1, z_2) + O(n^{-3/2}),$$

and we have

$$S_n^{(1)} = C_n(z_1, z_2) \frac{1}{n^2 z_1} \sum_\alpha \frac{\tau_\alpha}{(1 + \tau_\alpha f_n^\alpha(z_1))^2} + O(n^{-1/2}).$$

Summarizing, we get

$$T_n^{(1)} = C(z_1, z_2) \frac{c}{z_1} \int \frac{\tau \, d\sigma(\tau)}{(1 + \tau f(z_1))^2} + o(1). \tag{8.3}$$

Consider now $T_n^{(2)}$ of (8.2). By (4.5),

$$T_n^{(2)} = \frac{1}{nz_1} \sum_\alpha \mathbf{E}\{A_\alpha^{-1}(z_1)(B_\alpha/A_\alpha)^\circ(z_2)\}. \tag{8.4}$$

For shortness let for the moment $A_i = A_\alpha(z_i)$, $i = 1, 2$, $B_2 = B_\alpha(z_2)$. Iterating (4.12) with respect to A_1 and A_2 two times we get

$$\mathbf{E}\{(1/A_1)^\circ(B_2/A_2)^\circ\}$$
$$= \frac{\mathbf{E}\{(-A_1^\circ + A_1^{-1}A_1^{\circ 2})(B_2\mathbf{E}\{A_2\} - B_2 A_2^\circ + B_2 A_2^{-1}A_2^{\circ 2})^\circ\}}{\mathbf{E}\{A_1\}^2 \mathbf{E}\{A_2\}^2}$$
$$= \frac{-\mathbf{E}\{A_1^\circ B_2\}\mathbf{E}\{A_2\} + \mathbf{E}\{B_2\}\mathbf{E}\{A_1^\circ A_2\}}{\mathbf{E}\{A_1\}^2 \mathbf{E}\{A_2\}^2}$$
$$+ \frac{\mathbf{E}\{A_1^\circ B_2^\circ A_2^\circ - A_1^\circ B_2 A_2^{-1}A_2^{\circ 2} + A_1^{-1}A_1^{\circ 2}(B_2\mathbf{E}\{A_2\} - B_2 A_2^\circ + B_2 A_2^{-1}A_2^{\circ 2})^\circ\}}{\mathbf{E}\{A_1\}^2 \mathbf{E}\{A_2\}^2}.$$

Applying (1.22), (7.13), and using bounds (4.7), (4.8), (4.9) for $|B_2/A_2|$, $|A_i|^{-1}$, $|\mathbf{E}\{A_i\}|^{-1}$, $i = 1, 2$, one can show that the terms containing at least three centered factors A_1°, A_2°, B_2° are of the order $O(n^{-3/2})$. This implies that

$$\mathbf{E}\{(1/A_1)^\circ(B_2/A_2)^\circ\} = \frac{-\mathbf{E}\{A_1^\circ B_2\}\mathbf{E}\{A_2\} + \mathbf{E}\{B_2\}\mathbf{E}\{A_1^\circ A_2\}}{\mathbf{E}\{A_1\}^2 \mathbf{E}\{A_2\}^2} + O(n^{-3/2}).$$

Returning to the original notations and taking into account that

$$B_\alpha(z) = \partial A_\alpha(z)/\partial z,$$

we get

$$\mathbf{E}\{A_\alpha^{-1}(z_1)(B_\alpha/A_\alpha)^\circ(z_2)\} = -\frac{1}{\mathbf{E}\{A_\alpha(z_1)\}^2} \frac{\partial}{\partial z_2} \frac{\mathbf{E}\{A_\alpha^\circ(z_1)A_\alpha^\circ(z_2)\}}{\mathbf{E}\{A_\alpha(z_2)\}} + O(n^{-3/2}). \tag{8.5}$$

Denote for the moment

$$D = 2(a + b + 2).$$

It follows from (7.14) and (8.4–8.5) that

$$T_n^{(2)} = -\frac{Dc}{z_1} \int \frac{\tau^2 f(z_1)}{(1 + \tau f(z_1))^2} \frac{\partial}{\partial z_2} \frac{f(z_2)}{1 + \tau f(z_2)} d\sigma(\tau) + o(1).$$

This and (8.2–8.3) yield

$$C(z_1, z_2) = \frac{Dc}{c \int \tau(1 + \tau f(z_1))^{-2} d\sigma(\tau) - z_1} \int \frac{\tau^2 f(z_1)}{(1 + \tau f(z_1))^2} \frac{\partial}{\partial z_2} \frac{f(z_2)}{1 + \tau f(z_2)} d\sigma(\tau).$$

Note that by (1.10),

$$c \int \frac{\tau d\sigma(\tau)}{(1 + \tau f(z))^2} - z = \frac{f(z)}{f'(z)}.$$

Hence

$$C(z_1, z_2) = Dc \int \frac{f'(z_1)}{(1 + \tau f(z_1))^2} \frac{f'(z_2)}{(1 + \tau f(z_2))^2} \tau^2 d\sigma(\tau).$$

which completes the proof of the lemma.   □

9 Proof of Theorem 1.9

The proof essentially repeats the proofs of Theorem 1 of [25] and Theorem 1.8 of [13]; the technical details are provided by the calculations of the proof of Lemma 8.1. It suffices to show that if

$$Z_n(x) = \mathbf{E}\{e_n(x)\}, \quad e_n(x) = e^{ix\mathcal{N}_n^\circ[\varphi]/\sqrt{n}}, \tag{9.1}$$

then we have uniformly in $|x| \leq C$

$$\lim_{n \to \infty} Z_n(x) = \exp\{-x^2 V[\varphi]/2\}$$

with $V[\varphi]$ of (1.23). Define for every test functions $\varphi \in \mathcal{H}_s, s > 5/2$,

$$\varphi_\eta = P_\eta * \varphi, \tag{9.2}$$

where P_η is the Poisson kernel

$$P_\eta(x) = \frac{\eta}{\pi(x^2 + \eta^2)}, \tag{9.3}$$

and "$*$" denotes the convolution. We have

$$\lim_{\eta \downarrow 0} ||\varphi - \varphi_\eta||_s = 0. \tag{9.4}$$

Denote for the moment the characteristic function (9.1) by $Z_n[\varphi]$, to make explicit its dependence on the test function. Take any converging subsequence $\{Z_{n_j}[\varphi]\}_{j=1}^\infty$ Without loss of generality assume that the whole sequence $\{Z_{n_j}[\varphi_\eta]\}$ converges as $n_j \to \infty$. By (1.20), we have

$$|Z_{n_j}[\varphi] - Z_{n_j}[\varphi_\eta]| \le |x| n^{-1/2} \big(\mathbf{Var}\{\mathcal{N}_{n_j}[\varphi] - \mathcal{N}_{n_j}[\varphi_\eta]\} \big)^{1/2} \le C|x| ||\varphi - \varphi_\eta||_s,$$

hence

$$\lim_{\eta \downarrow 0} \lim_{n_j \to \infty} (Z_{n_j}[\varphi] - Z_{n_j}[\varphi_\eta]) = 0.$$

This and the equality $Z_{n_j}[\varphi] = (Z_{n_j}[\varphi] - Z_{n_j}[\varphi_\eta]) + Z_{n_j}[\varphi_\eta]$ imply that

$$\exists \lim_{\eta \downarrow 0} \lim_{n_j \to \infty} Z_{n_j}[\varphi_\eta] \quad \text{and} \quad \lim_{n_j \to \infty} Z_{n_j}[\varphi] = \lim_{\eta \downarrow 0} \lim_{n_j \to \infty} Z_{n_j}[\varphi_\eta]. \tag{9.5}$$

Thus it suffices to find the limit of

$$Z_{\eta n}(x) := Z_n[\varphi_\eta] = \mathbf{E}\{e_{\eta n}(x)\}, \quad \text{where} \quad e_{\eta n}(x) = e^{ix\mathcal{N}_n^\circ[\varphi_\eta]/\sqrt{n}},$$

as $n \to \infty$. It follows from (9.2) – (9.3) that

$$\mathcal{N}_n[\varphi_\eta] = \frac{1}{\pi} \int \varphi(\mu) \Im \gamma_n(z) d\mu, \quad z = \mu + i\eta. \tag{9.6}$$

This allows to write

$$\frac{d}{dx} Z_{\eta n}(x) = \frac{1}{2\pi} \int \varphi(\mu)(\mathcal{Y}_n(z, x) - \mathcal{Y}_n(\bar{z}, x)) d\mu, \tag{9.7}$$

where

$$\mathcal{Y}_n(z, x) = n^{-1/2} \mathbf{E}\{\gamma_n(z) e_{\eta n}^\circ(x)\}.$$

Since $|\mathcal{Y}_n(z, x)| \le 2n^{-1/2} \mathbf{Var}\{\gamma_n(z)\}^{1/2}$, it follows from the proof of Lemma 1.6 that for every $\eta > 0$ the integrals of $|\mathcal{Y}_n(z, x)|$ over μ are uniformly bounded in n. This and the fact that $\varphi \in L^2$ together with Lemma 9.1 below show that to find the limit of integrals in (9.7) it is enough to find the pointwise limit of $\mathcal{Y}_n(\mu + i\eta, x)$. We have

$$\mathcal{Y}_n(z, x) = -\frac{1}{zn^{1/2}} \sum_{\alpha=1}^m \big[\mathbf{E}\{A_\alpha^{-1}(z) e_{\eta n}^{\alpha\circ}(x)\} - \mathbf{E}\{A_\alpha^{-1}(z)(e_{\eta n}^\circ(x) - e_{\eta n}^{\alpha\circ}(x))\} \big],$$

where $e_{\eta n}^{\alpha}(x) = \exp\{ix\mathcal{N}_n^{\alpha\circ}[\varphi_\eta]/\sqrt{n}\}$ and $\mathcal{N}_n^{\alpha}[\varphi_\eta] = \mathrm{Tr}\,\varphi_\eta(M^\alpha)$. By (9.6),

$$
e_{\eta n} - e_{\eta n}^{\alpha} = \frac{ixe_{\eta n}^{\alpha}}{\sqrt{n\pi}}\int\varphi(\lambda_1)\Im(\gamma_n - \gamma_n^\alpha)^\circ(z_1)\mathrm{d}\lambda_1
$$
$$
+ O\left(\left|\frac{1}{\sqrt{n}}\int\varphi(\lambda_1)\Im(\gamma_n - \gamma_n^\alpha)^\circ(z_1)\mathrm{d}\lambda_1\right|^2\right),
$$

so that

$$
\mathbf{E}\{A_{\alpha n}^{-1}(z)(e_{\eta n} - e_{\eta n}^\alpha)^\circ(x)\} = \frac{ixe_{\eta n}^{\alpha}}{\sqrt{n\pi}}\int\varphi(\lambda_1)\Im(\gamma_n - \gamma_n^\alpha)^\circ(z_1)\mathrm{d}\lambda_1
$$
$$
+ \int\int O(R_n)\varphi(\lambda_1)\varphi(\lambda_2)\mathrm{d}\lambda_1\mathrm{d}\lambda_2,
$$

where $z_j = \lambda_j + i\eta$, $j = 1, 2$, and

$$
R_n = n^{-1}\mathbf{E}\{(A_{\alpha n}^{-1})^\circ(z)\Im(B_{\alpha n}A_{\alpha n}^{-1})^\circ(z_1)\Im(B_{\alpha n}A_{\alpha n}^{-1})^\circ(z_2)\}.
$$

Using the argument of the proof of the Lemma 8.1, it can be shown that $R_n = O(n^{-5/2})$. Hence,

$$
\mathcal{Y}_n(z, x) = -\frac{1}{zn^{1/2}}\sum_{\alpha=1}^m\mathbf{E}\{A_\alpha^{-1}(z)e_{\eta n}^{\alpha\circ}(x)\}
$$
$$
-\frac{ix}{zn\pi}\int\varphi(\lambda_1)\sum_{\alpha=1}^m\mathbf{E}\{e_{\eta n}^\alpha(x)(A_\alpha^{-1}(z))^\circ\Im(\gamma_n - \gamma_n^\alpha)^\circ(z_1)\}\mathrm{d}\lambda_1 + O(n^{-1}).
$$

Treating the r.h.s. similarly to $T_n^{(1)}$ and $T_n^{(2)}$ of (8.2), we get

$$
\mathcal{Y}_n(z, x) = \frac{xZ_{\eta n}(x)}{2\pi}\int\varphi(\lambda_1)\left[C(z, z_1) - C(z, \overline{z_1})\right]\mathrm{d}\lambda_1 + o(1), \qquad (9.8)
$$

where $C(z, z_1)$ is defined in (8.1). It follows from (9.7) and (9.8) that

$$
\frac{\mathrm{d}}{\mathrm{d}x}Z_{\eta n}(x) = -xV_\eta[\varphi]Z_{\eta n}(x) + o(1), \qquad (9.9)
$$

(see (1.23)) and finally

$$
\lim_{n\to\infty}Z_{\eta n}(x) = \exp\{-x^2V_\eta[\varphi]/2\}.
$$

Taking into account (9.5), we pass to the limit $\eta \downarrow 0$ and complete the proof of the theorem. $\qquad\square$

It remains to prove the following lemma.

Lemma 9.1 *Let $g \in L^2(\mathbb{R})$ and let $\{h_n\} \subset L^2(\mathbb{R})$ be a sequence of complex-valued functions such that*

$$\int |h_n|^2 dx < C \quad \text{and} \quad h_n \to h \quad a.e. \quad as \quad n \to \infty, \quad where \quad |h(x)| \leq \infty \quad a.e.$$

Then

$$\int g(x)h_n(x)dx \to \int g(x)h(x)dx \quad as \quad n \to \infty.$$

Proof According to the convergence theorem of Vitali (see, e.g., [24]), if $(X, \mathcal{F}, \mu)$ is a positive measure space and

$$\mu(X) < \infty,$$
$$\{F_n\}_n \quad \text{is uniformly integrable,}$$
$$F_n \to F \quad a.e. \quad as \quad n \to \infty, \quad |F(x)| \leq \infty \quad a.e.,$$

then $F \in L^1(\mu)$ and $\lim_{n\to\infty} \int_X |F_n - F| d\mu = 0$. Without loss of generality assume that $g(x) \neq 0$, $x \in \mathbb{R}$, and take

$$d\mu(x) = |g(x)|^2 dx, \quad F_n = gh_n/|g|^2, \quad F = gh/|g|^2.$$

Then

$$\mu(\mathbb{R}) = \int |g(x)|^2 dx < \infty,$$
$$\int_E |F_n(x)| d\mu(x) \leq ||h_n||_{L^2} \left(\int |g(x)|^2 dx \right)^{1/2} \leq C(\mu(E))^{1/2},$$
$$F_n \to F \quad a.e. \quad as \quad n \to \infty, \quad |F(x)| \leq \infty \quad a.e.$$

Hence, the conditions of Vitali's theorem are fulfilled and we get

$$\lim_{n\to\infty} \int |F_n - F| d\mu = \lim_{n\to\infty} \int |h_n - h||g| dx = 0,$$

which completes the proof of the lemma. □

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