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WELL-POSEDNESS OF A NONLINEAR INTERFACE PROBLEM DESCRIBED BY HEMIVARIATIONAL INEQUALITIES VIA BOUNDARY INTEGRAL OPERATORS

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Abstract
This paper is devoted to the well-posedness of a novel nonlinear interface problem on an unbounded domain with nonmonotone set-valued transmission conditions. This interface problem involves a nonlinear monotone partial differential equation in the interior domain and the Laplacian in the exterior domain. Such a scalar interface problem models nonmonotone frictional contact of elastic infinite media. The variational formulation of the interface problem leads to a hemivariational inequality (HVI), which however lives on the unbounded domain, and thus cannot be analyzed in a reflexive Banach space setting. Boundary integral methods lead to another HVI that is amenable to functional analytic methods using standard Sobolev spaces on the interior domain and Sobolev spaces of fractional order on the coupling boundary. Broadening the scope of the paper, we consider extended real-valued HVIs augmented by convex extended real-valued functions. Under a smallness hypothesis, we provide existence and uniqueness results and, moreover, establish a stability result for extended real-valued HVIs with respect to the extended real-valued function as a parameter. Based on the latter general stability result, we provide various stability results for the interface problem, as well as the stability of a related bilateral obstacle interface problem with respect to the obstacles.

Keywords Monotone operator · Nonmonotone transmission conditions · Unbounded domain · Extended real-valued hemivariational inequality · Existence · Uniqueness · Stability

Introduction

This paper is devoted to the well-posedness of a novel nonlinear interface problem on an unbounded domain with nonmonotone set-valued transmission conditions which is described by a hemivariational inequality (HVI) in a weak formulation.

The theory of HVIs was introduced and has been studied since 1980s by Panagiotopoulos [46], as a generalization of variational inequalities with the aim to model many problems coming from mechanics when the energy functionals are nonconvex,

Dedicated to Professor W.L. Wendland with high esteem.

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but locally Lipschitz, so the Clarke generalized differentiation calculus [11] can be used (see [18, 19, 41]). For more recent monographs on HVIs with application to contact problems, we refer to [39, 53].

While stability and sensitivity in variational inequalities and in quasi-variational inequalities have already been treated for a longer time (see in chronological order, e.g., the papers [1, 6, 10, 14, 29, 35, 37, 42]), stability in HVIs has been more recently studied (see, e.g., [3, 27, 30, 31, 54, 56, 57, 59]). Contrary to the work cited above, the underlying state problem of this paper is not a boundary value problem on a bounded domain, but an interface problem involving a pde on an unbounded domain. For the simplicity of presentation, we consider a scalar interface problem with a pde of the form div $(p(|\nabla u|) \cdot \nabla u)$ on the interior domain, where the nonlinearity p can be handled by the theory of monotone operators, and the Laplacian on the exterior domain, connected by nonmonotone set-valued transmission conditions as a novelty. This scalar problem models nonlinear contact problems with nonmonotone friction in infinite elastic media that arise in various fields of science and technology; let us mention geophysics (see, e.g., [50]), soil mechanics, in particular soil-structure interaction problems (see, e.g., [16]), and civil engineering of underground structures (see, e.g., [55]).

It should be underlined that such interface problems involving a pde on an unbounded domain are more difficult than standard boundary value problems on bounded domains, since a direct variational formulation of the former problems leads to a HVI, which lives on the unbounded domain, and thus cannot be analyzed in a reflexive Banach space setting. Thanks to boundary integral methods (see the monograph [28]), we provide another HVI that is amenable to functional analytic methods using standard Sobolev spaces on the interior domain and Sobolev spaces of fractional order on the coupling boundary. Let us note in passing that these integral methods lay the basis for the numerical treatment of such interface problems by the well-known coupling of boundary elements and finite elements (see [26, Chapter 12]).

A main novel ingredient of our analysis is a stability theorem that considerably improves a related result in the recent paper [51] and extends it to more general extended real-valued HVIs augmented by convex extended real-valued functions. This general stability theorem provides the key to a unified approach to various stability results for the interface problem with respect to the right-hand side, as well as stability for a related bilateral obstacle interface problem with respect to the obstacles.

The plan of the paper is as follows. The next Section 2 provides preliminaries and consists of three parts: a collection of some basic tools of Clarke's generalized differential calculus for the analysis of the nonmonotone transmission conditions, a description of the interface problem in strong form and in weak HVI formulation, and existence and uniqueness results for a class of abstract HVIs using an equilibrium approach. Section 3 establishes well-posedness results, in particular a stability theorem for a more general class of extended real-valued HVIs. Based on this general stability theorem, Section 4 presents a unified approach to various stability results for the interface problem as well as stability for a related bilateral obstacle interface problem with respect to the obstacles. The final Section 5 shortly summarizes our findings, gives some concluding remarks, and sketches some directions of further research.

Some preliminaries—Clarke's generalized differential calculus, the interface problem, and an equilibrium approach to HVIs

Some preliminaries from Clarke's generalized differential calculus

From Clarke's generalized differential calculus [11], we need the concept of the *generalized directional derivative* of a locally Lipschitz function $\phi : X \to \mathbb{R}$ on a real Banach space X at $x \in X$ in the direction $z \in X$ defined by

$$\phi^0(x;z) := \limsup_{y \to x; t \downarrow 0} \frac{\phi(y+tz) - \phi(y)}{t} .$$

Note that the function $z \in X \mapsto \phi^0(x;z)$ is finite, sublinear, and hence convex and continuous; further, the function $(x,z) \mapsto \phi^0(x;z)$ is upper semicontinuous. The *generalized gradient* of the function ϕ at x, denoted by (simply) $\partial\phi(x)$, is the unique nonempty weak* compact convex subset of the dual space X', whose support function is $\phi^0(x;.)$. Thus,

$$\xi \in \partial\phi(x) \Leftrightarrow \phi^0(x;z) \geq \langle \xi, z\rangle, \ \forall z \in X,$$
$$\phi^0(x;z) = \max\{\langle \xi, z\rangle \ : \ \xi \in \partial\phi(x)\}, \ \forall z \in X.$$

When X is finite dimensional, then, according to Rademacher's theorem, ϕ is differentiable almost everywhere, and the generalized gradient of ϕ at a point $x \in \mathbb{R}^n$ can be characterized by

$$\partial\phi(x) = \text{co}\,\{\xi \in \mathbb{R}^n \;:\; \xi = \lim_{k\to\infty} \nabla\phi(x_k),\; x_k \to x,\; \phi \text{ is differentiable at } x_k\},$$

where "co" denotes the convex hull.

The interface problem

Let $\Omega \subset \mathbb{R}^d$ $(d \geq 2)$ be a bounded domain with Lipschitz boundary $\Gamma = \text{cl}\,\Gamma_s \cup \text{cl}\,\Gamma_t$ with nonempty open disjoint boundary parts Γ_s and Γ_t. Let n denote the unit normal on Γ defined almost everywhere pointing from Ω into $\Omega^c := \mathbb{R}^d \setminus \overline{\Omega}$. Let the data $f \in L^2(\Omega)$, $u_0 \in H^{1/2}(\Gamma)$, and $q \in L^2(\Gamma)$ be given.
In the interior part Ω, consider the nonlinear partial differential equation

$$\text{div}\left(p(|\nabla u|)\cdot\nabla u\right) + f = 0 \qquad \text{in } \Omega, \tag{2.1}$$

where $p : [0,\infty) \to [0,\infty)$ is a continuous function with $t \cdot p(t)$ being monotonously increasing with t.
In the exterior part Ω^c, consider the Laplace equation

$$\Delta u = 0 \qquad \text{in } \Omega^c \tag{2.2}$$

with the radiation condition at infinity for $|x| \to \infty$,

$$u(x) = \left\{ \begin{array}{ll} a + o(1) & \text{if } d = 2, \\ O(|x|^{2-d}) & \text{if } d > 2, \end{array} \right\} \tag{2.3}$$

where a is a real constant for any u, but may vary with u.
With $u_1 := u|_\Omega$ and $u_2 := u|_{\Omega^c}$, the tractions on the coupling boundary Γ are given by the traces of $p(|\nabla u_1|)\frac{\partial u_1}{\partial n}$ and $-\frac{\partial u_2}{\partial n}$, respectively. Prescribe classical transmission conditions on Γ_t:

$$u_1|_{\Gamma_t} = u_2|_{\Gamma_t} + u_0|_{\Gamma_t} \quad \text{and} \quad p(|\nabla u_1|)\left.\frac{\partial u_1}{\partial n}\right|_{\Gamma_t} = \left.\frac{\partial u_2}{\partial n}\right|_{\Gamma_t} + q|_{\Gamma_t} \tag{2.4}$$

and on Γ_s analogously for the tractions:

$$p(|\nabla u_1|)\left.\frac{\partial u_1}{\partial n}\right|_{\Gamma_s} = \left.\frac{\partial u_2}{\partial n}\right|_{\Gamma_s} + q|_{\Gamma_s} \tag{2.5}$$

and the generally nonmonotone, set-valued transmission condition:

$$p(|\nabla u_1|)\left.\frac{\partial u_1}{\partial n}\right|_{\Gamma_s} \in \partial j(\cdot, u_0 + (u_2 - u_1)|_{\Gamma_s}). \tag{2.6}$$

Here, the function $j : \Gamma_s \times \mathbb{R} \to \mathbb{R}$ is such that $j(\cdot,\xi) : \Gamma_s \to \mathbb{R}$ is measurable on Γ_s for all $\xi \in \mathbb{R}$ and $j(s,\cdot) : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz for almost all (a.a.) $s \in \Gamma_s$ with $\partial j(s,\xi) := \partial j(s,\cdot)(\xi)$, the generalized gradient of $j(s,\cdot)$ at ξ.
Further, require the following growth condition on the so-called superpotential j: There exist positive constants $c_{j,1}$ and $c_{j,2}$ such that for a.a. $s \in \Gamma_s$, all $\xi \in \mathbb{R}$ and for all $\eta \in \partial j(s,\xi)$ the following inequalities hold:

$$(i) \quad |\eta| \leq c_{j,1}(1 + |\xi|)\,, (ii) \quad \eta\,\xi \geq -c_{j,2}|\xi|\,. \tag{2.7}$$

Note that it follows from Eq. (2.7) (i) and Eq. (2.7) (ii), respectively, that for a.a. $s \in \Gamma_s$

$$\left| j^0(s,\xi;\varsigma) \right| \leq c_{j,1}(1 + |\xi|)|\varsigma|, \quad \forall\, \xi, \varsigma \in \mathbb{R} \tag{2.8}$$

and

$$j^0(s,\xi;-\xi) \leq c_{j,2}|\xi|, \quad \forall \xi \in \mathbb{R}. \tag{2.9}$$

Altogether, the interface problem consists in finding $u_1 \in H^1(\Omega)$ and $u_2 \in H^1_{loc}(\Omega^c)$ that satisfy Eqs. (2.1)–(2.6) in a weak form.
To exhibit the relation of the above scalar interface problem to an elastic transmission problem with frictional contact, we insert the following remark.

Remark 1 In elasticity—to simplify focus to the case $d = 2$— instead of the unknown scalar field u, there is the displacement field u which decomposes in its normal component $u^n = u \cdot n$ and its tangential component $u^t = (u - u^n n) \cdot t$, where $t = (-n_2, n_1)^T$ for $n = (n_1, n_2)^T$. Similarly, as dual variable, the flux $q_\nu = \dfrac{\partial u}{\partial n}$ at the boundary is to be replaced by the boundary stress vector T with its normal component T^n and its tangential component T^t. This leads in local coordinates to $u_i = (u_i^t, u_i^n); T_i = (T_i^n, T_i^t)$ with $i = (1, 2)$ for the elastic body in the bounded domain ($i = 1$) and the exterior elastic medium ($i = 2$). Then, the set-valued transmission condition Eq. (2.6) includes a transmission condition of Tresca's type analogous to Tresca's friction boundary condition (given friction model) (see [13, 32]). Indeed, choose $j(\cdot,\xi) = g(\cdot)|\xi|$ with given nonnegative friction force $g \in L^\infty(\Gamma_s)$, then

$$\partial j\cdot,\xi = \begin{cases} -g & if \;\; \xi < 0 \\ [-g, g] & if \;\; \xi = 0 \\ g & if \;\; \xi > 0 \end{cases}$$

is monotone set-valued and with

$$\delta u := u_0 + (u_2^n - u_1^n)|_{\Gamma_s}$$

Equation (2.6) becomes

$$\begin{cases} |p(|T_1|)T_1^t| \leq g & \text{if} \;\; \delta u = 0, \\ p(|T_1|)T_1^t = g\dfrac{\delta u}{|\delta u|} & \text{if} \;\; \delta u \neq 0. \end{cases}$$

In this sense, Equation (2.6) gives a simplified (scalar) model of an elastic transmission problem with frictional contact.

To arrive at a first variational formulation of the interface problem in the form of a HVI, introduce some function spaces. For the bounded Lipschitz domain Ω, use the standard Sobolev space $H^s(\Omega)$ and the Sobolev spaces on the bounded Lipschitz boundary Γ (see [49, Sect 2.4.1]),

$$H^s(\Gamma) = \begin{cases} \{u|_\Gamma \,:\, u \in H^{s+1/2}(\mathbb{R}^d)\} & (0 < s \leq 1), \\ L^2(\Gamma) & (s = 0), \\ (H^{-s}(\Gamma))^* \text{ (dual space)} & (-1 \leq s < 0). \end{cases}$$

Further, for the unbounded domain $\Omega^c = \mathbb{R}^d \setminus \overline{\Omega}$ introduce the Fréchet space (see, e.g., [28, Section 4.1, (4.1.43)])

$$H^s_{loc}(\Omega^c) = \{u \in \mathcal{D}^*(\Omega^c) \,:\, \chi u \in H^s(\Omega^c) \; \forall \chi \in C_0^\infty(\mathbb{R}^d)\}\,.$$

By the trace theorem, $u|_\Gamma \in H^{1/2}(\Gamma)$ for $u \in H^1_{loc}(\Omega^c)$. Next, define $\Phi \,:\, H^1(\Omega) \times H^1_{loc}(\Omega^c) \to \mathbb{R} \cup \{\infty\}$ by

$$\Phi(u_1, u_2) := \int_\Omega g(|\nabla u_1|)\, dx + \frac{1}{2}\int_{\Omega^c} |\nabla u_2|^2\, dx - L(u_1, u_2|_\Gamma). \tag{2.10}$$

Here, the data $f \in L^2(\Omega), q \in L^2(\Gamma)$ enter the linear functional

$$L(u,v) := \int_\Omega f \cdot u\,dx + \int_\Gamma q \cdot v\,ds\,. \tag{2.11}$$

Further, in Eq. 2.10, the function g is given by p (see Eq. (2.1)) through

$$g : [0,\infty) \to [0,\infty), t \mapsto g(t) = \int_0^t s \cdot p(s)\,ds,$$

assume that p is C^1, $0 \le p(t) \le p_0 < \infty$, and $t \mapsto t \cdot p(t)$ is strictly monotonic increasing. Then, $0 \le g(t) \le \frac{1}{2}p_0 \cdot t^2$ and the real-valued functional

$$G(u) := \int_\Omega g(|\nabla u|)dx, \qquad u \in H^1(\Omega)$$

is strictly convex. The Gateaux derivative of G,

$$DG(u;v) = \int_\Omega p(|\nabla u|)(\nabla u)^T \cdot \nabla v\,dx \qquad u,v \in H^1(\Omega) \tag{2.12}$$

is Lipschitz continuous and strongly monotone in $H^1(\Omega)$ with respect to the semi-norm

$$|v|_{H^1(\Omega)} = \|\nabla\, v\|_{L^2(\Omega)},$$

that is, there exists a constant $c_G > 0$ such that

$$c_G\,|u-v|^2_{H^1(\Omega)} \le DG(u;u-v) - DG(v;u-v) \quad \forall u,v \in H^1(\Omega). \tag{2.13}$$

Analogously to [9, 38], we first define

$$\begin{aligned}\mathcal{L}_0 := \{v \in H^1_{loc}(\Omega^c) : \; & \Delta v = 0 \text{ in } H^{-1}(\Omega^c) \\ & (\text{and for } d = 2\; \exists a \in \mathbb{R} \text{ such that } v \text{ satisfies } (2.3))\},\end{aligned}$$

and then the affine, hence a convex set of admissible functions

$$C := \{(u_1,u_2) \in H^1(\Omega) \times H^1_{loc}(\Omega^c) \;:\; u_1|_{\Gamma_t} = u_2|_{\Gamma_t} + u_0|_{\Gamma_t} \text{ and } u_2 \in \mathcal{L}_0\}.$$

According to [9, Remark 4], C is closed in $H^1(\Omega) \times H^1_{loc}(\Omega^c)$. Further, it holds

$$\begin{aligned}D\Phi((\hat u_1,\hat u_2);(u_1,u_2)) = DG(\hat u_1;u_1) &+ \textstyle\int_{\Omega^c} \nabla \hat u_2 \cdot \nabla u_2\,dx \\ &- \textstyle\int_\Omega f \cdot u_1\,dx - \int_{\Gamma_t} q \cdot u_2|_{\Gamma_t}\,ds.\end{aligned}$$

Then, it can be proved [25, Theorem 1] that the interface problem Eqs. (2.1)–(2.6) is equivalent in the sense of distributions to the HVI problem (P_Φ): Find $(\hat u_1,\hat u_2) \in C$ such that for all $(u_1,u_2) \in C$, there holds for $\delta u_1 := u_1 - \hat u_1, \delta u_2 := u_2 - \hat u_2$,

$$D\Phi((\hat u_1,\hat u_2);(\delta u_1,\delta u_2)) + J^0(\gamma(\hat u_2 - \hat u_1 + u_0);\gamma(\delta u_2 - \delta u_1)) \ge 0\,. \tag{2.14}$$

However, since this HVI lives on the unbounded domain $\Omega \times \Omega^c$ (as the original problem), this HVI cannot be treated in a reflexive Banach space setting and therefore provides only an intermediate step in the analysis. Therefore, we employ the boundary integral operator theory [26, 28] to reformulate the interface problem Eqs. (2.1)–(2.6) in the weak sense as a boundary-domain variational inequality on $\Gamma \times \Omega$. From now on, concentrate the analysis to the case of dimension $d = 3$, since as already the distinction in the radiation condition Eq. (2.3) indicates, in the case $d = 2$, some peculiarities of boundary integral methods for exterior problems come up that need extra attention (see, e.g., [9, 26, Sec. 12.2]). As a result, we arrive at an equivalent hemivariational formulation of the original interface problem Eqs. (2.1)–(2.6) that lives on $\Omega \times \Gamma$ and consists of a weak formulation of the nonlinear differential operator in the bounded domain Ω, the Poincaré–Steklov operator on the bounded boundary Γ, and a nonsmooth functional on the boundary part Γ_s.

To this end recall, the Poincaré–Steklov operator for the exterior problem, $S : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$, is a selfadjoint operator with the defining property

$$S(u_2|_\Gamma) = -\partial_n u_2|_\Gamma \tag{2.15}$$

for solutions $u_2 \in \mathcal{L}_0$ of the Laplace equation on Ω^c. The operator S enjoys the important property that it can be expressed as

$$S = \frac{1}{2}[W + (I - K')V^{-1}(I - K)]\,,$$

where I, V, K, K', and W denote the identity, the single layer boundary integral operator, the double layer boundary integral operator, its formal adjoint, and the hypersingular integral operator, respectively; see [26, Sec. 12.2] for details.
Further, S gives rise to the positive definite bilinear form $\langle S\cdot, \cdot\rangle$, that is, there exists a constant $c_S > 0$ such that

$$\langle Sv, v\rangle \geq c_S \|v\|^2_{H^{1/2}(\Gamma)}\,, \quad \forall v \in H^{1/2}(\Gamma)\,, \tag{2.16}$$

where $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)}$ extends the L^2 duality on Γ.
Let $E := H^1(\Omega) \times \widetilde{H}^{1/2}(\Gamma_s)$ with $\widetilde{H}^{1/2}(\Gamma_s) := \{w \in H^{1/2}(\Gamma) | \operatorname{supp} w \subseteq \overline{\Gamma}_s\}$. Next, define the linear functional $\lambda \in E^*$ by

$$\lambda(u, v) := \int_\Omega f \cdot u\, dx + \langle q + Su_0, u|_\Gamma + v\rangle\,, \quad (u, v) \in E$$

Using the representation formula of potential theory (see [17, 38] for similar nonlinear interface problems), it can be proved [25, Theorem 2] that the intermediate HVI (P_Φ) is equivalent to the following HVI problem $(P_\mathcal{A})$: Find $(\hat{u}, \hat{v}) \in E$ such that for all $(u, v) \in E$,

$$\mathcal{A}(\hat{u}, \hat{v}; u - \hat{u}, v - \hat{v}) + J^0(\gamma\hat{v}; \gamma(v - \hat{v})) \geq \lambda(u - \hat{u}, v - \hat{v})\,, \tag{2.17}$$

where $\mathcal{A} : E \to E^*$ is defined for all $(u, v), (u', v') \in E$ by

$$\mathcal{A}(u, v)\,(u', v') = \mathcal{A}(u, v; u', v') := DG(u, u') + \langle S(u|_\Gamma + v), u'|_\Gamma + v'\rangle\,.$$

An equilibrium approach to a class of HVIs—existence and uniqueness results

Next, we describe the functional analytic setting for the interface problem and provide existence and uniqueness results using an equilibrium approach. To this end, let $X := L^2(\Gamma_s)$ and introduce the real-valued locally Lipschitz functional

$$J(y) := \int_{\Gamma_s} j(s, y(s))\, ds\,, \qquad y \in X\,. \tag{2.18}$$

Then, by Lebesgue's theorem of majorized convergence,

$$J^0(y; z) = \int_{\Gamma_s} j^0(s, y(s); z(s))\, ds\,, \qquad (y, z) \in X \times X\,, \tag{2.19}$$

where $j^0(s, \cdot\,; \cdot)$ denotes the generalized directional derivative of $j(s, \cdot)$.
As seen in the previous subsection, the weak formulation of the problem Eqs. (2.1)–(2.6) leads, in an abstract setting, to a hemivariational inequality (HVI) with a nonlinear operator $\mathcal{A}$ and the nonsmooth functional J: Find $\hat{v} \in \mathcal{C}$ such that

$$\mathcal{A}(\hat{v})(v - \hat{v}) + J^0(\gamma\hat{v}; \gamma v - \gamma\hat{v}) \geq \lambda(v - \hat{v}) \qquad \forall v \in \mathcal{C}. \tag{2.20}$$

Here, $\mathcal{C} \neq \emptyset$ is a closed convex subset of a real reflexive Banach space E, $\gamma := \gamma_{E\to X}$ is a linear continuous operator, the linear form λ belongs to the dual E^*. and the nonlinear monotone operator $\mathcal{A} : E \to E^*$ is Lipschitz continuous and strongly monotone with some monotonicity constant $c_\mathcal{A} > 0$, what results from the strong monotonicity of the nonlinear operator

DG in $H^1(\Omega)$ with respect to the semi-norm $|\cdot|_{H^1(\Omega)} = \|\nabla \cdot\|_{L^2(\Omega)}$ and the positive definiteness of the Poincaré–Steklov operator S (see [9, Lemma 4.1]).

On the other hand, by Eq. 2.8 and the compactness of the operator γ the real-valued upper semicontinuous bivariate function, shortly bifunction

$$\psi(v,w) := J^0(\gamma v;\gamma w - \gamma v)\,, \forall (v,w) \in E \times E$$

becomes pseudomonotone (see [44, Lemma 1], [24, Lemma 4.1]). The latter result also shows that Eq. (2.9) implies a linear growth of $\psi(\cdot, 0)$. This and the strong monotonicity of $\mathcal{A}$ imply coercivity. Therefore, by the theory of pseudomonotone VIs [20, Theorem 3], [58] (see [45] for the application to HVIs), we have solvability of Eq. 2.20.

Further, suppose that the generalized directional derivative J^0 satisfies the one-sided Lipschitz condition: There exists $c_J > 0$ such that

$$J^0(y_1;y_2 - y_1) + J^0(y_2;y_1 - y_2) \le c_J \|y_1 - y_2\|_X^2 \qquad \forall y_1, y_2 \in X\,. \tag{2.21}$$

Then, the smallness condition

$$c_J \|\gamma\|^2_{E\to X} < c_{\mathcal{A}} \tag{2.22}$$

implies unique solvability of Eq. 2.20 (see, e.g., [43, Theorem 5.1] and [53, Theorem 83]).

It is noteworthy that under the smallness condition Eq. (2.22) together with Eq. (2.21), fixed point arguments [8] or the theory of set-valued pseudomonotone operators [53] are not needed, but simpler monotonicity arguments are sufficient to conclude unique solvability. Moreover, the compactness of the linear operator γ is not needed either. In fact, Equation (2.20) can be framed as a *monotone equilibrium problem* in the sense of Blum-Oettli [5]:

Proposition 1 *Suppose Eqs. (2.21) and (2.22). Then, the bifunction $\varphi : \mathcal{C} \times \mathcal{C} \to \mathbb{R}$ defined by*

$$\varphi(v,w) := \mathcal{A}(v)(w - v) + J^0(\gamma v;\gamma w - \gamma v) - \lambda(w - v) \tag{2.23}$$

has the following properties:

$\varphi(v,v) = 0$ for all $v \in \mathcal{C}$.
$\varphi(v,\cdot)$ is convex and lower semicontinuous for all $v \in \mathcal{C}$.
There exists some $\mu > 0$ such that $\varphi(v,w) + \varphi(w,v) \le -\mu \|v - w\|_E^2$ for all $v, w \in \mathcal{C}$ (strong monotonicity).
The function $t \in [0,1] \mapsto \varphi(tw + (1-t)v, w)$ is upper semicontinous at $t = 0$ for all $v, w \in \mathcal{C}$ (hemicontinuity).

Proof Obviously, φ vanishes on the diagonal and is convex and lower semicontinuous with respect to the second variable. To show strong monotonicity, estimate

$$\begin{aligned}
&\varphi(v,w) + \varphi(w,v)\\
=&(\mathcal{A}(v) - \mathcal{A}(w))(w - v)\\
&+ J^0(\gamma v;\gamma w - \gamma v) + J^0(\gamma w;\gamma v - \gamma w)\\
\le& - c_{\mathcal{A}}\,\|v - w\|_E^2 + c_J \|\gamma v - \gamma w\|_X^2\\
\le& - (c_{\mathcal{A}} - c_J \|\gamma\|^2_{E\to X})\,\|v - w\|_E^2.
\end{aligned}$$

To show hemicontinuity, it is enough to consider the bifunction $(y,z) \in X \times X \mapsto J^0(y;z - y)$. Then, for $(y,z) \in X \times X$ fixed, $t \in [0,1]$, one has

$$J^0(y + t(z - y);z - (y + t(z - y))) = (1 - t)J^0(y + t(z - y);z - y)$$

and thus, hemicontinuity follows from upper semicontinuity of J^0,

$$\limsup_{t \downarrow 0} J^0(y + t(z - y); z - y) \leq J^0(y; z - y) .$$

□

Since strong monotonicity implies coercivity and uniqueness, the fundamental existence result [5, Theorem 1] applies to the HVI Eq. (2.20) to conclude the following:

Theorem 1 *Suppose Eqs*. (2.21) *and* (2.22). *Then*, *the HVI Eq*. (2.20) *is uniquely solvable*.

Thus, under the smallness condition, unique solvability holds for $(P_{\mathcal{A}})$.

Extended real-valued HVIs—well-posedness

In view of the subsequent study of stability in Section 4 for the interface problem which we have described in the previous section, we broaden the scope of analysis and consider extended real-valued HVIs: Find $\hat{v} \in \operatorname{dom} F$ such that

$$\mathcal{A}(\hat{v})(v - \hat{v}) + J^0(\gamma \hat{v}; \gamma v - \gamma \hat{v}) + F(v) - F(\hat{v}) \geq 0 \qquad \forall v \in V . \tag{3.1}$$

Here, V is a real reflexive Banach space, the nonlinear operator $\mathcal{A} : V \to V^*$ is a monotone operator, $\gamma := \gamma_{V \to X}$ with X a real Hilbert space (in the interface problem, we have $X = L^2(\Gamma_s)$) denotes a linear continuous operator, J^0 stands for the generalized directional derivative of a real-valued locally Lipschitz functional J, and now, in addition, $F : V \to \mathbb{R} \cup \{+\infty\}$ is a convex lower semicontinuous function that is supposed to be proper (i.e., $F \not\equiv \infty$ on $\mathcal{C}$). This means that the effective domain of F in the sense of convex analysis ([48]),

$$\operatorname{dom} F := \{v \in V : F(v) < +\infty\}$$

is nonempty, closed, and convex. To resume the HVI Eq. (2.20) of Section 2.2, let $F(v) := \lambda(v) + \chi_{\mathcal{C}}(v)$, where $\lambda \in V^*$ and

$$\chi_{\mathcal{C}}(v) := \begin{cases} 0 & \text{if } v \in \mathcal{C} \\ +\infty & \text{elsewhere} \end{cases}$$

is the indicator function on $\mathcal{C}$ in the sense of convex analysis ([48]).
Next, similarly to Eq. 2.23 in Sect. 2.2, define

$$\varphi(v, w) := \mathcal{A}(v)(w - v) + J^0(\gamma v; \gamma w - \gamma v) \tag{3.2}$$

and apply Proposition 1. Thus, under the assumptions Eqs. (2.21) and (2.22), the above HVI Eq. (3.1) falls into the framework of an *extended real-valued equilibrium problem of monotone type* in the sense of [23]. Clearly, strong monotonicity implies uniqueness. Note that by the separation theorem, it can be shown that any convex proper lower semicontinuous function $\phi : V \to \mathbb{R} \cup \{+\infty\}$ is conically minorized, that is, it enjoys the estimate

$$\phi(v) \geq -c_\phi(1 + \|v\|) , v \in V$$

with some $c_\phi > 0$. Hence, strong monotonicity implies the asymptotic coercivity condition in [23], too. Thus, the existence result [23, Theorem 5.9] applies to the HVI Eq. (3.1) to conclude the following:

Theorem 2 *Suppose Eqs*. (2.21) *and* (2.22). *Then*, *the HVI Eq*. (3.1) *is uniquely solvable*.

By this solvability result, we can introduce the solution map $\mathcal{S}$ by $\mathcal{S}(F) := \hat{v}$, the solution of Eq. 3.1. Next, we investigate the stability of the solution map $\mathcal{S}$ with respect to the extended real-valued function F. Here, we follow the concept of epi-convergence in the sense of Mosco [2, 40] ("Mosco convergence"). Let F_n $(n \in \mathbb{N})$, $F : V \to \mathbb{R} \cup \{+\infty\}$ be convex lower semicontinuous proper functions.

Then, F_n are called to converge to F in the Mosco sense, written $F_n \xrightarrow{\text{M}} F$, if and only if the subsequent two hypotheses hold:

(m1) If $v_n \in V$ $(n \in \mathbb{N})$ weakly converge to v for $n \to \infty$, then

$$F(v) \le \liminf_{n\to\infty} F_n(v_n)\,.$$

(m2) For any $v \in V$, there exist $v_n \in V$ $(n \in \mathbb{N})$ strongly converging to v for $n \to \infty$ such that

$$F(v) = \lim_{n\to\infty} F_n(v_n)\,.$$

In view of our later applications, it is not hard to require that the functions F_n are uniformly conically minorized, that is, there holds the estimate

$$F_n(v) \ge -d_0(1 + \|v\|),\ \forall n \in \mathbb{N}, v \in V \tag{3.3}$$

with some $d_0 \ge 0$. Moreover, similar to [51], in addition to the one-sided Lipschitz continuity Eq. (2.21), we assume that the local Lipschitz function J satisfies the following growth condition:

$$\|\zeta\|_{X^*} \le d_J(1 + \|z\|_X) \qquad \forall z \in X,\ \zeta \in \partial J(z) \tag{3.4}$$

for some $d_J > 0$, what is immediate from the growth condition Eq. (2.7) for the integrand j.
Now, we are in the position to state the main result of this section which extends the stability result of [21] for monotone variational inequalities to extended real-valued HVIs with an unperturbed bifunction φ in the coercive situation.

Theorem 3 *Suppose that the operator $\mathcal{A}$ is continuous and strongly monotone with monotonicity constant $c_{\mathcal{A}} > 0$, the linear operator γ is compact, and the generalized directional derivative J^0 satisfies the one-sided Lipschitz condition Eq.* (2.21) *and the growth condition Eq.* (3.4). *Moreover, suppose the smallness condition Eq.* (2.22). *Let $F, F_n : V \to \mathbb{R} \cup \{+\infty\}$ $(n \in \mathbb{N})$ be convex lower semicontinuous proper functions that satisfy the lower estimate Eq.* (3.3); *let $F_n \xrightarrow{\text{M}} F$. Then, strong convergence $\mathcal{S}(F_n) \to \mathcal{S}(F)$ holds.*

Proof We divide the proof into three parts. We first show that the $\hat{u}_n = \mathcal{S}(F_n)$ are bounded, before we can establish the convergence result. In the following, $c_0, c_1, \ldots$ are generic positive constants.

(1) *The sequence $\{\hat{u}_n\} \subset V$ is bounded.*

By definition, $\hat{u}_n$ satisfies for all $v \in V$,

$$\mathcal{A}(\hat{u}_n)(v - \hat{u}_n) + J^0(\gamma\hat{u}_n;\gamma v - \gamma\hat{u}_n) + F_n(v) - F_n(\hat{u}_n) \ge 0\,. \tag{3.5}$$

Now, let v_0 be an arbitrary element of dom F. Then, by Mosco convergence, (M2), there exist $v_n \in$ dom F_n $(n \in \mathbb{N})$ such that for $n \to \infty$ the strong convergences hold

$$v_n \to v_0;\, F_n(v_n) \to F(v_0)\,. \tag{3.6}$$

Let $n \in \mathbb{N}$. Then, insert $v = v_n$ in Eq. 3.5 and obtain

$$\mathcal{A}(\hat{u}_n)(\hat{u}_n - v_n) \le J^0(\gamma\hat{u}_n;\gamma v_n - \gamma\hat{u}_n) + F_n(v_n) - F_n(\hat{u}_n)\,.$$

Write $\mathcal{A}(\hat{u}_n) = \mathcal{A}(\hat{u}_n) - \mathcal{A}(v_n) + \mathcal{A}(v_n)$ and use the strong monotonicity of the operator $\mathcal{A}$ and the estimate Eq. (3.3) to get

$$\begin{aligned} &c_{\mathcal{A}}\|\hat{u}_n - v_n\|_V^2 \\ &\le \|\mathcal{A}(v_n)\|_{V^*}\|\hat{u}_n - v_n\|_V + F_n(v_n) + d_0(1 + \|\hat{u}_n\|) \\ &\quad + J^0(\gamma\hat{u}_n;\gamma v_n - \gamma\hat{u}_n). \end{aligned} \tag{3.7}$$

On the other hand, write

$$
\begin{aligned}
& J^0(\gamma \hat{u}_n;\gamma v_n - \gamma \hat{u}_n) \\
& = J^0(\gamma \hat{u}_n;\gamma v_n - \gamma \hat{u}_n) + J^0(\gamma v_n;\gamma \hat{u}_n - \gamma v_n) - J^0(\gamma v_n;\gamma \hat{u}_n - \gamma v_n).
\end{aligned}
$$

Hence, by the one-sided Lipschitz condition Eq. (2.21),

$$
J^0(\gamma \hat{u}_n;\gamma v_n - \gamma \hat{u}_n) \leq c_J \|\gamma\|^2 \|v_n - \hat{u}_n\|_V^2 - J^0(\gamma v_n;\gamma \hat{u}_n - \gamma v_n)\,. \tag{3.8}
$$

Further, by Eq. 3.4,

$$
\begin{aligned}
& - J^0(\gamma v_n;\gamma \hat{u}_n - \gamma v_n) \leq \max_{\zeta \in \partial J(\gamma v_n)} \|\zeta\|_{X^*} \|\gamma \hat{u}_n - \gamma v_n\|_X \\
& \leq d_J \|\gamma\|(1 + \|\gamma\|\|v_n\|_V)\|\hat{u}_n - v_n\|_V.
\end{aligned} \tag{3.9}
$$

By the convergences Eq. (3.6), $|F_n(u_n)| \leq c_0$, $\|\mathcal{A}(v_n)\|_{V^*} \leq c_1$, $\|v_n\|_V \leq c_2$. Thus, Eqs. (3.7), (3.8), and (3.9) result in

$$
(c_{\mathcal{A}} - c_J\|\gamma\|^2)\|\hat{u}_n - v_n\|_V^2 \leq c_0 + [c_1 + d_J\|\gamma\|(1 + c_2\|\gamma\|)]\|\hat{u}_n - v_n\|_V + d_0(1 + \|\hat{u}_n\|)\,.
$$

Hence, by the smallness condition Eq. (2.22), a contradiction argument proves the claimed boundedness of $\{\hat{u}_n\}$.

(2) $\hat{u}_n = \mathcal{S}(F_n)$ *converges weakly to* $\hat{u} = \mathcal{S}(F)$ *for* $n \to \infty$..

To prove this claim, we employ a “Minty trick” similar to the proof of [23, Prop.3.2] using the monotonicity of the operator $\mathcal{A}$.

Take $v \in V$ arbitrarily. By $(M2)$, there exist $v_n \in V$ $(n \in \mathbb{N})$ such that

$$
\lim_{n\to\infty} v_n = v;\ \lim_{n\to\infty} F_n(v_n) = F(v) \tag{3.10}
$$

We test the inequality Eq. (3.5) with v_n, use the monotonicity of the operator $\mathcal{A}$, and obtain

$$
\mathcal{A}(v_n)(v_n - \hat{u}_n) + J^0(\gamma \hat{u}_n;\gamma v_n - \gamma \hat{u}_n) \geq F_n(\hat{u}_n) - F_n(v_n)\,. \tag{3.11}
$$

On the other hand, by the previous step, there exists a subsequence $\{\hat{u}_{n_k}\}_{k\in\mathbb{N}}$ that converges weakly to some $\tilde{u} \in \operatorname{dom} F \subset V$. Further, since γ is completely continuous, $\gamma \hat{u}_{n_k} \to \gamma \tilde{u}$. Thus, the continuity of $\mathcal{A}$, the upper semicontinuity of $(y,z) \in X \times X \mapsto J^0(y;z)$, $(M1)$, and Eq. (3.10) entail together with Eq. (3.11)

$$
\begin{aligned}
& \mathcal{A}(v)(v - \tilde{u}) + J^0(\gamma \tilde{u};\gamma v - \gamma \tilde{u}) \\
& \geq \lim_{k\to\infty} \mathcal{A}(v_{n_k})(v_{n_k} - \hat{u}_{n_k}) + \limsup_{k\to\infty} J^0(\gamma \hat{u}_{n_k};\gamma v_{n_k} - \gamma \hat{u}_{n_k}) \\
& \geq \liminf_{n\to\infty} F_n(\hat{u}_n) - \lim_{n\to\infty} F_n(v_n) \\
& \geq F(\tilde{u}) - F(v).
\end{aligned}
$$

Hence, for $v \in \operatorname{dom} F$ fixed, for arbitrary $s \in [0,1)$ and $w_s := v + s(\tilde{u} - v) \in \operatorname{dom} F$ inserted above, the positive homogeneity of $J^0(\gamma \tilde{u};\cdot)$ and the convexity of F imply after division by the factor $(1-s) > 0$

$$
\mathcal{A}(w_s)(v - \tilde{u}) + J^0(\gamma \tilde{u};\gamma v - \gamma \tilde{u}) + F(v) \geq F(\tilde{u})\,.
$$

Letting $s \to 1$, hence $w_s \to \tilde{u}$, $\mathcal{A}(w_s) \to \mathcal{A}(\tilde{u})$ results in

$$
\mathcal{A}(\tilde{u})(v - \tilde{u}) + J^0(\gamma \tilde{u};\gamma v - \gamma \tilde{u}) + F(v) \geq F(\tilde{u}) \quad \forall v \in \operatorname{dom} F\,.
$$

This shows by uniqueness that $\tilde{u} = \mathcal{S}(F)$ and the entire sequence $\{\hat{u}_n\}$ converges weakly to $\hat{u} = \mathcal{S}(F)$.

(3) $\hat{u}_n = \mathcal{S}(F_n)$ *converges strongly to* $\hat{u} = \mathcal{S}(F)$ *for* $n \to \infty$.

By $(M2)$, there exist $u_n \in V$ $(n \in \mathbb{N})$ such that

$$
(i)\ \lim_{n\to\infty} u_n = \hat{u};(ii)\ \lim_{n\to\infty} F_n(u_n) = F(\hat{u})\,. \tag{3.12}
$$

Test the inequality Eq. (3.5) with u_n, use the strong monotonicity of the operator $\mathcal{A}$, and obtain

$$\mathcal{A}(u_n)(u_n - \hat{u}_n) + J^0(\gamma\hat{u}_n;\gamma u_n - \gamma\hat{u}_n) \\ + F_n(u_n) - F_n(\hat{u}_n) \geq c_{\mathcal{A}} \|u_n - \hat{u}_n\|^2. \tag{3.13}$$

Analyze the summands in Eq. 3.13 separately: By Eq. 3.12 (i), $\mathcal{A}(u_n) \to \mathcal{A}(\hat{u})$, hence

$$\lim_{n\to\infty} \mathcal{A}(u_n)(u_n - \hat{u}_n) = 0\,.$$

By the upper semicontinuity of $(y,z) \in X \times X \mapsto J^0(y;z)$, $(M1)$ and by the complete continuity of γ,

$$\limsup_{n\to\infty} J^0(\gamma\hat{u}_n;\gamma u_n - \gamma\hat{u}_n) \leq 0\,.$$

By Eq. 3.12 (ii) and by $M1$,

$$\limsup_{n\to\infty} [F_n(u_n) - F_n(\hat{u}_n)] \leq 0\,.$$

Thus, from Eq. (3.13) finally by the triangle inequality,

$$0 \leq \|\hat{u}_n - \hat{u}\| \leq \|\hat{u}_n - u_n\| + \|u_n - \hat{u}\| \to 0$$

and the theorem is proved. □

To conclude this section, let us compare the above Theorem 3 with a similar stability result of [51, Theorem 6]. There one has the special setting of $F(v) := \chi_{\mathcal{C}}(v) + \lambda(v)$, where $\mathcal{C} \subset V$ is closed convex and $\lambda \in V^*$ is given by $\lambda := \kappa^* f$ with $f \in X^*$, κ^* the adjoint to the linear operator $\kappa : V \to X$, which is assumed to be completely continuous or equivalently compact (see [51, (4.3)]).
Likewise, for $n \in \mathbb{N}$, one has convex closed sets $\mathcal{C}_n \subset V$ and linear forms $\lambda_n := \kappa^* f_n$ with $f_n \in X^*$ giving rise to $F_n(v) := \chi_{\mathcal{C}_n}(v) + \lambda_n(v)$.
Note by the Schauder theorem (see, e.g., [47, 3.7.17]), the adjoint $\kappa^* : X^* \to V^*$ is compact. Hence, the assumed weak convergence $f_n \rightharpoonup f$ in X^* entails the strong convergence $\lambda_n \to \lambda$ in V^*, thus further the lower estimate Eq. (3.3), in view of $\chi_{\mathcal{C}_n} \geq 0$. Moreover, the Mosco convergence $F_n \xrightarrow{\mathrm{M}} F$ follows at once from the assumed Mosco convergence $\mathcal{C}_n \xrightarrow{\mathrm{M}} \mathcal{C}$, namely from the hypotheses:

(m1) If $v_n \in \mathcal{C}_n$ $(n \in \mathbb{N})$ weakly converge to v for $n \to \infty$, then $v \in \mathcal{C}$.
(m2) For any $v \in \mathcal{C}$, there exist $v_n \in \mathcal{C}_n$ $(n \in \mathbb{N})$ strongly converging to v for $n \to \infty$.

On the other hand, an inspection of the above proof of Theorem 3 shows that is enough to demand for $\tilde{J}^0(v;w) = J^0(\gamma v;\gamma w)$, the generalized directional derivative of the real-valued locally Lipschitz functional $\tilde{J}(v) := J(\gamma v)$, that

$$u_n \rightharpoonup u \text{ in } V \text{ and } v_n \to v \text{ in } V \\ \Rightarrow \limsup \tilde{J}^0(u_n;v_n - u_n) \leq \tilde{J}^0(u;v-u).$$

This abstract condition [51, (4.2)] is derived in the above proof of Theorem 3 from the compactness of γ and the upper semicontinuity of $(y,z) \in X \times X \mapsto J^0(y;z)$.
Concerning the monotone operator $\mathcal{A} : V \to V^*$, we only require its norm continuity, not needing Lipschitz continuity. More importantly, we can also dispense with the condition [51, (4.1)]:

$$u_n \rightharpoonup u \text{ in } V \text{ and } v_n \to v \text{ in } V \\ \Rightarrow \limsup \langle \mathcal{A}u_n, u_n - v_n\rangle \geq \langle \mathcal{A}u, u - v\rangle.$$

It seems that this condition forces an elliptic operator, which stems from an elliptic pde on the domain, to be linear.

Stability results

In this section, we rely heavily on the stability result Theorem 3 and present a unified approach to various stability results for the interface problem, which was described in "Some preliminaries—Clarke's generalized differential calculus, the interface problem, and an equilibrium approach to HVIs," and for a related interface problem. For convenience, let us recall the boundary/domain HVI formulation $(P_{\mathcal{A}})$ of the interface problem: Find $(\hat{u}, \hat{v}) \in E$ such that for all $(u, v) \in E$,

$$\mathcal{A}(\hat{u}, \hat{v}; u - \hat{u}, v - \hat{v}) + J^0(\gamma\hat{v}; \gamma(v - \hat{v})) \geq \lambda(u - \hat{u}, v - \hat{v}), \tag{4.1}$$

where $E = H^1(\Omega) \times \widetilde{H}^{1/2}(\Gamma_s)$ with $\widetilde{H}^{1/2}(\Gamma_s) = \{w \in H^{1/2}(\Gamma) | \operatorname{supp} w \subseteq \overline{\Gamma}_s\}$ on the bounded domain Ω and the boundary part Γ_s. The operator $\mathcal{A}$ is given for all $(u, v), (u', v') \in E$ by

$$\mathcal{A}(u, v)(u', v') = \mathcal{A}(u, v; u', v') = DG(u, u') + \langle S(u|_\Gamma + v), u'|_\Gamma + v' \rangle$$

(see Eqs. (2.12), (2.15)). J^0 denotes the generalized directional derivative of the Lipschitz integral function J (see Eqs. (2.19), (2.18)) stemming from the generally nonmonotone, set-valued transmission condition Eq. (2.6). $\gamma : \widetilde{H}^{1/2}(\Gamma_s) \to L^2(\Gamma_s)$ denotes the linear continuous embedding operator which is compact. The linear functional $\lambda \in E^*$ is defined for $(u, v) \in E$ by

$$\lambda(u, v) = \int_\Omega f \cdot u \, dx + \langle q + Su_0, u|_\Gamma + v \rangle,$$

where $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)}$ extends the L^2 duality on Γ.
Just for simplicity, we set $u_0 := 0$ and impose for the data f, q that $f \in L^2(\Omega)$ and $q \in L^2(\Gamma)$. Thus, we can write

$$\lambda(u, v) = \langle f, \kappa u \rangle_{L^2(\Omega) \times L^2(\Omega)} + \langle q, \tau u + \iota v \rangle_{L^2(\Gamma) \times L^2(\Gamma)}, \tag{4.2}$$

where $\kappa : H^1(\Omega) \to L^2(\Omega)$, $\iota : H^{1/2}(\Gamma) \to L^2(\Gamma)$ are linear compact embedding operators and $\tau : H^1(\Omega) \to L^2(\Gamma)$ is a linear compact trace operator.

A first stability result for the interface problem

Here we consider stability with respect to the right-hand side $f \in L^2(\Omega)$ distributed on the domain Ω. Thus, in the abstract setting of "Extended real-valued HVIs—well-posedness," we choose the convex functional F as the linear functional

$$F(u, v) := \langle f, \kappa u \rangle_{L^2(\Omega) \times L^2(\Omega)} = (\kappa^* f)(u), \; F = (\kappa^* f, 0) \in E^*.$$

By the abstract existence and uniqueness result of Theorem 2, we have the control-to-state map $f \in L^2(\Omega) \mapsto \mathcal{S}(f) := (\hat{u}, \hat{v}) \in E$, the solution of Eq. 4.1.

Theorem 4 *Suppose that the generalized directional derivative J^0 satisfies the one-sided Lipschitz condition Eq. (2.21) and the growth condition Eq. (3.4). Moreover, suppose the smallness condition Eq. (2.22) with the monotonicity constant $c_{\mathcal{A}}$ of the operator $\mathcal{A}$. Let $\{f_n\}_{n\in\mathbb{N}} \subset L^2(\Omega)$ and $f_n \to \hat{f}$ in $L^2(\Omega)$ for $n \to \infty$ Then, there holds $\mathcal{S}(f_n) \to \mathcal{S}(\hat{f})$ in E. If only $f_n \rightharpoonup \hat{f}$ (weakly) in $L^2(\Omega)$, then there holds $\mathcal{S}(f_{n_k}) \to \mathcal{S}(\hat{f})$ in E for a subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$.*

Proof Since κ^* is completely continuous, we have strong convergence for a subsequence (sticking to the index n for simplicity of notation) $F_n := (\kappa^* f_n, 0) \to \hat{F} := (\kappa^* \hat{f}, 0)$ in E^*. Thus, the linear continuous functionals F_n satisfy the lower estimate Eq. (3.3). To show (m1), let $(u_n, v_n) \rightharpoonup (u, v)$ in E for $n \to \infty$. Then, clearly, $\hat{F}(u, v) = \lim_{n\to\infty} F_n(u_n, v_n)$. To show (m2), choose for any $(u, v) \in E$, simply $(u_n, v_n) := (u, v) \in E$. Then, clearly $(u_n, v_n) \to (u, v)$ and $\hat{F}(u, v) = \lim_{n\to\infty} F_n(u_n, v_n)$. Hence, $F_n \xrightarrow{M} \hat{F}$ and Theorem 3 applies; it yields $\mathcal{S}(f_n) \to \mathcal{S}(\hat{f})$ in E. □

Further stability results for the interface problem

Next, we consider the stability of the interface problem with respect to the right-hand side $q \in L^2(\Gamma)$ on the boundary Γ. Thus, in the abstract setting of "Extended real-valued HVIs—well-posedness," we choose the convex functional F as the linear functional

$$F(u, v) := \langle q, \tau u + \imath v \rangle_{L^2(\Gamma)\times L^2(\Gamma)} = (\tau^* q, \imath^* q)(u, v),\ F = (\tau^* q, \imath^* q) \in E^* .$$

By the abstract existence and uniqueness result of Theorem 2, we have the control-to-state map $q \in L^2(\Gamma) \mapsto \mathcal{S}(q) := (\hat{u}, \hat{v}) \in E$, the solution of Eq. 4.1.

Theorem 5 *Suppose that the generalized directional derivative J^0 satisfies the one-sided Lipschitz condition Eq. (2.21) and the growth condition Eq. (3.4). Moreover, suppose the smallness condition Eq. (2.22) with the monotonicity constant $c_{\mathcal{A}}$ of the operator $\mathcal{A}$. Let $\{q_n\}_{n\in\mathbb{N}} \subset L^2(\Gamma)$ and $q_n \to \hat{q}$ in $L^2(\Gamma)$ for $n \to \infty$. Then, there holds $\mathcal{S}(q_n) \to \mathcal{S}(\hat{q})$ in E. If only $q_n \rightharpoonup \hat{q}$ in $L^2(\Omega)$, then there holds $\mathcal{S}(q_{n_k}) \to \mathcal{S}(\hat{q})$ in E for a subsequence $\{q_{n_k}\}_{k\in\mathbb{N}}$.*

Proof The proof follows from arguments similar to those that were given in the proof of Theorem 4. So the details are omitted. □

Let us remark that we can also treat the simultaneous dependence of the interface problem on the parameters $f \in L^2(\Omega)$ and $q \in L^2(\Gamma)$. Then, we have the control-to-state map $(f, q) \in L^2(\Omega) \times L^2(\Gamma) \mapsto \mathcal{S}(f, q) := (\hat{u}, \hat{v})$, the solution of Eq. 4.1. By similar reasoning, we obtain an analogous stability result. The details are omitted.

A stability result for a bilateral obstacle interface problem

To conclude this section, we investigate a related bilateral obstacle interface problem similar to [36]. First, for the strong formulation, we modify the nonlinear partial differential equation Eq. (2.1) in the interior part $\Omega \subset \mathbb{R}^3$ to the obstacle problem: Find $u = u(x) \in [\underline{u}(x), \overline{u}(x)]$ such that

$$\left.\begin{aligned} -\mathrm{div}\left(p(|\nabla u|)\cdot\nabla u\right) &\ge f \quad \text{if } u = \underline{u} && \text{a.e. in } \Omega,\\ -\mathrm{div}\left(p(|\nabla u|)\cdot\nabla u\right) &= f \quad \text{if } \underline{u} < u < \overline{u} && \text{a.e. in } \Omega,\\ -\mathrm{div}\left(p(|\nabla u|)\cdot\nabla u\right) &\le f \quad \text{if } u = \overline{u} && \text{a.e. in } \Omega, \end{aligned}\right\} \tag{4.3}$$

where the obstacle functions $\underline{u}, \overline{u} \in H^1(\Omega)$ with $\underline{u} \le \overline{u}$ a.e. in Ω are given. In the exterior part Ω^c, we consider still the Laplace equation Eq. (2.2) with the radiation condition Eq. (2.3). The transmission conditions Eqs. (2.4), (2.5), and (2.6) remain in force. The variational analysis described in Section 2.2 easily modifies to arrive at the following HVI problem $(P_{\mathcal{A}\Leftrightarrow\mathcal{C}})$: Find $(\hat{u}, \hat{v}) \in \mathcal{C}$ such that for all $(u, v) \in \mathcal{C}$,

$$\mathcal{A}(\hat{u}, \hat{v}; u - \hat{u}, v - \hat{v}) + J^0(\gamma\hat{v}; \gamma(v - \hat{v})) \ge \lambda(u - \hat{u}, v - \hat{v}), \tag{4.4}$$

where the operator $\mathcal{A}$, the generalized directional derivative J^0, the linear continuous embedding operator γ, and the linear functional λ are defined as before and above in this section, whereas now $E := H^1(\Omega) \times H^1(\Omega)$ and the constraint set

$$\mathcal{C} := \mathcal{C}_{\underline{u},\overline{u}} := \{(u, v) \in E \,|\, \underline{u} \le u \le \overline{u} \text{ a.e. in } \Omega\} \tag{4.5}$$

is closed and convex. This gives rise to the closed convex functional

$$F := F_{\underline{u},\overline{u}} := \chi_{\mathcal{C}} = \chi_{\mathcal{C}_{\underline{u},\overline{u}}} .$$

Here, we consider the dependence on the obstacles $\underline{u}, \overline{u}$ distributed on the domain Ω and introduce the admissible set

$$U_{\mathrm{ad}} := \{(\underline{u}, \overline{u}) \in E |\, \underline{u} \le \overline{u} \text{ a.e. in } \Omega\}\,.$$

By the abstract existence and uniqueness result of Theorem 2, we have the control-to-state map $(\underline{u}, \overline{u}) \in U_{ad} \mapsto \mathcal{S}(\underline{u}, \overline{u}) := (\hat{u}, \hat{v}) \in \mathcal{C}$, the solution of Eq. 4.4.
An essential ingredient in the subsequent proof of the subsequent stability result is the Mosco convergence of constraint sets. For that latter result, we exploit the lattice structure of $H^1(\Omega)$. Namely, since Ω is supposed to be a Lipschitz domain, $H^1(\Omega)$ is a *Dirichlet space* ([4, Theorem 5.23], [33, Corollary A.6]) in the following sense: Let $\theta : \mathbb{R} \to \mathbb{R}$ be a uniformly Lipschitz function such that the derivative θ' exists except at finitely many points and that $\theta(0) = 0$; then the induced map θ^* on $H^1(\Omega)$ given by $w \in H^1(\Omega) \mapsto \theta \circ w$ is a continuous map into $H^1(\Omega)$. In particular, the map $w \in H^1(\Omega) \mapsto w^+ = \max(0, w) = \frac{1}{2}(w + |w|)$ is a continuous map into $H^1(\Omega)$.
Now, we are in the position to establish the following stability result.

Theorem 6 *Suppose that the generalized directional derivative J^0 satisfies the one-sided Lipschitz condition Eq. (2.21) and the growth condition Eq. (3.4). Moreover, suppose the smallness condition Eq. (2.22) with the monotonicity constant $c_{\mathcal{A}}$ of the operator $\mathcal{A}$. Let $\{(\underline{u}_n, \overline{u}_n)\}_{n\in\mathbb{N}} \subset U_{ad}$ and $(\underline{u}_n, \overline{u}_n) \to (\underline{u}_\infty, \overline{u}_\infty)$ in E for $n \to \infty$. Then, there holds $\mathcal{S}(\underline{u}_n, \overline{u}_n) \to \mathcal{S}(\underline{u}_\infty, \overline{u}_\infty)$*

Proof We can pass to a subsequence of $\{(\underline{u}_n, \overline{u})\}_{n\in\mathbb{N}}$, also denoted $\{(\underline{u}_n, \overline{u}_n)\}_{n\in\mathbb{N}}$ such that

$$\underline{u}_n \to \underline{u}_\infty \,\&\, \overline{u}_n \to \overline{u}_\infty \quad \text{ a.e. in } \Omega\,. \tag{4.6}$$

We claim the Mosco convergence $\mathcal{C}_n \xrightarrow{\mathrm{M}} \mathcal{C}_\infty$ for the constraint sets

$$\mathcal{C}_n := \mathcal{C}_{\underline{u}_n, \overline{u}_n} := \{(u, v) \in E |\, \underline{u}_n \le u \le \overline{u}_n \text{ a.e. in } \Omega\},$$
$$\mathcal{C}_\infty := \mathcal{C}_{\underline{u}_\infty, \overline{u}_\infty} := \{(u, v) \in E |\, \underline{u}_\infty \le u \le \overline{u}_\infty \text{ a.e. in } \Omega\}.$$

To show (m1), let $(u_n, v_n) \in \mathcal{C}_n$ such that $(u_n, v_n) \rightharpoonup (u, v)$ in E for $n \to \infty$. For some subsequence $u_n \to u$ a.e. in Ω. Thus by Eq. 4.6, $\underline{u}_\infty \le u \le \overline{u}_\infty$ a.e. in Ω. Hence, $(u, v) \in \mathcal{C}$ as required and (m1) is proven.

To show (m2), we exploit the above-mentioned lattice structure of $H^1(\Omega)$ and employ a cutting technique. Let $(u, v) \in \mathcal{C}_\infty$. Then, $\underline{u}_\infty \le u \le \overline{u}_\infty$ a.e. in Ω. This means

$$\max(\underline{u}_\infty, \min(\overline{u}_\infty, u)) = \max(\underline{u}_\infty, u) = u\,.$$

Then, set

$$u_n := \max(\underline{u}_n, \min(\overline{u}_n, u)),\; v_n := v\,.$$

By construction $(u_n, v_n) \in \mathcal{C}_n$, moreover, by Eq. 4.6, $\min(\overline{u}_n, u) \to \min(\overline{u}_\infty, u)$ and $u_n \to u$ in $H^1(\Omega)$. Hence, $(u_n, v_n) \to (u, v)$ in E as required. Thus, (m2) and the claimed Mosco convergence for the constraint sets are proved. This entails $F_n \xrightarrow{\mathrm{M}} F_\infty$, where $F_n := \chi_{\mathcal{C}_n}, F_\infty := \chi_{\mathcal{C}_\infty}$. Therefore, in virtue of Theorem 3, $\mathcal{S}(\underline{u}_n, \overline{u}_n) \to \mathcal{S}(\underline{u}_\infty, \overline{u}_\infty)$, and the theorem is proved. □

Conclusions and outlook

This paper has shown how various techniques from different fields of mathematical analysis can be combined to arrive at well-posedness results for a nonlinear interface problem that models nonmonotone frictional contact of elastic infinite media. In particular, we established a stability result for extended real-valued hemivariational inequalities that extends and considerably improves the stability result of [51]. Based on this general stability result, we could present a unified approach to stability results for the interface problem and for a related obstacle interface problem.
In this paper, we focused to the simplest pde on the exterior domain given by the Laplacian, used the fundamental solution from potential theory, and arrived at a computable HVI that lives on the interior domain and on the coupling boundary, only. Thus, the present paper showcases how this HVI approach to interface problems on unbounded domains with non-monotone set-valued transmission conditions could be extended to more involved related contact problems for systems

of pdes from mathematical physics, as soon as a fundamental solution to the linear pde system on the exterior domain is available; let us mention linear elasticity [26, Chapter 5], thermoelasticity (see, e.g., [12]), piezoelectric elasticity (see, e.g., [7]), and hemitropic elasticity (see, e.g., [15]). Also, in fluid mechanics, the presented HVI approach based on the boundary layer potential method could be combined with additional fixed point arguments to arrive at well-posedness results for transmission problems for the Stokes and Darcy–Forchheimer–Brinkman pde system (see [34]), with nonmonotone transmission conditions. In this paper, we dealt with the primal HVI formulation of the interface problem. On the other hand, mixed variational formulations (see, e.g., [3, 22, 27, 52]) are another direction of research.

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Declarations

Conflict of interest The author declares no competing interests.

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