Journal of Mathematical Sciences
https://doi.org/10.1007/s10958-024-07036-w

The Lichnerowicz Laplacian Acting on Symmetric Tensor Fields — The Bochner Technique Point of View

J. Mikeš[1] · S. E. Stepanov[2,3] · I. I. Tsyganok[3]

Accepted: 15 March 2024

Abstract
In this paper, we prove vanishing theorems for the null space of the Laplacian admitting the Weizenböck decomposition and acting on the space of smooth sections of a Riemannian bundle and, in particular, the space of smooth sections of a bundle of symmetric tensors over complete and closed Riemannian manifolds. In addition, we give a general estimate for eigenvalues of the Laplacian admitting the Weizenböck decomposition and acting on the space of smooth sections of a bundle of symmetric tensors over closed Riemannian manifolds.

Keywords Riemannian manifold · Laplacian · Weizenböck decomposition · Symmetric tensors · Vanishing theorem · Eigenvalues of the Laplacian

2020 Mathematics Subject Classification 53B05 · 35R01

The Bochner technique is considered part

of the basis vocabulary of every geometer.

Wu H.-H.

✉ J. Mikeš
josef.mikes@upol.cz

S. E. Stepanov
s.e.stepanov@mail.ru

I. I. Tsyganok
i.i.tsyganok@mail.ru

1 Palacky University, 17. listopadu, 12, 77146 Olomouc, Czech Republic

2 All Russian Institute for Technical Information of the Russian Academy of Sciences, Usievicha 20, 125190 Moscow, Russia

3 Finance University, Leningradsky Prospect, 49–55, 125468 Moscow, Russia

Published online: 01 May 2024

Introduction

The Lichnerowicz Laplacian acting on smooth sections of a tensor bundle over a Riemannian manifold differs from the usual Laplacian acting on functions by the Weitzenböck decomposition formula involving the Riemann curvature tensor (see, for example, ([14], p. 344); ([4], pp. 54-58); [11]). The Laplacian of this type is the simplest elliptic operator and is at the core of Hodge theory as well as the results of de Rham cohomology.

On the other hand, one of the oldest and most important techniques in modern Riemannian geometry is that of the *Bochner technique* (see, for example, ([14], pp. 333-364); [15, 19]). Furthermore, in the well-known monograph ([4], p. 53) the following was written: the Bochner technique is a method of proving vanishing theorems for null space of a Laplace operator admitting a Weitzenböck decomposition and further of estimating its lowest eigenvalue. Therefore, the Bochner technique has a broad scope of applications.

Our paper is structured as follows. The second section of the article discusses the basic information about the Laplacian admitting the Weizenböck decomposition and acting on the space of smooth sections of a Riemannian bundle (see ([15], p.104) and [9]). In particular, we prove several new vanishing theorems for the null space of the Laplace operator acting on these sections and obtain new estimates for its smallest eigenvalue. As an application of the obtained results, in the third section of the article, we consider the kernel and give a general estimate for eigenvalues of the Laplacian admitting the Weizenböck decomposition and acting on traceless q-tensors (special cases were considered in [6]). In addition, in these paper sections, we will show how the Bochner technique works for these operators. This topic is of exceptional importance (see, for example, [13] and its References).

In the presented paper, we will continue the research begun in our article [17].

Preliminary information about a Laplace operator admitting a Weizenböck decomposition

2.1. Let (M, g) be an n-dimensional connected complete Riemannian manifold with the Levi-Civita connection ∇. Further, let E be a Riemannian fiber bundle of rank r over (M, g), i.e., a fiber bundle equipped with a scalar product and a compatible connection (see ([2], p. 378)). At the same time, the scalar product and compatible connection of E (as well as of M) will be denoted by the same symbols g and ∇. Let $P(TM)$ denote the principal $O(n)$-bundle of orthonormal frames on (M, g) or, in particular, $SO(n)$-bundle if (M, g) has an orientation. In this case, we can consider $E \to M$ as a Riemannian vector bundle, associated with $P(TM)$ via an orthogonal representation $\rho : O(n) \to O(E)$.

Moreover, we can define the L^2-global scalar product on C^∞-sections of E by the formula $\langle \theta, \eta \rangle = \int_M g(\theta, \eta) dv_g$ for $\theta, \eta \in C^\infty(E)$ and we introduce the associated Hilbert space $L^2(E)$. Then, using the L^2-structures on $C^\infty(E)$, we can define the *Bochner* (or *connection*) *Laplacian* by the formula $\bar{\Delta} = \nabla^* \nabla$ where ∇^* is the self-adjoint operator with respect to the L^2-global scalar product of the compatible connection $\nabla : C^\infty(E) \to C^\infty(T^*M \otimes E)$ (see ([14], p. 378) and ([2], p. 378)). Due to the above definitions and notations, we can define the natural Laplacian Δ_E: $C^\infty(E) \to C^\infty(E)$ which satisfies the Weitzenböck decomposition formula (see [9])

$$\Delta_E = \bar{\Delta} + t\, \mathfrak{R}, \tag{1}$$

where $t \in \mathbb{R}$ is a suitable constant and $\mathfrak{R}$ is the *Weitzenböck curvature operator* that defines a symmetric endomorphism $\mathfrak{R}_x$: $E_x \to E_x$ for all $x \in M$ (see also ([2], p. 378-379)). The Weitzenböck curvature operator $\mathfrak{R}$ is defined as follows (see [9]): the Riemann curvature tensor R of (M, g) at each its point lies in $\mathrm{Sym}^2(\mathfrak{so}(n))$ for the Lie algebra $\mathfrak{so}(n) \cong \Lambda^2\mathbb{R}^n$. Applying the representation $\rho : \mathfrak{so}(n) \to \mathrm{End}\, E$, we get the Weitzenböck curvature operator $\mathfrak{R}$ in $\mathrm{Sym}^2(\mathrm{End}\, E)$ and then composition of endomorphisms gives a self-adjoint endomorphism of E.

Various geometrical problems give rise to different values of $t \in \mathbb{R}$, (see [9]). Namely, the Laplacian Δ_E is a second-order elliptic linear differential operator on $C^\infty(E)$ which is symmetric with respect to the L^2-global scalar product. On a closed (i.e., compact without boundary) manifold (M, g), for fixed $t \in \mathbb{R}$, we have the orthogonal (with respect to the L^2-global scalar product) decomposition formula (see ([2], p. 464))

$$C^\infty(E) = \ker \Delta_E \oplus \mathrm{Im} \Delta_E, \tag{2}$$

where the first component $\ker \Delta_E$ is the kernel of the Laplacian Δ_E. Since Δ_E is elliptic, the dimension of the kernel of Δ_E is finite. In accordance with ([15], p. 104), its smooth sections will be called Δ_E *-harmonic*. Moreover, we define the vector space of Δ_E-harmonic C^∞-sections of $E \to M$ by $\ker \Delta_E = \{\varphi \in C^\infty(E)\colon \Delta_E \varphi = 0\}$ for an arbitrary $t \neq 0$. On the other hand, for $t = 0$ any smooth Δ_E-harmonic section satisfies the equation $\nabla\varphi = 0$.
If (M, g) is a closed manifold, then multiplying the Weitzenböck formula (1) by φ and integrating over M, gives (see also ([2], p. 389))

$$\begin{aligned}\int_M g(\Delta_E\varphi, \varphi) dv_g &= \int_M \{g(\bar{\Delta}\varphi, \varphi) + t\, g(\mathfrak{R}(\varphi), \varphi)\} dv_g \\ &= \int_M \|\nabla\varphi\|^2 dv_g + t\int_M g(\mathfrak{R}(\varphi), \varphi)\} dv_g \geq t\int_M g(\mathfrak{R}(\varphi), \varphi)\} dv_g,\end{aligned} \tag{3}$$

where $\varphi \in C^\infty(E)$. In this case, if $t > 0$ and $\mathfrak{R}$ is non-negative endomorphism at each point of M, then the right side of inequality (3) is non-negative. Hence $\Delta_E\varphi = 0$ implies both $\nabla\varphi = 0$ and $\mathfrak{R}(\varphi) = 0$. Furthermore, if $\mathfrak{R}$ is strictly positive at some point of M, then clearly $\varphi = 0$. Therefore, we can formulate the following theorem.

Theorem 2.1 *Let Δ_E be the Laplacian on C^∞-sections of a Riemannian fiber bundle $E \to M$ of rank r over a closed Riemannian manifold (M, g), satisfying (1).*

(i) *If $t > 0$ and the Weitzenböck curvature operator $\mathfrak{R}$ satisfies $\mathfrak{R} \geq 0$, then $\nabla\varphi = 0$ for an arbitrary $\varphi \in \ker \Delta_E$ and $\dim_{\mathbb{R}} \ker \Delta_E \leq r$. In particular, if $\mathfrak{R} \geq 0$, and there exists an x in M such that $\mathfrak{R} > 0$, then $\dim_{\mathbb{R}} \ker \Delta_E = 0$.*
(ii) *On the other hand, if $t < 0$ and the Weitzenböck curvature operator $\mathfrak{R}$ satisfies $\mathfrak{R} \leq 0$, then $\nabla\varphi = 0$ for an arbitrary $\varphi \in \ker \Delta_E$ and $\dim_{\mathbb{R}} \ker \Delta_L \leq r$. In particular, if $\mathfrak{R} \leq 0$, and there exists an x in M such that $\mathfrak{R} < 0$, then $\dim_{\mathbb{R}} \ker \Delta_E = 0$.*

The Weitzenböck curvature operator $\mathfrak{R}$ has the following representation (see ([5], p. 1214), [9])

$$\mathfrak{R} = -\sum_{a,b} R_{ab} \mathrm{d}\varrho(E_a)\mathrm{d}\varrho(E_b),$$

where $\{E_a\}$ is an orthonormal basis of the Lie algebra $\mathfrak{so}(n) \cong \Lambda^2\mathbb{R}^n$, such that the curvature operator of the first kind $\hat{R} : \Lambda^2 M \to \Lambda^2 M$ of (M, g) is identified by $\hat{R} = \sum_{a,b} R_{ab} E_a \otimes E_b$ (see ([14], p. 345); [5, 9]). As observed by Hitchin (see [9]), the curvature operator of the first kind $\hat{R} : \Lambda^2 M \to \Lambda^2 M$ of (M, g) is positive semidefinite (resp. positive) if and only if the Weitzenböck curvature operator $\mathfrak{R}$ is positive semidefinite (resp. positive) for all irreducible representations $\varrho : SO(n) \to O(E)$. Or in other words, if $\varrho : \mathfrak{so}(n) \to \mathrm{End}\, E$ has no trivial factors and $\hat{R}$ is positive definite, then $\mathfrak{R}$ is also positive definite. In particular, if $\hat{R}$ is positive semidefinite, then so is $\mathfrak{R}$ (see [5, 9]). Due to the above definitions and notations, we can formulate a corollary.

Corollary 2.1 *Let Δ_E be the Laplacian on C^∞-sections of a Riemannian fiber bundle $E \to M$ of rank r over a closed Riemannian manifold (M, g), satisfying (1) for $t > 0$. If the representation $\varrho : \mathfrak{so}(n) \to \mathrm{End}\, E$ has no trivial factors and the curvature operator of the first kind $\hat{R} : \Lambda^2 M \to \Lambda^2 M$ of (M, g) is positive semidefinite, then $\nabla\varphi = 0$ for an arbitrary $\varphi \in \ker\Delta_E$ and $\dim_{\mathbb{R}}\ker\Delta_E \leq r$. In particular, if $\hat{R}$ is positive definite, $\dim_{\mathbb{R}}\ker\Delta_E = 0$.*

The above arguments belong to the *classical Bochner technique* (see, for example, ([14], pp. 333-364)). In what follows, we will assume that (M, g) is a complete noncompact Riemannian manifold. First of all, we define the vector space of $L^q(E)$-sections of $E \to M$ by the condition $L^q(\ker \Delta_E) = \{\varphi \in \ker \Delta_E\colon \|\varphi\| \in L^q(M)\}$, where $\|\varphi\|^2 = g(\varphi, \varphi)$.
Let $\varphi \in \ker \Delta_E$, then from the second Kato inequality (see ([2], p. 380))

$$\|\varphi\| \cdot \Delta\|\varphi\| \geq -g(\bar{\Delta}\varphi, \varphi),$$

where $\Delta = div \circ grad$ is the *Beltrami Laplacian* on function and Weitzenböck decomposition (1), we deduce

$$\|\varphi\| \cdot \Delta \|\varphi\| \geq t\, g(\mathfrak{R}(\varphi), \varphi). \tag{4}$$

In this case, if $t > 0$ and $\mathfrak{R}_x$ is a non-negative symmetric endomorphism at each point x of M, then the right side of the above inequality is non-negative. In this case, by the S.-T. Yau theorem in the above inequality, we conclude that for any positive number $q > 1$ either $\int_M \|\varphi\|^q dv_g = \infty$ or $\|\varphi\| = C$ for some constant $C \geq 0$ (see [20, 21]). In particular, if $\|\varphi\| \in L^q(M)$ for some the positive number $q > 1$ and the volume of (M, g) is infinite, then the constant C is zero. On the other hand, for the case when $\|\varphi\| = C$, $t > 0$ and $\mathfrak{R}$ is non-negative, we conclude from (4) that $\nabla\varphi = 0$. Then, the following statement holds.

Theorem 2.2 *Let (M, g) be a complete noncompact Riemannian manifold and Δ_E be the natural Laplacian on C^∞-sections of a Riemannian fiber bundle $E \to M$ over (M, g), satisfying (1).*

(i) *If $t > 0$ and $\mathfrak{R}_x$ is a non-negative symmetric endomorphism of E_x at each point x of M, then $\nabla\varphi = 0$ for an arbitrary $\varphi \in L^q(\ker \Delta_E)$ and for any positive number $q > 1$. In particular, if (M, g) has infinite volume, then $L^q(\ker \Delta_E)$ is trivial for any number $q > 1$.*

(ii) *On the other hand, if $t < 0$ and $\mathfrak{R}_x$ is a non-negative symmetric endomorphism of E_x at each point x of M, then $\nabla\varphi = 0$ for an arbitrary $\varphi \in L^q(\ker \Delta_E)$ and for any positive number $q > 1$. In particular, if (M, g) has infinite volume, then $L^q(\ker \Delta_E)$ is trivial for any number $q > 1$.*

2.2. Let (M, g) be an n-dimensional $(n \geq 2)$ closed Riemannian manifold. An eigensection of the natural Laplacian Δ_E is $\varphi \in C^\infty(E)$ satisfying the condition $\Delta_L \varphi_x = \lambda\, \varphi_x$ for some $\lambda \in \mathbb{R}$ at each point $x \in M$, where the constant λ is called the *eigenvalue* of Δ_E corresponding to φ. Since Δ_E is an elliptic operator, we can conclude that on a closed manifold (M, g) the space $E(\lambda)$ of Δ_E eigensection of associated with the eigenvalue λ has a finite dimension. An estimate for the dimension of this space can be found in ([2], p. 389). Moreover, the set of all eigenvalues of the Laplacian Δ_E is discrete and forms a nondecreasing sequence of the eigenvalues $\{\lambda_a\}_{a \geq 1}$ counted with multiplicities (see [2]).
We define the real numbers $\mathfrak{R}_{\min} = \inf\{\mathfrak{R}_x : x \in M\}$ and $\mathfrak{R}_{\max} = \sup\{\mathfrak{R}_x : x \in M\}$, then $\mathfrak{R}_{\min} \leq \mathfrak{R}_x \leq \mathfrak{R}_{\max}$ for any $x \in M$. Next, let t be a fixed positive number, then integrating the Weitzenböck formula (1), we obtain (see also ([2], p. 394))

$$\int_M g(\Delta_L \varphi, \varphi) dv_g \geq t\, \mathfrak{R}_{\min} \int_M \|\varphi\|^2 dv_g,$$

Then, from the above inequality, we deduce $\lambda \geq t\, \mathfrak{R}_{\min}$ for any $\varphi \in E(\lambda)$. As a result, we can formulate the following

Theorem 2.3 *Let $\Delta_L \colon C^\infty(E) \to C^\infty(E)$ be the natural Laplacian on a closed Riemannian manifold (M, g) satisfying the Weitzenbock decomposition formula $\Delta_L = \bar{\Delta} + t\, \mathfrak{R}$ for $t \in \mathbb{R}$. Then, for all $a \geq 1$ the eigenvalues λ_a of Δ_L satisfy the inequalities*

(i) $\lambda_a \geq t\, \mathfrak{R}_{\min}$ *for a fixed positive number t;*

(ii) $\lambda_a \geq t\, \mathfrak{R}_{\max}$ *for a fixed negative number t.*

2.3. Let $t > 0$ and $E = \otimes^p T^* M$, then the natural Laplacian on C^∞-sections of E satisfying (1) is the *Lichnerowicz Laplacian* defined in ([14], p. 344). In special cases when $t = 1$ the Lichnerowicz-type Laplacian is the ordinary Lichnerowicz Laplacian defined in ([11], p. 315) and ([4], p. 54) by the Weitzenböck decomposition formula $\Delta_L = \bar{\Delta} + \mathfrak{R}$, where the Weitzenböck curvature operator $\mathfrak{R}$ depends linearly on the Riemann curvature tensor R and the Ricci tensor Ric of (M, g).
Remark In particular, if $E = C^\infty(\Lambda^q M)$ for an arbitrary $1 \leq q \leq n-1$, where $\Lambda^q M$ is the vector bundle of differential q-forms on M, then the Lichnerowicz Laplacian Δ_L is the well-studied *Hodge Laplacian* Δ_H acting on C^∞-sections of $\Lambda^q M$ on (M, g) (see ([14], p. 237)). It admits decomposition (1) with $t = 1$, where $\mathfrak{R}$ is an algebraic operator, which is the restriction of the Weitzenböck curvature operator to differential forms. Further in this paper, we will consider the Lichnerowicz Laplacian Δ_L acting on C^∞-sections of the bundle $S^q M$ of covariant symmetric q-tensors (see ([3], pp. 387-388)).

The theory of such Lichnerovich Laplacians has hardly been studied for the case $q > 2$. An exception is the studies of Boucetta (see, for example, [6]).

On the kernel and estimate for the eigenvalues of the Lichnerowicz Laplacian acting on symmetric tensors

3.1. In this section, we consider the Lichnerowicz Laplacian $\Delta_L\colon C^\infty(S^qM)\to C^\infty(S^qM)$. In this case, we have the Weitzenböck decomposition formula (1) for a suitable constant $t > 0$ and the curvature operator $\mathfrak{R}$ given by the formula (see ([4], p. 54); ([11], p. 315))

$$\mathfrak{R}(\varphi)_{i_1\dots i_q} = \sum_{a=1}^{q} g^{jk}R_{i_aj}\varphi_{i_1\dots i_{a-1}ki_{a+1}\dots i_q} - \sum_{a,b=1,a\neq b}^{q} g^{jk}g^{lm}R_{i_aji_bl}\varphi_{i_1\dots i_{a-1}ki_{a+1}\dots i_{b-1}mi_{b+1}\dots i_q}$$

for any $i, j, k, l, \dots \in \{1, 2, \dots, n\}$ and for local components $\varphi_{i_1\dots i_q}$, R_{ik} and R_{ikjl} of $\varphi \in C^\infty(S^qM)$ the Ricci tensor Ric and the curvature tensor R, respectively. These components are defined by the formulas (see ([10], pp. 203; 249))

$$\varphi_{i_1\dots i_q} = \varphi(e_{i_1}, \dots, e_{i_q}), \quad R^l_{kji}e_l = R(e_i, e_j)e_k, \quad R_{ijkl} = g_{im}R^m_{jkl},$$

where $g_{ij} = g(e_i, e_j)$ for any frame $\{e_1, \dots, e_n\}$ of T_xM at an arbitrary point x. In addition, we have that $R_{lkij} + R_{ijkl} = 0$, $R_{kl} = R^i_{kil}$, $R^k_j = g^{ki}R_{ij}$, where $(g_{ij}) = (g^{ij})^{-1}$.

Remark Some writers denote our $R_{ijkl} = g_{im}R^m_{jkl}$ by $R_{jkli} = g_{im}R_{jkl}{}^m$ for $R_{ijk}{}^m e_m = R(e_i, e_j)e_k$.

In this case, the Weitzenböck curvature operator $\mathfrak{R}\colon \otimes^q T^*M \to \otimes^q T^*M$ satisfies the following identities (see ([11], p. 315)):

$$g(\mathfrak{R}(\varphi), \varphi') = g(\varphi, \mathfrak{R}(\varphi')), \quad trace_g\mathfrak{R}(\varphi) = \mathfrak{R}(trace_g(\varphi))$$

for any $\varphi, \varphi' \in C^\infty(S^qM)$.

By direct calculations from the Weitzenböck decomposition formula (1), we obtain the following formula:

$$\frac{1}{2}\,\Delta\|\varphi\|^2 = -g(\Delta_L\varphi, \varphi) + \|\nabla\varphi\|^2 + t\,Q(\varphi), \tag{5}$$

where $Q(\varphi) = g(\mathfrak{R}(\varphi), \varphi)$ is a quadratic form $Q\colon S^qM \otimes S^qM \to \mathbb{R}$ such that (see also [16])

$$\begin{aligned} Q(\varphi) &= \mathfrak{R}(\varphi)_{i_1,\dots,i_q}\varphi^{i_1,\dots,i_q} \\ &= q\cdot R_{ij}\varphi^{ikk_3\dots k_q}\varphi^j_{\ kk_3\dots k_q} - q(q-1)R_{ijkl}\varphi^{ikk_3\dots k_q}\varphi^{jl}{}_{kk_3\dots k_q}, \end{aligned} \tag{6}$$

where $\varphi^{i_1,\dots,i_q} = \varphi_{j_1,\dots,j_q}g^{i_1j_1}\dots g^{i_qj_q}$ are local contravariant components of an arbitrary $\varphi \in S^qM$.

On the other hand, we recall that the point-wise symmetric curvature operator $\mathring{R}\colon S^2_0M \to S^2_0M$ defined on traceless symmetric two-tensor fields by equations

$$\mathring{R}(\varphi)_{ij} = R_{iklj}\varphi^{kl} \tag{7}$$

for the local contravariant components $\varphi^{kl} = \varphi_{ij}g^{ik}g^{jl}$ of an arbitrary $\varphi \in S^2_0M$ is called as the *curvature operator of the second kind* (see [7]).

Remark In the monograph ([4], p. 52), the curvature operator of the second kind was defined by the formula $\mathring{R}(\varphi)_{ij} = R_{ikjl}\varphi^{kl}$ since the local components R_{jkli} of the Riemann curvature tensor were defined there by the identities $R_{ijkl} = g(R(e_i, e_j)e_k, e_l)$.

We say that $\overset{\circ}{R} \geq 0$ (resp., $\overset{\circ}{R} > 0$) if the eigenvalues of $\overset{\circ}{R}$ as a bilinear form on $S_0^2 M$ are non-negative (resp., strictly positive). At an arbitrary point $x \in M$, we choose orthogonal unit vectors $X, Y \in T_x M$ at an arbitrary point $x \in M$ and define the symmetric 2-tensor field $\theta = X \otimes Y + Y \otimes X$, then by direct calculation, we obtain

$$\begin{aligned} g(\overset{\circ}{R}(\theta), \theta) &= R_{iklj}(X^i Y^j + X^j Y^i)(X^k Y^l + X^l Y^k) \\ &= R_{iklj} X^i Y^j X^l Y^k + R_{iklj} X^j Y^i X^k Y^l \\ &= 2\, sec(X, Y). \end{aligned} \tag{8}$$

Therefore, if the operator $\overset{\circ}{R}$ is nonnegative (resp. strictly positive) defined on any section of the bundle $S_0^2 M$, then the sectional curvature of (M, g) is everywhere non-negative (resp. positive) (see also ([7], p. 196)). In addition, if $X \in T_x M$ is a unit vector and we complete it to an orthonormal basis $\{X, e_2, \dots, e_n\}$ for $T_x M$, then

$$Ric(X, X) = \sum_{i=2}^{n} sec(X, e_i). \tag{9}$$

Therefore, the Ricci curvature of (M, g) is everywhere non-negative (resp. positive) if the operator $\overset{\circ}{R}$ is non-negative (resp. strictly positive), defined on any section of the bundle $S_0^2 M$.

Remark We recall here that closed n-dimensional Riemannian manifolds, whose curvature operator of the second kind acting on symmetric traceless tensors is everywhere positive definite, are diffeomorphic to a spherical space form $\mathbb{S}/\Gamma$ (see [7]).

Next, if we suppose that $\overset{\circ}{R} \geq 0$ (resp., $\overset{\circ}{R} > 0$), then $R_{ijkl}\varphi^{ik}\varphi^{jl} \leq 0$ (resp., $R_{ijkl}\varphi^{ik}\varphi^{jl} < 0$) for an arbitrary $\varphi \in S_0^2 M$ with local components φ_{ij} and, therefore, $R_{ijkl}\varphi^{ikk_3\dots k_q}\varphi^{jl}_{k_3\dots k_q} \leq 0$ (resp., $R_{ijkl}\varphi^{ikk_3\dots k_q}\varphi^{jl}_{k_3\dots k_q} < 0$) for an arbitrary $\varphi \in S_0^q M$ with local components $\varphi_{k_1\dots k_q}$ for $q \geq 2$ (see also [15]). Furthermore, $R_{ij}\varphi^{ik_2\dots k_q}\varphi^{j}_{k_2\dots k_q} \geq 0$ (resp., $R_{ij}\varphi^{ik_2\dots k_q}\varphi^{j}_{k_2\dots k_q} > 0$) for an arbitrary $\varphi \in S_0^q M$ with local components $\varphi_{k_1\dots k_q}$ for $q \geq 2$.

As a result, from the above, we conclude that if (M, g) is a Riemannian manifold with positive semidefinite (resp. strictly positive) curvature operator of the second kind, then $Q(\varphi) = g(\mathfrak{R}(\varphi), \varphi) \geq 0$ (resp. $Q(\varphi) = g(\mathfrak{R}(\varphi), \varphi) > 0$).

Let (M, g) be a closed Riemannian manifold and $\Delta_L \varphi = 0$ for some $\varphi \in C^\infty(S_0^q M)$, then from (5), we deduce the following integral formulas

$$0 = \int_M (\|\nabla\varphi\|^2 + t\, Q(\varphi))dv_g \geq t \int_M Q(\varphi) dv_g.$$

It is obvious that $\overset{\circ}{R}$ is positive-semidefinite on (M, g) and $\overset{\circ}{R} > 0$ at some point at (M, g), then there are no non-zero Δ_L -harmonic traceless symmetric q-tensor fields ($q \geq 2$).

Theorem 3.1 *Let (M, g) be a closed Riemannian manifold. If the curvature operator of the second kind $\overset{\circ}{R}$ is positive-semidefinite on (M, g), then $\dim_{\mathbb{R}} \Delta_L \leq \binom{n+q-1}{q}$. Moreover, if $R > 0$ at every point of (M, g), then $\dim_{\mathbb{R}} \ker \Delta_L = 0$.*

Remark A form $\omega \in C^\infty(\Lambda^q M)$ is *harmonic* if it is in the kernel of the Hodge Laplacian: $\Delta_H \omega = 0$. The main motivation for considering this operator is that the dimension of the vector space of harmonic q-forms, denoted $\mathfrak{H}^q(M, \mathbb{R})$, is a topological invariant, and in fact equals to the q-th de Rham Betti number $b_q(M, \mathbb{R})$ of M. Recall that the *curvature operator of the first kind* is the self-adjoint endomorphism $\hat{R}$ of $\Lambda^2 M$, defined by: $g(\hat{R}(X \wedge Y), Z \wedge W) = R(X, Y, Z, W)$, where R is the Riemann tensor and X, Y, Z, W are tangent vectors (see ([14], p. 116)). Then, if $\hat{R}$, is positive-semidefinite on (M, g) then $dim_{\mathbb{R}} \Delta_H \leq \binom{n}{q}$. Moreover, if $\hat{R} > 0$ at some point at (M, g), then there are no non-zero harmonic q-forms ($1 \leq q \leq n-1$). In this case $dim_{\mathbb{R}} \ker \Delta_H = b_q(M, \mathbb{R}) = 0$ (see ([14], p. 351)).

On the other hand, it is obvious that (4) is valid for any $q \geq 2$. In this case, from (4), we obtain the inequality

$$\|\varphi\|\Delta\|\varphi\| \geq t\, Q(\varphi), \tag{10}$$

for any $\varphi \in C^\infty(S_0^q M) \cap \ker \Delta_L$. If the curvature operator of the second kind of (M, g) is non-negative at an arbitrary point $x \in M$, then $Q(\varphi) = g(\mathfrak{R}(\varphi), \varphi) \geq 0$. At the same time, if a Riemannian manifold has a non-negative curvature operator of the second kind, then it has non-negative sectional curvature (see, our formula (8)). On the other hand, if a complete non-compact Riemannian manifold has non-negative sectional curvature, then it must have infinite volume (see [20]). Therefore, we can reformulate our Theorem 2.2 in the following form.

Theorem 3.2 *Let (M, g) be a connected noncompact and complete Riemannian manifold with non-negative curvature operator of the second kind of (M, g). Then, the vector space $L^q(\ker \Delta_L)$ is trivial for any $1 < p < \infty$.*

Remark This assertion is an analog of the vanishing theorem for q-forms of the space $L^2(\ker \Delta_H)$ on complete Riemannian manifolds (see [1]).
Let (M, g) be a connected closed Riemannian manifold and φ be a Δ_L- harmonic traceless symmetric q-tensor field $(q \geq 2)$ defined on (M, g), then proceeding from the formula (5) and using the *strong maximum principle* (see ([14], p. 75)), we conclude that the following corollary holds.

Corollary 3.1 *Let (M, g) be a connected closed Riemannian manifold and φ be a Δ_L-harmonic traceless symmetric q-tensor field $(q \geq 2)$ defined on (M, g). If the curvature operator of the second kind of (M, g) is positive semi-define at every points, then $\|\varphi\|^2$ is a constant function and φ is invariant under parallel translation in (M, g).*

3.2. We denote by S_0^qM the subbundle of the bundle S^qM on a Riemannian manifold defined by the condition $trace_g\varphi = \sum_{i=1}^n \varphi(e_i, e_j, X_3, \dots, X_q) = 0$ for $\varphi \in S_0^qM$ and orthonormal basis $\{e_1 \dots, e_n\}$ of T_xM at an arbitrary point $x \in M$ (see the details of the theory in our monograph [12]). The Laplacian Δ_L maps $C^\infty(S^qM)$ into itself. This property is a corollary of (3) for any $\varphi \in C^\infty(S^qM)$. In particular, the following equation holds

$$trace_g(\Delta_L\varphi) = \Delta_L(trace_g\varphi) = 0 \tag{11}$$

for an arbitrary $\varphi \in C^\infty(S_0^qM)$ and $q \geq 2$. Then, we can conclude that $\Delta_L\colon \varphi \in C^\infty(S_0^qM) \to \varphi \in C^\infty(S_0^qM)$ since $trace_g\varphi = 0$. Furthermore, we also conclude that $trace_g\varphi \in \ker \Delta_L$ for an arbitrary $\varphi \in C^\infty(S^qM)$ such that $\varphi \in \ker \Delta_L$.

Lemma 3.1 *Let $\Delta_L\colon C^\infty(S^qM) \to C^\infty(S^qM)$ be the Lichnerowicz Laplacian acting on C^∞-sections of the bundle of covariant symmetric p-tensor fields S^qM $(q \geq 2)$ on a Riemannian manifold (M, g). Then,*

(i) *$trace_g\varphi \in \ker \Delta_L$ for an arbitrary $\varphi \in C^\infty(S^qM)$ such that $\varphi \in \ker \Delta_L$;*
(ii) *Δ_L maps S_0^qM into itself for the subbundle S_0^qM of traceless covariant symmetric p-tensor fields on (M, g).*

If $\varphi \in C^\infty(S_0^qM)$ and the sectional curvature of (M, g) is nonnegative at an arbitrary point $x \in M$, then $Q(\varphi) = g(\mathfrak{R}(\varphi), \varphi) \geq 0$ (see [5]). In this case, for any $\varphi \in C^\infty(S_0^qM) \cap \ker \Delta_L$, we can reformulate our Theorem 2.2 in the following form.

Theorem 3.3 *Let (M, g) be a connected noncompact and complete Riemannian manifold with non-negative sectional curvature. Then, there is no non-zero Δ_L-harmonic traceless symmetric q-tensor field $(q \geq 2)$ such that it lies in L^p for some $1 < p < \infty$.*

The following corollary is obvious (see also the proof of Theorem 2.2).

Corollary 3.2 *Let (M, g) be a connected closed Riemannian manifold and φ be a Δ_L-harmonic traceless symmetric q-tensor field $(q \geq 2)$ defined on (M, g). If the section curvature of (M, g) is positive semi-define at every points, then $\|\varphi\|^2$ is a constant function and φ is invariant under parallel translation in (M, g).*

In what follows, we will assume that (M, g) is a closed Riemannian manifold. In this case, the Lichnerowicz Laplacian Δ_L: $C^\infty(S^qM) \to C^\infty(S^qM)$ has a discrete spectrum $Spec^{(q)}\Delta_L$ on (M, g) since it is an elliptic formally self-adjoint second-order differential operator. If $\lambda_a^{(q)} \in Spec^{(q)}\Delta_L$ is an eigenvalue of Δ_L corresponding to an eigentensor $\varphi \in C^\infty(S^qM)$ for the case $q \geq 2$, then from (5), we obtain the integral inequality

$$\lambda_a^{(q)} \int_M \|\varphi\|^2 dv_g \geq t \int_M Q(\varphi) dv_g.$$

Let $\Delta_L: C^\infty(S_0^qM) \to C^\infty(S_0^qM)$ be the Lichnerowicz Laplacian acting on the bundle S_0^qM of traceless symmetric q-tensors over (M, g). Then, if we suppose that the curvature operator of the second kind $\mathring{R}: S_0^2M \to S_0^2M$ of (M, g) is positive semidefinite (resp. strictly positive), then from the above formula, we conclude that $\lambda_a^{(q)} \geq 0$ (resp. $\lambda_a^{(q)} > 0$). Namely, assume that the curvature operator of the second kind $\mathring{R}: S_0^2M \to S_0^2M$ of (M, g) satisfies the inequality $g(\mathring{R}(\varphi), \varphi) \geq K \|\varphi\|^2$ for certain positive number K and for any symmetric 2-tensor field φ. Taking two mutually orthogonal unite vector fields X and Y and putting $\varphi = X \otimes Y + Y \otimes X$ we find, from (8),

$$sec(X, Y) \geq K > 0. \tag{12}$$

Then, from (9) and (12), we obtain $Ric(X, X) \geq (n-1)K > 0$. Thus,

$$\begin{aligned} Q(\varphi) &= \mathfrak{R}(\varphi)_{i_1\ldots i_q}\varphi^{i_1\ldots i_q} \\ &= qR_{ij}\varphi^{ik_2\ldots k_q}\varphi^j_{k_2\ldots k_q} - q(q-1)R_{ijkl}\varphi^{ikk_3\ldots k_q}\varphi^{jl}_{k_3\ldots k_q} \geq q(n-1)K\|\varphi\|^2 + q(q-1)K\|\varphi\|^2 \\ &= q(n+q-2)K\|\varphi\|^2 \end{aligned}$$

which is positive for $n \geq 2$ and $q \geq 1$. In this case, from Theorem 2.3, we can conclude that the following corollary holds.

Corollary 3.3 *Let (M, g) be an n-dimensional closed Riemannian manifold with the curvature operator of the second kind $\mathring{R}: S_0^2M \to S_0^2M$ satisfying the inequalities $\mathring{R} \geq 2K > 0$ for certain positive number K. At the same time, let $\Delta_L: C^\infty(S_0^qM) \to C^\infty(S_0^qM)$ be the Lichnerowicz Laplacian that satisfies the Weitzenböck decomposition $\Delta_L = \bar{\Delta} + t\,\mathfrak{R}$ for a fixed positive number t. Then, for an arbitrary $a \geq 1$ the eigenvalues $\lambda_a^{(q)}$ of Δ_L satisfy the inequality $\lambda_a^{(q)} \geq q(n+q-2)\,tK$.*

Remark Having discussed above the kernel of the Hodge Laplacian Δ_H, we now turn our attention to its first positive eigenvalue, which we will denote by the symbol: $\mu_1^{(q)}$. Then, if (M, g) is an n-dimensional closed manifold with the curvature operator of the first kind $\hat{R}$ satisfying the inequalities $\hat{R} \geq K > 0$ for certain positive number K, then $\mu_1^{(q)} \geq C(n, q)K$, where $C(n, q) = \min\{q(n-q+1); (q+1)(n-q)\}$ for all $q = 1, \ldots, n-1$ (see [8]).
In particular, let (M, g) is a manifold with constant sectional curvature $K > 0$ (see ([10], p. 203)), then its Riemann and Ricci curvature tensors are given by the identities $R_{ijkl} = K(g_{ik}g_{jl} - g_{il}g_{jk})$ and $R_{ij} = K(n-1)g_{ij}$, respectively. Then, using these equalities and the equality (7) for any symmetric form $\varphi \in S_0^qM$, we have

$$\begin{aligned} g(\mathfrak{R}(\varphi), \varphi) &= qR_{ij}\varphi^{ik_2\ldots k_q}\varphi^j_{k_2\ldots k_q} - q(q-1)R_{ijkl}\varphi^{ikk_2\ldots k_q}\varphi^{jl}_{k_2\ldots k_q} \\ &= (n-1)K\|\varphi\|^2 + q(q-1)K\|\varphi\|^2 \\ &= (n+q-2)K\|\varphi\|^2 \geq 0, \end{aligned}$$

where equality is possible in the case $\varphi = 0$. In this case, we deduce from (5) the integral inequality

$$(-\lambda_a^{(q)} + q(n+q-2)tK) \int_M \|\varphi\|^2 dv_g \leq 0.$$

Then, from the previous inequality, we conclude that $\lambda_a^{(q)} \geq q(n+q-2)\,tK$ for the first non-zero eigenvalues $\lambda_1^{(q)}$ of the Lichnerowicz Laplacian $\Delta_L: C^\infty(S_0^qM) \to C^\infty(S_0^qM)$ acting on the bundle S_0^qM of traceless symmetric q-tensors on a Riemannian manifold with constant sectional curvature $K > 0$. Thus, we have proved the following theorem.

Theorem 3.4 *Let (M, g) be an n-dimensional closed Riemannian manifold with constant sectional curvature $K > 0$ and let $\Delta_L: C^\infty(S_0^qM) \to C^\infty(S_0^qM)$ be the Lichnerowicz Laplacian acting on the bundle S_0^qM of traceless symmetric q-tensors over (M, g). Then, the first non-zero eigenvalue $\lambda_1^{(q)}$ of Δ_L satisfies the inequality $\lambda_1^{(q)} \geq q(n+q-2)\,tK$.*

Acknowledgements The research of the first author was supported by project IGA PrF 2024011 Palacky University Olomouc.

Funding Open access publishing supported by the National Technical Library in Prague.

Data availability We do not analyze or generate any datasets, because our work proceeds within a theoretical and mathematical approach. One can obtain the relevant materials from the references below.

Declarations

Conflict of interest The authors declare no competing interests.

REFERENCES

1. M.T. Anderson, L^2-harmonic forms on complete Riemannian manifolds, Lect. Notes Math., 1339, 1-19 (2006).
2. P. H. Bérard, From vanishing theorems to estimating theorems: the Bochner technique revisited, Bull. AMS, 19:2, 371–406 (1988).
3. M. Berger and D. Ebine, Some decomposition of the space of symmetric tensors of a Riemannian manifold, J. Diff. Geom., 3, 379–392, (1969).
4. A.L. Besse, Einstein Manifolds, Springer, Berlin (1987).
5. R.G. Bettiol and A.E. Mendes, Sectional curvature and Weitzenböck formula, Indiana Univ. J., 71, 1209–1242 (2022).
6. M. Boucetta, Spectra and symmetric eigentensors of the Lichnerowicz Laplacian on $P^n(\mathbb{C})$, J. Geom. Phys., 60:10, 1352–1369 (2010).
7. X. Cao, M.J. Gusky and H. Tran, Curvature of the second kind and conjecture of Nishikawa, Comment. Math. Helv., 98:1, 195–216 (2023).
8. S. Gallot and D. Meyer, Operateur de courbure et laplacien des formes différentielles d'une varieté riemannianne, J. Math. Pures. Appl., 54:3, 259–284 (1975).
9. N. Hitchin, A note on vanishing theorems, In: Geometry and Analysis on Manifolds, Progr. Math., 308, 373–382 (2015).
10. S. Kobayashi and K. Nomizu, Foundations of Differential Geometry, Vol. I, Intersci. Publ., New York (1963).
11. A. Lichnerowicz, Propagateurs et commutateurs en relativité generate, Publ. Math. de l'IHÉS, 10:1, 293–344 (1961).
12. J. Mikeš et al., Differential Geometry of Special Mappings, Palacky Univ. Press, Olomouc (2019).
13. J. Nienhaus, P. Petersen, M. Wink, Betti numbers and the curvature operator of the second kind, J. London Math. Soc., II, 108:4, 1642–1668 (2023).
14. P. Petersen, Riemannian Geometry, 3d ed., Springer, New York (2016).
15. S. Pigola, M. Rigoli and A. G. Setti, Vanishing and Finiteness Results in Geometric Analysis. A Generalization of the Bochner Technique, Birkhäuser Verlag, Berlin (2008).
16. S. E. Stepanov, Symmetric tensor fields on a compact Riemannian manifold, Math. Notes, 52:3-4, 1048–1050 (1992).
17. S. E. Stepanov, J. Mikeš and V. Rovenski, An example of Lichnerowicz-type Laplacian, Ann. Global Anal. Geom., 58, 19–34, (2020).
18. Sh. Tachibana and K. Ogiue, Les variétés riemanniennes dont l'opérateur de coubure restreint est positif sont des sphéres d'homologie réelle, C. R. Acad. Sci. Paris, 289, 29–30 (1979).
19. H.-H. Wu, The Bochner Technique in Differential Geometry, Harwood Acad. Publ. (1988).
20. S.-T. Yau, Some function-theoretic properties of complete Riemannian manifolds and their applications to geometry, Indiana Univ. Math. J., 25, 659–670 (1976).
21. S.-T. Yau, Erratum: Some function-theoretic properties of complete Riemannian manifold and their applications to geometry, Vol. 25 (1976), 659–670, Indiana Univ. Math. J., 31:4, 607 (1982).