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TOPOLOGICAL MODULES GEOMETRY

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Abstract
The extremal structure of zero-neighbourhoods of a topological module is analyzed reaching unexpected conclusions when the module topology is not Hausdorff. These results motivate us to introduce the notion of metric modules, which are modules endowed with a translation-invariant metric, turning them into an (additive) topological group. We study the central and diametral points of additively symmetric subsets and find examples of convex sets which are not symmetric translates (translates of addivitely symmetric subsets). As a consequence of all of these, it seems natural to transport the well-known Bishop-Phelps property from the category of real topological vector spaces to general topological modules over topological rings. Then we stick to particular topological rings, the unital C^*-algebras, showing that the subset of positive elements lying below the unity is an effect algebra. We also prove that every continuous linear operator on a Hausdorff locally convex topological vector space that commutes with all continuous linear projections of one-dimensional range is a multiple of the identity. Finally, we discuss how to transport the previous result to C^*-algebras.

Keywords Topological module · Topological ring · C^*-algebra · Effect algebra

Introduction

Classical geometrical properties of convex sets in a normed space [9] have been studied in more particular contexts such as normed algebras [33] and more general contexts such as topological vector spaces [31, 14, 22]. Recently, a strong trend has been consolidating which consists in transporting the classical notions of Geometry of Banach Spaces [8, 4, 3, 25, 26, 20] to the category of topological modules over a topological ring [19, 16, 17, 18, 43, 21]. This manuscript contributes to that trend by providing new and unexpected results dealing with the extremal structure of zero-neighbourhoods and additively symmetric subsets of topological modules. Applications to C^*-algebras are also provided.
If X is a bounded pseudometric space, then the diameter of X is defined as $\mathrm{d}(X) := \sup d(X \times X) = \sup\{d(x, y) : x, y \in X\}$. For every $x \in X$, the x-radius of X is defined as $\mathrm{r}_x(X) := \sup\{d(x, y) : y \in X\}$. The radius of X is defined as $\mathrm{r}(X) := \inf\{\mathrm{r}_x(X) : x \in X\}$. It is not hard to check that $\mathrm{d}(X) \leq 2\mathrm{r}(X)$. Indeed, for every $\varepsilon > 0$ there exists $x_0 \in X$ such that $\mathrm{r}(X) > \mathrm{r}_{x_0}(X) - \frac{\varepsilon}{2}$, thus, for every $x, y \in X$, $d(x, y) \leq d(x, x_0) + d(x_0, y) \leq \mathrm{r}_{x_0}(X) + \mathrm{r}_{x_0}(X) < 2\mathrm{r}(X) + \varepsilon$, meaning that $\mathrm{d}(X) \leq 2\mathrm{r}(X)$. We refer the reader to [23] for a wider perspective on such topics.

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The center of X [7] is defined as $X_c := \{x \in X : r_x(X) = r(X)\}$. The points of X_c will be called central points. The diametrality of X is defined as $X_d := \{x \in X : r_x(X) = d(X)\}$. The points of X_d will be called diametral points. And X is said to be diametral provided that $X_d = X$. Notice that

$$X_c = \bigcap_{y \in X} \mathsf{B}_X(y, r(X)) = \bigcap_{y \in X} \left(\bigcap_{n \in \mathbb{N}} \mathsf{B}_X\left(y, r(X) + \frac{1}{n}\right)\right),$$

that is, X_c is admissible if it is nonempty. Also, notice that if $X_d \neq X$ and $X_c \neq \emptyset$, then $d(X_c) < d(X)$. These notions have strong applications on Fixed-Point Metric Theory [30].

All rings considered in this manuscript are assumed to be associative and unitary, and all modules over a ring are considered left and unital by default (notice that the dual of a left module is a right module). Let R be a ring and M an R-module. A subset B of M is said to be (additively) symmetric provided that $B = -B$. A translation-invariant pseudometric on M is a pseudometric d on M satisfying that $d(m+p, n+p) = d(m, n)$ for every $m, n, p \in M$. Such a module will be called a pseudometric module. Observe that a translation-invariant pseudometric turns a module into an additive topological group. The above metric notions (center and diametrality) are invariant under translations in a pseudometric module M. Indeed, if $A \subseteq M$ is bounded, then $r_a(A) = r_{m+a}(m + A)$ for all $m \in M$ and all $a \in A$. Therefore, $(m + A)_c = m + A_c$ and $(m + A)_d = m + A_d$ for all $m \in M$.

Given a topological space X and an element $x \in X$, the filter of all neighbourhoods of x is denoted by $\mathcal{N}_x(X)$, or simply by $\mathcal{N}_x$ if there is no confusion with X. The intersection of all neighbourhoods of x, denoted by

$$X_{(x)} := \bigcap_{V \in \mathcal{N}_x(X)} V = \{y \in X : x \in \mathrm{cl}(\{y\})\},$$

plays an important role when it comes to separation properties. Indeed, if X is a regular topological space, then the following conditions are equivalent for all $x, y \in X$:

1. $y \in \bigcap_{V \in \mathcal{N}_x(X)} V$.
2. $x \in \bigcap_{V \in \mathcal{N}_y(X)} V$.
3. $y \in \mathrm{cl}(\{x\})$.
4. $x \in \mathrm{cl}(\{y\})$.

As a consequence, $X_{(x)} = \mathrm{cl}(\{x\})$, hence $X_{(x)}$ is contained in any closed subset of X that contains x.

Let R be a ring. The following characterization of ring topology [49, 50, 10] will be very much employed. If τ is a ring topology on R and $\mathcal{B}$ is a base of neighbourhoods of 0, then the following is verified:

- For every $V \in \mathcal{B}$, there exists $U \in \mathcal{B}$ with $U + U \subseteq V$.
- For every $V \in \mathcal{B}$, there exists $U \in \mathcal{B}$ with $-U \subseteq V$.
- For every $V \in \mathcal{B}$, there exists $U \in \mathcal{B}$ with $UU \subseteq V$.
- For every $V \in \mathcal{B}$ and every $r \in R$, there exists $U \in \mathcal{B}$ with $rU \cup Ur \subseteq V$.

Conversely, if $\mathcal{B}$ is a filter base of $\mathcal{P}(R)$ verifying all four properties above, then there exists a unique ring topology on R such that $\mathcal{B}$ is a base of 0-neighbourhoods. This topology is given by

$$\tau := \{A \subseteq R : \forall a \in A \ \ \exists U \in \mathcal{B} \ \ a + U \subseteq A\} \cup \{\emptyset\}.$$

Notice that topological rings are uniform spaces, therefore, they are regular topological spaces, but they are not necessarily Hausdorff. In fact, it is not hard to show that a topological ring R is Hausdorff if and only if $\bigcap_{V \in \mathcal{N}_0(R)} V = \{0\}$. Thus, throughout this manuscript, topological rings will not be assumed Hausdorff by default. Topological rings satisfying that zero belongs to the closure of the invertibles will be called practical rings [19]. Next, let M be an R-module for R a topological ring. According to [49, 50], if τ is a module topology on M and $\mathcal{B}$ is a base of neighbourhoods of 0 in M, then the following is verified:

- For every $V \in \mathcal{B}$, there exists $U \in \mathcal{B}$ with $U + U \subseteq V$.
- For every $V \in \mathcal{B}$, there exists $U \in \mathcal{B}$ with $-U \subseteq V$.
- For every $V \in \mathcal{B}$, there exist $U \in \mathcal{B}$ and $W \in \mathcal{N}_0(R)$ with $WU \subseteq V$.
- For every $V \in \mathcal{B}$ and every $r \in R$, there exists $U \in \mathcal{B}$ with $rU \subseteq V$.
- For every $V \in \mathcal{B}$ and every $m \in M$, there exists $W \in \mathcal{N}_0(R)$ with $Wm \subseteq V$.

Conversely, if $\mathcal{B}$ is a filter base of $\mathcal{P}(M)$ verifying all five properties above, then there exists a unique module topology on M such that $\mathcal{B}$ is a base of zero-neighbourhoods. This topology is given by

$$\tau := \{A \subseteq M : \forall a \in A \ \ \exists U \in \mathcal{B} \ \ a + U \subseteq A\} \cup \{\varnothing\}.$$

Topological modules are also uniform spaces, therefore, they are regular topological spaces, but they are not necessarily Hausdorff. In fact, a topological module M is Hausdorff if and only if $\bigcap_{V\in\mathcal{N}_0(M)} V = \{0\}$. Thus, throughout this manuscript, topological modules will not be assumed Hausdorff by default. The notion of boundedness in this setting will be crucial for our results in this manuscript. A subset B of a topological module M over a topological ring R is said to be bounded if for every neighbourhood $V \subseteq M$ of 0 there exists an invertible $u \in \mathcal{U}(R)$ verifying that $B \subseteq uV$. The topological dual of a topological module M over a topological ring R is by definition the module of all continuous R-linear maps from M to R and is usually denoted by M^*. It is worth mentioning that, since M is assumed to be a left R-module, then M^* acquires structure of right R-module. From now on, if there is no confusion with the underlying R, we will simply say "linear" instead of "R-linear". The reader is referred to [49, 50, 2, 10, 15] for a wider perspective on topological rings and modules.

Results

The intersection of all neighbourhoods of zero

If M is a topological module over a topological ring R and $\mathcal{N}_0(M)$ denotes the filter of all neighbourhoods of 0 in M, then the intersection of all 0-neighbourhoods of M is commonly denoted as O_M, that is, $O_M := M_{(0)} := \bigcap_{V\in\mathcal{N}_0(M)} V$. Observe that O_M is a bounded and closed submodule of M whose inherited topology is the trivial topology [19, Theorem 2]. It is well known that M is Hausdorff if and only if $O_M = \{0\}$ [49, 50, 2, 10].

Theorem 1 *If M is a topological module over a topological ring R, then O_M is contained in every closed submodule N of M.*

Proof Let N be any closed submodule of M. Since $\{0\} \subseteq N$, we have that $\mathrm{cl}(\{0\}) \subseteq N$. Finally, any topological group is regular, in particular, so is any topological module, therefore, $O_M = \mathrm{cl}(\{0\}) \subseteq N$ by bearing in mind the observation provided in the Introudction about regular topological spaces. □

Theorem 1 motivates the definition of "topological kernel".

Definition 1 (Topological kernel) Let M, N be topological modules over a topological ring R. The topological kernel of a linear operator $T : M \to N$ is defined as $\ker_{\mathrm{t}}(T) := T^{-1}\left(O_N\right) = \{m \in M : T(m) \in \bigcap_{V\in\mathcal{N}_0(N)} V\}$.

Observe that if M, N are seminormed modules over a seminormed ring R, then $O_N = \bigcap_{V\in\mathcal{N}_0(N)} V = \{n \in N : \|n\| = 0\}$ in view of [19, Theorem 2], hence $\ker_{\mathrm{t}}(T) = \{m \in M : \|T(m)\| = 0\}$.

Theorem 2 *Let M, N be topological modules over a topological ring R. Consider a linear operator $T : M \to N$. Then $\ker_{\mathrm{t}}(T)$ is a submodule of M. Even more, if T is continuous, then $\ker_{\mathrm{t}}(T)$ is closed and contains O_M.*

Proof In virtue of [19, Theorem 2], O_N is a closed submodule of N, so $\ker_{\mathrm{t}}(T) := T^{-1}\left(O_N\right)$ is a submodule of M. Suppose that T is continuous. Then $\ker_{\mathrm{t}}(T)$ is closed in M because O_N is a closed submodule of N. Thus, according to Theorem 1, $\ker_{\mathrm{t}}(T)$

contains O_M. However, we will provide a different proof of the fact that $\ker_\mathfrak{t}(T)$ contains O_M without having to rely on properties of regular topological spaces. Indeed, the continuity of T allows that T^{-1} preserves neighbourhoods of 0, hence

$$\left\{T^{-1}(V) : V \in \mathcal{N}_0(N)\right\} \subseteq \mathcal{N}_0(M).$$

Therefore,

$$O_M = \bigcap_{W\in\mathcal{N}_0(M)} W \subseteq \bigcap_{V\in\mathcal{N}_0(N)} T^{-1}(V) = T^{-1}\left(\bigcap_{V\in\mathcal{N}_0(N)} V\right) = T^{-1}(O_N) = \ker_\mathfrak{t}(T).$$

□

Corollary 1 *Let M, N be topological modules over a topological ring R. If $T : M \to N$ is linear and continuous and N is Hausdorff, then $O_M \subseteq \ker(T)$.*

Proof Since N is Hausdorff, $O_N = \{0\}$, hence $\ker_\mathfrak{t}(T) = \ker(T)$. By Theorem 2, $O_M \subseteq \ker_\mathfrak{t}(T) = \ker(T)$. □

According to [19, Theorem 2], the inherited topology of O_M is the trivial topology. Therefore, if O_M is linearly complemented in M, that is, if there exists a linear projection $P : M \to M$ whose range is O_M, then P is continuous. Our final result in this subsection is aimed at unveiling more properties of this projection map by proving that $I - P$ is a closed map over its range in the sense that it maps closed subsets of M onto closed subsets of $\ker(P)$.

Theorem 3 *Let M be a topological module over a topological ring R. Suppose that O_M is linearly complemented in M and let $P : M \to M$ be a linear projection whose range is O_M. Let $B \subseteq M$. If $z \in \mathrm{cl}(B)$, then $z - P(z) \in \mathrm{cl}(B)$. As a consequence, if B is closed in M, then $(I - P)(B)$ is closed in $\ker(P)$.*

Proof Let us show first that $z - P(z) \in \mathrm{cl}(B)$. Fix any arbitrary 0-neighbourhood V of M. There exists another 0-neighbourhood W of M such that $W + W \subseteq V$. Observe that $z + W \subseteq (z - P(z)) + V$. Indeed, if $w \in W$, then $z + w = (z - P(z)) + P(z) + w \in (z - P(z)) + W + W \subseteq (z - P(z)) + V$. At this point, it is sufficient to realize that $(z + W) \cap B \neq \varnothing$ by hypothesis, meaning that $((z - P(z)) + V) \cap B \neq \varnothing$. This shows that $z - P(z) \in \mathrm{cl}(B)$. Next, assume that B is closed in M. For every $b \in B$, $b - P(b) \in \mathrm{cl}(B) = B$, that is, $(I - P)(B) \subseteq B$, meaning that $(I - P)(B) = B \cap \ker(P)$, hence $(I - P)(B)$ is closed in $\ker(P)$. □

Extreme points

In [16, Definition 3.12] and [18, Definition 4.1], the notion of 2-extreme point for rings and modules was introduced, which we recall in the next definition.

Definition 2 (2-Extreme point) Let M be a module over a ring R. Let $e \in A \subseteq M$. We will say that e is a of A provided that the condition $2e = a + b$, with $a, b \in A$, implies that $a = b = e$. The set of 2-extreme points of A is denoted by $\mathrm{ext}_2(A)$.

Our next results are aimed at showing that if a topological module is not Hausdorff, then every regular closed neighbourhood of zero does not contain 2-extreme points.

Lemma 1 *Let M be a topological module over a topological ring R. If U is an open neighbourhood of 0 in M, then $v + U = U$ for all $v \in \bigcap_{V\in\mathcal{N}_0(M)} V$. If U is a regular closed neighbourhood of 0 in M, then $v + U = U$ for all $v \in \bigcap_{V\in\mathcal{N}_0(M)} V$.*

Proof First of all, let us assume that U is open. For every $u \in U$, $U - u \in \mathcal{N}_0(M)$, hence $v \in U - u$, meaning that $v + U \subseteq U$. By the same argument, $-v + U \subseteq U$. As a consequence, we obtain the desired result. Next, let us assume that U is regular closed. Since $\mathrm{int}(U)$ is open, we know that $\mathrm{int}(U + v) = \mathrm{int}(U) + v = \mathrm{int}(U)$. By taking closures, we obtain that $U + v = \mathrm{cl}(\mathrm{int}(U)) + v = \mathrm{cl}(\mathrm{int}(U + v)) = \mathrm{cl}(\mathrm{int}(U)) = U$. □

Theorem 4 *Let M be a topological module over a topological ring R. If M is not Hausdorff, then every regular closed 0-neighbourhood of X is free of 2-extreme points.*

Proof Let U be any regular closed 0-neighbourhood of M. Fix any arbitrary $u \in U$. We will show that u is not a 2-extreme point of U. Take any $v \in \bigcap_{V \in \mathcal{N}_0(M)} V$ such that $v \neq 0$. By Lemma 1, $v + u, -v + u \in U$. Then $2u = (v + u) + (-v + u)$ with $u \neq v + u$, meaning that $u \notin \mathrm{ext}_2(U)$. □

Countable biorthogonal systems

As mentioned in the Introduction, the topological dual of a topological module M over a topological ring R is by definition the module of all continuous R-linear maps from M to R and is usually denoted by M^*. Notice that, since M is assumed to be a left R-module, then M^* acquires structure of right R-module. If there is no confusion with the underlying R, we will simply say "linear" instead of "R-linear".
The linear span of a subset G of a topological module M is by definition the set of all linear combinations of elements of G and is usually denoted by $\mathrm{span}(G)$, that is, $\mathrm{span}(G) := \{r_1 g_1 + \cdots + r_k g_k : k \in \mathbb{N}, r_1, \ldots, r_k \in R, g_1, \ldots, g_k \in G\}$. To avoid confusion, whenever we say that G spans M, we mean that $\mathrm{span}(G) = M$, and whenever we say that G generates M, we mean that $\overline{\mathrm{span}}(G) = M$.

Proposition 5 *Let M be a topological module over a topological ring R. If A is a dense subset of R and $G \subseteq M$ satisfies that* $\mathrm{span}(G) = M$*, then the set*

$$S := \left\{ \sum_{i=1}^{n} a_i g_i : n \in \mathbb{N}, a_1, \ldots, a_n \in A, g_1, \ldots, g_n \in G \right\}$$

is dense in M.

Proof Fix an arbitrary nonempty open subset $U \subseteq M$. Take any $u \in U$. We can write $u = \sum_{i=1}^{n} r_i g_i$ with $r_1, \ldots, r_n \in R$ and $g_1, \ldots, g_n \in G$. There exists a 0-neighbourhood V of M such that $u + V \subseteq U$. There also exists a 0-neighbourhood $W \subseteq M$ satisfying that $W + \overset{n}{\cdots} + W \subseteq V$. For every $i \in \{1, \ldots, n\}$, we can find a 0-neighbourhood $O_i \subseteq R$ satisfying that $O_i g_i \subseteq W$. For every $i \in \{1, \ldots, n\}$, let $a_i \in A \cap \left(r_i + O_i\right)$. Then

$$\sum_{i=1}^{n} a_i g_i = u + \sum_{i=1}^{n} (a_i - r_i) g_i \in u + \sum_{i=1}^{n} O_i g_i \subseteq u + \left(W + \overset{n}{\cdots} + W\right) \subseteq u + V \subseteq U.$$

□

From Proposition 5 it is directly inferred that M is separable if so is R and G is countable. Recall that a topological ring is said to be practical [19] whenever the invertibles approach zero.

Lemma 2 *Let M be a first countable topological module over a practical topological ring. Let $\mathcal{B} = \{U_n : n \in \mathbb{N}\}$ be a countable nested base of 0-neighbourhoods of M. If $u_n \in U_n$ for each $n \in \mathbb{N}$, then $\{u_n : n \in \mathbb{N}\}$ is bounded.*

Proof Fix any arbitrary 0-neighbourhood V of M. There are $W_0 \subseteq R$ a 0-neighbourhood in R and $V_0 \subseteq M$ a 0-neighbourhood in M satisfying that $W_0 V_0 \subseteq V$. There also exists $W_{00} \subseteq R$ a 0-neighbourhood in R with $W_{00} W_{00} \subseteq W_0$. Since $\mathcal{B}$ is nested, there exists $n_0 \in \mathbb{N}$ such that $U_n \subseteq V_0$ for all $n \geq n_0$. For every $n \in \{1, \ldots, n_0 - 1\}$, there exists a 0-neighbourhood $W_n \subseteq R$ such that $W_n u_n \subseteq V_0$. Since R is practical, there exists $r \in \mathcal{U}(R) \cap (W_{00} \cap W_0 \cap W_1 \cap \cdots \cap W_{n_0 - 1})$. Finally, observe that if $n \in \{1, \ldots, n_0 - 1\}$, then $r^2 u_n = r(r u_n) \in r(W_n u_n) \subseteq r V_0 \subseteq W_0 V_0 \subseteq V$, and if $n \geq n_0$, then $r^2 u_n \in W_{00} W_{00} U_n \subseteq W_0 U_n \subseteq W_0 V_0 \subseteq V$, in other words, $r^2 \{u_n : n \in \mathbb{N}\} \subseteq V$, reaching the conclusion that $\{u_n : n \in \mathbb{N}\}$ is bounded because $r^2 \in \mathcal{U}(R)$. □

Lemma 2 does not remain true if $\mathcal{B}$ is not nested. Indeed, consider $\mathcal{B} := \left\{ \left[-\frac{1}{n}, \frac{1}{n}\right] : n \in \mathbb{N} \right\} \cup \{[-n, n] : n \in \mathbb{N}\}$ is a countable base of 0-neighbourhoods in $\mathbb{R}$ but taking $u_n := n$ for all $n \in \mathbb{N}$ we conclude that $\{u_n : n \in \mathbb{N}\}$ is unbounded. In general, an important difference between Banach spaces and general topological modules is that finite-dimensional subspaces of a Banach space are always closed, while finitely spanned submodules of a topological module (even of a Hilbert C^*-module) need not be closed. This is one of the reasons why some notions and results from Operator Theory on Banach spaces can not directly be transferred to operators on topological modules and why Operator Theory on topological modules sometimes requires another approach than Operator Theory on Banach spaces. Fredholm and semi-Fredholm Theory on the standard Hilbert module over an unital C^*-algebra is one of the examples illustrating how the proofs and the approach in this setting are very different from the situation of the classical Fredholm and semi-Fredholm Theory on Banach spaces. Although Hilbert C^*-modules are also Banach spaces, semi-Fredholm Theory in the sense of [28, 27, 38, 40] differs very much from the classical semi-Fredholm Theory exactly due to the fact that finitely spanned submodules may behave much differently from finite-dimensional subspaces.

Let M be a topological module over a topological ring R. Given a subset $B \subseteq M$, the annihilator of B is defined as $B^\perp := \{m^* \in M^* : B \subseteq \ker(m^*)\}$. The preannihilator of a subset $A \subseteq M^*$ is defined as $A^\top := \bigcap_{a^* \in A} \ker(a^*)$.

Next notions to be recalled require the hypothesis of Hausdorffness. Let M be a Hausdorff topological module over a Hausdorff topological ring R. We say that M is torsionless [19] provided that $A^\perp = M^*$ implies that $A = \{0\}$ for every nonempty subset A of M. We say that M is strongly torsionless [19] provided that for every proper closed submodule N of M and every $m \in M \setminus N$, there exists $m^* \in N^\perp \setminus \{m\}^\perp$. And M is said to be w^*-strongly torsionless [19] provided that for every proper w^*-closed submodule N of M^* and every $m^* \in M^* \setminus N$, there exists $m \in N^\top \setminus \{m^*\}^\top$.

Our next result shows that, under the hypotheses of Hausdorffness and strongly torsionless, finitely spanned submodules are closed.

Theorem 6 *Let M be a Hausdorff topological module over a Hausdorff topological division ring R. Then*:

1. *If M is torsionless, then the principal submodules of M (i.e. those spanned by one element) are closed.*
2. *If M is strongly torsionless, then the finitely spanned submodules of M are closed.*

Proof We will proceed by induction on the number of generators.

1. Fix an arbitrary $m \in M$ and consider Rm. If $Rm = \{0\}$, then it is closed because M is Hausdorff. Assume that $Rm \neq \{0\}$. Since M is torsionless, there exists $m^* \in M^*$ such that $m^*(Rm) \neq \{0\}$. Now, if $(r_i m)_{i\in I}$ is a net of Rm converging to some $n \in M$, then $\left(r_i m^*(m)\right)_{i\in I}$ converges to $m^*(n)$. Note that $m^*(m) \neq 0$, hence $(r_i)_{i\in I}$ converges to $m^*(n)m^*(m)^{-1}$. As a consequence, $(r_i m)_{i\in I}$ converges to $m^*(n)m^*(m)^{-1}m$. Since R is Hausdorff, we conclude that $n = m^*(n)m^*(m)^{-1}m$.
2. According to [19, Lemma 4(1)], M is torsionless, so the principal submodules of M are closed. Let $m_1, \ldots, m_k \in M$ with $k > 1$. We will show that $\mathrm{span}\{m_1, \ldots, m_k\}$ is closed. By induction on k, we assume that $\mathrm{span}\{m_1, \ldots, m_{k-1}\}$ is closed. We also may assume that $m_k \notin \mathrm{span}\{m_1, \ldots, m_{k-1}\}$. Since M is strongly torsionless, there exists $m^* \in M^*$ with $m^*(m_k) \neq 0$ and $m^*\left(\mathrm{span}\{m_1, \ldots, m_{k-1}\}\right) = \{0\}$. Next, take a net $(r_{1i}m_1 + \cdots + r_{ki}m_k)_{i\in I}$ of $\mathrm{span}\{m_1, \ldots, m_k\}$ converging to some $n \in M$. The net $\left(r_{ki}m^*(m_k)\right)_{i\in I}$ converges to $m^*(n)$, hence $(r_{ki})_{i\in I}$ converges to $m^*(n)m^*(m_k)^{-1}$. As a consequence, $(r_{ki}m_k)_{i\in I}$ converges to $m^*(n)m^*(m_k)^{-1}m_k$. Then $(r_{1i}m_1 + \cdots + r_{k-1i}m_k)_{i\in I}$ converging to $n - m^*(n)m^*(m_k)^{-1}m_k$, in other words, $n - m^*(n)m^*(m_k)^{-1}m_k \in \mathrm{span}\{m_1, \ldots, m_{k-1}\}$, meaning that $n \in \mathrm{span}\{m_1, \ldots, m_k\}$. As a consequence, $\mathrm{span}\{m_1, \ldots, m_k\}$ is closed. □

Let R be a topological ring and M a topological R-module. A biorthogonal system in M is a family of pairs $\left(e_i, e_i^*\right)_{i\in I} \subseteq M \times M^*$ such that $e_i^*(e_j) = \delta_{ij}$ for all $i, j \in I$. Next theorem provides a sufficient condition on a topological module to assure the existence of a biorthogonal system. Within its proof, bear in mind that if M is a left R-module, then M^* is a right R-module, hence elements of M^* will be multiplied from the right by elements of the ring R, like m^*r.

Theorem 7 *Let R be a Hausdorff topological division ring and M a Hausdorff topological R-module. If M is strongly torsionless and linearly spanned by a countable subset, that is, $M = span(G)$ for a countable subset G of M, then there* *exists a countable biorthogonal system* $\left(e_i, e_i^*\right)_{i\in N}$, *with* $\left(e_i, e_i^*\right) \in M \times M^*$ *for all* $i \in N$, *such that* $M = \operatorname{span}\left\{e_i : i \in N\right\}$. *If, in addition, M is w*-strongly torsionless, then* $M^* = \overline{\operatorname{span}}^{\,\omega^*}\left\{e_i^* : i \in N\right\}$.

Proof Since R is a division ring, Zorn's Lemma assures the existence of a countable basis $\left(u_i\right)_{i\in N}$ in M in the sense that it is linearly independent and spans M, that is, $M = \operatorname{span}\{u_i : i \in N\}$ (in fact, $\left(u_i\right)_{i\in N}$ can be chosen to be contained in G). According to Theorem 6, the finitely spanned submodules of M are closed. Therefore, the upcoming sequences can be recursively constructed following a Gram-Schmidt process: $e_1 := u_1$ and $e_1^* := u_1^* u_1^*(e_1)^{-1}$, where $u_1^* \in \{0\}^\perp \setminus \{e_1\}^\perp$, that is, $u_1^* \in M^*$ and $u_1^*(e_1) \neq 0$, and for $i \geq 2$,

$$e_i := u_i - \sum_{k=1}^{i-1} e_k^*\left(u_i\right) e_k \text{ and } e_i^* := u_i^* u_i^*(e_i)^{-1},$$

where $u_i^* \in \left(\operatorname{span}\{e_1, \ldots, e_{i-1}\}\right)^\perp \setminus \{e_i\}^\perp$ due to the fact that $\operatorname{span}\{e_1, \ldots, e_{i-1}\}$ is closed and M is strongly torsionless. Observe that $\left(e_i, e_i^*\right)_{i\in N}$ is a biorthogonal system such that $M = \operatorname{span}\left\{e_i : i \in N\right\}$ because $\operatorname{span}\left\{e_1, \ldots, e_i\right\} = \operatorname{span}\left\{u_1, \ldots, u_i\right\}$ for all $i \in N$. Finally, let us assume that M is w^*-strongly torsionless. According to [19, Lemma 5] and by bearing in mind that $M = \operatorname{span}\left\{e_i : i \in N\right\}$, we conclude that

$$\overline{\operatorname{span}}^{\,\omega^*}\left\{e_i^* : i \in N\right\} = \left(\left\{e_i^* : i \in N\right\}^\top\right)^\perp = \{0\}^\perp = M^*.$$

□

In accordance with [38, Definition 1.4.1], the notion "countably generated" means that a module is the closure of the linear span of a countable subset. Under the settings of Theorem 7, if M is assumed to be countably generated, that is, $M = \overline{\operatorname{span}}(G)$ for a countable subset G of M, then the constructed countable biorthogonal system $\left(e_i, e_i^*\right)_{i\in N}$ verifies that $M = \overline{\operatorname{span}}\left\{e_i : i \in N\right\}$ since $\operatorname{span}\{e_1, \ldots, e_i\} = \operatorname{span}\{u_1, \ldots, u_i\}$ for all $i \in N$.
The assumptions in Theorem 7 that R is a division ring and that every finitely spanned submodule of M is closed seem to be quite strong in order to assure the existence of countable biorthogonal systems. Indeed, if A is a unital C^*-algebra and if we consider the Hilbert A-module $\ell_2(A)$, then this module satisfies that A is not a division ring and its finitely spanned submodules are not necessarily closed, however $\ell_2(A)$ has a natural orthonormal basis [38].

Metric modules

An old problem in Geometry of Banach Spaces [39, 42] consists on determining the best translation possible that turns a convex set into the most symmetric possible. Motivated by this problem, we show the existence of convex sets which are not symmetric translates, that is, no translate of the convex set is symmetric.

Lemma 3 *Let M be a pseudometric module over a ring R. For every* $p, q \in M$ *and every* $n \in \mathbb{N}$, $d(np, nq) \leq nd(p, q)$.

Proof We will prove it first for $q = 0$. Note that $d(m+m, 0) \leq d(m+m, m) + d(m, 0) = d(m, 0) + d(m, 0) = 2d(m, 0)$. By induction, we obtain the desired result for $q = 0$. Finally, for any $p, q \in M$, $d(np, nq) = d(n(p-q), 0) \leq nd(p-q, 0) = nd(p, q)$.

Notice that, under the settings of Lemma 3, if $d(np, 0) = nd(p, 0)$ for all $p \in M$ and all $n \in \mathbb{N}$, then $d(np, nq) = nd(p, q)$ for every $p, q \in M$ and every $n \in \mathbb{N}$. A pseudometric d satisfying that $d(np, nq) = nd(p, q)$ for every $p, q \in M$ and every $n \in \mathbb{N}$ will be called a homogeneous pseudometric.

Theorem 8 *Let R be a ring and M a homogeneous pseudometric R-module. If* $A \subseteq M$ *is bounded and symmetric and* $0 \in A$, *then* $A_{\mathrm{c}} \supseteq \mathsf{B}_A(0, 0)$ *and* $\mathrm{r}(A) = \frac{1}{2}\mathrm{d}(A)$.

Proof We will first prove that $\mathrm{r}_0(A) \leq \mathrm{r}_a(A)$ for every $a \in A$. Indeed, fix an arbitrary $a \in A$. Take any $b \in A$. If $d(a,b) \leq d(0,b)$, then

$$2d(0,b) = d(0,b+b) = d(-b,b) \leq d(-b,a) + d(a,b) \leq d(-b,a) + d(0,b),$$

meaning that $d(0,b) \leq d(-b,a) \leq \mathrm{r}_a(A)$. Otherwise, $d(0,b) < d(a,b) \leq \mathrm{r}_a(A)$. In any case, $d(0,b) \leq \mathrm{r}_a(A)$, hence $\mathrm{r}_0(A) \leq \mathrm{r}_a(A)$. As a consequence, $\mathrm{r}(A) = \mathrm{r}_0(A)$ and $0 \in A_{\mathrm{c}}$. Next, notice that if a in metric space X two elements $x, y \in X$ satisfy that $d(x,y) = 0$, then $d(x,z) = d(y,z)$ for all $z \in X$, therefore, $\mathrm{r}_x(X) = \mathrm{r}_y(X)$ if X is bounded. As a consequence, $\mathsf{B}_A(0,0) \subseteq A_{\mathrm{c}}$. Moreover, for every $\varepsilon > 0$ we can find $b \in A$ such that $\mathrm{r}_0(A) < d(0,b) + \varepsilon$, meaning that

$$\mathrm{r}(A) = \mathrm{r}_0(A) < d(0,b) + \varepsilon = \frac{1}{2}d(-b,b) + \varepsilon \leq \frac{1}{2}\mathrm{d}(A) + \varepsilon,$$

which implies that $\mathrm{r}(A) \leq \frac{1}{2}\mathrm{d}(A)$. Finally, simply note that the reverse inequality, $\mathrm{r}(A) \geq \frac{1}{2}\mathrm{d}(A)$, holds in any bounded pseudometric space as highlighted in the Introduction, reaching the desired result. □

Corollary 2 *Let R be a ring and M a homogeneous pseudometric R-module. If $A \subseteq M$ is bounded and there exists $m \in M$ such that $m + A$ is symmetric and $0 \in m + A$, then $-m \in A_{\mathrm{c}}$.*

Proof In accordance with Theorem 8, $0 \in (m+A)_{\mathrm{c}} = m + A_{\mathrm{c}}$, hence $-m \in A_{\mathrm{c}}$. □

We will apply the previous results to find a compact convex subset of a Hilbert space which is not a symmetric translate. Recall that a convex subset of a real vector space is symmetric if and only if it is balanced. We refer the reader to Appendix A for the related basic notions.

Example 1 Let X be a real Hilbert space of $\dim(X) \geq 2$. Let $T := \mathrm{co}\{x,y,z\}$ where $x, y, z \in X$ are different, not aligned, and equidistant, that is, $\|x-y\| = \|x-z\| = \|y-z\|$. It is trivial to observe that $T_{\mathrm{c}} = \{b\}$, where b is the centroid of the equilateral triangle T, that is, $b = \frac{1}{3}z + \frac{2}{3}\left(\frac{1}{2}x + \frac{1}{2}y\right)$. If there exists $h \in X$ such that $h + T$ is balanced, then $0 \in (h+T)_{\mathrm{c}} = h + T_{\mathrm{c}} = h + \{b\} = \{h+b\}$, hence $0 = h + b$, so $h = -b$. Note that $-b + T = \mathrm{co}(\{x-b, y-b, z-b\})$. By assumption, $b - z \in -b + T$ because $-b + T$ is symmetric, in other words, $b - z = \lambda_1(x-b) + \lambda_2(y-b) + \lambda_3(z-b)$ for some $\lambda_1, \lambda_2, \lambda_3 \geq 0$ with $\lambda_1 + \lambda_2 + \lambda_3 = 1$, meaning that $b - z = \lambda_1 x + \lambda_2 y + \lambda_3 z - b$, hence $2b = \lambda_1 x + \lambda_2 y + (\lambda_3 + 1)z$. On the other hand, observe that $3b = x + y + z$. Then

$$2x + 2y + 2z = 3\lambda_1 x + 3\lambda_2 y + 3(\lambda_3 + 1)z,$$

from which we obtain that

$$z = \frac{2 - 3\lambda_1}{3\lambda_3 + 1}x + \frac{2 - 3\lambda_2}{3\lambda_3 + 1}y.$$

Finally, it only suffices to notice that

$$\begin{aligned} \frac{2-3\lambda_1}{3\lambda_3+1} + \frac{2-3\lambda_2}{3\lambda_3+1} &= \frac{2 - 3\lambda_1 + 2 - 3\lambda_2}{3\lambda_3+1} = \frac{4 - 3(\lambda_1+\lambda_2)}{3\lambda_3+1} \\ &= \frac{4 - 3(1-\lambda_3)}{3\lambda_3+1} = \frac{1+3\lambda_3}{3\lambda_3+1} = 1, \end{aligned}$$

reaching the contradiction that x, y, z are aligned.

Bishop-Phelps property

Let M be a topological module over a topological ring R and I a nonempty set. The set of all maps from I to M, M^I, acquires structure of R-module. Every submodule F of M^I can be endowed with a module topology called "convergence linear topology". Indeed, take $\mathcal{G} \subseteq \mathcal{P}(I)$ upward directed (like, for instance, a bornology on I) satisfying that $f(G)$ is bounded in M for all $G \in \mathcal{G}$ and all $f \in F$. For every $G \in \mathcal{G}$ and every 0-neighbourhood $U \subseteq M$, the sets $\mathcal{U}(G, U) := \{f \in F : f(G) \subseteq U\}$ form a basis of 0-neighbourhoods for a module topology on F called the convergence linear topology of F generated by $\mathcal{G}$. Observe that if $\mathcal{G}$ is the set of finite subsets of I, then we obtain the pointwise convergence topology on F or, equivalently, the inherited product topology on F. The dual module of M, M^*, which is a right R-module, can be endowed with a convergence linear topology since $M^* \subseteq R^M$. For instance, if R is a practical topological ring, in other words, $0 \in \mathrm{cl}(\mathcal{U}(R))$, then every finite subset of M is bounded (see Appendix B), hence the set of finite subsets of M defines a convergence linear topology on M^* which is the pointwise convergence topology on M^*, also known as the w^*-topology. If we take the family $\mathcal{B}\wr_M$ of all bounded subsets of M, then we obtain a stronger convergence linear topology on M^* called the $\mathcal{B}\wr_M$-topology.

Remark 1 Let M be a topological module over a topological ring R, I a nonempty set and $\mathcal{G} \subseteq \mathcal{P}(I)$ upward directed. If $E \subseteq F \subseteq M^I$ are submodules such that $f(G)$ is bounded for every $G \in \mathcal{G}$ and every $f \in F$, then the inherited convergence linear topology of E from F is precisely the convergence linear topology of E in view of the fact that $\mathcal{U}_E(G, U) = \mathcal{U}_F(G, U) \cap E$ for all $G \in \mathcal{G}$ and all $U \in \mathcal{N}_0(M)$. On the other hand, let $\mathcal{G}_1, \mathcal{G}_2 \subseteq \mathcal{P}(I)$ be upward directed. If $\mathcal{G}_1 \subseteq \mathcal{G}_2$, then the convergence linear topology generated by $\mathcal{G}_1$ is clearly coarser than the convergence linear topology generated by $\mathcal{G}_2$. If for every $G_2 \in \mathcal{G}_2$ there exists $G_1 \in \mathcal{G}_1$ such that $G_2 \subseteq G_1$, then the convergence linear topology generated by $\mathcal{G}_1$ is finer than the convergence linear topology generated by $\mathcal{G}_2$.

Next notion is borrowed from Metric Fixed-Point Theory [5], particularly from the notion of convexity structure.

Definition 3 (Structure) Let R be a topological ring and M a topological R-module. A nonempty subset $\mathcal{G}$ of $\mathcal{P}(M) \setminus \{\varnothing\}$ is said to be a structure on M provided that $\mathcal{G}$ is upward directed, closed under nonempty intersections, and every element of $\mathcal{G}$ is bounded and closed. The convergence linear topology on M^* generated by $\mathcal{G}$ is called the $\mathcal{G}$-topology.

Notice that if R is practical topological ring and M is Hausdorff topological module, then the set of finite subsets of M is a structure. The following notion is borrowed from the Banach Space Geometry Theory [9, 6] and relies on the topological notion of semiopenness and semicontinuity. Levine [32] introduced the concept of semiopen sets and semicontinuity in general topological spaces, and they are now the research topics of many topologists worldwide, such as the semiseparation axioms and the binary topological spaces [24, 1, 47, 32]. A subset A of a topological space X is said to be semiopen [32] if there exists an open set U such that $U \subseteq A \subseteq \mathrm{cl}(U)$. Obviously, any open set is semiopen, but the converse does not hold in general. The complement of a semiopen set is said to be semiclosed. A topological space is called semicompact [51] provided that any cover of semiopen sets has a finite subcover. This notion of semicompactness is different from an earlier one provided by Zippin [51]. We will stick with Ganster semicompactness to introduce the Bishop-Phelps property.

Definition 4 (Bishop-Phelps property) A topological module M over a topological ring R is said to have the Bishop-Phelps property with respect to a structure $\mathcal{G} \subseteq \mathcal{P}(M) \setminus \{\varnothing\}$ if for every $G \in \mathcal{G}$, the set $\mathsf{SA}(G) := \{f \in M^* : f(G)$is semiclosed and semicompact in $R\}$ is dense in M^* for the $\mathcal{G}$-topology (convergence linear topology on M^* generated by $\mathcal{G}$).

Next theorem shows that the Bishop-Phelps property is hereditary to closed complemented submodules (keep in mind that, in non-Hausdorff topological modules, the range of a continuous linear projection does not need to be closed). Observe that if $\mathcal{G} \subseteq \mathcal{P}(M) \setminus \{\varnothing\}$ is a structure in a topological module M, then $\mathcal{G}_N := \{G \cap N : G \in \mathcal{G}, G \cap N \neq \varnothing\}$ is a structure in any submodule N of M provided that $\mathcal{G}_N \neq \varnothing$.

Theorem 9 *Let M be a topological module over a topological ring R. Let N be a closed complemented submodule of M. Let $\mathcal{G} \subseteq \mathcal{P}(M) \setminus \{\varnothing\}$ be a structure in M containing $\mathcal{G}_N \neq \varnothing$. If M has the Bishop-Phelps property with respect to $\mathcal{G}$, then so does N with respect to $\mathcal{G}_N$.*

Proof Take any $G \in \mathcal{G}$. We will show that $\mathsf{SA}_{N^*}(G \cap N)$ is dense in N^* for the $\mathcal{G}_N$-topology. Indeed, fix an arbitrary $H \in \mathcal{G}$, an arbitrary 0-neighbourhood $U \subseteq R$, and an arbitrary $n^* \in N^*$. We will prove that $\mathsf{SA}_{N^*}(G \cap N) \cap \left[n^* + \mathcal{U}_{N^*}(H \cap N, U)\right] \neq \varnothing$. Indeed, by hypothesis, $G \cap N \in \mathcal{G}$, hence $\mathsf{SA}_{M^*}(G \cap N)$ is dense in M^* for the $\mathcal{G}$-topology, that is, at

$\mathsf{SA}_{M^*}(G \cap N) \cap \left[m^* + \mathcal{U}_{M^*}(H, U)\right] \neq \emptyset$, where $m^* := P^*(n^*) = n^* \circ P$ and $P : M \to N$ is a continuous linear projection of M onto N. Take $f \in \mathsf{SA}_{M^*}(G \cap N) \cap \left[m^* + \mathcal{U}_{M^*}(H, U)\right]$. Observe that $f|_N(h) - n^*(h) = f(h) - m^*(h) \in U$ for every $h \in H \cap N$, meaning that $f|_N \in n^* + \mathcal{U}_{N^*}(H, U)$. It only remains to show that $f|_N \in \mathsf{SA}_{N^*}(G \cap N)$ which is immediate since $f \in \mathsf{SA}_{M^*}(G \cap N)$. □

Conclusion

The transportation of classical properties and notions of Operator Theory and Geometry of Banach Spaces to the scope of topological rings and algebras is an ongoing and successful trend that is enriching the literature of Functional Analysis. This manuscript adds up to that trend by considering extremal and geometric properties of neighbourhoods of zero and by relating effect algebras with C^*-algebras.

Appendix A The centroid of a triangle

Let X be a real vector space of dimension at least 2. Let $x, y, z \in X$ be different and not aligned. Consider the triangle $T := \mathrm{co}\{x, y, z\}$. The centroid of T is algebraically defined as $b = \frac{1}{3}z + \frac{2}{3}\left(\frac{1}{2}x + \frac{1}{2}y\right) \in T$. Notice that $3b = z + x + y$, hence $b = \frac{1}{3}x + \frac{2}{3}\left(\frac{1}{2}y + \frac{1}{2}z\right) = \frac{1}{3}y + \frac{2}{3}\left(\frac{1}{2}x + \frac{1}{2}z\right)$. Suppose now that X is normed. We can consider the number $r := \max\{\|x - b\|, \|y - b\|, \|z - b\|\}$. Consider the closed ball $\mathsf{B}_X(b, r)$ of center b and radius r. We will show that $T \subseteq \mathsf{B}_X(b, r)$. Since $\mathsf{B}_X(b, r)$ is convex, it only suffices to prove that $x, y, z \in \mathsf{B}_X(b, r)$, which is immediate by the choice of r. As a consequence, if $t \in T$, then $\|t - b\| \leq r$ because $T \subseteq \mathsf{B}_X(b, r)$, meaning that $\mathrm{r}_b(T) = r$.

Appendix B Boundedness over practical rings

Let M be a topological module over a topological ring R. A subset $A \subseteq X$ is said to be bounded if for each 0-neighbourhood U in M there is an invertible $u \in \mathcal{U}(R)$ such that $A \subseteq uU$. The collection of all bounded subsets of a topological module M over a topological ring is usually denoted by $\mathcal{B}_0(M)$.

Lemma 4 *Let R be a practical topological ring and M a topological R-module. Let $A, B \subseteq M$ bounded and $\{m_1, \ldots, m_k\} \subseteq M$. Then:*

1. *$\{m_1, \ldots, m_k\}$ is bounded.*
2. *$A + B$ is bounded.*
3. *$A \cup B$ is bounded.*

Proof

1. Fix an arbitrary 0-neighbourhood $U \subseteq M$. For every $i \in \{1, \ldots, k\}$, there exists a 0-neighbourhood $V_i \subseteq R$ such that $V_i m_i \subseteq M$. Take $V := V_1 \cap \cdots \cap V_k$. Since R is practical, there exists an invertible $v \in V \cap \mathcal{U}(R)$. Then $vm_i \in V_i m_i \subseteq U$ for every $i \in \{1, \ldots, k\}$, meaning that $m_i \in v^{-1}U$ for every $i \in \{1, \ldots, k\}$.
2. Fix an arbitrary 0-neighbourhood U in M. Take V a 0-neighbourhood in M with $V + V \subseteq U$. There are neighbourhoods W_1, W_2 of 0 in R and M, respectively, such that $W_1 W_2 \subseteq V$. Since A, B are bounded, we can find invertibles $r, s \in R$ with $A \subseteq rW_2$ and $B \subseteq sW_2$. Next, there exist neighbourhoods E, F of 0 in R such that $Er \subseteq W_1$ and $Fs \subseteq W_1$. Since R is practical, there exists $t \in \mathcal{U}(R)$ with $t \in E \cap F$. Then $tr \in Er \subseteq W_1$ and $ts \in Fs \subseteq W_1$. This means that

$$t(rW_2 + sW_2) = (tr)W_2 + (ts)W_2 \subseteq W_1 W_2 + W_1 W_2 \subseteq V + V \subseteq U,$$

in other words,

$$A + B \subseteq rW_2 + sW_2 \subseteq t^{-1}U.$$

This proves that $A + B$ is bounded.

3. Observe that $A \cup \{0\}$ and $B \cup \{0\}$ are trivially bounded. Next, $A \cup B \subseteq (A \cup \{0\}) + (B \cup \{0\})$ and $(A \cup \{0\}) + (B \cup \{0\})$ is bounded. □

Let R be a practical topological ring and M a nontrivial topological R-module. The set of all bounded subsets of M, $\mathcal{B}_0(M)$, is a bornology of $\mathcal{P}(M)$ in view of Lemma 4. The collection

$$\left\{uV : u \in \mathcal{U}(R), V \in \mathcal{N}_0(M) \cap \mathcal{B}_0(M)\right\}$$

is clearly a bornology base of $\mathcal{B}_0(M)$ provided that it is nonempty. In fact, if there exists $V \in \mathcal{N}_0(M) \cap \mathcal{B}_0(M)$, then $\{uV : u \in \mathcal{U}(R)\}$ is both a base of the filter $\mathcal{N}_0(M)$ and a base of the bornology $\mathcal{B}_0(M)$.

Appendix C The center of $\mathcal{B}(H)$

It is a well-known result in Operator Theory that if a bounded operator on a complex Hilbert space H commutes with all orthogonal projections of one-dimensional range, then such operator is central, that is, belongs to the center of $\mathcal{B}(H)$, hence it commutes with all elements of $\mathcal{B}(H)$ and thus it must be a multiple of the identity. This fact strongly relies on the Hahn-Banach Theorem. Since the Hahn-Banach Theorem also works on Hausdorff locally convex topological vector spaces, a similar result can be accomplished for continuous linear operators on Hausdorff locally convex topological vector spaces. For the sake of completeness, we include a summary of the proof.

Remark 2 Let X be a Hausdorff locally convex topological vector space. If $T \in \mathcal{B}(X)$ commutes with all rank-one projections, then T is a multiple of the identity. Indeed, fix arbitrary elements $x \in X \setminus \{0\}$ and $y \in X$. There exists $x^* \in X^*$ satisfying that $x^*(x) = 1$ in view of the Hahn-Banach Theorem. Consider the rank-one projection $\pi_{\mathbb{K}x}$ of X onto $\mathbb{K}x$ along $\ker(x^*)$ given by $\pi_{\mathbb{K}x}(z) := x^*(z)x$ for all $z \in X$. By hypothesis

$$x^*(y)T(x) = T(x^*(y)x) = T(\pi_{\mathbb{K}x}(y)) = \pi_{\mathbb{K}x}(T(y)) = x^*(T(y))x.$$

In particular, if $x = y$, then $T(x) = x^*(T(x))x$. This means that every nonzero vector of X is an eigenvector of T. As a consequence, T is a multiple of the identity.

Appendix D Effect algebras vs. C^*-algebras

Effect algebras and Boolean algebras are examples of universal algebras which are in between Group Theory and Order Theory. They have diverse origins. For example, Boolean algebras have historically been involved in Measure Theory, Electronics, Computer Sciences, etc. However, effect algebras were originated from Quantum Mechanics. We refer the reader to [46, 48, 41] for a wide perspective on effect algebras and Boolean algebras. Effect algebras were introduced for the first time in [12] in the context of Quantum Mechanics.

Definition 5 (Effect algebra) An effect algebra is a universal algebra $(L, \oplus, 0, 1, {}^\perp)$, where

$$\begin{aligned} \oplus : \Sigma \subseteq L \times L &\to L \\ (p, q) &\mapsto p \oplus q \end{aligned}$$

is a partially defined binary internal operation,

$$\begin{array}{rl} {}^{\perp} : L & \to L \\ p & \mapsto p^{\perp} \end{array}$$

is a unary internal operation (called orthocomplementation), and

$$\begin{array}{rl} 0 : L & \to L \\ p & \mapsto 0 \end{array} \text{ and } \begin{array}{rl} 1 : L & \to L \\ p & \mapsto 1 \end{array}$$

are nullary internal operations satisfying the following conditions for all $p, q, r \in L$:

- *Commutativity:* Σ is a symmetric binary relation on L and if $(p, q) \in \Sigma$, then $p \oplus q = q \oplus p$.
- *Associativity:* If $(q, r), (p, q \oplus r) \in \Sigma$, then $(p, q), (p \oplus q, r) \in \Sigma$ and $(p \oplus q) \oplus r = p \oplus (q \oplus r)$.
- *Orthocomplementation:* $p^{\perp}$ is the only element in L such that $\left(p, p^{\perp}\right) \in \Sigma$ and $p \oplus p^{\perp} = 1$.
- *Zero-One Law:* $1 \neq 0$ and 0 is the only element in L such that $(1, 0) \in \Sigma$.

The first example of effect algebra [12], which also motivated the above definition, is given by the set of positive selfadjoint bounded operators on a complex Hilbert space.

Example 2 Let H be a complex Hilbert space. Then $(\mathcal{E}(H), \oplus, 0, I, {}^{\perp})$ is an effect algebra, where $\mathcal{E}(H) := \{T \in \mathcal{B}(H) : T = T^*,\ 0 \leq T \leq I\}$ and the partially defined binary internal operation is given by $T \oplus S := T + S \Leftrightarrow 0 \leq T + S \leq I$ and orthocomplementation defined by $T^{\perp} := I - T$.

The idea behind all of these is the First Postulate of Quantum Mechanics [34, 44, 36, 35, 37, 45, 11], which establishes that a quantum mechanical system is represented by an infinite dimensional separable complex Hilbert space H. The observable measurements or magnitudes are given by the (bounded or unbounded) selfadjoint operators on H. The unsharp measurements, also called effects, are represented by the positive selfadjoint bounded operators lying below the identity. If an observable magnitude is represented by a selfadjoint bounded operator T, then $\|T\|$ represents the intensity of the magnitude. The quantum bits are the unit vectors of the Hilbert space H and the quantum states are the complex rays spanned by the quantum bits. There are two main universal algebras involved in this frame: C^*-algebras and effect algebras. C^*-algebras model the behavior of observable measurements. Effect algebras model the behavior of unsharp measurements. Our next remark is the expected result that the set of positive selfadjoint elements of a unital C^*-algebra that are less than or equal to the unity acquires structure of effect algebra. An element a of a C^*-algebra A is called selfadjoint provided that $a = a^*$. The subset of selfadjoint elements of A is denoted by A_{sa}. If A is unital, then a selfadjoint element $a \in A$ is called positive provided that $\sigma(a) \subseteq [0, \infty)$. If a is positive, then it is commonly written that $a \geq 0$ and $A^+ := \{a \in A : a \geq 0\}$. The binary internal relation on A_{sa} given by $a \leq b$ if and only if $b - a \geq 0$ is a partial order relation on A_{sa}. Observe that A_{sa} is a real vector subspace of A and the previous order relation is compatible with addition and multiplication by positive scalars.

Remark 3 If A is a unital C^*-algebra, then $\mathcal{E}(A) := \{a \in A^* : a = a^* \text{ and } 0 \leq a \leq 1\}$ is an effect algebra with the partially defined binary internal operation given by $a \oplus b := a + b \Leftrightarrow 0 \leq a + b \leq 1$ and orthocomplementation given by $a^{\perp} := 1 - a$. Indeed, notice that $\oplus$ is clearly associative whenever it is defined. Indeed, let $a, b, c \in \mathcal{E}(A)$ with $b + c \in \mathcal{E}(A)$ and $a + (b + c) \in \mathcal{E}(A)$. We have to show that $a + b \in \mathcal{E}(A)$. On the one hand, $a + b \geq 0$ because $a, b \geq 0$. On the other hand, $(a + b) + c = a + (b + c) \in \mathcal{E}(A)$ so $(a + b) + c \leq 1$, that is, $1 - [(a + b) + c] \geq 0$, so by compatibility with addition $1 - (a + b) = 1 - [(a + b) + c] + c \geq c \geq 0$, in other words, $a + b \in \mathcal{E}(A)$. In a similar way, $\oplus$ is commutative whenever it is defined. The uniqueness of the orthocomplement is also inferred from the uniqueness of the opposite. Finally, the Zero-One Law is also trivially verified.

Remark 3 can be easily generalized to ordered modules over ordered rings. An ordered ring is a ring endowed with a partial order compatible with addition and multiplication by positive elements. An ordered module over an ordered ring is a module over an ordered ring endowed with a partial order compatible with addition and multiplication by positive elements of the ring.

Theorem 10 *If M is an ordered module over a unital ordered ring R and $m \in M$ is so that $m > 0$, then $[0, m]$ is an effect algebra with the partially defined binary internal operation given by $p \oplus q := p + q \Leftrightarrow p + q \leq m$ for all $p, q \in [0, m]$ and orthocomplementation given by $p^{\perp} := m - p$ for all $p \in [0, m]$.*

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Declarations

Conflicts of interest The author declares no conflict of interest.

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