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REMARKS ON GENERALIZED HILBERT OPERATORS

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Abstract

Given a regular measure $\eta \in M([0,1))$ and an analytic function $g \in \mathcal{H}(\mathbb{D})$, we define $H(\eta, g)(z) = \int_0^1 g(tz)d\eta(t)$ and study its boundedness from $X \times Y$ into Z where $X \subset M([0,1))$ and $Y, Z \subset \mathcal{H}(\mathbb{D})$ are the Hardy spaces. We shall analyze the case $X = L^p([0,1))$ and characterize the functions $g \in \mathcal{H}(\mathbb{D})$ such that H_g maps $L^p([0,1))$ into $H^p(\mathbb{D})$ where $H_g(\eta) = H(\eta, g)$.

Keywords Generalized Hilbert operator · Hardy spaces · Mixed norm spaces

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Introduction

We denote by $\mathbb{D}$ the unit disc in the complex plane $\mathbb{C}$ and by $\mathcal{H}(\mathbb{D})$ the space of all analytic functions in $\mathbb{D}$. Throughout the paper, η stands for a complex measure in $M([0,1))$ with $\|\eta\|_1 = \int_0^1 d|\eta|(t)$ and $g \in \mathcal{H}(\mathbb{D})$. We denote by $\hat{g}(n)$ the Taylor coefficients of $g \in \mathcal{H}(\mathbb{D})$, i.e., $g(z) = \sum_{n=0}^\infty \hat{g}(n)z^n$ and by $\mu_n(\eta)$ the moments of η, that is $\mu_n(\eta) = \int_0^1 t^n d\eta(t)$ for $n \geq 0$. Also we write $h * g$ for the Hadamard product $h * g(z) = \sum_{n=0}^\infty \hat{h}(n)\hat{g}(n)z^n$ between $h, g \in \mathcal{H}(\mathbb{D})$. We denote $K_\lambda(z) = \frac{1}{(1-z)^{1+\lambda}}$ for $\lambda > -1$ and $D_\lambda g = K_\lambda * g$. In particular $Dg(z) = \sum_{n=0}^\infty (n+1)\hat{g}(n)z^n$.
We use the notations $dm(\xi) = \frac{d\theta}{2\pi}$ and $dA = \frac{dxdy}{\pi}$ for the normalized Lebesgue measures on $\partial\mathbb{D}$ and $\mathbb{D}$, respectively. As usual, for $f \in \mathcal{H}(\mathbb{D})$, $1 \leq p \leq \infty$ and $0 < r < 1$ we write $f_r(z) = f(rz)$ and $M_p(f,r) = (\int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi})^{1/p}$. For $\alpha > 0$, $\beta > -1$ and $1 \leq p \leq \infty$, we denote by $H^p(\mathbb{D})$, $H^p_\alpha(\mathbb{D})$ and $A^p_\beta(\mathbb{D})$ the Hardy spaces, and weighted Hardy and weighted Bergman spaces consisting on functions $f \in \mathcal{H}(\mathbb{D})$ such that

$$\|f\|_{H^p} = \sup_{0<r<1} M_p(f,r) < \infty,$$

$$\|f\|_{H^p_\alpha} = \sup_{0<r<1} (1-r)^\alpha M_p(f,r) < \infty$$

and

$$\|f\|_{A^p_\beta} = \left(\int_{\mathbb{D}} |f(z)|^p (1-|z|^2)^\beta dA(z)\right)^{1/p} < \infty.$$

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Let us also recall the mixed norm spaces of analytic functions to be used later on: given $1 \leq p, q \leq \infty$ and $\alpha > 0$, we denote by $H(p, q, \alpha)$ the space of functions $f \in \mathcal{H}(\mathbb{D})$ such that

$$\|f\|_{p,q,\alpha} = \left(\int_0^1 (1-r)^{q\alpha-1} M_p^q(f, r) dr \right)^{1/q} < \infty.$$

We say that $f \in \mathcal{D}(p, q, \alpha)$ if $Df \in H(p, q, \alpha)$ and $\|f\|_{\mathcal{D}(p,q,\alpha)} = \|Df\|_{p,q,\alpha}$. Note that $H(p, \infty, \alpha) = H_\alpha^p(\mathbb{D})$, and that for $0 < \alpha < 1$ the space

$$\mathcal{D}(p, \infty, \alpha) = \{f \in \mathcal{H}(\mathbb{D}) : M_p(Df, r) = O((1-r)^{-\alpha})\}$$

coincides with the Lipschitz classes $\Lambda(p, 1-\alpha)$ (see [8, Theorem 5.4]).
Of course, one has the identification $H^2(\mathbb{D}) = \mathcal{D}(2, 2, 1)$ and, due to the work of Hardy, Littlewood, Paley, and Flett, we have the following embeddings:

$$\mathcal{D}(p, p, 1) \subset H^p \subset \mathcal{D}(p, 2, 1), 1 \leq p \leq 2 \tag{1}$$

and

$$\mathcal{D}(p, 2, 1) \subset H^p \subset \mathcal{D}(p, p, 1), p \geq 2. \tag{2}$$

The reader is referred to the books [8, 14] for these results and for the original references.
We now mention the operators we shall be dealing with. Recall that the Hilbert matrix can be viewed as an operator acting on spaces of analytic functions by its action on the Taylor coefficients:

$$(a_n)_{n=0}^\infty \to \left(\sum_{k=0}^\infty \frac{a_k}{n+k+1} \right)_{n=0}^\infty$$

which can be formally rephrased as

$$\mathcal{H}(f)(z) = \int_0^1 \frac{f(t)}{1-tz} dt, \quad f \in \mathcal{H}(\mathbb{D})$$

whenever the integral makes sense. Of course, the above operators are well defined for $f(z) = \sum_{n=0}^\infty a_n z^n$ whenever $\sum_{n=0}^\infty \frac{|a_n|}{n+1} < \infty$ or whenever $f \in L^1([0, 1))$, respectively.
A possible generalization of the above mapping was given in [10] by considering a positive measure $\mu \in M([0, 1))$ and defining the operator

$$(a_n)_{n=0}^\infty \to \left(\sum_{k=0}^\infty \mu_{k,n} a_k \right)_{n=0}^\infty$$

where $\mu_{k,n} = \int_0^1 t^{k+n} d\mu(t)$. This operator can be seen as an integral operator in the case that μ is a Carleson measure, namely

$$\mathcal{H}_\mu(f)(z) = \int_0^1 \frac{f(t)}{1-tz} d\mu(t), \quad f \in H^1(\mathbb{D}).$$

The conditions on a positive measure $\mu \in M([0, 1))$ to obtain that $\mathcal{H}_\mu$ is bounded on different spaces of analytic functions have been studied by several authors (see [4, 10, 13]).
Another generalization of the operator $\mathcal{H}$ was given in [11]. The authors introduced, for a given $g \in \mathcal{H}(\mathbb{D})$, the operator

$$\mathcal{H}_g(f) = \int_0^1 f(t) g'(tz) dt.$$

This clearly generalizes $\mathcal{H}$, which corresponds to $g(z) = \log(\frac{1}{1-z})$. The study of functions $g \in \mathcal{H}(\mathbb{D})$, such that $\mathcal{H}_g$ is bounded between Hardy, Bergman, and weighted Bergman spaces, has been developed in different papers (see [11, 16]).

In this paper, we shall consider the bilinear map $\mathcal{H} : M([0,1)) \times \mathcal{H}(\mathbb{D}) \to \mathcal{H}(\mathbb{D})$ given by

$$H(\eta, g)(z) = \sum_{n=0}^{\infty} \mu_n(\eta)\hat{g}(n)z^n = \int_0^1 g(tz)d\eta(t) \tag{3}$$

for any $\eta \in M([0,1))$ and $g \in \mathcal{H}(\mathbb{D})$. We shall write $H_g(f) = H(f(t)dt, g)$ and $H_\eta(g) = H(\eta, g)$.
Of course, denoting $K_0(z) = \frac{1}{1-z}$ we have $\mathcal{H}(f) = H_{K_0}(f(t)dt)$, $\mathcal{H}_\mu(f) = H_{K_0}(f(t)d\mu(t))$ and $\mathcal{H}_g(f) = H(f(t)dt, g')$ for any $f \in H^1(\mathbb{D})$ and $g \in \mathcal{H}(\mathbb{D})$.
There are many results in the literature concerning the boundedness of H_η where $d\eta(t) = f(t)dt$ for some $f \in \mathcal{H}(\mathbb{D})$ belonging to the classical spaces of analytic functions. In this paper, we shall analyze the situation for general measures $d\eta$ and in particular for $d\eta(t) = f(t)dt$ where $f \in L^p([0,1))$ is not necessarily a holomorphic function. Let us recall some of the known results when acting on Hardy spaces

$$\mathcal{H} : H^p(\mathbb{D}) \to H^p(\mathbb{D}), \quad 1 < p < \infty, \tag{4}$$

[6, *Theorem* 1.1],

$$\mathcal{H} : H^\infty(\mathbb{D}) \to BMOA \tag{5}$$

[15, *Theorem* 1.2],

$$\mathcal{H} : H^p_\alpha(\mathbb{D}) \to H^p_\alpha(\mathbb{D}), \quad 1 \le p < \infty, \alpha < 1/p' \tag{6}$$

[15, *Theorem* 3.1],

$$\mathcal{H}_g : H^p(\mathbb{D}) \to H^p(\mathbb{D}) \text{iff} g \in \Lambda^p_{1/p}(\mathbb{D}), \quad 1 < p \le 2 \tag{7}$$

[11, *Theorem* 1].
If $1 < \max\{q, 2\} < p$, then

$$g \in \Lambda^q_{1/q}(\mathbb{D}) \Longrightarrow \mathcal{H}_g : H^p(\mathbb{D}) \to H^p(\mathbb{D}) \tag{8}$$

[11, *Theorem* 2].
We shall study some conditions on η and g to obtain that $H(\eta, g) \in H^p(\mathbb{D})$ and to get conditions on g to obtain that $H_g : L^p([0,1)) \to H^q(\mathbb{D})$ for some values of p and q. Our techniques are based on the formula

$$H_g(\eta)(z) = \mathcal{H}(\eta) * g(z), \tag{9}$$

combined with some results on $\mathcal{H}(\eta)$, multipliers between spaces of analytic functions $(X, Y) = \{g \in \mathcal{H}(\mathbb{D}) : g * f \in Y \quad \forall f \in X\}$, and inclusions between mixed norm spaces and Hardy spaces.
The paper is divided into two more sections. In the first one, we show that for very general Banach spaces $Y \subset \mathcal{H}(\mathbb{D})$ we have that $H_g : M([0,1)) \to Y$ is equivalent to $H_g : L^1([0,1)) \to Y$ and it holds only when $g \in Y$ (see Theorem 3.1). Then, we study some couples $X \subset M([0,1))$ and $Y \subset \mathcal{H}(\mathbb{D})$ such that $H : X \times Y \to H^p(\mathbb{D})$ for $1 \le p \le \infty$. In section "3," we study the functions $g \in \mathcal{H}(\mathbb{D})$ such that $H_g : L^p([0,1)) \to H^q(\mathbb{D})$ for some values of p and q. We get some independent proofs of some results in [6] and [11] and extend them to L^p-spaces instead of $H^p(\mathbb{D})$.
As usual, p' stands for the conjugate exponent $1/p + 1/p' = 1$ and C denotes a constant that may vary from line to line.

Results on the Hilbert matrix operator

We start mentioning some general facts on the Hilbert matrix operator which follows the ideas from [15].

Proposition 2.1 *Let* $\eta \in M([0,1))$. *Then*

(i) $\mathcal{H}(\eta)(z) = \frac{F_\eta(z)}{1-z}$ *where* $F_\eta \in A(\mathbb{T})$, *i.e.* $\sum_{n=0}^{\infty} |\hat{F}_\eta(n)| < \infty$.
(ii) $\mathcal{H}(\eta) \in A^1_\beta(\mathbb{D})$ *for any* $\beta > -1$.
(iii) *If* $\log(\frac{1}{1-t}) \in L^1(|\eta|)$ *then* $\mathcal{H}(\eta) \in H^1(\mathbb{D})$.
(iv) *If* $\eta \geq 0$ *and* $\mathcal{H}(\eta) \in H^1(\mathbb{D})$ *then* $\log(\frac{1}{1-t}) \in L^1(\eta)$.

Proof

(i) Let us write

$$F_\eta(z) = (1-z)\mathcal{H}(\eta)(z) = \int_0^1 \frac{1-z}{1-tz} d\eta(t) = \sum_{n=0}^{\infty} (\mu_n(\eta) - \mu_{n+1}(\eta))z^n.$$

Clearly,

$$\sum_{n=0}^{\infty} |\mu_n(\eta) - \mu_{n+1}(\eta)| \leq \sum_{n=0}^{\infty} \int_0^1 t^n(1-t)d|\eta|(t) = \|\eta\|_1.$$

(ii) From (i), we have

$$\begin{aligned}\int_{\mathbb{D}} |\mathcal{H}(\eta)(z)|(1-|z|)^\beta dA(z) \leq& \int_0^1 \frac{|F_\eta(z)|}{|1-z|}(1-|z|)^\beta dA(z)\\ \leq& C\int_0^1 (\sum_{n=0}^{\infty} |\hat{F}_\eta(n)|r^n) M_1(K_0, r)(1-r)^\beta dr\\ \leq& C\sum_{n=0}^{\infty} |\hat{F}_\eta(n)|(\int_0^1 r^n \log(\frac{1}{1-r})(1-r)^\beta dr)\\ \leq& C\sum_{n=0}^{\infty} |\hat{F}_\eta(n)|(\sum_{k=0}^{\infty} \frac{1}{k+1}(\int_0^1 r^{n+k}(1-r)^\beta dr)\\ \leq& C\sum_{n=0}^{\infty} |\hat{F}_\eta(n)|(\sum_{k=0}^{\infty} \frac{1}{(k+1)(n+k+1)^{\beta+1}})\\ \leq& C(\beta)\sum_{n=0}^{\infty} |\hat{F}_\eta(n)| \leq C(\beta)\|\eta\|_1.\end{aligned}$$

(iii) Assume that $\log(\frac{1}{1-t}) \in L^1(|\eta|)$. Note that

$$|\mathcal{H}(\eta)(z)| \leq \int_0^1 \frac{d|\eta|(t)}{|1-tz|} dt$$

implies that

$$M_1(\mathcal{H}(\eta), r) \leq C\int_0^1 \log(\frac{1}{1-rt}) d|\eta|(t). \tag{10}$$

Therefore, for any $0 < r < 1$

$$\|\mathcal{H}(\eta)\|_{H^1} = \sup_{0<r<1} M_1(\mathcal{H}(\eta), r) \leq C\int_0^1 \log \frac{1}{1-t} d|\eta|(t) \leq C.$$

(iv) Assume $\eta \geq 0$ and $\mathcal{H}(\eta) \in H^1(\mathbb{D})$. From Féjer-Riesz's inequality (see [8, Theorem 3.13]), it follows:

$$\int_0^1 \log(\frac{1}{1-t})d\eta(t) = \int_0^1 \int_0^1 \frac{r}{1-rt} dr d\eta(t) = \int_0^1 \mathcal{H}(\eta)(r) r dr \leq C\|\mathcal{H}(\eta)\|_1.$$

□

The case $X = M([0,1))$

Let us start by showing a result for a very general class of Banach spaces of the analytic functions. We deal here with Banach spaces $Y \subset \mathcal{H}(\mathbb{D})$ containing the monomials $u_n(z) = z^n$ for $n \geq 0$ and satisfying

$$\lim_{n\to\infty} \sqrt[1/n]{\|u_n\|_Y} = 1. \tag{11}$$

Note that this condition guarantees that $g_r \in Y$ for any $g \in \mathcal{H}(\mathbb{D})$ and for any $0 < r < 1$. Moreover, the mapping $r \to g_r$ is continuous from $[0, 1)$ into Y. This is due to the fact that the series $\sum_{n=0}^{\infty} \hat{g}(n) u_n z^n$ defines a Y-valued homorphic function defined in the disc because if $g \in \mathcal{H}(\mathbb{D})$ then (11) implies that

$$\limsup_{n\to\infty} \sqrt[1/n]{|\hat{g}(n)|\|u_n\|_Y} = \limsup_{n\to\infty} \sqrt[1/n]{|\hat{g}(n)|} \geq 1.$$

We shall also assume the following properties (see [3] or [14, Chapter 9]):
(P1) There exists $M > 0$ such that $\sup_{0<r<1} \|g_r\|_Y \leq M\|g\|_Y$ for any $g \in Y$.
(P2) Y satisfies the Fatou property, i.e., there exists $A > 0$ such that for any sequence $(g_n) \subset Y$ with $\sup_n \|g_n\|_Y \leq 1$ and $g_n \to g$ in $\mathcal{H}(\mathbb{D})$ one has that $g \in Y$ and $\|g\|_Y \leq A$.

Proposition 3.1 *Let $Y \subset \mathcal{H}(\mathbb{D})$ be a Banach space satisfying (11), (P1) and (P2), and let $g \in \mathcal{H}(\mathbb{D})$. Then, the following conditions are equivalent:*

(1) *H_g is bounded from $M([0, 1))$ to Y.*
(2) *H_g is bounded from $L^1([0, 1))$ to Y.*
(3) *$g \in Y$.*

Proof (1) $\Longrightarrow$ (2) It is obvious.

(2) $\Longrightarrow$ (3) Let $0 < r < 1$. Then, for any $\varepsilon > 0$, we can write

$$H_g(\frac{1}{\varepsilon}\chi_{[r,r+\varepsilon]}dm)(z) = \frac{1}{\varepsilon}\int_r^{r+\varepsilon} g(tz)dt, \quad z \in \mathbb{D}.$$

Observe now that $h_n = H_g(n\chi_{[r,r+1/n]}dm) \in Y$ for all $n \in \mathbb{N}$, $g_r = \lim_{n\to\infty} h_n$ in $\mathcal{H}(\mathbb{D})$ and

$$\|H_g(n\chi_{[r,r+1/n]}dm)\|_Y \leq \|H_g\|_{\mathcal{L}(L^1,Y)}, \quad n \in \mathbb{N}$$

Hence, using the Fatou property, one gets $\|g_r\|_Y \leq \|H_g\|_{\mathcal{L}(L^1,Y)}$. Since $g_{r_n} \to g$ in $\mathcal{H}(\mathbb{D})$ for any $r_n \to 1$, due to the Fatou property again we conclude that $g \in Y$.

(3) $\Longrightarrow$ (1) From (P2), we have that the mapping $t \to g_t$ is Bochner integrable with respect to $|\eta|$ and

$$\|H_g(\eta)_r\|_Y \leq \int_0^1 \|g_{tr}\|_Y d|\eta|(t) \leq A\|g\|_Y\|\eta\|_1.$$

Therefore, using the Fatou property again, we obtain $\|H_g(\eta)\|_Y \leq K\|g\|_Y\|\eta\|_1$ for some constant $K > 0$.

□

Note that from Proposition 3.1, we have $H : M([0,1)) \times H^p(\mathbb{D}) \to H^p(\mathbb{D})$ for any $1 \le p \le \infty$. We shall now consider other possible pairs of Banach spaces such that H maps $X \times Y$ into $H^p(\mathbb{D})$.
We shall make use of the following result, which can be found in [8, Theorem 6.7] for $p = 2$ and that was proved by Stein-Zygmund for $p > 2$ (see [2, 14, 17]).

Lemma 3.2 *Let $p \ge 2$ and $F \in \mathcal{H}(\mathbb{D})$. The following ones are equivalent*:

(i) *There exists $C > 0$ such that $\|F * f\|_{H^p} \le C\|f\|_{H^1}$ for any $f \in H^p(\mathbb{D})$*
(ii) $M_p(DF, r) = O(\frac{1}{1-r})$.

Theorem 3.3 *Let $p \ge 2$ and set, for $\alpha > 0$,*

$$X_\alpha = \{\eta \in M([0,1)); \sup_{n\ge 0}(n+1)^\alpha \mu_n(|\eta|) < \infty\}.$$

Then $H : X_{1/p'} \times H^1(\mathbb{D}) \to H^p(\mathbb{D})$ is bounded.

Proof Let $\eta \in X_{1/p'}$. Let us first show that

$$\sup_{0<r<1} (1-r)M_p(D\mathcal{H}(\eta), r) < \infty.$$

Since $D\mathcal{H}(\eta) = \int_0^1 \frac{d\eta(t)}{(1-tz)^2}$, we obtain that

$$M_p(D\mathcal{H}(\eta), r) \le \int_0^1 M_p(K_1, rt)d|\eta|(t) \le C\int_0^1 \frac{d|\eta|(t)}{(1-rt)^{1+\frac{1}{p'}}}.$$

Taking into account that $\frac{1}{(1-r)^{1+\gamma}} \approx \sum_{n=0}^\infty (n+1)^\gamma r^n$ for $\gamma > 0$, we get the following estimates:

$$\int_0^1 \frac{d|\eta|(t)}{(1-rt)^{1+\frac{1}{p'}}} \approx \sum_{n=0}^\infty (n+1)^{1/p'}\mu_n(|\eta|)r^n \le \frac{C}{1-r}.$$

Now we can use Lemma 3.2, together with the fact $H_\eta(g) = \mathcal{H}(\eta) * g$, to obtain

$$\|H(\eta, g)\|_{H^p} \le C\|g\|_{H^1} \sup_{0<r<1}(1-r)M_p(D\mathcal{H}(\eta), r)$$

and the proof is completed. ◻

Theorem 3.4 *Let $p \ge 2$ and set*

$$M_{log} = \{\eta \in M([0,1)) : \int_0^1 \log(\frac{1}{1-t})d|\eta|(t) < \infty\}$$

and denote the Zygmund class by

$$\mathcal{Z}_p = \{f \in \mathcal{H}(\mathbb{D}) : \sup_{0<r<1}(1-r)M_p(D^2 f, r) < \infty\}.$$

Then $H : M_{log} \times \mathcal{Z}_p \to H^p(\mathbb{D})$ is bounded.

Proof Combine Proposition 2.1 and Lemma 3.2 as in the previous theorem.

The case $X = L^p([0,1))$

In this section, we consider the case $d\eta(t) = f(t)dt$ where $f \in L^p([0,1))$.

Lemma 4.1 *Let* $1 < p \leq \infty$, $1 \leq q \leq \infty$ *and* $\frac{1}{q'} + \frac{1}{p} > 0$. *If* $f \in L^p(0,1)$, *then* $\mathcal{H}(f) \in \mathcal{D}(q,p,\frac{1}{q'}+\frac{1}{p})$ *and*

$$\|\mathcal{H}(f)\|_{\mathcal{D}(q,p,\frac{1}{q'}+\frac{1}{p})} \leq C(p,q)\|f\|_{L^p}.$$

Proof As abovementioned, we use that $D\mathcal{H}(f)(z) = \int_0^1 f(t)K_1(tz)dt$ for $z \in \mathbb{D}$. Therefore,

$$M_q(D\mathcal{H}(f),r) \leq \int_0^1 |f(t)|M_q(K_1,tr)dt \leq C\int_0^1 \frac{|f(t)|}{(1-rt)^{1+\frac{1}{q'}}}dt.$$

If $p = \infty$ then $q > 1$ and for $f \in L^\infty([0,1))$ we have

$$M_q(D\mathcal{H}(f),r) \leq C\frac{\|f\|_{L^\infty}}{(1-r)^{1/q'}}.$$

This gives that $\mathcal{H}(f) \in \mathcal{D}(q,\infty,\frac{1}{q'})$.

For $1 < p < \infty$, we use the estimate

$$(1-r)^{1/q'}M_q(D\mathcal{H}(f),r) \leq C\Big(\int_0^r \frac{|f(t)|}{1-t}dt + \frac{1}{1-r}\int_r^1 |f(t)|dt\Big). \tag{12}$$

It is well known and it follows easily from interpolation that the mapping $f \to \frac{1}{1-r}\int_r^1 f(t)dt$ defines a bounded operator on $L^p([0,1))$ for $1 < p \leq \infty$. Hence, using the adjoint operator, also the mapping $f \to \int_0^r \frac{f(t)}{1-t}dt$ defines a bounded operator on $L^p([0,1))$ for $1 < p < \infty$. Therefore, for any $f \in L^p([0,1))$ we have

$$\int_0^1 (1-r)^{p/q'}M_q^p(D\mathcal{H}(f),r)dr \leq C\|f\|^p_{L^p(0,1)}.$$

That gives that $\mathcal{H}(f) \in \mathcal{D}(q,p,\alpha)$ for $\alpha = \frac{1}{p} + \frac{1}{q'}$ and the desired estimate for the norms. □

Corollary 4.2 *Let* $1 < p \leq \infty$.

(1) *If* $f \in L^p([0,1))$ *for some* $p < \infty$, *then* $\int_0^1 M_1^p(D\mathcal{H}(f),r)dr < \infty$.
(2) *If* $f \in L^p([0,1))$ *for some* $1 < p \leq 2$, *then* $\mathcal{H}(f) \in H^p(\mathbb{D})$.
(3) *If* $f \in L^\infty([0,1))$, *then* $\mathcal{H}(f) \in \cap_{q>1}\Lambda(q,\frac{1}{q})$.

Corollary 4.3 ([6, 15]) *Let* $1 < p < \infty$. *Then* $\mathcal{H} : H^p(\mathbb{D}) \to H^p(\mathbb{D})$ *and* $\mathcal{H} : H^\infty(\mathbb{D}) \to BMOA$

Proof We first show that $\mathcal{H} : H^\infty(\mathbb{D}) \to BMOA$. This follows trivially from (3) in Corollary 4.2 since $\Lambda(2,\frac{1}{2}) \subset BMOA$.

For $1 < p \leq 2$, we have that $H^p(\mathbb{D}) \subset L^p([0,1))$ (due to Fèjer-Riesz's inequality) and the result follows using Lemma 4.1 for $q = p$ and the fact that $D(p,p,1) \subset H^p(\mathbb{D})$.

Finally, the case $2 < p < \infty$ follows from duality. Note that $\mathcal{H}^* = \mathcal{H}$ as an operator acting on Hardy spaces. Indeed, if $f \in H^p(\mathbb{D})$ and $g \in H^{p'}(\mathbb{D})$, we have

$$\begin{aligned}\int_{\mathbb{T}} \mathcal{H}(f)(\xi)\overline{g(\xi)}\frac{d\xi}{\xi} &= \int_{\mathbb{T}}(\int_0^1 \int_{\mathbb{T}} \frac{f(\rho)}{1-t\bar{\rho}}\frac{d\rho}{\rho})\frac{dt}{1-t\xi})\overline{g(\xi)}\frac{d\xi}{\xi} \\ &= \int_{\mathbb{T}}(\overline{\int_0^1 \int_{\mathbb{T}} \frac{g(\xi)}{1-t\bar{\xi}}\frac{d\xi}{\xi})\frac{dt}{1-t\rho}})f(\rho)\frac{d\rho}{\rho} \\ &= \int_{\mathbb{T}} \overline{\mathcal{H}(g)(\rho)}f(\rho)\frac{d\rho}{\rho}\end{aligned}$$

Therefore, $\mathcal{H}^*$ is bounded on $H^{p'}(\mathbb{D})$ for $1 < p' < 2$ and then we obtain the result. □

Remark 4.1 The case $2 < p < \infty$ also follows from interpolation using that $[H^2(\mathbb{D}), BMOA]_\theta = H^p(\mathbb{D})$ for $\frac{1}{p} = \frac{1-\theta}{2}$.

Remark 4.2 Let $d\eta(t) = f(t)dt$ for some $f \in L^p([0,1))$, since $\mu_n(|f|) \le \|f\|_p(n+1)^{-1/p'}$, we have that $L^p([0,1)) \subset X_{1/p'}$ and invoking Theorem 3.3 we obtain $H : L^p([0,1)) \times H^1(\mathbb{D}) \to H^p(\mathbb{D})$ is bounded for $p \ge 2$.

On the other hand,

$$|H_g(f)(z)| \le \|f\|_{p'}(\int_0^1 |g(tz)|^p dt)^{1/p}.$$

Therefore,

$$M_p(H_g(f), r) \le \|f\|_{p'}(\int_0^1 M_p^p(g, rt))dt)^{1/p} \le \|f\|_{p'}\|g\|_{A^p}.$$

This shows that $H : L^{p'}([0,1)) \times A^p(\mathbb{D}) \to H^p(\mathbb{D})$ is bounded for $p > 1$.

Our objective now is to get the "best" space $Y \subset \mathcal{H}(\mathbb{D})$ such that $H : L^p([0,1)) \times Y \to H^q(\mathbb{D})$ is bounded. We start with the case $p = q = 2$.

Theorem 4.4 *Let $g \in \mathcal{H}(\mathbb{D})$. The following ones are equivalent*:

(1) $H_g : L^2([0,1)) \to H^2(\mathbb{D})$ *is bounded.*
(2) $H_g : H^2(\mathbb{D}) \to H^2(\mathbb{D})$ *is bounded.*
(3) $g \in H^2_{1/2}(\mathbb{D})$.

In particular $H : L^2([0,1)) \times H^2_{1/2}(\mathbb{D}) \to H^2(\mathbb{D})$ is bounded.

Proof (1) $\Longrightarrow$ (2) It follows from Féjer-Riesz's inequality.

(2) $\Longrightarrow$ (3) Let us assume that H_g is bounded on $H^2(\mathbb{D})$. For $0 < r < 1$ consider $C_r(z) = K_0(rz) = \frac{1}{1-rz} = \sum_{n=0}^\infty r^n z^n$. We have $\|C_r\|_{H^2} = (1-r^2)^{-1/2}$, $\mu_n(C_r) = \sum_{k=0}^\infty \frac{r^k}{n+k+1}$ and

$$H_g(C_r) = \sum_{n=0}^\infty \hat{g}(n)(\sum_{k=0}^\infty \frac{r^k}{n+k+1})z^n.$$

Note that

$$\mu_n(C_r) \ge (\sum_{k=n}^{2n} \frac{r^k}{n+k+1}) \ge \frac{1}{2}r^{2n}. \tag{13}$$

Therefore, for each $0 < r < 1$, we have

$$\begin{aligned}\|H_g\|^2_{\mathcal{L}(H^2,H^2)}\|C_r\|^2_{H^2} \geq& \|H_g(C_r)\|^2_{H^2}\\ =&\sum_{n=0}^{\infty}|\hat{g}(n)|^2|\mu_n(C_r)|^2\\ \geq&\frac{1}{2}\sum_{n=0}^{\infty}|\hat{g}(n)|^2r^{2n}\\ \geq&\frac{1}{2}M_2^2(g,r).\end{aligned}$$

This shows that $g \in H^2_{1/2}(\mathbb{D})$.

(3) $\Longrightarrow$ (1) Let $\sup_{0<r<1}(1-r)^{1/2}M_2(g,r)) = A$ and $f \in L^2([0,1))$. Combining the fact $H^2(\mathbb{D}) = \mathcal{D}(2,2,1)$ and Lemma 4.1, we obtain that

$$\begin{aligned}\|H_g(f)\|^2_{H^2} \approx& \int_0^1(1-r)M_2^2(DH_g(f),r)dr\\ \approx&\int_0^1(1-r^2)M_2^2(g*D\mathcal{H}(f),r^2)rdr\\ \leq& C\int_0^1(1-r)M_2^2(g,r)M_1^2(D\mathcal{H}(f),r)dr\\ \leq& CA^2\int_0^1 M_1^2(D\mathcal{H}(f),r)dr\\ \leq& CA^2\|f\|^2_{L^2}.\end{aligned}$$

This completes the proof.

Corollary 4.5 [11, *Theorem* 1] *Let* $g \in \mathcal{H}(\mathbb{D})$. *Then* $\mathcal{H}_g : H^2(\mathbb{D}) \to H^2(\mathbb{D})$ *iff* $g \in \Lambda(2,1/2)$.

Theorem 4.4 can be easily generalized for L^q spaces and $D(2,q,\alpha)$.

Theorem 4.6 *Let* $g \in \mathcal{H}(\mathbb{D}), 1 < q < \infty$ *and* $\alpha > 1/q$. *The following ones are equivalent*

(1) $H_g : L^q([0,1)) \to \mathcal{D}(2,q,\alpha)$ *is bounded.*
(2) $H_g : H^q(\mathbb{D}) \to \mathcal{D}(2,q,\alpha)$ *is bounded.*
(3) $g \in H^2_{\alpha-1/q}(\mathbb{D})$.

Proof (1) $\Longrightarrow$ (2) It follows as above stated using now that $H^q(\mathbb{D}) \subset L^q([0,1))$.

(2) $\Longrightarrow$ (3) Let us assume that H_g is bounded from $H^q(\mathbb{D})$ into $\mathcal{D}(2,q,\alpha)$. Note that $DH_g(C_r) = H_{Dg}(C_r)$ and arguing as in Theorem 4.4 we have that for $0 < s < 1$

$$M_2(DH_g(C_r),s) = M_2(H_{Dg}(C_r),s) \geq \frac{1}{2}M_2(Dg,rs) \tag{14}$$

Therefore,

$$\begin{aligned}\|H_g\|^q_{\mathcal{L}(H^q,\mathcal{D}(2,q,\alpha))}\|C_r\|^q_{H^q} \geq& \|H_g(C_r)\|^q_{\mathcal{D}(2,q,\alpha)}\\ =&\|H_{Dg}(C_r)\|^q_{H(2,q,\alpha)}\\ \geq&\frac{1}{2^q}\|(Dg)_r\|^q_{H(2,q,\alpha)}.\end{aligned}$$

Since $\|C_r\|_{H^q} \approx \frac{1}{(1-r)^{1/q'}}$ it implies that $\|(Dg)_r\|_{H(2,q,\alpha)} \le \frac{C}{(1-r)^{1/q'}}$ for some constant $C > 0$. Therefore, for any $0 < s, r < 1$

$$(1-s)^{\alpha q} M_2^q(Dg, rs) \le \alpha q \int_s^1 (1-t)^{\alpha q - 1} M_2^q(Dg, rt)dt, \le \alpha q \|(Dg)_r\|^q_{H(2,q,\alpha)}.$$

This gives $M_2(Dg, r^2) \le \frac{C}{(1-r)^{\alpha+1/q'}}$. Now since $\alpha + 1/q' > 1$, this is equivalent to $(1-r)^{\alpha-1/q} M_2(g, r) \le C$. Therefore, $g \in H^2_{\alpha-1/q}(\mathbb{D})$.

(3) $\Longrightarrow$ (1) Set $A = \sup_{0<r<1}(1-r)^{\alpha-1/q} M_2(g, r)$. We can use Lemma 4.1 for $f \in L^q([0,1))$ to obtain

$$\begin{aligned}\int_0^1 (1-r)^{\alpha q-1} M_2^q(DH_g(f), r)dr \approx &\int_0^1 (1-r^2)^{\alpha q-1} M_2^q(g * D\mathcal{H}(f), r^2) r dr \\ \le &C \int_0^1 \Big((1-r)^{\alpha-1/q} M_2(g,r)\Big)^q M_1^q(D\mathcal{H}(f), r)dr \\ \le &CA^q \int_0^1 M_1^q(D\mathcal{H}(f), r)dr \\ \le &CA^q \|f\|^q_{L^q}.\end{aligned}$$

This shows that $H_g(f) \in \mathcal{D}(2, q, \alpha)$ and the proof is complete now. □

Theorem 4.7 *Let $1 < q \le p < \infty$, $\alpha > \frac{1}{p}$ and $\frac{1}{t} = \frac{1}{q} - \frac{1}{p}$. Then $H : L^p([0,1)) \times H(p, t, \alpha - \frac{1}{p}) \to \mathcal{D}(p, q, \alpha)$ is bounded.*

Proof Let $f \in L^p([0,1))$. Due to Lemma 4.1, we have $\mathcal{H}(f) \in \mathcal{D}(1, p, \frac{1}{p})$. Assume first that $q < p$ and use Hölder inequality, for p/q and $p/(p-q)$ to obtain the following estimate:

$$\begin{aligned}&\int_0^1 (1-r)^{\alpha q-1} M_p^q(r, D(g * \mathcal{H}(f))dr \\ \le &C \int_0^1 (1-r)^{\alpha q-1} M_p^q(g, r) M_1^q(D\mathcal{H}(f), r)dr \\ \le &C\Big(\int_0^1 (1-r)^{(\alpha q-1)p/(p-q)} M_p^{pq/(p-q)}(g, r)dr\Big)^{1-q/p} \Big(\int_0^1 M_1^p(D\mathcal{H}(f), r)dr\Big)^{q/p} \\ \le &C\Big(\int_0^1 (1-r)^{(\alpha-1/p)t-1} M_p^t(g, r)dr\Big)^{q/t} \|\mathcal{H}(f)\|^q_{\mathcal{D}(1,p,\frac{1}{p})} \\ \le &C\|g\|^q_{p,t,\alpha-1/p} \|f\|^q_{L^p}.\end{aligned}$$

The case $p = q$ (hence, $t = \infty$) follows using the same argument but a simpler one. □

Corollary 4.8 *Let $g \in \mathcal{H}(\mathbb{D})$.*

(i) *If $1 < p \le 2$ and $g \in \Lambda(p, \frac{1}{p})$, then $\mathcal{H}_g : L^p(0,1) \to H^p$ is bounded (see [11]).*
(ii) *If $2 < p < \infty$ and $g \in \mathcal{D}(p, t, \frac{1}{p'})$ where $\frac{1}{t} = \frac{1}{2} - \frac{1}{p}$, then $\mathcal{H}_g : L^p(0,1) \to H^p$ is bounded.*

Proof

(i) Recall that due to Hardy-Littlewood $g \in \Lambda(p, \frac{1}{p})$ corresponds to $g \in \mathcal{D}(p, \infty, \frac{1}{p'})$. Now use Theorem 4.7 together with the inclusion $\mathcal{D}(p, p, 1) \subset H^p$ for $1 < p \le 2$.
(ii) Recall that $\mathcal{D}(p, 2, 1) \subset H^p$ for $p > 2$ and use Theorem 4.7 for $q = 2$ and $\alpha = 1$.

□

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