A SEPARATION THEOREM FOR NONCONVEX SETS AND ITS APPLICATIONS

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ABSTRACT. We prove theorems on separation by sphere or (in a general case) by the boundary of a shifted quasiball of two closed disjoint subsets of a Banach space one of which is prox-regular or weakly convex and the other is a summand of a ball or quasiball. These separation theorems are applied for proving some theorems on the continuity of the intersection of two multifunctions, the values of one of them being prox-regular or weakly convex (nonconvex, in general), and the values of the other being convex and summands of a ball or quasiball. As a corollary, a theorem on the continuity of a multifunction with values bounded by the graphs of two functions is obtained.

1. Introduction

The Hahn–Banach theorem on separation of two convex disjoint sets by a hyperplane is well-known in functional analysis. This theorem serves as a basis of the duality theory, which has many applications in optimization methods and other branches of mathematics. It is easy to see that two nonconvex disjoint sets in the general case cannot be separated by a hyperplane. But if one of the sets is prox-regular, and the other is a summand of a ball of a sufficiently small radius, then these sets can be separated by a sphere.

Let E be a real normed vector space. For a set $A \subset E$, by $\text{int } A$, $\overline{A}$, and ∂A we denote the interior, the closure, and the boundary of $A \subset E$, respectively. We use $\langle p, x \rangle$ to denote the value of the functional $p \in E^*$ at a vector $x \in E$. For $r \geq 0$ and $a \in E$, we define the *ball* with center a and radius r as

$$\mathfrak{B}_r(a) = \{x \in E: \|x - a\| \leq r\}.$$

By $\mathfrak{B}_r$ denote the ball with center at the zero element of the space E : $\mathfrak{B}_r = \mathfrak{B}_r(0)$. The *Minkowski sum* of sets $A \subset E$ and $B \subset E$ is, by definition, the set

$$A + B = \{a + b: a \in A, b \in B\}.$$

A set $C \subset E$ is called a *summand* of a set $M \subset E$ if there exists a set $C_1 \subset E$ such that $C + C_1 = M$.

The *distance from a point $x \in E$ to a set $A \subset E$* is defined as

$$\varrho(x, A) = \inf_{a \in A} \|x - a\|. \quad (1.1)$$

The *set of best approximations* or the *metric projection* of a point $x \in E$ on a set $A \subset E$ is defined as

$$P(x, A) = \{a \in A: \|x - a\| = \varrho(x, A)\}. \quad (1.2)$$

By $\Omega_R(A)$ we denote the *R -tube* around $A \subset E$:

$$\Omega_R(A) = \{x \in E: 0 < \varrho(x, A) < R\}.$$

The notions of prox-regular sets and weakly convex sets considered in this paper date back to sets with positive reach, which were introduced by H. Federer [12]. A set $A \subset \mathbb{R}^n$ is called a *set with positive reach* if $\text{reach}(A) > 0$, where

$$\text{reach}(A) = \sup\{R > 0 \mid P(x, A) \text{ is a singleton for all } x \in \Omega_R(A)\}.$$

Such sets are also called *sets with Chebyshev layer* [2].

J.-P. Vial [23] introduced another definition of weakly convex sets. It is easy to see that a closed set A in a topological vector space is convex if and only if for any two points $x_0, x_1 \in A$, there exists a point $x \in [x_0; x_1] \cap A$ that does not coincide with the points x_0 and x_1 . Substituting in this condition the segment $[x_0; x_1]$ by some special set $D_R(x_0, x_1)$, which is called a strongly convex segment, we obtain the definition of R -weak convexity (see [23]).

Clark, Stern and Wolenski [9], when extending the notion of positive reach sets to Hilbert spaces, introduced the notion of *proximally smooth sets* as the sets $A \subset E$ for which the distance function $\varrho(\cdot, A)$ is continuously differentiable on the set $\Omega_R(A)$ for some $R > 0$. R. Poliquin and R.-T. Rockafellar in [21] introduced the notion of prox-regularity in finite-dimensional Euclidean space, and then this notion was extended to the Hilbert space setting. R. Poliquin, R.-T. Rockafellar, and L. Thibault [22] showed that in a Hilbert space this class coincides with the class of proximally smooth sets. In [6, 7], F. Bernard, L. Thibault, and N. Zlateva studied the *uniformly R -prox-regular sets* in the Banach space E , i.e., sets $A \subset E$ such that

$$P(a + Rz, A) = \{a\} \quad \forall a \in A, \quad \forall z \in N(a, A): \|z\| < 1,$$

where

$$N(a, A) = \{z \in E: \exists t > 0: a \in P(a + tz, A)\}$$

is the *cone of proximal normals* to the set A in the point a .

The *convexity modulus* of a space E is determined by the equality

$$\delta(t) = \inf \left\{ 1 - \frac{\|x + y\|}{2} : x, y \in \mathfrak{B}_1, \|x - y\| \geq t \right\}, \quad t \in [0, 2]. \quad (1.3)$$

The notion of convexity modulus was introduced by J. Clarkson [10]. The space E is called *uniformly convex* if $\delta(t) > 0$ for all $t \in (0, 2]$.

The *smoothness modulus* of the space E is defined as (see [11])

$$\beta(t) = \sup \left\{ \frac{\|x + y\| + \|x - y\|}{2} - 1 : x, y \in E, \|x\| = 1, \|y\| \leq t \right\}, \quad t \geq 0. \quad (1.4)$$

The space E is called *uniformly smooth* if

$$\lim_{t \rightarrow +0} \frac{\beta(t)}{t} = 0.$$

In [3], it was shown that in a uniformly convex and uniformly smooth Banach space E the class of R -prox-regular sets coincides with the class of R -Vial-weakly-convex sets if and only if the unit ball in the Banach space is a generating set (the Hilbert space is an example of a space with the unit ball a generating set). Some properties of R -Vial-weakly convex sets in spaces of type $C(Q)$ (which are neither uniformly smooth nor uniformly convex) are considered in [1].

2. Theorems on Separation and on the Continuity of the Intersection for Prox-Regular Sets

Theorem 2.1 (on separation). *Let E be a uniformly convex and uniformly smooth Banach space, $0 < r < R$, a set $A \subset E$ be closed and uniformly R -prox-regular, a set $C \subset E$ be convex, closed and a summand of the ball $\mathfrak{B}_r$, $\text{int } C \neq \emptyset$, and $A \cap \text{int } C = \emptyset$. Then (see Fig. 1) there exist points $a, c \in E$ such that*

$$\text{int } C \subset \text{int } \mathfrak{B}_r(c) \subset \text{int } \mathfrak{B}_R(a) \subset E \setminus A.$$

Theorem 2.1 generalizes Theorem 1.18.2 of [13], which was obtained for the Hilbert space setting. Further it will be shown that Theorem 2.1 may be used, for example, to obtain sufficient continuity conditions for the intersection of two multifunctions.

Recall that the *deviation* of a set $A \subset E$ from a set $B \subset E$ is defined as

$$h^+(A, B) = \sup_{a \in A} \varrho(a, B). \quad (2.1)$$

Fig. 1

The *Hausdorff distance* between sets $A \subset E$ and $B \subset E$ is

$$h(A, B) = \max\{h^+(A, B), h^+(B, A)\}.$$

Let (T, ϱ_T) be a metric space. A multifunction $A: T \rightarrow 2^E$ is called (Hausdorff) *continuous* at $t_0 \in T$ if $h(A(t), A(t_0)) \rightarrow 0$ as $t \rightarrow t_0$.

In the metric space (T, ϱ_T) let us be given multifunctions $A: T \rightarrow 2^E$ and $C: T \rightarrow 2^E$, continuous at $t_0 \in T$ and let $F(t) = A(t) \cap C(t) \neq \emptyset$ for any $t \in T$. Even in the case where the values of the multifunctions $A(\cdot)$ and $C(\cdot)$ are convex compact sets, the multifunction $F(\cdot)$ may fail to be continuous at t_0 . In [4], M. V. Balashov and D. Repovš showed that if the values of a multifunction $A(\cdot)$ are convex and closed and if the values of a multifunction $C(\cdot)$ are uniformly convex, then the multifunction $F(\cdot)$ is continuous at t_0 . The same authors in [5] showed that for spaces with the modulus of convexity of second order in this theorem, the convexity assumption for the multifunction $A(\cdot)$ can be weakened by substituting this assumption by some assumption in terms of modulus of nonconvexity, introduced by the authors.

The next theorem holds in any uniformly convex and uniformly smooth Banach space (not only in spaces with the modulus of convexity of second order), and gives sufficient condition for the continuity of the intersection of two multifunctions in terms of prox-regularity condition.

Theorem 2.2 (on the continuity of the intersection of multifunctions). *Let E be a uniformly convex and uniformly smooth Banach space, (T, ϱ_T) be a metric space, multifunctions $A: T \rightarrow 2^E$ and $C: T \rightarrow 2^E$ be continuous at $t_0 \in T$. Let $0 < r < R$ and for any $t \in T$ the set $A(t) \subset E$ be closed and uniformly R -prox-regular, the set $C(t) \subset E$ be convex, closed and be a summand of the ball $\mathfrak{B}_r$, and $F(t) = A(t) \cap C(t) \neq \emptyset$. Then the multifunction $F: T \rightarrow 2^E$ is continuous at t_0 .*

Further we will consider the notion of weakly convex sets with respect to (w.r.t.) a quasiball, i.e., w.r.t. a nonsymmetric seminorm. This notion generalizes the notion of prox-regularity of sets. We shall prove Theorem 3.1 on separation for spaces with nonsymmetric seminorm. This theorem will be found useful in obtaining Theorem 3.2 on the continuity of the intersection of multifunctions (a proof of Theorem 3.2 is based on Lemma 7.3, which depends on Theorem 3.1). Theorems 3.1 and 3.2 imply, in particular, Theorems 2.1 and 2.2.

A result close to Theorem 2.2 was obtained in [16, Theorem 4.3] using a different method. It is important that a quasiball may be an unbounded set, for example, the epigraph of a convex function. This last aspect allows the application of the obtained results to the study of the continuity of a multifunction bounded by graphs of functions (see Theorem 4.3).

3. Theorems on Separation and Continuity of the Intersection for Weakly Convex Sets

Let E be a Banach space. The *quasiball* $M \subset E$ is a closed convex set M , $M \neq E$, such that $0 \in \text{int } M$.

The *Minkowski function* of a quasiball $M \subset E$ is, by definition, the function $\mu_M: E \rightarrow [0; +\infty)$ defined by

$$\mu_M(x) = \inf\{t > 0: x \in tM\} \quad \forall x \in E.$$

A function $\mu: E \rightarrow \mathbb{R}$ is called *sublinear* if it is *positive homogeneous*,

$$\mu(\lambda x) = \lambda \mu(x) \quad \forall x \in E \quad \forall \lambda \geq 0,$$

and *subadditive*,

$$\mu(x + y) \leq \mu(x) + \mu(y) \quad \forall x, y \in E.$$

A sublinear nonnegative function is called a *nonsymmetric seminorm*.

Remark 3.1. A continuous function $\mu: E \rightarrow [0; +\infty)$ is a nonsymmetric seminorm if and only if it is the Minkowski function of a quasiball.

Remark 3.2. For any quasiball $M \subset E$ and for any vector $x \in E$, the inequality $\mu_M(x) \leq 1$ is equivalent to the inclusion $x \in M$, and the equality $\mu_M(x) = 1$ is equivalent to the inclusion $x \in \partial M$.

Let $M \subset E$ be a quasiball. The M -distance from a set $C \subset E$ to a set $A \subset E$ is defined as

$$\varrho_M(C, A) = \inf_{\substack{a \in A \\ c \in C}} \mu_M(c - a). \quad (3.1)$$

In particular, the M -distance from a point $x \in E$ to a set $A \subset E$ is defined as

$$\varrho_M(x, A) = \inf_{a \in A} \mu_M(x - a). \quad (3.2)$$

For any quasiball $M \subset E$, we denote

$$\sigma_M = \inf_{x \in \partial M} \|x\|. \quad (3.3)$$

Remark 3.3. For any quasiball $M \subset E$ the inequality $\sigma_M > 0$ and the inclusion $\mathfrak{B}_{\sigma_M} \subset M$ hold.

Remark 3.4. The Minkowski function of any quasiball $M \subset E$ satisfies the Lipschitz condition with the constant $1/\sigma_M$ on E . For any set $A \subset E$ the function $\varrho_M(\cdot, A)$ satisfies the Lipschitz condition on E with the same constant.

The M -projection of a point $x \in E$ on a set $A \subset E$ is, by definition,

$$P_M(x, A) = A \cap (x - \varrho_M(x, A)M). \quad (3.4)$$

By definition, the *cone of proximal normals* to a set $A \subset E$ at a point $a \in A$ w.r.t. a quasiball $M \subset E$ is as follows:

$$N_M(a, A) = \{z \in E: \exists t > 0: a \in P_M(a + tz, A)\}. \quad (3.5)$$

We set

$$N_M^1(a, A) = \{z \in N_M(a, A): \mu_M(z) = 1\}. \quad (3.6)$$

A set $A \subset E$ is called *weakly convex w.r.t. a quasiball $M \subset E$* if

$$a + z \notin A + \text{int } M \quad \forall a \in A \quad \forall z \in N_M^1(a, A). \quad (3.7)$$

Remark 3.5. If $R > 0$, $M = \mathfrak{B}_R$, then for any $x \in E$, $A \subset E$, the equalities

$$\mu_M(x) = \frac{\|x\|}{R}, \quad \varrho_M(x, A) = \frac{\varrho(x, A)}{R}, \quad P_M(x, A) = P(x, A), \quad N_M(x, A) = N(x, A)$$

hold.

A set $M \subset E$ is called *strictly convex* if for any distinct points $x, y \in M$ the inclusion $(x+y)/2 \in \text{int } M$ holds.

Remark 3.6. If $R > 0$ and $M = \mathfrak{B}_R$ is a strongly convex ball, then the class of sets, weakly convex w.r.t. a quasiball M coincides with the class of uniformly R -prox-regular sets.

A set $M \subset E$ is called *boundedly uniformly convex* if $\delta_M(\varepsilon, R) > 0$ for any positive numbers R and ε , where

$$\delta_M(\varepsilon, R) = \sup \left\{ \delta \in \left[0, \frac{\varepsilon}{2}\right] : \|x - y\| \geq \varepsilon \Rightarrow \mathfrak{B}_\delta \left(\frac{x + y}{2} \right) \subset M \quad \forall x, y \in M \cap \mathfrak{B}_R \right\}. \quad (3.8)$$

Remark 3.7. If a set $M \subset E$ is boundedly uniformly convex, then it is strictly convex.

Remark 3.8. If $M = \mathfrak{B}_r$, then $\delta_M(\varepsilon, R) = \delta(\varepsilon)$ for $R \geq r > 0$, $\varepsilon > 0$. Therefore, a ball in a uniformly convex space E is boundedly uniformly convex.

Remark 3.9. A boundedly uniformly convex set may fail to be bounded. For example, if E is a uniformly convex space, then the set $\{(x, y) \in E \times \mathbb{R} : y \geq \|x\|^2\}$ is boundedly uniformly convex in the space $E \times \mathbb{R}$ with the norm $\|(x, y)\| = \|x\| + |y|$.

A quasiball $M \subset E$ is called *boundedly uniformly smooth* if

$$\lim_{t \rightarrow +0} \frac{\beta_M(t, R)}{t} = 0 \quad \forall R > \sigma_M, \quad (3.9)$$

where σ_M is defined by (3.3), and, for any $t \geq 0$ and $R > \sigma_M$,

$$\beta_M(t, R) = \sup \left\{ \frac{\mu_M(x + ty) + \mu_M(x - ty)}{2} - 1 : x \in \partial M \cap \mathfrak{B}_R, y \in \mathfrak{B}_1 \right\}. \quad (3.10)$$

A set $M \subset E$ is called *parabolic* if, for any vector $b \in E$, the set $(b + (1/2)M) \setminus M$ is bounded. A set $M \subset E$ is called *strongly parabolic* if, for any $R > 0$, the set $(\mathfrak{B}_R + (1/2)M) \setminus M$ is bounded.

A set $A \subset E$ is called *closed w.r.t. a quasiball M* or *M -closed* if for any $x \in E \setminus A$ the inequality $\varrho_M(x, A) > 0$ holds.

A set $A \subset E$ is called *M -quasibounded* if it is M -closed and for any $R \geq 0$ the inequality $\kappa(R) < +\infty$ holds, where

$$\kappa(R) = \sup\{\|z\| : z \in N_M^1(a, A), a \in A \cap \mathfrak{B}_R\}. \quad (3.11)$$

Let $M \subset E$ be a quasiball. By $\mathcal{WC}(M)$ denote the class of closed subsets of the space E that are weakly convex w.r.t. the quasiball M ; by $\mathcal{SC}(M)$ we denote the class of convex closed subsets of the space E that are summands of the quasiball M .

Theorem 3.1 (on separation). *Let $M \subset E$ be a parabolic and boundedly uniformly convex quasiball. Let $r \in (0, 1)$, $A \in \mathcal{WC}(M)$, $C \in \mathcal{SC}(-rM)$. In addition, let at least one of the following conditions hold:*

- (1) $\varrho_M(C, A) > 0$;
- (2) $\text{int } C \neq \emptyset$, $A \cap \text{int } C = \emptyset$, the quasiball M is boundedly uniformly smooth, the set A is M -quasibounded.

Then (see Fig. 2) there exist $a, c \in E$ such that

$$\text{int } C \subset c - \text{int } rM \subset a - \text{int } M \subset E \setminus A.$$

Fig. 2

Remark 3.10. The condition of M -quasiboundedness of the set A in condition (2) of Theorem 3.1 cannot be relaxed. Indeed, let us consider in Euclidean space $E = \mathbb{R}^2$ the quasiball

$$M = \{(x, y) \in \mathbb{R}^2 : y \geq x^2 - 1\}$$

and the sets

$$A = \{(x, y) \in \mathbb{R}^2 : y \geq 2x^2 - 1, x \leq 0\}, \quad C = \{(x, y) \in \mathbb{R}^2 : y \leq 1 - 2x^2, x \geq 0\}.$$

Then for $r = 1/2$ the assumptions of condition (2) of Theorem 3.1 hold except for the M -quasiboundedness of the set A . Note that the sets A and C cannot be separated by the boundary of the shifted quasiball M (see Fig. 3).

Fig. 3

Given any $R > 0$, consider the following notion, which generalizes the notion of Hausdorff distance to the setting of possibly unbounded sets:

$$h_R(A_1, A_2) = \max\{h^+(A_1 \cap \mathfrak{B}_R, A_2), h^+(A_2 \cap \mathfrak{B}_R, A_1)\}. \quad (3.12)$$

Let (T, ϱ_T) be a metric space. A multifunction $F: T \rightarrow 2^E$ is called R -continuous at $t_0 \in T$ if for any $R > 0$ the relation $h_R(F(t), F(t_0)) \rightarrow 0$ holds as $t \rightarrow t_0$.

Remark 3.11. If a multifunction is Hausdorff continuous at some point, then it is R -continuous at this point. The inverse statement does not hold. For example, the multifunction $F: (0, +\infty) \rightarrow 2^{\mathbb{R}^2}$, where $F(t) = \{(x, y) \in \mathbb{R}^2 : y \geq tx^2\}$ is R -continuous, but it is not Hausdorff continuous at any $t_0 \in (0, +\infty)$.

A family $\{A(t)\}_{t \in T}$ of subsets of a space E is called M -equi-quasibounded if for any $R > 0$,

$$\sup_{t \in T} \sup_{a \in A(t) \cap \mathfrak{B}_R} \sup_{z \in N_M^1(a, A(t))} \|z\| < +\infty. \quad (3.13)$$

Remark 3.12. If a quasiball $M \subset E$ is bounded, then any closed set $A \subset E$ is M -quasibounded and any family of closed subsets of the space E is M -equi-quasibounded.

An unbounded set $M \subset E$ is called *coercive* if there exists a vector $v_M \in \partial \mathfrak{B}_1$ such that for any sequence $\{z_k\} \subset M$ that satisfies the condition

$$\lim_{k \rightarrow \infty} \|z_k\| = +\infty$$

the relation

$$\lim_{k \rightarrow \infty} \frac{z_k}{\|z_k\|} = v_M$$

holds. The vector v_M is called the *asymptotic direction* of set M .

Theorem 3.2 (continuity of the intersection of multifunctions). *Let $M \subset E$ be a boundedly uniformly convex and boundedly uniformly smooth quasiball. Assume that the quasiball M is either bounded (here we suppose that $v_M = 0$) or is strongly parabolic and coercive, where v_M is its asymptotic direction. Let multifunctions $A: T \rightarrow 2^E$ and $C: T \rightarrow 2^E$ be R -continuous at $t_0 \in T$. Assume that there exist $r \in (0, 1)$ such that for any $t \in T$ the inclusion $C(t) \in \mathcal{SC}(-rM)$ holds. Next, for all $t \in T$, the conditions $A(t) \in \mathcal{WC}(M)$ and $A(t) + \lambda v_M \subset A(t)$ hold for any $\lambda > 0$, $A(t) \cap C(t) \neq \emptyset$, $\text{int } C(t) \neq \emptyset$. Assume that the family $\{A(t)\}_{t \in T}$ is M -equi-quasibounded. Then the multifunction $F(t) = A(t) \cap C(t)$ is Hausdorff continuous at $t_0 \in T$.*

In view of Remarks 3.6, 3.11, and 3.12, Theorems 2.1 and 2.2 follow from Theorems 3.1 and 3.2, respectively.

4. Theorem on the Continuity of a Multifunction Bounded by the Graphs of Two Functions

Recall that the *epigraph* of a function $f: E \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$ is defined as

$$\text{epi } f = \{(x, y) \in E \times \mathbb{R} \mid x \in E, y \geq f(x)\};$$

and the *effective set* of f is defined as

$$\text{dom } f = \{x \in E \mid f(x) \in \mathbb{R}\}.$$

The *infimal convolution* or, in other words, the *episum* of a function $f: E \rightarrow \mathbb{R} \cup \{+\infty\}$ and $g: E \rightarrow \mathbb{R} \cup \{+\infty\}$ is the function

$$(f \boxplus g)(x) = \inf_{u \in E} (f(x - u) + g(u)), \quad x \in E.$$

Remark 4.1. The epigraph of the episum of two functions to within the closure coincides with the Minkowski sum of the epigraph of these functions:

$$\text{epi } f + \text{epi } g \subset \text{epi}(f \boxplus g) \subset \overline{\text{epi } f + \text{epi } g}.$$

Recall the *subdifferential* of a convex function $f: E \rightarrow \mathbb{R} \cup \{+\infty\}$ at $x_0 \in \text{dom } f$ is the set

$$\partial f(x_0) = \{p \in E^*: f(x) - f(x_0) \geq \langle p, x - x_0 \rangle \forall x \in E\}. \quad (4.1)$$

Assume that we are given a convex lower semicontinuous function $\gamma: E \rightarrow \mathbb{R} \cup \{+\infty\}$. By $\pi_\gamma f(x)$ we denote the γ -*predifferential* of a function $f: E \rightarrow \mathbb{R} \cup \{+\infty\}$ at $x \in \text{dom } f$:

$$\pi_\gamma f(x) = \{u \in \text{dom } \gamma: \exists r > 0: (f \boxplus \gamma_r)(x + ru) = f(x) + \gamma_r(ru)\}, \quad (4.2)$$

where $\gamma_r(x) = r\gamma(x/r)$ for all $x \in E$.

A function $f: E \rightarrow \mathbb{R} \cup \{+\infty\}$ is called *strongly convex* w.r.t. the function γ if f is convex, there exist $x_0 \in \text{dom } \gamma$ and $u_0 \in \text{dom } f$ such that $\partial f(u_0) \cap \partial \gamma(x_0) \neq \emptyset$, and

$$\begin{aligned} \gamma(x_1 + w) - \gamma(x_1) &\leq f(u_1 + w) - f(u_1) \\ &\forall w \in E, \forall u_1 \in \text{dom } f, \forall x_1 \in \text{dom } \gamma: \partial f(u_1) \cap \partial \gamma(x_1) \neq \emptyset. \end{aligned} \quad (4.3)$$

We let $\mathcal{SC}(\gamma)$ denote the class of lower semicontinuous functions that are strongly convex w.r.t. a function γ .

A function $f: E \rightarrow \mathbb{R} \cup \{+\infty\}$ is called *weakly convex* w.r.t. a function γ if

$$(f \boxplus \gamma)(x + u) = f(x) + \gamma(u) \quad \forall x \in \text{dom } f, \forall u \in \pi_\gamma f(x). \quad (4.4)$$

We let $\mathcal{WC}(\gamma)$ denote the class of lower semicontinuous functions that are weakly convex w.r.t. a function γ .

A function $\gamma: E \rightarrow \mathbb{R} \cup \{+\infty\}$ is called *coercive* if

$$\lim_{\|x\| \rightarrow \infty} \frac{\gamma(x)}{\|x\|} = +\infty.$$

A function $\gamma: E \rightarrow \mathbb{R} \cup \{+\infty\}$ is called *uniformly convex* on a convex set $X \subset \text{dom } \gamma$ if for any $\varepsilon > 0$ the inequality $\delta_\gamma(\varepsilon, X) > 0$ holds, where

$$\delta_\gamma(\varepsilon, X) = \inf_{\substack{x_1, x_2 \in X \\ \|x_1 - x_2\| \geq \varepsilon}} \left(\frac{\gamma(x_1) + \gamma(x_2)}{2} - \gamma\left(\frac{x_1 + x_2}{2}\right) \right), \quad \varepsilon > 0.$$

A function $\gamma: E \rightarrow \mathbb{R} \cup \{+\infty\}$ is called *uniformly smooth* on a convex set $X \subset \text{dom } \gamma$ if

$$\lim_{\varepsilon \rightarrow +0} \frac{\beta_\gamma(\varepsilon, X)}{\varepsilon} = 0,$$

where

$$\beta_\gamma(\varepsilon, X) = \sup_{\substack{x_1, x_2 \in X \\ \|x_1 - x_2\| \leq \varepsilon}} \left| \frac{\gamma(x_1) + \gamma(x_2)}{2} - \gamma\left(\frac{x_1 + x_2}{2}\right) \right|, \quad \varepsilon > 0. \quad (4.5)$$

Assume the norm in the space $E_1 = E \times \mathbb{R}$ is correlated with the norm of E so that for any $x \in E$ and $y \in \mathbb{R}$ the norm of $z = (x, y) \in E_1$ satisfies the inequalities

$$\max\{\|x\|_E, |y|\} \leq \|z\|_{E_1} \leq \|x\|_E + |y|. \quad (4.6)$$

As is customary, in the notation for norms we shall omit the subscripts E and E_1 whenever they are understood.

The following two theorems put forward a connection between the classes of functions $\mathcal{SC}(\gamma)$, $\mathcal{WC}(\gamma)$ and the classes of sets $\mathcal{SC}(\text{epi } \gamma)$, $\mathcal{WC}(\text{epi } \gamma)$, respectively. Theorem 4.1 will be proved in Sec. 8, Theorem 4.2 is contained in [17, Theorem 3.5].

Theorem 4.1. *Let a function $\gamma: E \rightarrow \mathbb{R}$ be continuous, coercive, and uniformly convex on any ball $\mathfrak{B}_R$, $R > 0$, and let $\gamma(0) < 0$. Then for any $f: E \rightarrow \mathbb{R} \cup \{+\infty\}$ the inclusions $f \in \mathcal{SC}(\gamma)$ and $\text{epi } f \in \mathcal{SC}(\text{epi } \gamma)$ are equivalent.*

Theorem 4.2 ([17, Theorem 3.5]). *Let a function $\gamma: E \rightarrow \mathbb{R} \cup \{+\infty\}$ be convex and lower semicontinuous, $\gamma(0) < 0$, and $0 \in \text{int dom } \gamma$. Then, for any function $f: E \rightarrow \mathbb{R} \cup \{+\infty\}$, the inclusions $f \in \mathcal{WC}(\gamma)$ and $\text{epi } f \in \mathcal{WC}(\text{epi } \gamma)$ are equivalent.*

With the help of Theorems 3.2, 4.1, and 4.2 we shall prove the following theorem on the continuity of a multifunction whose values are bounded by the graphs of two functions.

Theorem 4.3. *Assume we are given functions $\alpha: T \times E \rightarrow \mathbb{R}$, $\omega: T \times E \rightarrow \mathbb{R} \cup \{+\infty\}$, and $\gamma: E \rightarrow \mathbb{R}$, satisfying the following conditions:*

- (1) *a function γ is convex, coercive, uniformly convex on any bounded set, and uniformly smooth on any bounded set;*
- (2) *$t \mapsto \text{epi } \alpha(t, \cdot)$ and $t \mapsto \text{epi } \omega(t, \cdot)$ are R -continuous at $t_0 \in T$;*
- (3) *$\alpha(t, \cdot) \in \mathcal{WC}(\gamma)$ for any $t \in T$;*
- (4) *there exists $r \in (0, 1)$ such that, for any $t \in T$, the inclusion $-\omega(t, \cdot) \in \mathcal{SC}(\gamma_r^-)$ holds, where $\gamma_r^-(x) = r\gamma(-x/r)$ for any $x \in E$;*
- (5) *$\text{int epi } \omega(t, \cdot) \neq \emptyset$ for any $t \in T$;*
- (6) *for any $t \in T$, there exists $x \in E$ such that $\alpha(t, x) \leq \omega(t, x)$;*
- (7) *for any $R > 0$, there exists $L_\alpha(R)$ such that for any $t \in T$ the function $\alpha(t, \cdot)$ satisfies the Lipschitz condition with constant $L_\alpha(R)$ on the ball $\mathfrak{B}_R$.*

Fig. 4

Then the multifunction

$$t \mapsto \{(x, y) \in E \times \mathbb{R} : \alpha(t, x) \leq y \leq \omega(t, x)\}$$

is Hausdorff continuous at $t_0 \in T$.

5. Auxiliary Results

The *support function* of a set $M \subset E$ is defined as

$$s(p, M) = \sup_{x \in M} \langle p, x \rangle, \quad p \in E^*. \quad (5.1)$$

Lemma 5.1. *Let $M \subset E$ be a quasiball, $p \in E^*$, $s(p, M) < +\infty$. Then*

$$\langle p, x \rangle \leq \mu_M(x) s(p, M) \quad \forall x \in E, \quad \forall p \in E^*.$$

Proof. We fix an arbitrary vector $x \in E$ and a functional $p \in E^*$ such that $s(p, M) < +\infty$. If $\mu_M(x) = 0$, then for any $t > 0$ due to Remark 3.2 the inclusion $x/t \in M$ holds. As

$$\sup_{t>0} \left\langle p, \frac{x}{t} \right\rangle \leq s(p, M) < +\infty,$$

then $\langle p, x \rangle \leq 0$, the required inequality holding. Let $\mu_M(x) > 0$. Then the vector $x_1 = x/\mu_M(x)$ satisfies the inclusion $x_1 \in M$ and, therefore,

$$\frac{\langle p, x \rangle}{\mu_M(x)} = \langle p, x_1 \rangle \leq s(p, M). \quad \square$$

A set $C \subset E$ is said to satisfy the *supporting condition of strong convexity* w.r.t. a quasiball $M \subset E$ if

$$C - c \subset M - z \quad \forall c \in C, \quad \forall z \in N_M^1(c, C). \quad (5.2)$$

Lemma 5.2 ([20, Lemma 3.1]). *Let a set $C \subset E$ be a summand of a strongly convex quasiball $M \subset E$. Then the set C satisfies the supporting condition of strong convexity w.r.t. the quasiball M .*

We shall require the following properties of weakly convex sets.

Lemma 5.3 ([18, Lemma 3.3]). *Let $M \subset E$ be a parabolic and a boundedly uniformly convex quasiball, let $A \in \mathcal{WC}(M)$ and $x \in E$ be such that $\varrho_M(x, A) > 0$. Then $A + \text{int } M \neq E$.*

Theorem 5.1 (on the Chebyshev layer [18, Theorem 4.2]). *Let $M \subset E$ be a parabolic and boundedly uniformly convex quasiball. Let $A \in \mathcal{WC}(RM)$, $R > 0$. Let $x \in E$ be such that $0 < \varrho_M(x, A) < R$. Then $P_M(x, A)$ is a singleton.*

Theorem 5.2 (on closest points [18, Theorem 4.3]). *Let $M \subset E$ be a parabolic and boundedly uniformly convex quasiball and let $A \in \mathcal{WC}(RM)$. Assume that a set C is convex, closed and satisfies the supporting condition of strong convexity w.r.t. the quasiball $(-rM)$, $0 < \varrho_M(C, A) < R - r$, where $0 < r < R$. Then the minimum $\min_{a \in A, c \in C} \mu_M(c - a)$ is attained at a single pair of points.*

Lemma 5.4 ([19, Lemma 3.9]). *Let $M \subset E$ be a parabolic and boundedly uniformly convex quasiball. Let the set $C \subset E$ satisfy the supporting condition of strong convexity w.r.t. the quasiball $(-rM)$. Assume that there exist a vector $c_1 \in E$ such that $c_1 + C \subset -rM$. Let a set $A \in \mathcal{WC}(RM)$ be M -quasibounded, where $0 < r < R$, and let $\varrho_M(C, A) = 0$, $A \cap \text{int } C = \emptyset$, and $\text{int } C \neq \emptyset$. Then the set $A \cap C$ is a singleton.*

Lemma 5.5.

- (1) *Let a set $M \subset E$ be convex and parabolic, $0 < \lambda_1 < \lambda_2$, $x_1, x_2 \in E$. Then the set $(\lambda_1 M + x_1) \setminus (\lambda_2 \text{int } M + x_2)$ is bounded.*
- (2) *Let a set $M \subset E$ be convex and strongly parabolic, $0 < \lambda_1 < \lambda_2$, $R > 0$, then the set $(\lambda_1 M + \mathfrak{B}_R) \setminus (\lambda_2 \text{int } M)$ is bounded.*

The first assertion of Lemma 5.5 is implied by Lemma 5.1 of [15]. The second assertion in Lemma 5.5 is proved similarly.

Lemma 5.6. *Let $M \subset E$ be a quasiball, $A \in \mathcal{WC}(M)$, $a_0 \in \partial A$, $z \in N_M^1(a_0, A) \cap \mathfrak{B}_R$, where $R > \sigma_M$. Let $p \in E^*$, $\langle p, z \rangle = s(p, M) = 1$. Then*

$$\langle p, a - a_0 \rangle \leq 2\beta_M(\|a - a_0\|, R) \quad \forall a \in A,$$

where β_M is the modulus of smoothness of M , as determined by equality (3.10).

Proof. We fix an arbitrary $a \in A$. As $z \in N_M^1(a_0, A) \cap \mathfrak{B}_R$, $A \in \mathcal{WC}(M)$, then, according to (3.7), we have that $a_0 + z \notin A + \text{int } M$. Therefore, $a_0 + z - a \notin \text{int } M$; i.e., $\mu_M(a_0 + z - a) \geq 1$. Using (3.10) and the inclusion $z \in N_M^1(a_0, A) \subset \partial M$, we find that

$$\beta_M(\|a - a_0\|, R) \geq \frac{\mu_M(z + a_0 - a) + \mu_M(z + a - a_0)}{2} - 1 \geq \frac{\mu_M(z + a - a_0) - 1}{2}. \quad (5.3)$$

On the other hand, Lemma 5.1 and the equality $s(p, M) = 1$ imply that $\langle p, z + a - a_0 \rangle \leq \mu_M(z + a - a_0)$. Now the required inequality follows from the equality $\langle p, z \rangle = 1$ and inequality (5.3). $\square$

Remark 5.1. For any $R > \sigma_M$, the function $\beta_M(\cdot, R)$ is a convex function as the supremum of convex functions. This and the equality $\beta_M(0, R) = 0$ and the inequality $\beta_M(t, R) \geq 0$ imply that the function $t \mapsto \beta_M(t, R)/t$ nonstrictly increases for any $R > \sigma_M$.

Lemma 5.7. *Let $M \subset E$ be a parabolic and boundedly uniformly convex quasiball. Assume that an M -quasibounded set $A \in \mathcal{WC}(M)$ contains points x_0 and x_1 such that, for any $\lambda \in (0, 1)$, the inequality $\|x_1 - x_0\| \min\{\lambda, 1 - \lambda\} < \sigma_M$ holds, where the number σ_M is defined by (3.3). Let $x_\lambda = (1 - \lambda)x_0 + \lambda x_1$, $a \in P_M(x_\lambda, A)$, $R \geq \max\{\sigma_M + 1, \kappa(\|a\|)\}$, $\varkappa = \kappa(\|a\|)/\sigma_M$, where the function $\kappa(\cdot)$ is defined by equality (3.11). Then*

$$\varrho_M(x_\lambda, A) \leq 4\lambda(1 - \lambda)\beta_M((1 + \varkappa)\|x_1 - x_0\|, R).$$

Proof. If $\varrho_M(x_\lambda, A) = 0$, then the required inequality holds trivially. Now suppose that $\varrho_M(x_\lambda, A) > 0$. According to (3.5) and (3.6), the vector $z = (x_\lambda - a)/\varrho_M(x_\lambda, A)$ satisfies the inclusion $z \in N_M^1(a, A)$. Using (3.11), we find that

$$\|z\| \leq \kappa(\|a\|) \leq R. \quad (5.4)$$

As $z \in \partial M$, then by the Hahn–Banach separation theorem, there exists a functional $p \in E^*$ such that $\langle p, z \rangle = s(p, M) = 1$. Lemma 5.6 implies that

$$\langle p, x_0 - a \rangle \leq 2\beta_M(\|x_0 - a\|, R), \quad \langle p, x_1 - a \rangle \leq 2\beta_M(\|x_1 - a\|, R).$$

Therefore,

$$\langle p, x_\lambda - a \rangle = (1 - \lambda)\langle p, x_0 - a \rangle + \lambda\langle p, x_1 - a \rangle \leq 2(1 - \lambda)\beta_M(\|x_0 - a\|, R) + 2\lambda\beta_M(\|x_1 - a\|, R). \quad (5.5)$$

According to Remark 3.4, we get

$$\varrho_M(x_\lambda, A) \leq \min\{\mu_M(x_\lambda - x_0), \mu_M(x_\lambda - x_1)\} \leq \frac{\min\{\|x_\lambda - x_0\|, \|x_\lambda - x_1\|\}}{\sigma_M} = \frac{\|x_1 - x_0\|}{\sigma_M} \min\{\lambda, 1 - \lambda\} < 1,$$

and so, by inequality (5.4),

$$\|x_\lambda - a\| \leq \kappa(\|a\|)\varrho_M(x_\lambda, A) \leq \varkappa\|x_1 - x_0\| \min\{\lambda, 1 - \lambda\}.$$

Therefore,

$$\begin{aligned} \|x_0 - a\| &\leq \|x_0 - x_\lambda\| + \|x_\lambda - a\| \leq \lambda(1 + \varkappa)\|x_1 - x_0\|, \\ \|x_1 - a\| &\leq \|x_1 - x_\lambda\| + \|x_\lambda - a\| \leq (1 - \lambda)(1 + \varkappa)\|x_1 - x_0\|. \end{aligned}$$

Hence, using inequalities (5.5),

$$\langle p, x_\lambda - a \rangle \leq 2(1 - \lambda)\beta_M(\lambda(1 + \varkappa)\|x_1 - x_0\|, R) + 2\lambda\beta_M((1 - \lambda)(1 + \varkappa)\|x_1 - x_0\|, R).$$

Employing Remark 5.1, we obtain the inequality

$$\langle p, x_\lambda - a \rangle \leq 4\lambda(1 - \lambda)\beta_M((1 + \varkappa)\|x_1 - x_0\|, R). \quad (5.6)$$

As $\langle p, z \rangle = 1$, then $\langle p, x_\lambda - a \rangle = \varrho_M(x_\lambda, A)$. Now the required inequality follows from (5.6). $\square$

Lemma 5.8. *Let $M \subset E$ be a parabolic and boundedly uniformly convex quasiball. Assume that a set $A \in \mathcal{WC}(M)$ is M -closed, $a_0 \in \partial A$. Then*

$$\eta := \sup\{\|a\| : a \in P_M(x, A), x \in \mathfrak{B}_{(1/4)\sigma_M}(a_0)\} < +\infty. \quad (5.7)$$

Proof. Let $x \in \mathfrak{B}_{(1/4)\sigma_M}(a_0)$ and $a \in P_M(x, A)$. Then

$$\mu_M(a_0 - a) \leq \mu_M(x - a) + \mu_M(a_0 - x) = \varrho_M(x, A) + \mu_M(a_0 - x) \leq \mu_M(x - a_0) + \mu_M(a_0 - x).$$

The last inequality follows from the fact that $a_0 \in \partial A$. As $\{x - a_0, a_0 - x\} \subset \mathfrak{B}_{(1/4)\sigma_M} \subset (1/4)M$, then $\mu_M(x - a_0) \leq 1/4$ and $\mu_M(a_0 - x) \leq 1/4$. Therefore, $\mu_M(a_0 - a) \leq 1/2$, and so, $a \in A \cap (a_0 - (1/2)M)$. Therefore, it suffices to show that the set $A \cap (a_0 - (1/2)M)$ is bounded.

As the set A is M -closed and $A \neq E$, there exists $x_0 \in E$ such that $\varrho_M(x_0, A) > 0$. Lemma 5.3 implies that there exists $w \in E \setminus (A + \text{int } M)$. Therefore, $A \subset E \setminus (w - \text{int } M)$, and so, $A \cap (a_0 - (1/2)M) \subset (a_0 - (1/2)M) \setminus (w - \text{int } M)$. Using the parabolicity of the set M , Lemma 5.5 secures the boundedness of the set $A \cap (a_0 - (1/2)M)$. $\square$

6. Proof of Theorem 3.1

The *contingent cone* to a set $A \subset E$ at $a_0 \in E$, which was introduced in [8], is defined as

$$T(a_0, A) = \bigcap_{\varepsilon > 0} \bigcap_{\delta > 0} \bigcup_{t \in (0, \delta]} \left(\frac{1}{t}(A - a_0) + \mathfrak{B}_\varepsilon \right).$$

Remark 6.1. A vector v belongs to $T(a_0, A)$ if and only if there exist sequences $\{v_k\} \subset E$ and $\{t_k\} \subset (0, +\infty)$ such that $\lim_{k \rightarrow \infty} v_k = v$, $\lim_{k \rightarrow \infty} t_k = 0$ and $a_0 + t_k v_k \in A$ for all $k \in \mathbb{N}$.

Lemma 6.1. *Let $M \subset E$ be parabolic, boundedly uniformly convex, and boundedly uniformly smooth. Assume that a set $A \in \mathcal{WC}(M)$ is M -quasibounded, $a_0 \in \partial A$. Then the cone $T(a_0, A)$ is convex.*

Proof. Let $u, v \in T(a_0, A)$. We have to prove the inclusion

$$u + v \in T(a_0, A). \quad (6.1)$$

According to Remark 6.1, there exist sequences $\{u_k\} \subset E$, $\{v_k\} \subset E$, $\{t_k\} \subset (0, +\infty)$, and $\{\tau_k\} \subset (0, +\infty)$ such that

$$\lim_{k \rightarrow \infty} u_k = u, \quad \lim_{k \rightarrow \infty} v_k = v, \quad \lim_{k \rightarrow \infty} t_k = 0, \quad \lim_{k \rightarrow \infty} \tau_k = 0, \quad (6.2)$$

$a_0 + t_k u_k \in A$ and $a_0 + \tau_k v_k \in A$ for all $k \in \mathbb{N}$. For any $k \in \mathbb{N}$ we denote

$$\xi_k = \frac{t_k \tau_k}{t_k + \tau_k}, \quad y_k = a_0 + \xi_k(u_k + v_k), \quad \varrho_k = \varrho_M(y_k, A).$$

If for any index k_0 there exists an index $k \geq k_0$ such that $y_k \in A$, then Remark 6.1 and relation (6.2) imply inclusion (6.1). Therefore, we suppose that $y_k \notin A$ for all k starting from k_0 . Using the M -quasi-boundedness of the set A , we obtain the inequality $\varrho_k > 0$ for all $k \geq k_0$.

As $0 < \xi_k < t_k \rightarrow 0$, $\tau_k \rightarrow 0$ for $k \rightarrow \infty$, and since the sequences $\{u_k\}$ and $\{v_k\}$ are bounded, we have that $y_k \rightarrow a_0$ and $\varrho_k \rightarrow 0$ as $k \rightarrow \infty$. Therefore, there exists an index $k_1 \geq k_0$ such that

$$\varrho_k < 1, \quad y_k \in \mathfrak{B}_{(1/4)\sigma_M}(a_0), \quad \|t_k u_k - \tau_k v_k\| < \sigma_M \quad \forall k \geq k_1.$$

Theorem 5.1 implies that for any $k \geq k_1$ there exists a point $a_k \in P_M(y_k, A)$. According to relation (5.7), we have that $\sup_{k \geq k_1} \|a_k\| \leq \eta < +\infty$. For any fixed $k \geq k_1$, Lemma 5.7, as applied to $x_0 = a_0 + t_k u_k$,

$x_1 = a_0 + \tau_k v_k$, and $\lambda = t_k/(t_k + \tau_k)$, implies that

$$\varrho_k \leq 4\xi_k \varepsilon_k, \quad (6.3)$$

where

$$\varepsilon_k = \frac{\beta_M((1 + \varkappa)\|t_k u_k - \tau_k v_k\|, R)}{t_k + \tau_k}, \quad \varkappa = \frac{\kappa(\eta)}{\sigma_M}, \quad R = \max\{\sigma_M + 1, \kappa(\eta)\}.$$

Using relation (3.9) and the boundedness of sequences $\{u_k\}$ and $\{v_k\}$, we find that

$$\lim_{k \rightarrow \infty} \varepsilon_k = 0. \quad (6.4)$$

Given $k \geq k_1$, we set $z_k = (y_k - a_k)/\varrho_k$. Since $a_k \in P_M(y_k, A)$, we have that $z_k \in N_M^1(a_k, A)$. This and the M -quasiboundedness of A imply that $\sup_{k \geq k_1} \|z_k\| = C < +\infty$. Using inequality (6.3), we find that, for any $k \geq k_1$,

$$\|y_k - a_k\| = \|z_k\| \varrho_k \leq C \varrho_k \leq 4C \xi_k \varepsilon_k.$$

Denoting $w_k = (a_k - a_0)/\xi_k$, we obtain the inequality $\|u_k + v_k - w_k\| \leq 4C \varepsilon_k$ as $k \geq k_1$. Using (6.4) and (6.2), this gives $w_k \rightarrow u + v$ as $k \rightarrow \infty$. As $a_k = a_0 + \xi_k w_k \in A$, then relation (6.1) holds. $\square$

Lemma 6.2. *Let a set $A \subset E$ and a convex set $C \subset E$ be such that $0 \in A \cap C$, $\text{int } C \neq \emptyset$, and $A \cap \text{int } C = \emptyset$. Then $T(0, A) \cap \text{int } C = \emptyset$.*

Proof. Suppose, on the contrary, that there exists a vector $v \in T(0, A) \cap \text{int } C$. Since $v \in \text{int } C$, there exists $\delta > 0$ such that $\mathfrak{B}_\delta(v) \subset \text{int } C$. Next, since $v \in T(0, A)$, there exists $u \in \mathfrak{B}_\delta(v)$ and $t \in (0, 1)$ such that $tu \in A$. On the other hand, the inclusions $u \in \mathfrak{B}_\delta(v) \subset \text{int } C$, $0 \in C$ and the convexity of C imply that $tu \in \text{int } C$. However, this contradicts the assumption $A \cap \text{int } C = \emptyset$. $\square$

Lemma 6.3. *Let $M \subset E$ be a parabolic, boundedly uniformly convex, and boundedly uniformly smooth quasiball. Assume that a set $A \in \mathcal{WC}(M)$ is M -quasibounded and $0 \in \partial A$. Then there exist positive numbers δ , C_1 , and C_2 and a number $R > \sigma_M$ such that for any nonzero vector $a \in A \cap \mathfrak{B}_\delta$ there exists a nonzero vector $v \in T(0, A)$ that satisfies the inequality*

$$\|v - a\| \leq C_1 \beta_M(C_2 \|a\|, R), \quad (6.5)$$

where the function $\beta_M(\cdot)$ is defined by equality (3.10).

Proof. Employing the function $\kappa(\cdot)$ as defined by (3.11) and the number η as given by (5.7), we set

$$\varkappa = \frac{\kappa(\eta)}{\sigma_M}, \quad R = \max\{\sigma_M + 1, \kappa(\eta)\}. \quad (6.6)$$

For any $t > 0$, we define

$$\gamma_M(t) = \frac{\beta_M((1 + \varkappa)t, R)}{t}.$$

Remark 5.1 implies that function $\gamma_M(\cdot)$ nonstrictly increases. Relation (3.9) implies that $\lim_{t \rightarrow +0} \gamma_M(t) = 0$. Therefore, there exists a positive number $\delta \leq \min\{\sigma_M/2, \eta\}$ such that

$$8R\gamma_M(\delta) \leq 1. \quad (6.7)$$

We fix an arbitrary nonzero vector $a \in A \cap \mathfrak{B}_\delta$. Suppose that $a_0 = a$ and $\delta_0 = \|a_0\|$. As $\lim_{t \rightarrow +0} \gamma_M(t) = 0$, then for any $k \in \mathbb{N}$ there exists a number $\delta_k \in (0, \delta]$ such that

$$\gamma_M(\delta_k) \leq \frac{\gamma_M(\delta_0)}{2^k}. \quad (6.8)$$

Assume that for some $k \in \mathbb{N} \cup \{0\}$ we are given a vector $a_k \in A$ such that

$$0 < \|a_k\| \leq \delta_k. \quad (6.9)$$

We fix any $\lambda_k \in (0, 1)$ such that

$$2\lambda_k\|a_k\| < \delta_{k+1}. \quad (6.10)$$

As

$$\|\lambda_k a_k\| < \frac{\delta_{k+1}}{2} \leq \frac{\delta}{2} \leq \frac{\sigma_M}{4},$$

Remark 3.4 implies the inequalities

$$\varrho_M(\lambda_k a_k, A) \leq \frac{\|\lambda_k a_k\|}{\sigma_M} < \frac{1}{4}.$$

Therefore, according to Theorem 5.1, there exists $a_{k+1} \in P_M(\lambda_k a_k, A)$. The inequality $\|\lambda_k a_k\| < \sigma_M/4$ and equality (5.7) imply the inequality $\|a_{k+1}\| \leq \eta$. By Lemma 5.7, as applied to $x_0 = 0$, $x_1 = a_k$, $\lambda = \lambda_k$, and $a = a_{k+1}$, and equality (6.6), we see that

$$\varrho_M(\lambda_k a_k, A) \leq 4\lambda_k \beta_M((1 + \varkappa)\|a_k\|, R) = 4\lambda_k \|a_k\| \gamma_M(\|a_k\|). \quad (6.11)$$

Equality (3.11) implies that

$$\|\lambda_k a_k - a_{k+1}\| \leq \kappa(\eta) \mu_M(\lambda_k a_k - a_{k+1}) \leq R \mu_M(\lambda_k a_k - a_{k+1}).$$

This, inequality (6.11), and the equality $\mu_M(\lambda_k a_k - a_{k+1}) = \varrho_M(\lambda_k a_k, A)$ imply that

$$\|\lambda_k a_k - a_{k+1}\| \leq R \mu_M(\lambda_k a_k - a_{k+1}) \leq 4R \lambda_k \|a_k\| \gamma_M(\|a_k\|). \quad (6.12)$$

According to Remark 5.1, the inequality $\gamma_M(\|a_k\|) \leq \gamma_M(\delta)$ holds. Hence inequality (6.7) yields

$$\|\lambda_k a_k - a_{k+1}\| \leq \frac{\lambda_k \|a_k\|}{2}.$$

Therefore, using inequality (6.10), we obtain the series of inequalities

$$0 < \frac{1}{2} \lambda_k \|a_k\| \leq \|a_{k+1}\| \leq \frac{3}{2} \lambda_k \|a_k\| < \delta_{k+1}.$$

Therefore, $0 < \|a_{k+1}\| \leq \delta_{k+1}$ and we may continue the process of constructing the sequence a_k . Thus, the recursively constructed sequence $\{a_k\}_{k=0}^\infty \subset A$ is such that, for any $k \in \mathbb{N} \cup \{0\}$, inequalities (6.9) and (6.12) hold.

For any $k \in \mathbb{N} \cup \{0\}$, using the inequalities

$$\left\| \frac{a_{k+1}}{\|a_{k+1}\|} - \frac{a_k}{\|a_k\|} \right\| \leq \left\| \frac{a_{k+1}}{\|a_{k+1}\|} - \frac{a_{k+1}}{\lambda_k \|a_k\|} \right\| + \left\| \frac{a_{k+1}}{\lambda_k \|a_k\|} - \frac{a_k}{\|a_k\|} \right\| \leq \frac{2\|\lambda_k a_k - a_{k+1}\|}{\lambda_k \|a_k\|},$$

relations (6.8), (6.9), and (6.12) and Remark 5.1, we find that

$$\left\| \frac{a_{k+1}}{\|a_{k+1}\|} - \frac{a_k}{\|a_k\|} \right\| \leq 8R\gamma_M(\|a_k\|) \leq 8R\gamma_M(\delta_k) \leq \frac{8R\gamma_M(\delta_0)}{2^k}.$$

Therefore,

$$\left\| \frac{a_m}{\|a_m\|} - \frac{a_n}{\|a_n\|} \right\| \leq \sum_{k=n}^{m-1} \left\| \frac{a_{k+1}}{\|a_{k+1}\|} - \frac{a_k}{\|a_k\|} \right\| \leq \frac{R\gamma_M(\delta_0)}{2^{n-4}} \quad \forall n \in \mathbb{N} \cap \{0\}, \quad \forall m > n.$$

Thus, the sequence $\{a_k/\|a_k\|\}$ is a Cauchy sequence, which converges to some $v_0 \in E$. Moreover,

$$\left\| v_0 - \frac{a}{\delta_0} \right\| \leq 16R\gamma_M(\delta_0) = \frac{16R\beta_M((1+\varkappa)\delta_0, R)}{\delta_0}.$$

Therefore, the vector $v = \delta_0 v_0$ satisfies inequality (6.5) as $C_1 = 16R$ and $C_2 = 1 + \varkappa$. Since $a_k \in A$, $a_k/\|a_k\| \rightarrow v_0$, $\|a_k\| \rightarrow 0$ as $k \rightarrow \infty$, we have that $v_0 \in T(0, A)$, and, therefore, $v \in T(0, A)$. $\square$

Lemma 6.4 ([15, Lemma 5.2]). *Let a set $M \subset E$ be parabolic. Then*

- (1) *for any functional $p \in \text{dom } s(\cdot, M) \setminus \{0\}$, the set $\{x \in M : \langle p, x \rangle \geq -1\}$ is bounded;*
- (2) *the set $\text{dom } s(\cdot, M) \setminus \{0\}$ is open.*

Inclusion (21) from [17], and Lemmas 4.4 and 7.2 from the same paper imply the following two lemmas.

Lemma 6.5. *If in a space E there exists a parabolic boundedly uniformly convex quasiball, then the space E is reflexive.*

Lemma 6.6. *Let $M \subset E$ be a boundedly uniformly convex and parabolic quasiball. Let a functional $p \in E^* \setminus \{0\}$ and a bounded sequence $\{w_k\} \subset \partial M$ be such that*

$$\lim_{k \rightarrow \infty} \langle p, w_k \rangle = s(p, M) < +\infty.$$

Then the sequence $\{w_k\}$ converges.

Lemma 6.7. *Let $M \subset E$ be a parabolic, boundedly uniformly convex, and boundedly uniformly smooth quasiball, let a set $A \in \mathcal{WC}(M)$ be M -quasibounded, $0 \in \partial A$. Next, let a functional $p \in E^*$ satisfy the equality $s(p, T(0, A)) = 0$. Then*

- (1) $s(p, M) < +\infty$;
- (2) *if $z \in \partial M$ satisfies the equalities $\langle p, z \rangle = s(p, M) = 1$, then $0 \in P_M(z, A)$.*

Proof. We fix an arbitrary $z \in M$ such that $\langle p, z \rangle > 0$. If, for any k_0 , there exists a number $k \geq k_0$ such that $z/k \in A$, then $z \in T(0, A)$. Therefore, $0 < \langle p, z \rangle \leq s(p, T(0, A)) = 0$. This contradiction proves the existence of a number k_0 such that $z/k \notin A$ for all $k \geq k_0$. This and the M -quasiboundedness of set A imply that $\varrho_M(z/k, A) > 0$ as $k \geq k_0$. As $0 \in A$, we get

$$\varrho_M\left(\frac{z}{k}, A\right) \leq \mu_M\left(\frac{z}{k}\right) \leq \frac{1}{k} \quad \forall k \in \mathbb{N}. \quad (6.13)$$

Theorem 5.1 shows that, for any $k > k_0$, there exists $a_k \in P_M(z/k, A)$. For any $k > k_0$, we set

$$w_k = \frac{z/k - a_k}{\mu_M(z/k - a_k)}. \quad (6.14)$$

Choose an index $k_1 > k_0$ such that $\|z\|/k_1 < \sigma_M/4$. Then, according to equality (5.7), we get that $\sup_{k \geq k_1} \|a_k\| \leq \eta$. Using (3.11) and the M -quasiboundedness of set A , we have

$$\|w_k\| \leq \kappa(\eta) < +\infty \quad \forall k \geq k_1.$$

Hence, employing (6.13) and (6.14) and the equality

$$\mu_M \left(\frac{z}{k} - a_k \right) = \varrho_M \left(\frac{z}{k}, A \right),$$

we find that

$$\left\| \frac{z}{k} - a_k \right\| \leq \frac{\kappa(\eta)}{k} \quad \forall k \geq k_1. \quad (6.15)$$

Therefore,

$$\|a_k\| \leq \left\| \frac{z}{k} - a_k \right\| + \frac{\|z\|}{k} \leq \frac{\kappa(\eta) + \|z\|}{k} \rightarrow 0, \quad k \rightarrow \infty. \quad (6.16)$$

Lemma 6.3 shows that there exist indexes $k_2 \geq k_1$, positive numbers C_1 and C_2 , and a number $R > \sigma_M$ such that, for any $k \geq k_2$, there exists a vector $v_k \in T(0, A)$ satisfying the inequality

$$\|a_k - v_k\| \leq \varepsilon_k := C_1 \beta_M(C_2 \|a_k\|, R). \quad (6.17)$$

Using (3.9) and (6.16), we obtain that

$$k\varepsilon_k = C_1 k \beta_M(C_2 \|a_k\|, R) \leq C_1 k \beta_M \left(\frac{C_2(\kappa(\eta) + \|z\|)}{k}, R \right) \rightarrow 0, \quad k \rightarrow \infty. \quad (6.18)$$

As $v_k \in T(0, A)$, we have that $\langle p, v_k \rangle \leq s(p, T(0, A)) = 0$. Therefore, according to inequality (6.17),

$$\langle p, a_k \rangle \leq \langle p, a_k - v_k \rangle \leq \|p\| \cdot \|a_k - v_k\| \leq \varepsilon_k \|p\| \quad \forall k \geq k_2.$$

So,

$$\left\langle p, \frac{z}{k} \right\rangle \leq \left\langle p, \frac{z}{k} - a_k \right\rangle + \varepsilon_k \|p\| \quad \forall k \geq k_2. \quad (6.19)$$

Using inequality (6.15), this establishes

$$\langle p, z \rangle \leq \left(\left\| \frac{z}{k} - a_k \right\| + \varepsilon_k \right) k \|p\| \leq (\kappa(\eta) + k\varepsilon_k) \|p\| \quad \forall k \geq k_2.$$

Making $k \rightarrow \infty$ and taking into account (6.18), we find that

$$\langle p, z \rangle \leq \kappa(\eta) \|p\|, \quad (6.20)$$

where $\kappa(\cdot)$ and the number η are defined by (3.11) and (5.7), respectively. Therefore, $\kappa(\eta)$ does not depend on z . If $\langle p, z \rangle \leq 0$, then inequality (6.20) also holds. Thus, inequality (6.20) holds for any $z \in M$. Therefore, $s(p, M) \leq \kappa(\eta) \|p\| < +\infty$. This proves the first assertion of the lemma.

Let us prove the second assertion. Let $z \in \partial M$ satisfy the equalities $\langle p, z \rangle = s(p, M) = 1$. Using inequality (6.19), we have that

$$\left\langle p, \frac{z}{k} - a_k \right\rangle \geq \left\langle p, \frac{z}{k} \right\rangle - \varepsilon_k \|p\| = \frac{1 - k\varepsilon_k \|p\|}{k} \quad \forall k \geq k_2.$$

Relation (6.18) implies that there exists $k_3 \geq k_2$ such that $k\varepsilon_k \|p\| < 1$ for all $k \geq k_3$. Using the equality

$$\mu_M \left(\frac{z}{k} - a_k \right) = \varrho_M \left(\frac{z}{k}, A \right)$$

and (6.13), (6.14), we obtain that

$$s(p, M) \geq \langle p, w_k \rangle = \frac{\langle p, z/k - a_k \rangle}{\mu_M(z/k - a_k)} \geq \frac{1 - k\varepsilon_k \|p\|}{k \mu_M(z/k - a_k)} \geq 1 - k\varepsilon_k \|p\| \quad \forall k \geq k_2.$$

We have $w_k \in M$, and hence by (6.18)

$$s(p, M) \geq \langle p, w_k \rangle \geq 1 - k\varepsilon_k \|p\| \rightarrow 1 = s(p, M), \quad k \rightarrow \infty.$$

The sequence $\{w_k\}$ is bounded by Lemma 6.4(1). Therefore, in view of Lemma 6.6, this sequence converges to some vector $w \in E$. In addition, $w \in M$ and $\langle p, w \rangle = s(p, M) = \langle p, z \rangle$. Taking into consideration the strict convexity of M , we get the equality $w = z$. So,

$$\lim_{k \rightarrow \infty} w_k = z. \quad (6.21)$$

As $a_k \in P_M(z/k, A)$, then using (3.5), (3.6), we have $w_k \in N_M^1(a_k, A)$. Thus, taking into consideration the inclusion $A \in \mathcal{WC}(M)$ and formulas (3.7), we prove that $a_k + w_k \notin A + \text{int } M$, i.e., $\varrho_M(a_k + w_k, A) = 1$. Using (6.16), (6.21), and the continuity of the function $\varrho_M(\cdot, A)$, we get the equality $\varrho_M(z, A) = 1$, i.e., $0 \in P_M(z, A)$. $\square$

For any set $X \subset E$ and any functional $p \in E^*$, we consider the set

$$\text{Exp}(p, X) = \{x \in X : \langle p, x \rangle = s(p, X)\}. \quad (6.22)$$

Remark 6.2. Definition (6.22) and formulas $s(p, X_1 + X_2) = s(p, X_1) + s(p, X_2)$ imply that

$$\text{Exp}(p, X_1 + X_2) = \text{Exp}(p, X_1) + \text{Exp}(p, X_2) \quad \forall X_1, X_2 \subset E, \quad \forall p \in E^*.$$

Proof of Theorem 3.1. Lemma 5.2 implies that the set C satisfies the strong convexity supporting condition w.r.t. the quasiball $(-rM)$.

(1) Let $\varrho_M(C, A) > 0$.

(a) First, we suppose that $\varrho_M(C, A) < 1 - r$. By Theorem 5.2, the minimum

$$\min_{\substack{a \in A \\ c \in C}} \mu_M(c - a)$$

is attained at a single pair of points $a_0 \in A$, $c_0 \in C$. Therefore,

$$a_0 \in P_M(c_0, A), \quad c_0 \in P_{-M}(a_0, C).$$

Denote

$$w = \frac{c_0 - a_0}{\mu_M(c_0 - a_0)}.$$

As $A \in \mathcal{WC}(M)$, then, according to (3.7), we have that

$$a_0 + w - \text{int } M \subset E \setminus A.$$

As C satisfies the strong convexity supporting condition w.r.t. the quasiball $(-rM)$, it follows from (5.2) that

$$C \subset c_0 - r(M - w).$$

Let $c = c_0 + rw$ and $a = a_0 + w$. Then $C \subset c - rM$ and $a - \text{int } M \subset E \setminus A$. It remains to prove that $a - c \in \text{int}(1 - r)M$. From the definition of the vector w we have that

$$\mu_M(a - c) = \mu_M(a_0 - c_0 + (1 - r)w) = \mu_M\left((1 - r - \mu_M(c_0 - a_0))w\right) = 1 - r - \mu_M(c_0 - a_0) < 1 - r,$$

i.e., $a - c \in \text{int}(1 - r)M$.

(b) Now let $\varrho_M(C, A) \geq 1 - r$. The inclusion $C \in \mathcal{SC}(-rM)$ implies that there exists a set $C' \subset E$ such that $C + C' = -rM$. Given $c_0 \in C'$, for any $t \in [0, 1]$ consider the set $C_t = C + t(C' - c_0)$. Suppose that $\varrho_M(C_1, A) \geq 1 - r$. As $C_1 = C_t|_{t=1} = -c_0 - rM$, then $\varrho_M(-c_0 - rM, A) \geq 1 - r$. Therefore, $(-c_0 - rM) \cap (A + (1 - r)\text{int } M) = \emptyset$, and so, $-c_0 - \text{int } M \subset E \setminus A$. Assuming that $a = c = -c_0$, we obtain the required inclusions. Now assume that $\varrho_M(C_1, A) < 1 - r$. As the function $t \mapsto \varrho_M(C_t, A)$ is continuous and $\varrho_M(C_0, A) = \varrho_M(C, A) \geq 1 - r > 0$, there exists $\tau \in (0, 1]$ such that $0 < \varrho_M(C_\tau, A) < 1 - r$. Note that $C_\tau \in \mathcal{SC}(-rM)$. In view of the assertion obtained at Step (a), there exist $a, c \in E$ such that

$$\text{int } C_\tau \subset c - \text{int } rM \subset a - \text{int } M \subset E \setminus A.$$

Now the required inclusions follow since $C \subset C_\tau$.

(2) Let all the assumptions of assertion (2) of Theorem 3.1 hold. If $\varrho_M(C, A) > 0$, then the necessary inclusions were obtained in the previous point. Therefore, we suppose that $\varrho_M(C, A) = 0$. Lemma 5.4 shows that $A \cap C$ consists of a single element. Without loss of generality, we assume that $A \cap C = \{0\}$. By Lemma 6.2 we have that $T(0, A) \cap \text{int } C = \emptyset$. Next, Lemma 6.1 shows that the cone $T(0, A)$ is convex. The Hahn–Banach separation theorem implies that there exists a nonzero functional $p \in E^*$ such that $s(p, T(0, A)) \leq -s(-p, C)$. Using the equality $A \cap C = \{0\}$, we find that $s(p, T(0, A)) = 0$, $s(-p, C) = 0$. By Lemma 6.7(1), the inequality $s(p, M) < +\infty$ holds; i.e., $p \in \text{dom } s(\cdot, M)$. Using Lemma 6.4(2), we get

that $p \in \text{int dom } s(\cdot, M)$. Next, employing Lemma 6.5, we obtain the reflexivity of the space E . Thus, the function $s(\cdot, M)$ has a nonempty subdifferential at p . This means that there exists a vector $z \in \partial M$ such that $\langle p, z \rangle = s(p, M)$. Using Lemma 6.7(2), we have that $0 \in P_M(z, A)$. Therefore,

$$z - \text{int } M \subset E \setminus A. \quad (6.23)$$

As $C \in \mathcal{SC}(-rM)$, there exists a set $C_1 \subset E$ such that $C + C_1 = -rM$. As $z \in M$, $\langle p, z \rangle = s(p, M)$, and the set M is strictly convex, then $\text{Exp}(p, M) = \{z\}$ by (6.22); i.e., $\text{Exp}(-p, -rM) = \{-rz\}$. So, by Remark 6.2 there exists a single pair of points $c_0 \in \text{Exp}(-p, C)$, $c_1 \in \text{Exp}(-p, C_1)$, and now $c_0 + c_1 = -rz$. As $0 \in C$ and $s(-p, C) = 0$, we have that $0 \in \text{Exp}(-p, C)$. Therefore, $c_0 = 0$. Thus, $-rz = c_0 + c_1 = c_1 \in C_1$, and so, $C - rz \subset C + C_1 = -rM$. Thus, using (6.23) and the fact that $z \in M$, we obtain

$$\text{int } C \subset rz - \text{int } rM \subset z - \text{int } M \subset E \setminus A.$$

Now the required assertion follows if we put $c = rz$, $a = z$. □

7. Proof of Theorem 3.2

Lemma 7.1. *Let a quasiball $M \subset E$, vectors $x, y \in E$, and a number $d > 0$ be such that*

$$0 < \|x\| \leq d \cdot \mu_M(x), \quad 0 < \|y\| \leq d \cdot \mu_M(y).$$

Then for any $r \in (0, 1]$,

$$\frac{\min\{\mu_M(x), \mu_M(y)\}}{rd} \cdot \delta_M \left(r \left\| \frac{x}{\mu_M(x)} - \frac{y}{\mu_M(y)} \right\|, 2d \right) \leq \mu_M(x) + \mu_M(y) - \mu_M(x + y), \quad (7.1)$$

where $\delta_M(\cdot, \cdot)$ is defined by (3.8).

Proof. We set

$$a = \frac{x}{\mu_M(x)}, \quad b = \frac{y}{\mu_M(y)}, \quad c = \frac{a + b}{2}, \quad \delta = \delta_M(r\|a - b\|, 2d).$$

Let us show that

$$\mu_M(c) \leq 1 - \frac{\delta}{2rd}. \quad (7.2)$$

As $\|a\| \leq d$ and $\|b\| \leq d$, equality (3.8) implies that

$$\delta \leq \frac{r\|a - b\|}{2} \leq rd.$$

Therefore, if $\mu_M(c) \leq 1/2$, then (7.2) holds. Now assume that $\mu_M(c) > 1/2$. We set

$$c_0 = \frac{c}{\mu_M(c)}, \quad a_r = (1 - r)c_0 + ra, \quad b_r = (1 - r)c_0 + rb, \quad c_r = \frac{a_r + b_r}{2}.$$

As $\|c_0\| \leq 2\|c\| \leq 2d$, we have that $a_r, b_r \in M \cap \mathfrak{B}_{2d}$. Using (3.8) with $R = 2d$, $x = a_r$, $y = b_r$, and $\varepsilon = r\|a - b\| = \|x - y\|$ and noting that $c_0 \in \partial M$, we get that $\|c_0 - c_r\| \geq \delta$. As $c_0 - c_r = r(c_0 - c)$, we see that

$$\|c_0 - c\| \geq \frac{\delta}{r}.$$

This and the equality $c = \mu_M(c)c_0$ show that

$$\|c_0\| \cdot |1 - \mu_M(c)| \geq \frac{\delta}{r}.$$

Taking into consideration inequalities $\|c_0\| \leq 2d$, $\mu_M(c) \leq 1$, we obtain (7.2).

We put $\mu_1 = \mu_M(x)$, $\mu_2 = \mu_M(y)$. Without loss of generality, we assume that $\mu_2 \leq \mu_1$. Further, using Remark 3.1 and taking into account that $\mu_M(a) = 1$, we have that

$$\mu_M(x + y) = \mu_M((\mu_1 - \mu_2)a + \mu_2(a + b)) \leq (\mu_1 - \mu_2) \cdot \mu_M(a) + \mu_2 \cdot \mu_M(a + b) = \mu_1 - \mu_2 + 2\mu_2 \cdot \mu_M(c).$$

Using (7.2), this gives

$$\mu_M(x+y) \leq \mu_1 + \mu_2 - \frac{\mu_2 \delta}{rd} = \mu_1 + \mu_2 - \frac{\delta}{rd} \min\{\mu_1, \mu_2\}. \quad \square$$

Along with the M -projection (3.4) we consider the *approximate M -projection* of a point $x \in E$ on a set $A \subset E$ with accuracy $\varepsilon > 0$:

$$P_M^\varepsilon(x, A) = A \cap (x - (\varrho_M(x, A) + \varepsilon)M). \quad (7.3)$$

Lemma 7.2. *Let a parabolic quasiball $M \subset E$ be boundedly uniformly convex. Let a set $A \in \mathcal{WC}(M)$ and a point $x_0 \in A$ satisfy the relations $0 < \varrho_M(x_0, A) = r < 1$. Assume that positive numbers ε , ε_0 , and d and points $a_0 \in P_M(x_0, A)$, $a \in P_M^\varepsilon(x_0, A)$ be such that*

$$\max\{\|a - x_0\|, \|a_0 - x_0\|\} \leq dr, \quad \varepsilon d < (1 - r)\delta_M(\varepsilon_0, 2d).$$

Then $\|a - a_0\| < (3/2)\varepsilon_0$.

Proof. As $a_0 \in P_M(x_0, A)$, we have that $\mu_M(x_0 - a_0) = r$. Denote

$$z = \frac{x_0 - a_0}{r}.$$

In view of (3.6), we have that $z \in N_M^1(a_0, A)$. As $A \in \mathcal{WC}(M)$, then, according to (3.7), we obtain that $\varrho_M(a_0 + z, A) = 1$.

We set $x = x_0 - a$ and $y = (1 - r)z$. Using (7.3), we find that $r \leq \mu_M(x) \leq r + \varepsilon$. As $\mu_M(y) = 1 - r$, we have

$$\mu_M(x + y) = \mu_M(a_0 + z - a) \geq \varrho_M(a_0 + z, A) = 1 \geq \mu_M(x) + \mu_M(y) - \varepsilon.$$

Next, since $\max\{\|a - x_0\|, \|a_0 - x_0\|\} \leq dr$ and $r \leq \mu_M(x)$, we see that $\|x\| \leq dr \leq d \cdot \mu_M(x)$, $\|z\| \leq d$, and $\|y\| = (1 - r)\|z\| \leq d \cdot \mu_M(y)$. Lemma 7.1 and the inequality

$$\varepsilon < \frac{(1 - r)\delta_M(\varepsilon_0, 2d)}{d}$$

imply that

$$\begin{aligned} \delta_M\left(r \left\| \frac{x}{\mu_M(x)} - \frac{y}{\mu_M(y)} \right\|, 2d\right) &\leq \frac{rd}{\min\{\mu_M(x), \mu_M(y)\}} (\mu_M(x) + \mu_M(y) - \mu_M(x + y)) \\ &\leq \frac{\varepsilon rd}{\min\{r, 1 - r\}} \leq \frac{\varepsilon d}{1 - r} < \delta_M(\varepsilon_0, 2d). \end{aligned} \quad (7.4)$$

So,

$$r \left\| \frac{x}{\mu_M(x)} - \frac{y}{\mu_M(y)} \right\| < \varepsilon_0.$$

We have

$$\left\| \frac{rx}{\mu_M(x)} - x \right\| = \frac{\|x\|}{\mu_M(x)} (\mu_M(x) - r) \leq \varepsilon d,$$

and so

$$\|a - a_0\| = \|(x_0 - a_0) - (x_0 - a)\| = \left\| \frac{ry}{\mu_M(y)} - x \right\| < \varepsilon_0 + \varepsilon d.$$

Equality (3.8) implies that $\delta_M(\varepsilon_0, 2d) \leq \varepsilon_0/2$. Therefore, $\varepsilon d < \delta_M(\varepsilon_0, 2d) \leq \varepsilon_0/2$ and so, $\|a - a_0\| < (3/2)\varepsilon_0$. $\square$

Lemma 7.3. *Let a quasiball $M \subset E$ be uniformly smooth and boundedly uniformly convex. Let a quasiball M be either bounded (in this case we assume that $v_M = 0$), or parabolic and coercive, and v_M be its asymptotic direction. Let the set $A \in \mathcal{WC}(M)$ satisfy the inclusion $A + \lambda v_M \subset A$ for any $\lambda > 0$. Let $r \in (0, 1)$, $C \in \mathcal{SC}(-rM)$, $A \cap C \neq \emptyset$, and $\text{int } C \neq \emptyset$. Then*

$$\kappa = \sup\{\|z\| : z \in N_M^1(a, A), a \in A \cap C\} < +\infty. \quad (7.5)$$

Assume that ε_0 and ε be positive numbers such that

$$R = \sup_{x \in A \cap (C + \mathfrak{B}_\varepsilon)} \|x\| < +\infty, \quad d = \kappa + \frac{2R}{r}, \quad \varepsilon < \frac{(1-r)\sigma_M \delta_M(\varepsilon_0, 2d)}{d}. \quad (7.6)$$

Let $a \in A \cap (C + \mathfrak{B}_\varepsilon)$. Then there exists $a_0 \in A \cap C$ such that $\|a - a_0\| < (3/2)\varepsilon_0$.

Proof. We have $a \in A \cap (C + \mathfrak{B}_\varepsilon)$, and hence there exists $c_1 \in C$ such that $\|c_1 - a\| \leq \varepsilon$. For any $t \in [0, 1]$, we consider the set $C_t = (1-t)c_1 + tC$. Let

$$\tau = \inf\{t \in [0, 1] : A \cap C_t \neq \emptyset\}.$$

Since $A \cap C_1 = A \cap C \neq \emptyset$, the set $\{t \in [0, 1] : A \cap C_t \neq \emptyset\}$ is nonempty and $\tau \leq 1$. Consider the case $\tau = 0$. Let us show that in this case

$$c_1 \in A. \quad (7.7)$$

In the case $\tau = 0$, there exists a sequence $t_k > 0$ that tends to zero and a sequence a_k such that $a_k \in A \cap C_{t_k}$ for all $k \in \mathbb{N}$. As $C \in \mathcal{SC}(-rM)$, there exists a vector $d \in E$ such that $C \subset d - rM$. We have

$$a_k \in C_{t_k} = (1-t_k)c_1 + t_kC \subset (1-t_k)c_1 + t_kd - rt_kM,$$

and hence there exists $z_k \in M$ such that $a_k = (1-t_k)c_1 + t_kd - rt_kz_k$. As $a_k \in A \cap C$, and the set $A \cap C$ is bounded by the first relation in (7.6), the sequence $\{t_k\|z_k\|\}$ is bounded. Therefore, there exists a sequence $\{t_{k_j}\|z_{k_j}\|\}$ that converges to β . Note that $\beta \geq 0$. If $\beta = 0$, then

$$\lim_{j \rightarrow \infty} a_{k_j} = c_1.$$

Therefore, (7.7) holds. Consider the case $\beta > 0$. If the quasiball M is bounded, then $\beta = 0$ and, therefore, in this case the quasiball M is parabolic and coercive. Thus,

$$\lim_{j \rightarrow \infty} \frac{z_{k_j}}{\|z_{k_j}\|} = v_M.$$

Therefore,

$$\lim_{j \rightarrow \infty} a_{k_j} = c_1 - r\beta v_M.$$

Since A is closed, we get that $c_1 - r\beta v_M \in A$. As $A + r\beta v_M \subset A$, inclusion (7.7) holds. In this case, we put $a_0 = c_1$.

Now assume that $\tau \in (0, 1]$. Then $A \cap \text{int } C_\tau = \emptyset$ and $\varrho_M(C_\tau, A) = 0$. By Lemma 5.4 there exists $a_0 \in A \cap C_\tau$. Theorem 3.1 produces $c_0 \in E$ such that

$$\text{int } C_\tau \subset c_0 - \text{int } \tau rM \subset E \setminus A.$$

We set

$$z = \frac{c_0 - a_0}{\tau r}.$$

As $a_0 \in C_\tau \subset c_0 - \tau rM$, then $z \in M$. As $c_0 - \text{int } \tau rM \subset E \setminus A$, we have

$$\varrho_M(c_0, A) \geq \tau r \geq \tau r \mu_M(z) = \mu_M(c_0 - a_0).$$

Therefore, $a_0 \in P_M(c_0, A)$ and $z \in \partial M$. In view of (3.6), $z \in N_M^1(a_0, A)$. Using (7.5), we have $\|z\| \leq \kappa$. By the first relation in (7.6), $\|a - a_0\| \leq 2R$. We set $b = a_0 + rz$. Then, using the second equality in (7.6), we find that

$$\max\{\|a - b\|, \|a_0 - b\|\} \leq r\|z\| + \|a - a_0\| \leq r\kappa + 2R = rd.$$

As $A \in \mathcal{WC}(M) \subset \mathcal{WC}(rM)$, we have that $a_0 \in P_M(b, A)$ and $\varrho_M(b, M) = r$. Note that

$$c_1 \in C_\tau \subset c_0 - \tau rM = a_0 + \tau r(z - M) \subset a_0 + r(z - M) = b - rM.$$

Using Remark 3.3, this gives $\mathfrak{B}_{\sigma_M} \subset M$, and so,

$$a \in \mathfrak{B}_\varepsilon(c_1) \subset b - (rM + \mathfrak{B}_\varepsilon) \subset b - (r + \varepsilon')M,$$

where $\varepsilon' = \varepsilon/\sigma_M$. This and equalities (7.3) and $\varrho_M(b, M) = r$ imply that $a \in P_M^{\varepsilon'}(b, A)$. Now the required inequality follows from Lemma 7.2. $\square$

Lemma 7.4. *Let a parabolic quasiball $M \subset E$ be boundedly uniformly convex. Let a set $A \in \mathcal{WC}(M)$ be M -quasibounded and $a_0 \in \partial A$. Then there exists a sequence $\{a_k\} \subset \partial A$ such that $N_M^1(a_k, A) \neq \emptyset$ and $a_k \rightarrow a_0$ as $k \rightarrow \infty$.*

Proof. As $a_0 \in \partial A$, there exists a sequence $\{x_k\} \subset E \setminus A$ such that $x_k \rightarrow a_0$ as $k \rightarrow \infty$. As $x_k \notin A$ and the set A is M -quasibounded, we have that $\varrho_M(x_k, A) > 0$ for all $k \in \mathbb{N}$. In view of the continuity of the function $\varrho_M(\cdot, A)$, we have that $\varrho_M(x_k, A) \rightarrow \varrho_M(a_0, A) = 0$ as $k \rightarrow \infty$. Therefore, there exists an index k_0 such that $\varrho_M(x_k, A) < 1/2$ as $k \geq k_0$. According to Theorem 5.1, for any $k \geq k_0$, there exists $a_k \in P_M(x_k, A)$. According to equality (3.6), for any $k \geq k_0$ the vector

$$z_k = \frac{x_k - a_k}{\mu_M(x_k - a_k)}$$

satisfies the inclusion $z_k \in N_M^1(a_k, A)$. It remains to prove that $a_k \rightarrow a_0$ as $k \rightarrow \infty$.

We put $w = a_{k_0} + z_{k_0}$. As $A \in \mathcal{WC}(M)$, relation (3.7) implies that $(w - \text{int } M) \cap A = \emptyset$. Therefore, $a_k \notin w - \text{int } M$ for any $k \geq k_0$. On the other hand,

$$a_k \in x_k - \varrho_M(x_k, A)M \subset x_k - \frac{1}{2}M$$

as $k \geq k_0$. As $x_k \rightarrow a_0$, it follows from Lemma 5.8 that the sequence $\{a_k\}$ is bounded. Hence, the M -quasiboundedness of set A imply the boundedness of the sequence $\{z_k\}$. As $x_k \rightarrow a_0$ and $\mu_M(x_k - a_0) = \varrho_M(x_k, A) \rightarrow 0$, then $a_k = x_k - \mu_M(x_k - a_k)z_k \rightarrow a_0$ as $k \rightarrow \infty$. $\square$

Let (T, ϱ_T) be a metric space. For any $t_0 \in T$ and $\delta > 0$, the set

$$U_\delta(t_0) = \{t \in T : \varrho_T(t, t_0) < \delta\}$$

is called the δ -neighborhood of t_0 .

Lemma 7.5. *Let a parabolic quasiball $M \subset E$ be boundedly uniformly convex. Let a multifunction $A : T \rightarrow 2^E$ be R -continuous at $t_0 \in T$, and let the family $\{A(t)\}_{t \in T}$ be M -equi-quasibounded. Next, let $A(t) \in \mathcal{WC}(M)$ for any $t \in T$. Then there exists $\delta > 0$ and a bounded function $w : U_\delta(t_0) \rightarrow E$ such that*

$$A(t) \cap (w(t) - \text{int } M) = \emptyset \quad \forall t \in U_\delta(t_0). \quad (7.8)$$

Proof. We fix $a_0 \in \partial A(t_0)$. By the R -continuity of the multifunction A there exist $\delta > 0$ and a function $a_1 : U_\delta(t_0) \rightarrow E$ such that $a_1(t) \in \partial A(t) \cap \mathfrak{B}_1(a_0)$ for any $t \in U_\delta(t_0)$. By Lemma 7.4 for any $t \in U_\delta(t_0)$ there exists $a(t) \in \partial A(t) \cap \mathfrak{B}_1(a_1(t))$ such that $N_M^1(a(t), A(t)) \neq \emptyset$. Therefore, for any $t \in U_\delta(t_0)$ there exists a vector $z(t) \in N_M^1(a(t), A(t))$. As $A(t) \in \mathcal{WC}(M)$, we have that $A(t) \cap (a(t) + z(t) - \text{int } M) = \emptyset$ by (3.7). The M -equi-quasiboundedness of the family $\{A(t)\}_{t \in T}$ implies the boundedness of the function $z : U_\delta(t_0) \rightarrow E$. Assuming that $w(t) = a(t) + z(t)$, we obtain the required equality. $\square$

Lemma 7.6. *Let a quasiball $M \subset E$ be boundedly uniformly convex. Let a multifunction $C : T \rightarrow 2^E$ be R -continuous at $t_0 \in T$ and let $C(t) \in \mathcal{SC}(M)$ for any $t \in T$. Then there exists a function $c_1 : T \rightarrow E$, continuous at t_0 , such that $C(t) + c_1(t) \subset M$ for any $t \in T$.*

Proof. As $C(t) \in \mathcal{SC}(M)$ for all $t \in T$, for any $t \in T$ there exists a set $C_1(t) \subset E$ such that $C(t) + C_1(t) = M$. We fix a point $z \in \partial M$. Then for any $t \in T$ there exist $c(t) \in C(t)$ and $c_1(t) \in C_1(t)$ such that

$$z = c(t) + c_1(t). \quad (7.9)$$

Hence, for any $t \in T$, $C(t) + c_1(t) \subset C(t) + C_1(t) = M$. It remains to prove the continuity of the function $c_1(\cdot)$ at t_0 . Suppose on the contrary that there exist $\varepsilon \in (0, 1/2)$ and a sequence $\{t_k\} \subset T$ such that $\varrho_T(t_k, t_0) \rightarrow 0$ as $k \rightarrow \infty$ and $\|c_1(t_k) - c_1(t_0)\| \geq \varepsilon$ for any $k \in \mathbb{N}$. This and equality (7.9) imply that

$$\|c(t_k) - c(t_0)\| \geq \varepsilon \quad \forall k \in \mathbb{N}. \quad (7.10)$$

The multifunction $C(\cdot)$ is R -continuous at $t_0 \in T$ and $\varrho_T(t_k, t_0) \rightarrow 0$ as $k \rightarrow \infty$. Hence there exists a sequence $\{a_k\} \subset E$ such that $a_k \in C(t_k)$ for all $k \in \mathbb{N}$ and

$$\lim_{k \rightarrow \infty} a_k = c(t_0). \quad (7.11)$$

For any $k \in \mathbb{N}$, we consider the point

$$\tilde{c}_k = \begin{cases} c(t_k), & \|c(t_k) - a_k\| \leq 1, \\ a_k + \frac{c(t_k) - a_k}{\|c(t_k) - a_k\|}, & \|c(t_k) - a_k\| > 1. \end{cases} \quad (7.12)$$

The convexity of the set $C(t_k)$ implies that for any $k \in \mathbb{N}$ the inclusion $\tilde{c}_k \in C(t_k)$ holds. In view of (7.11), there exists an index k_0 such that $\|a_k - c(t_0)\| < 1/2$ for all $k \geq k_0$. Let us show that

$$\|\tilde{c}_k - c(t_0)\| \geq \varepsilon \quad \forall k \geq k_0. \quad (7.13)$$

Indeed, we fix an index $k \geq k_0$. If $\|c(t_k) - a_k\| \leq 1$, then $\tilde{c}_k = c(t_k)$ and inequality (7.13) follows from (7.10). Otherwise, $\|c(t_k) - a_k\| > 1$ and so by (7.12) we have $\|\tilde{c}_k - a_k\| = 1$. Using the inequality $\|a_k - c(t_0)\| < 1/2$, we get

$$\|\tilde{c}_k - c(t_0)\| \geq \|\tilde{c}_k - a_k\| - \|a_k - c(t_0)\| > \frac{1}{2} > \varepsilon,$$

and inequality (7.13) once again holds.

The multifunction $C(\cdot)$ is R -continuous at $t_0 \in T$ and $\varrho_T(t_k, t_0) \rightarrow 0$ as $k \rightarrow \infty$, the sequence $\{\tilde{c}_k\}$ is bounded, and hence there exists a sequence $\{b_k\} \subset C(t_0)$ such that

$$\lim_{k \rightarrow \infty} \|b_k - \tilde{c}_k\| = 0. \quad (7.14)$$

Hence by (7.13) there exists an index $k_1 \geq k_0$ such that

$$\|b_k - c(t_0)\| \geq \frac{\varepsilon}{2} \quad \forall k \geq k_1.$$

Therefore, for the sequence $z_k = z + b_k - c(t_0)$, we have

$$\|z_k - z\| = \|b_k - c(t_0)\| \geq \frac{\varepsilon}{2} \quad \forall k \geq k_1. \quad (7.15)$$

Equality (7.9) implies that $z - c(t_0) = c_1(t_0) \in C_1(t_0)$. Therefore,

$$z_k \in b_k + C_1(t_0) \subset C(t_0) + C_1(t_0) = M.$$

Using the bounded uniform convexity of the quasiball M and (7.15), we see that there exists $\delta > 0$ such that

$$\mathfrak{B}_\delta \left(\frac{z_k + z}{2} \right) \subset M \quad \forall k \geq k_1. \quad (7.16)$$

As $z \in \partial M$, the Hahn–Banach theorem implies that there exists a functional $p \in E^* \setminus \{0\}$ such that $\langle p, z \rangle = s(p, M)$. We have that $a_k + c_1(t_k) \in C(t_k) + C_1(t_k) = M$, and so

$$\langle p, a_k + c_1(t_k) \rangle \leq s(p, M) = \langle p, z \rangle = \langle p, c(t_k) + c_1(t_k) \rangle.$$

Therefore, $\langle p, a_k \rangle \leq \langle p, c(t_k) \rangle$ and so by (7.12) we have that $\langle p, \tilde{c}_k \rangle \geq \langle p, a_k \rangle$. Using relations (7.11) and (7.14), we find that

$$\liminf_{k \rightarrow \infty} \langle p, b_k \rangle = \liminf_{k \rightarrow \infty} \langle p, \tilde{c}_k \rangle \geq \liminf_{k \rightarrow \infty} \langle p, a_k \rangle = \langle p, c(t_0) \rangle.$$

Therefore,

$$\liminf_{k \rightarrow \infty} \langle p, z_k \rangle \geq \langle p, z \rangle.$$

On the other hand, (7.16) implies that

$$\frac{1}{2} \langle p, z_k + z \rangle + \|p\| \delta \leq s(p, M)$$

for any $k \geq k_1$. Thus,

$$\langle p, z \rangle \leq \frac{1}{2} \liminf_{k \rightarrow \infty} \langle p, z_k + z \rangle \leq s(p, M) - \|p\|\delta,$$

which contradicts the equality $\langle p, z \rangle = s(p, M)$. $\square$

Proof of Theorem 3.2. Lemma 7.5 implies that in some δ -neighborhood of t_0 the bounded function $w: U_\delta(t_0) \rightarrow E$ is defined, which satisfies condition (7.8). By Lemma 7.6 there exists a function $c_1: T \rightarrow E$, continuous at t_0 , such that

$$C(t) + c_1(t) \subset -rM \quad \forall t \in T. \quad (7.17)$$

We fix any $r_1 \in (r, 1)$. Let $\varepsilon_1 = (r_1 - r)\sigma_M$. Then, by Remark 3.3, we have that $\mathfrak{B}_{\sigma_M} \subset M$, and so, using (7.17),

$$C(t) + \mathfrak{B}_{\varepsilon_1} \subset -c_1(t) - rM + \mathfrak{B}_{(r_1-r)\sigma_M} \subset -c_1(t) - r_1M \quad \forall t \in T.$$

This and equality (7.8) imply that

$$A(t) \cap (C(t) + \mathfrak{B}_{\varepsilon_1}) \subset (-c_1(t) - r_1M) \setminus (w(t) - \text{int } M) \quad \forall t \in U_\delta(t_0). \quad (7.18)$$

The continuity of the function $c_1(\cdot)$ at t_0 implies the boundedness of this function in $U_{\delta_1}(t_0)$ for some $\delta_1 \in (0, \delta]$. By Lemma 5.5(2) and inclusion (7.18), we obtain that

$$R := \sup_{t \in U_{\delta_1}(t_0)} \sup_{x \in A(t) \cap (C(t) + \mathfrak{B}_{\varepsilon_1})} \|x\| < +\infty.$$

Using the M -equi-quasiboundedness of the family $\{A(t)\}_{t \in T}$ (see (3.13)), we have

$$\kappa := \sup_{t \in U_{\delta_1}(t_0)} \sup_{a \in A(t) \cap C(t)} \sup_{z \in N_M^1(a, A(t))} \|z\| < +\infty.$$

We fix an arbitrary $\varepsilon_0 > 0$ and define

$$d = \kappa + \frac{2R}{r}, \quad \varepsilon = \min \left\{ \frac{\varepsilon_0}{2}, \varepsilon_1, \frac{(1-r)\sigma_M \delta_M(\varepsilon_0, 2d)}{2d} \right\}. \quad (7.19)$$

By the R -continuity of the multifunctions $A(\cdot)$ and $C(\cdot)$ at t_0 , there exists $\delta_2 \in (0, \delta_1]$ such that

$$h_R(A(t), A(t_0)) + h_R(C(t), C(t_0)) + 2\varrho_T(t, t_0) \leq \varepsilon \quad \forall t \in U_{\delta_2}(t_0). \quad (7.20)$$

We claim that

$$h(F(t), F(t_0)) \leq 2\varepsilon_0 \quad \forall t \in U_{\delta_2}(t_0). \quad (7.21)$$

Let $t_1, t_2 \in U_{\delta_2}(t_0)$ be given points and $x_1 \in F(t_1) = A(t_1) \cap C(t_1)$ be a vector. By the definitions of (2.1) and (3.12), there exists a vector $x'_1 \in A(t_2)$ such that

$$\|x_1 - x'_1\| \leq \varepsilon' := h_R(A(t_1), A(t_2)) + \varrho_T(t_1, t_2).$$

We set

$$\varepsilon'' = h_R(C(t_1), C(t_2)) + \varrho_T(t_1, t_2).$$

As $x_1 \in C(t_1) \cap \mathfrak{B}_R \subset C(t_2) + \mathfrak{B}_{\varepsilon''}$, we have that $x'_1 \in A(t_2) \cap (C(t_2) + \mathfrak{B}_{\varepsilon' + \varepsilon''})$. Inequality (7.20) implies that $\varepsilon' + \varepsilon'' \leq \varepsilon$. Hence, $x'_1 \in A(t_2) \cap (C(t_2) + \mathfrak{B}_\varepsilon)$ and so by Lemma 7.3 there exists $x_2 \in F(t_2) = A(t_2) \cap C(t_2)$ that satisfies the inequality $\|x_2 - x'_1\| < (3/2)\varepsilon_0$. Therefore,

$$\|x_2 - x_1\| \leq \|x_2 - x'_1\| + \|x_1 - x'_1\| < \frac{3}{2}\varepsilon_0 + \varepsilon' \leq \frac{3}{2}\varepsilon_0 + \varepsilon \leq 2\varepsilon_0.$$

As the vector $x_1 \in F(t_1)$ is arbitrary, it follows from formula (2.1) that $h^+(F(t_1), F(t_2)) \leq 2\varepsilon_0$. Similarly, $h^+(F(t_2), F(t_1)) \leq 2\varepsilon_0$, and so, $h(F(t_1), F(t_2)) \leq 2\varepsilon_0$. This proves (7.21). From (7.21), since the number ε_0 is arbitrary, we get the continuity of the multifunction F at t_0 . $\square$

8. Proof of Theorems 4.1 and 4.3

Lemma 8.1. *Let a function $\gamma: E \rightarrow \mathbb{R}$ be coercive. Then the set $\text{epi } \gamma$ is coercive, the vector $(0_E, 1)$ is the asymptotic direction of this set (here 0_E is the zero element of the space E).*

Proof. Let a sequence $z_k = (x_k, y_k) \in \text{epi } \gamma$ satisfy the relation $\lim_{k \rightarrow \infty} \|z_k\| = +\infty$. We claim that

$$\lim_{k \rightarrow \infty} \frac{y_k}{\|x_k\|} = +\infty. \quad (8.1)$$

Suppose the contrary. Then there exists a sequence $\{z_{k_j}\} = \{(x_{k_j}, y_{k_j})\}$ such that

$$\sup_{j \in \mathbb{N}} \frac{|y_{k_j}|}{\|x_{k_j}\|} < +\infty. \quad (8.2)$$

If the sequence $\{x_{k_j}\}$ is unbounded, then this contradicts the coerciveness of the function γ . If the sequence $\{x_{k_j}\}$ is bounded, then in view of (8.2) we get the boundedness of the sequence $\{y_{k_j}\}$, which contradicts the assumption $\lim_{k \rightarrow \infty} \|z_k\| = +\infty$. This proves (8.1). Therefore, there exists an index k_0 such that $y_k > 0$ for $k \geq k_0$. According to (4.6), for $k \geq k_0$ we have that

$$1 \leq \frac{\|z_k\|}{y_k} \leq 1 + \frac{\|x_k\|}{y_k}.$$

Using (8.1), we obtain that

$$\lim_{k \rightarrow \infty} \frac{\|z_k\|}{y_k} = 1. \quad (8.3)$$

The inequalities

$$\left\| \frac{z_k}{\|z_k\|} - (0_E, 1) \right\| \leq \frac{\|x_k\| + |y_k - \|z_k\||}{\|z_k\|} \leq \frac{\|x_k\| + |y_k - \|z_k\||}{|y_k|} = \frac{\|x_k\|}{|y_k|} + \left| 1 - \frac{\|z_k\|}{y_k} \right|$$

and relations (8.1) and (8.3) imply the equality

$$\lim_{k \rightarrow \infty} \left\| \frac{z_k}{\|z_k\|} - (0_E, 1) \right\| = 0. \quad \square$$

Lemma 8.2 ([17, Lemma 10.1]). *If a function $\gamma: E \rightarrow \mathbb{R}$ is uniformly convex on any ball $\mathfrak{B}_R$, $R > 0$, then the set $\text{epi } \gamma$ is boundedly uniformly convex.*

Lemma 8.3.

- (1) *If a function $\gamma: E \rightarrow \mathbb{R}$ is convex, continuous, and coercive, then the set $\text{epi } \gamma$ is parabolic.*
- (2) *If, in addition, a function γ is bounded on every bounded set, then the set $\text{epi } \gamma$ is strongly parabolic.*

The first assertion of Lemma 8.3 was proved in [15, Lemma 2.5], the second assertion is contained in [14, Theorem 2.1].

Lemma 8.4. *Let a function $\gamma: E \rightarrow \mathbb{R}$ be convex, continuous, and uniformly smooth on any ball $\mathfrak{B}_R$, $R > 0$, let $\gamma(0) < 0$. Then the set $M = \text{epi } \gamma$ is a boundedly uniformly smooth quasiball.*

Proof. The convexity and continuity of the function γ imply the convexity and closeness of the set M . Let 0_E be the zero element of the space E , $0_{E \times \mathbb{R}}$ be the zero element of the space $E \times \mathbb{R}$. As function γ is continuous and $\gamma(0_E) < 0$, we have that $0_{E \times \mathbb{R}} \in \text{int } M$. Thus, M is a quasiball. Let us show that

$$\sigma_M \beta_M(t, R) \leq \frac{3}{2} \beta_\gamma \left(2t \left(1 + \frac{4R}{\sigma_M} \right), \mathfrak{B}_{4R} \right) \quad \forall R > \sigma_M, \quad \forall t \in \left(0, \frac{\sigma_M}{2} \right), \quad (8.4)$$

where $\beta_M(\cdot, \cdot)$ is defined by (3.10), the function $\beta_\gamma(\cdot, \cdot)$ is given by (4.5), and the number σ_M , by (3.3).

We fix an arbitrary $R > \sigma_M$, $t \in (0, \sigma_M/2)$. If $\beta_M(t, R) = 0$, then (8.4) holds. Therefore, we will suppose that $\beta_M(t, R) > 0$. We fix $\delta \in (0, \beta_M(t, R))$. In view of (3.10), there exist vectors $u \in \partial M \cap \mathfrak{B}_R \subset E \times \mathbb{R}$ and $v \in \mathfrak{B}_1 \subset E \times \mathbb{R}$ such that

$$\frac{\mu_M(u + tv) + \mu_M(u - tv)}{2} - 1 > \beta_M(t, R) - \delta. \quad (8.5)$$

Assume, for definiteness, that $\mu_M(u + tv) \geq \mu_M(u - tv)$. Using the convexity of the Minkowski functional, we get

$$\mu_M(u + tv) \geq \frac{1}{2}(\mu_M(u + tv) + \mu_M(u - tv)) \geq \mu_M(u) = 1.$$

As $|t| < \sigma_M/2$, we have by Remark 3.4 that

$$\frac{1}{2} \leq \mu_M(u) - \frac{|t| \cdot \|v\|}{\sigma_M} \leq \mu_M(u - tv) \leq \mu_M(u + tv) \leq \mu_M(u) + \frac{|t| \cdot \|v\|}{\sigma_M} \leq \frac{3}{2}.$$

As $\|u\| \leq R$ and $|t| \leq R$, we get that $\|u + tv\| \leq 2R$ and $\|u - tv\| \leq 2R$. So,

$$\mu_M(u + tv) \in \left[1, \frac{3}{2}\right], \quad \mu_M(u - tv) \in \left[\frac{1}{2}, \frac{3}{2}\right], \quad \|u + tv\| \leq 2R, \quad \|u - tv\| \leq 2R. \quad (8.6)$$

We denote

$$z_1 = \frac{u + tv}{\mu_M(u + tv)}, \quad z_2 = \frac{u - tv}{\mu_M(u - tv)}. \quad (8.7)$$

By Remark 3.4 we have that

$$|\mu_M(u + tv) - \mu_M(u - tv)| \leq \frac{2t}{\sigma_M}.$$

Therefore,

$$\begin{aligned} \|z_1 - z_2\| &\leq \left\| \frac{u + tv}{\mu_M(u + tv)} - \frac{u - tv}{\mu_M(u - tv)} \right\| + \|u - tv\| \left| \frac{1}{\mu_M(u + tv)} - \frac{1}{\mu_M(u - tv)} \right| \\ &\leq \frac{2t}{\mu_M(u + tv)} + \frac{2t\|u - tv\|}{\sigma_M \mu_M(u + tv) \mu_M(u - tv)} \leq 2t \left(1 + \frac{4R}{\sigma_M} \right), \end{aligned} \quad (8.8)$$

where the last inequality follows from (8.6). Using Remark 3.1, we get

$$\begin{aligned} \mu_M(z_1 + z_2) &\leq \mu_M \left(\frac{u + tv}{\mu_M(u + tv)} + \frac{u - tv}{\mu_M(u - tv)} \right) + \mu_M(u - tv) \left(\frac{1}{\mu_M(u - tv)} - \frac{1}{\mu_M(u + tv)} \right) \\ &= \frac{2}{\mu_M(u + tv)} + \frac{\mu_M(u + tv) - \mu_M(u - tv)}{\mu_M(u + tv)} = 2 - \frac{\mu_M(u + tv) + \mu_M(u - tv) - 2}{\mu_M(u + tv)}. \end{aligned}$$

According to inequality (8.5), we obtain that

$$\mu_M(z_1 + z_2) < 2 - \frac{2(\beta_M(t, R) - \delta)}{\mu_M(u + tv)}.$$

Since $\mu_M(u + tv) \leq 3/2$ by (8.6), we get

$$\frac{\mu_M(z_1 + z_2)}{2} < 1 - \frac{2}{3}(\beta_M(t, R) - \delta). \quad (8.9)$$

As $\mu_M(z_1) = \mu_M(z_2) = 1$, we have $z_1, z_2 \in \partial M = \partial \text{epi } \gamma$. Therefore, there exist $x_1, x_2 \in E$ such that

$$z_1 = (x_1, f(x_1)), \quad z_2 = (x_2, f(x_2)).$$

We have $\|z_1\| \leq 4R$, $\|z_2\| \leq 4R$ by (8.6) and (8.7). Using (4.6) and (8.8), we obtain that

$$\|x_1\| \leq 4R, \quad \|x_2\| \leq 4R, \quad \|x_1 - x_2\| \leq 2t \left(1 + \frac{4R}{\sigma_M} \right).$$

Therefore, according to inequality (4.5),

$$\left| \frac{f(x_1) + f(x_2)}{2} - f\left(\frac{x_1 + x_2}{2}\right) \right| \leq \hat{\beta} := \beta_\gamma \left(2t \left(1 + \frac{4R}{\sigma_M} \right), \mathfrak{B}_{4R} \right).$$

Hence, by (4.6), the point

$$z = \left(\frac{x_1 + x_2}{2}, f\left(\frac{x_1 + x_2}{2}\right) \right)$$

satisfies the inequality

$$\left| \frac{z_1 + z_2}{2} - z \right| \leq \hat{\beta}.$$

As $z \in \partial M$, we get that $\mu_M(z) = 1$, and so, by Remark 3.4,

$$\mu_M \left(\frac{z_1 + z_2}{2} \right) \geq 1 - \frac{\hat{\beta}}{\sigma_M}.$$

Using (8.9), we get

$$1 - \frac{\hat{\beta}}{\sigma_M} \leq \mu_M \left(\frac{z_1 + z_2}{2} \right) < 1 - \frac{2}{3}(\beta_M(t, R) - \delta).$$

Making $\delta \rightarrow +0$, we arrive at (8.4). The function $\gamma(\cdot)$ is uniformly smooth on the ball $\mathfrak{B}_{4R}$, and hence (8.4) implies that

$$\lim_{t \rightarrow +0} \frac{\beta_M(t, R)}{t} = 0.$$

The number $R > \sigma_M$ was arbitrary, and so the set M is boundedly uniformly convex. $\square$

Lemma 8.5. *Let a function $\gamma: E \rightarrow \mathbb{R}$ be convex, coercive, and bounded on any bounded set, let $\gamma(0) < 0$. Assume that for a given function $\alpha: T \times E \rightarrow \mathbb{R}$ and for any $R > 0$ there exists a number $L_\alpha(R)$ such that for any $t \in T$ the function $\alpha(t, \cdot)$ satisfies the Lipschitz condition with the constant $L_\alpha(R)$ on the ball $\mathfrak{B}_R$. Let $M = \text{epi } \gamma$, $A(t) = \text{epi } \alpha(t, \cdot)$ for any $t \in T$. Then the family $\{A(t)\}_{t \in T}$ is M -equi-quasibounded.*

Proof. Fix an arbitrary $R > 0$. As function γ is coercive, there exists $r_\gamma > 0$ such that

$$\frac{\gamma(x) - \gamma(0)}{\|x\|} > L_\alpha(R + 1) \quad \forall x \in E \setminus \mathfrak{B}_{r_\gamma}. \quad (8.10)$$

The function γ is bounded on any ball $\mathfrak{B}_{r_\gamma}$, and hence,

$$C_\gamma := \sup_{x \in \mathfrak{B}_{r_\gamma}} \gamma(x) < +\infty.$$

We claim that

$$\sup_{t \in T} \sup_{a \in A(t) \cap \mathfrak{B}_R} \sup_{z \in N_M^1(a, A(t))} \|z\| \leq C_\gamma + r_\gamma. \quad (8.11)$$

We fix arbitrary $t \in T$, $a = (a_x, a_y) \in A(t) \cap \mathfrak{B}_R$, and $z = (z_x, z_y) \in N_M^1(a, A(t))$. As $a \in \partial A(t)$ and $z \in \partial M$, then $a_y = \alpha(t, a_x)$ and $z_y = \gamma(z_x)$. As $z \in N_M^1(a, A(t))$, we obtain from (3.4)–(3.6) that there exists $\tau > 0$ such that $a + \tau z \notin A(t) + \text{int } \tau M$. Note that for any $\varepsilon > 0$ the inclusion $(a_x - \varepsilon \tau z_x, \alpha(t, a_x - \varepsilon \tau z_x)) \in A(t)$ holds. Therefore,

$$(a_x + \tau z_x, \alpha(t, a_x) + \tau \gamma(z_x)) = a + \tau z \notin (a_x - \varepsilon \tau z_x, \alpha(t, a_x - \varepsilon \tau z_x)) + \text{int } \tau M,$$

i.e.,

$$\left((1 + \varepsilon)z_x, \frac{\alpha(t, a_x) - \alpha(t, a_x - \varepsilon \tau z_x)}{\tau} + \gamma(z_x) \right) \notin \text{int } M.$$

Hence, using the convexity of the function γ ,

$$\varepsilon(\gamma(z_x) - \gamma(0)) \leq \gamma((1 + \varepsilon)z_x) - \gamma(z_x) \leq \frac{\alpha(t, a_x) - \alpha(t, a_x - \varepsilon \tau z_x)}{\tau}. \quad (8.12)$$

As $a = (a_x, a_y) \in \mathfrak{B}_R$, we have $\|a_x\| \leq R$. Using the Lipschitz condition for the function $\alpha(t, \cdot)$ on the ball $\mathfrak{B}_{R+1}$ and inequality (8.12) for sufficiently small $\varepsilon > 0$, we obtain

$$\gamma(z_x) - \gamma(0) \leq L_\alpha(R+1)\|z_x\|.$$

This and inequality (8.10) imply that $\|z_x\| \leq r_\gamma$. Therefore, $|z_y| = |\gamma(z_x)| \leq C_\gamma$. Thus, $\|z\| \leq \|z_x\| + |z_y| \leq C_\gamma + r_\gamma$, proving (8.11). Now the assertion of the lemma follows from the definition of M -equi-quasiboundedness (see (3.13)). $\square$

Lemma 8.6. *Let E be a reflexive Banach space and a function $f: E \rightarrow \mathbb{R} \cup \{+\infty\}$ be convex, lower semicontinuous, and coercive.*

(1) *If a sequence $\{x_n\}$ is such that*

$$\inf_{n \in \mathbb{N}} (\langle p, x_n \rangle - f(x_n)) > -\infty \quad (8.13)$$

for some functional $p \in E^$, then there exists a sequence $\{x_{n_k}\}$ that weakly converges to some $x_0 \in E$, and also*

$$f(x_0) \leq \liminf_{k \rightarrow \infty} f(x_{n_k}).$$

(2) *For any $p \in E^*$ the maximum $\max_{x \in E} (\langle p, x \rangle - f(x))$ is attained.*

Proof. (1) Suppose that the sequence $\{x_n\}$ satisfies condition (8.13). If this sequence is unbounded, then there exists a subsequence $\{x_{n_k}\}$ such that

$$\lim_{k \rightarrow \infty} \|x_{n_k}\| = +\infty.$$

Then, in view of the coerciveness of function f , we have that

$$\frac{\langle p, x_{n_k} \rangle - f(x_{n_k})}{\|x_{n_k}\|} \leq \|p\| - \frac{f(x_{n_k})}{\|x_{n_k}\|} \rightarrow -\infty \text{ as } k \rightarrow \infty,$$

which contradicts condition (8.13). Therefore, the sequence $\{x_n\}$ is bounded. According to the Banach–Alaoglu theorem, there exists a subsequence $\{x_{n_k}\}$ that weakly converges to $x_0 \in E$. As the set $\text{epi } f$ is convex and closed, by Mazur’s theorem this set is weakly closed. So,

$$f(x_0) \leq \liminf_{k \rightarrow \infty} f(x_{n_k}).$$

(2) Consider the sequence $\{x_n\}$ such that

$$\lim_{n \rightarrow \infty} (\langle p, x_n \rangle - f(x_n)) = \sup_{x \in E} (\langle p, x \rangle - f(x)).$$

Then condition (8.13) holds, and so by (1), there exists a subsequence $\{x_{n_k}\}$ that weakly converges to some $x_0 \in E$, and also

$$f(x_0) \leq \liminf_{k \rightarrow \infty} f(x_{n_k}).$$

Therefore,

$$\langle p, x_0 \rangle - f(x_0) \geq \lim_{k \rightarrow \infty} (\langle p, x_{n_k} \rangle - f(x_{n_k})) = \sup_{x \in E} (\langle p, x \rangle - f(x)). \quad \square$$

Proof of Theorem 4.1. Lemmas 8.2 and 8.3(1) imply that the set $\text{epi } \gamma$ is a parabolic boundedly uniformly convex quasiball in the space $E \times \mathbb{R}$. By Lemma 6.5, the space $E \times \mathbb{R}$ (and, therefore, the space E) is reflexive. Applying Lemma 8.6(2) to the functional $p = 0$, we see that the minimum $\min_{x \in E} \gamma(x)$ is attained at some point $x_0 \in E$. We denote $C = \text{epi } f$ and $M = \text{epi } \gamma$.

(1) Let the set C be a summand of M ; i.e., there exists a set $C' \subset E \times \mathbb{R}$ such that $M = C + C'$. By Remark 6.2, as applied to functional $p \in (E \times \mathbb{R})^*$ given by the equality $\langle p, (x, y) \rangle = -y$, for all $(x, y) \in E \times \mathbb{R}$, the point $z_0 = (x_0, \gamma(x_0))$ can be presented as $z_0 = z + z'$, where $z = (u_0, f(u_0))$, $z' \in C'$, $f(u_0) = \min_{u \in E} f(u)$. Therefore, $0 \in \partial f(u_0) \cap \partial \gamma(x_0)$.

We fix some arbitrary points $u_1 \in \text{dom } f$, $x_1 \in \text{dom } \gamma$ such that there exists a functional $p_1 \in \partial f(u_1) \cap \partial \gamma(x_1)$. As $p_1 \in \partial \gamma(x_1)$, and since the function γ is strictly convex, we see that $\max_{(x,y) \in \text{epi } \gamma} (\langle p_1, x \rangle - y)$ is attained at a single pair of points $z_1 := (x_1, \gamma(x_1))$. By Remark 6.2, the representation $z_1 = z + z'$ holds, where $z = (u_1, f(u_1))$, $z' \in C'$. Therefore, $C + z_1 - z = C + z' \subset C + C' = M$. Thus, $(u, f(u)) + z_1 - z \in M$ for any point $u \in \text{dom } f$; i.e.,

$$(u + x_1 - u_1, f(u) + \gamma(x_1) - f(u_1)) \in M = \text{epi } \gamma,$$

and, therefore,

$$\gamma(u + x_1 - u_1) \leq f(u) + \gamma(x_1) - f(u_1). \quad (8.14)$$

Since (8.14) automatically holds for $u \in E \setminus \text{dom } f$, it also holds for all $u \in E$. Changing the variable $u = u_1 + w$, $w \in E$, we obtain (4.3). Therefore, $f \in \mathcal{SC}(\gamma)$.

(2) Let $f \in \mathcal{SC}(\gamma)$. By the definition there exist points $x_0 \in \text{dom } \gamma$ and $u_0 \in \text{dom } f$ such that $\partial f(u_0) \cap \partial \gamma(x_0) \neq \emptyset$ and relation (4.3) holds. In particular,

$$\gamma(x_0 + w) - \gamma(x_0) \leq f(u_0 + w) - f(u_0) \quad \forall w \in E.$$

This and the coerciveness of function γ imply the coerciveness of f .

Consider the function

$$g(v) = \sup_{u \in E} (\gamma(v + u) - f(u)), \quad v \in E. \quad (8.15)$$

The function g is convex and lower-semicontinuous as the supremum of a family of convex lower-semicontinuous functions. As $g(v) \geq \gamma(v + u_0) - f(u_0)$, and for any $v \in E$, the function γ is coercive, the function g is also coercive. Applying Lemma 8.6(2) to the functional $p = 0$, we obtain that the functions f are g bounded from below. We denote

$$C = \text{epi } f, \quad C' = \text{epi } g, \quad M' = C + C'.$$

The set M' is convex as the Minkowski sum of two convex sets. Let us show that the set M' is closed. Let the sequence $\{z_n\} \subset M'$ converge to some $z^0 = (x^0, y^0) \in E \times \mathbb{R}$. As $z_n \in M' = C + C'$, it follows that for any $n \in \mathbb{N}$ we can represent $z_n = (x_n, y_n) + (x'_n, y'_n)$, where $y_n \geq f(x_n)$ and $y'_n \geq g(x'_n)$. As the functions f and g are bounded from below, the sequences $\{y_n\}$ and $\{y'_n\}$ are bounded from below. This and the relations $\lim_{n \rightarrow \infty} (y_n + y'_n) = y^0$ imply that these sequences are bounded from above and below. Thus, the sequences $\{f(x_n)\}$ and $\{g(x'_n)\}$ are bounded from above. Applying Lemma 8.6(1) for $p = 0$, we see that there exists a subsequence $\{z_{n_k}\}$ such that the sequence $\{x_{n_k}\}$ converges weakly to some vector u^0 , the sequence $\{x'_{n_k}\}$, to vector v^0 , and also

$$f(u^0) \leq \liminf_{k \rightarrow \infty} f(x_{n_k}), \quad g(v^0) \leq \liminf_{k \rightarrow \infty} g(x'_{n_k}).$$

As $\lim_{n \rightarrow \infty} (x_n + x'_n) = x^0$, we have $u^0 + v^0 = x^0$. Next, we have that

$$f(u^0) + g(v^0) \leq \liminf_{k \rightarrow \infty} (y_{n_k} + y'_{n_k}) = y^0$$

and

$$(u^0 + v^0, f(u^0) + g(v^0)) \in C + C' = M',$$

and so $z^0 = (x^0, y^0) \in M'$. Thus, we have proved that the set M' is closed.

To complete the proof, it suffices to show that $M' = M$.

First, let $(x, y) \in C$ and $(x', y') \in C'$. Then $y \geq f(x)$ and $y' \geq g(x')$. In view of equality (8.15), we have that $y' \geq \gamma(x + x') - f(x)$. Therefore, $y + y' \geq \gamma(x + x')$, i.e., $(x + x', y + y') \in \text{epi } \gamma = M$. Thus, the inclusion $M' \subset M$ is proved. Now suppose that the inverse inclusion does not hold; i.e., there exists $\hat{z} = (\hat{x}, \hat{y}) \in M \setminus M'$.

We have that $\hat{z} \notin M'$, and so in view of the Hahn–Banach separation theorem, there exists a functional $\tilde{p} \in (E \times \mathbb{R})^*$ such that $s(\tilde{p}, M') < \langle \tilde{p}, \hat{z} \rangle$; i.e., there exists a functional $p \in E^*$ and $q \in \mathbb{R}$, $\varepsilon > 0$, such that

$$\sup_{(x,y) \in M'} (\langle p, x \rangle - qy) + \varepsilon < \langle p, \hat{x} \rangle - q\hat{y}. \quad (8.16)$$

We have that $(x, y + t) \in M'$ for any point $(x, y) \in M'$ and any $t > 0$, and so (8.16) implies that $q \geq 0$. Let us show that we can assume that $q > 0$. Indeed, suppose that $q = 0$; i.e.,

$$\sup_{(x,y) \in M'} \langle p, x \rangle + \varepsilon < \langle p, \hat{x} \rangle.$$

As the functions f and g are bounded from below, we have that

$$\sup_{(x,y) \in M'} (\hat{y} - y) = \hat{y} - \inf_{u \in E} f(u) - \inf_{v \in E} g(v) < +\infty.$$

Thus, there exists a number $q_1 > 0$ such that

$$q_1 \sup_{(x,y) \in M'} (\hat{y} - y) < \frac{\varepsilon}{2}.$$

Therefore,

$$\sup_{(x,y) \in M'} (\langle p, x \rangle - q_1 y) + \frac{\varepsilon}{2} < \langle p, \hat{x} \rangle - q_1 \hat{y}.$$

Thus, there exists a functional $p \in E^*$ and numbers $q > 0$, $\varepsilon > 0$, such that (8.16) holds. Dividing this inequality by q and setting $p_0 = p/q$, we obtain that

$$\sup_{(x,y) \in M'} (\langle p_0, x \rangle - y) < \langle p_0, \hat{x} \rangle - \hat{y}.$$

As $\hat{z} = (\hat{x}, \hat{y}) \in M$, for any functional $\tilde{p}_0 \in (E \times \mathbb{R})^*$, as given by the equality

$$\langle \tilde{p}_0, (x, y) \rangle = \langle p_0, x \rangle - y, \quad x \in E, \quad y \in \mathbb{R}, \quad (8.17)$$

we have that

$$s(\tilde{p}_0, M') < \langle \tilde{p}_0, \hat{z} \rangle \leq s(\tilde{p}_0, M). \quad (8.18)$$

By Lemma 8.6(2), there exist vectors $\hat{u}_0 \in \text{dom } f$ and $\hat{x}_0 \in \text{dom } \gamma$ such that

$$\max_{u \in E} (\langle p_0, u \rangle - f(u)) = \langle p_0, \hat{u}_0 \rangle - f(\hat{u}_0), \quad \max_{x \in E} (\langle p_0, x \rangle - \gamma(x)) = \langle p_0, \hat{x}_0 \rangle - \gamma(\hat{x}_0). \quad (8.19)$$

By the definition of subdifferential (4.1), we have the inclusion $p_0 \in \partial f(\hat{u}_0) \cap \partial \gamma(\hat{x}_0)$. Using (4.3), this gives

$$\gamma(\hat{x}_0 + w) - \gamma(\hat{x}_0) \leq f(\hat{u}_0 + w) - f(\hat{u}_0) \quad \forall w \in E.$$

This and equality (8.15) imply that

$$g(\hat{x}_0 - \hat{u}_0) = \gamma(\hat{x}_0) - f(\hat{u}_0). \quad (8.20)$$

Moreover, equality (8.15) implies that $g(v) \geq \gamma(v + \hat{u}_0) - f(\hat{u}_0)$ for any $v \in E$. Therefore, by to the second equality in (8.19) and equality (8.20), we have

$$g(v) \geq \gamma(\hat{x}_0) - f(\hat{u}_0) + \langle p_0, v - \hat{x}_0 + \hat{u}_0 \rangle = g(\hat{x}_0 - \hat{u}_0) + \langle p_0, v - \hat{x}_0 + \hat{u}_0 \rangle \quad \forall v \in E.$$

Supposing that

$$\hat{v}_0 = \hat{x}_0 - \hat{u}_0, \quad (8.21)$$

we obtain

$$\max_{v \in E} (\langle p_0, v \rangle - g(v)) = \langle p_0, \hat{v}_0 \rangle - g(\hat{v}_0). \quad (8.22)$$

Equalities (8.19) and (8.22) mean that for the functional $\tilde{p}_0$, as given by relation (8.17), the equalities

$$s(\tilde{p}_0, \text{epi } f) = \langle \tilde{p}_0, (\hat{u}_0, f(\hat{u}_0)) \rangle, \quad s(\tilde{p}_0, \text{epi } \gamma) = \langle \tilde{p}_0, (\hat{x}_0, \gamma(\hat{x}_0)) \rangle, \quad s(\tilde{p}_0, \text{epi } g) = \langle \tilde{p}_0, (\hat{v}_0, g(\hat{v}_0)) \rangle$$

hold.

Equalities (8.20) and (8.21) imply that

$$(\hat{x}_0, \gamma(\hat{x}_0)) = (\hat{u}_0, f(\hat{u}_0)) + (\hat{v}_0, g(\hat{v}_0)).$$

Therefore,

$$s(\tilde{p}_0, \text{epi } \gamma) = s(\tilde{p}_0, \text{epi } f) + s(\tilde{p}_0, \text{epi } g),$$

i.e.,

$$s(\tilde{p}_0, M) = s(\tilde{p}_0, C) + s(\tilde{p}_0, C') = s(\tilde{p}_0, M'),$$

which contradicts inequalities (8.18). $\square$

Proof of Theorem 4.3. Let the assumptions of Theorem 4.3 hold. These conditions are preserved if the function γ is substituted by the function $\tilde{\gamma}(x) = \gamma(x) - \gamma(0) - 1$, for which the inequality $\tilde{\gamma}(0) < 0$ holds. Therefore, without loss of generality, we shall assume that $\gamma(0) < 0$.

By Lemmas 8.1–8.4, the set $M = \text{epi } \gamma$ is a coercive, strongly parabolic, boundedly uniformly convex, and boundedly uniformly smooth quasiball, and besides, $v_M = (0_E, 1)$ is the asymptotic direction of M .

By the hypothesis, the multifunctions $t \mapsto A(t) = \text{epi } \alpha(t, \cdot)$ and $t \mapsto \text{epi } \omega(t, \cdot)$ are R -continuous at t_0 . Therefore, the multifunction $t \mapsto C(t) = \{(x, y) \in E \times \mathbb{R} : y \leq \omega(t, x)\}$ has the same property.

For any $t \in T$, we consider the set $C^-(t) = \text{epi}(-\omega(t, \cdot))$. We note that

$$C^-(t) = \{(x, -y) \in E \times \mathbb{R} : (x, y) \in C(t)\}. \quad (8.23)$$

Since $-\omega(t, \cdot) \in \mathcal{SC}(\gamma_r^-)$, in view of Theorem 4.1 we have that $C^-(t) \in \mathcal{SC}(\text{epi } \gamma_r^-)$ for any $t \in T$; i.e., the set $C^-(t)$ is convex and closed, and there exists a set $C_1^-(t) \subset E \times \mathbb{R}$ such that $C^-(t) + C_1^-(t) = \text{epi } \gamma_r^-$. This and equalities (8.23) imply that the set $C(t)$ is convex and closed and there exists a set

$$C_1(t) = \{(x, y) \in E \times \mathbb{R} : (x, -y) \in C_1^-(t)\}$$

such that $C(t) + C_1(t) = -\text{epi } \gamma_r$, where $\gamma_r(x) = r\gamma(x/r)$ for all $x \in E$. Thus, $C(t) + C_1(t) = -r \text{epi } \gamma$; i.e., $C(t) \in \mathcal{SC}(-rM)$ for all $t \in T$.

By Theorem 4.2, we have $A(t) \in \mathcal{WC}(M)$ for any $t \in T$. As $A(t) = \text{epi } \alpha(t, \cdot)$ and $v_M = (0_E, 1)$, we see that $A(t) + \lambda v_M \subset A(t)$ for all $\lambda > 0$, $t \in T$.

As for any $t \in T$ there exists $x \in E$ such that $\alpha(t, x) \leq \omega(t, x)$, we get that $A(t) \cap C(t) \neq \emptyset$ for all $t \in T$.

By Lemma 8.5, the family $\{A(t)\}_{t \in T}$ is M -equi-quasibounded.

Now by Theorem 3.2 the multifunction

$$t \mapsto A(t) \cap C(t) = \{(x, y) \in E \times \mathbb{R} : \alpha(t, x) \leq y \leq \omega(t, x)\}$$

is Hausdorff continuous at $t_0 \in T$. $\square$

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