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A CRITERION FOR APPROXIMABILITY BY HARMONIC FUNCTIONS IN LIPSCHITZ SPACES

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Let X be a compact subset of $\mathbb{R}^3$, and let f be a function that is harmonic inside X and belongs to the Lipschitz space $C^\gamma(X)$, $0<\gamma<1$. A criterion for approximability of f on X in $C^\gamma(X)$ by functions that are harmonic on neighborhoods of X is obtained in terms of the Hausdorff content of order $1+\gamma$. The proof is completely constructive, and Vitushkin's method of singularities separation and approximation by parts is applied. Bibliography: 15 titles.

1. Introduction

Let $0<\gamma<1$ and let $X\subset\mathbb{R}^3$ be a compact set. Recall (see, for example, [1, Chap. 3, Sec. 1]) that the Lipschitz space $\mathrm{Lip}^\gamma(X)$ consists of functions $f:X\to\mathbb{R}$ such that the inequality

$$|f(x)-f(y)|\le c|x-y|^\gamma \tag{1.1}$$

holds for all $x,y\in X$, where $c=c(f,X,\gamma)<\infty$. The infimum of values c in formula (1.1) defines a seminorm $\|f\|_{\gamma,X}$. If $x_0\in X$ is a fixed point, then $\mathrm{Lip}^\gamma(X)$ is a Banach space with norm $|f(x_0)|+\|f\|_{\gamma,X}$.

The definition of the space $\mathrm{Lip}^\gamma(\mathbb{R}^3)$ with seminorm $\|f\|_\gamma=\|f\|_{\gamma,\mathbb{R}^3}$ is similar; in formula (1.1), we assume that $x,y\in\mathbb{R}^3$ and, in addition, that $\|f\|_{\mathrm{L}_\infty}<\infty$.

Recall that by the Whitney extension theorem (see, for example, [2, Chap. 6, Theorem 3]), any function $f\in\mathrm{Lip}^\gamma(X)$ can be extended to a function from the class $\mathrm{Lip}^\gamma(\mathbb{R}^3)$ with compact support and belonging to class C^∞ outside X so that $\|f\|_\gamma\le A\|f\|_{\gamma,X}$, where $A\ge 1$ is an absolute constant. For this reason, in what follows we assume that any function from $\mathrm{Lip}^\gamma(X)$ is extended to the whole space $\mathbb{R}^3$ according to the Whitney theorem.

The space $C^\gamma(X)$ is the subspace of $\mathrm{Lip}^\gamma(X)$ consisting of functions f for which the inequality

$$\frac{|f(x)-f(y)|}{|x-y|^\gamma}\le\epsilon_1(\delta)$$

holds for $x,y\in X$ such that $|x-y|\le\delta$ and $x\ne y$, where $\lim\limits_{\delta\to 0}\epsilon_1(\delta)=0$, $\epsilon_1=\epsilon_1(f,X,\gamma)$.

It follows, for example, from Theorem 3 and formula (15) of [2, Chap. 6] that a function $f\in C^\gamma(X)$ extended by the Whitney theorem satisfies the condition

$$\frac{|f(x)-f(y)|}{|x-y|^\gamma}\le\epsilon_{f,\gamma}(\delta),\quad\text{where}\quad\lim_{\delta\to 0}\epsilon_{f,\gamma}(\delta)=0, \tag{1.2}$$

for all $x,y\in\mathbb{R}^3$ such that $x\ne y$ and $|x-y|\le\delta$.

Further, $C^\gamma(\mathbb{R}^3)$ is the subspace of $\mathrm{Lip}^\gamma(\mathbb{R}^3)$ consisting of functions f that satisfy condition (1.2).

Let X^o be the set of interior points of a compact set X and let Δ be the Laplace operator in the space $\mathbb{R}^3$; denote by $h_\gamma(X)$ and $H_\gamma(X)$ the following classes of functions:

$$h_\gamma(X)=C^\gamma(X)\cap\{f|_X:\Delta f=0\ \text{ in }\ X^o\},$$

and $H_\gamma(X)$ is the closure in $C^\gamma(X)$ of the set of functions

$$\{f|_X:\Delta f=0\text{ in a neighborhood of }X\}$$

(the neighborhood of X depends on the function f); clearly, $H_\gamma(X)\subset h_\gamma(X)$.

The goal of this paper is to describe functions of the class $H_\gamma(X)$ in terms of the functional $M^{1+\gamma}(\cdot)$, i.e., the Hausdorff content of order $1+\gamma$. By definition (see, for example, [3, Chap. 2]), for a bounded set $U\subset\mathbb{R}^3$ and $t>0$,

$$M^t(U)=\inf\sum_k(r_k)^t, \tag{1.3}$$

where the infimum is taken over all coverings of U by not more than countable families of balls B_k of radii r_k (it is irrelevant whether the balls are open or closed).

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The following statement holds.

Denote by $B = B(a, r)$ the open ball centered at $a \in \mathbb{R}^3$ of radius r.

Theorem 1. *Assume that there exists a constant $k \geq 1$ and a function $\epsilon(r)$ with $\epsilon(r) \searrow 0$ as $r \searrow 0$ such that for any ball $B = B(a, r)$ with boundary ∂B, the following estimate holds:*

$$\left| \frac{1}{\sigma(\partial B)} \int\limits_{\partial B} f(x) d\sigma_x - \frac{1}{m(B)} \int\limits_{B} f(x)\, dm_x \right| \leq \epsilon(r) r^{-1} M^{1+\gamma}(kB \setminus X), \tag{1.4}$$

where $\sigma_{(\cdot)}$ is the surface measure on ∂B, $m_{(\cdot)}$ is Lebesgue measure in $\mathbb{R}^3$, and $kB = B(a, kr)$. Then $f \in H_\gamma(X)$.

Conversely, if $f \in H_\gamma(X)$, then estimate (1.4) *holds with $k = 1$ and $\epsilon = A\epsilon_{f,\gamma}$, where $A > 0$ is an absolute constant and $\epsilon_{f,\gamma}$ satisfies* (1.2).

In what follows, for brevity we denote the averages of the function f in the left-hand side of inequality (1.4) by $f_{\mathrm{av}}(\partial B)$ and $f_{\mathrm{av}}(B)$, respectively.

Note that condition (1.4) is an analog of [4, Theorem 1.1, condition (iv)].

The key role in the proof of Theorem 1 is played by the following lemma (further, $Q(a, s)$ is the cube whose sides are parallel to coordinate axes, a is the center of the cube, s is the side length, $kQ = Q(a, ks)$ for $k > 0$, and $\mathrm{Spt}\,(\cdot)$ is the closure of the support of a function).

Lemma 1.1. *Let f be a function from $h_\gamma(X)$ extended according to the Whitney theorem. If there exists a function $\varepsilon : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\lim_{s \to 0} \varepsilon(s) = 0$ and the estimate*

$$\left| \int\limits_{Q} f(x) \Delta\psi(x)\, dm_x \right| \leq \varepsilon(s) \|\nabla^3 \psi\|_{\mathrm{L}_\infty} s^3 M^{1+\gamma}((9/8)Q \setminus X) \tag{1.5}$$

holds for any open cube $Q = Q(a, s)$ and any function $\psi \in C^\infty(\mathbb{R}^3)$ with $\mathrm{Spt}\,(\psi) \subset Q$, then $f \in H_\gamma(X)$.

The statement of Lemma 1.1 is close to statements of A. G. Vitushkin's lemma on uniform analytic approximations (see [5, Chap. 4, Sec. 2, Lemma 1]), of P. V. Paramonov's theorem on harmonic approximations in C^1-norm (see [6, Theorem 5.1]), and of A. O'Farrell's theorem on approximation by analytic functions in Lipschitz norms for $0 < \gamma < 1$ (see [1, Chap. 3, Theorem 1.1]).

Remark 1.1. One can show that condition (1.5) is not only sufficient for the inclusion $f \in H_\gamma(X)$ but necessary as well; in this case, one may take $\varepsilon(s) = A\epsilon_{f,\gamma}(s)$, where $\epsilon_{f,\gamma}$ satisfies (1.2) and $A > 0$ is an absolute constant. This fact is standard; its proof is similar to the reasoning applied in the proof of Theorem 1.1 of [1, Chap. 3] or of Lemma 4.2 below.

Remark 1.2. One can show that the conclusion of Lemma 1.1 still holds if the set $(9/8)Q \setminus X$ in the right-hand side of inequality (1.5) is replaced by $kQ \setminus X$ with a fixed $k \geq 1$.

Remark 1.3. Theorem 1 and Lemma 1.1 can be carried over to the case of harmonic functions in the space $\mathbb{R}^d$ with $d \geq 3$; in this case, the content $M^{d-2+\gamma}$ in the right-hand side of inequality (1.5) is replaced by $M^{1+\gamma}$, and the expression $r^{-1}M^{1+\gamma}(\cdot)$ in the right-hand side of inequality (1.4) is replaced by $r^{2-d}M^{d-2+\gamma}(\cdot)$.

Recall that a description of compact sets X for which $h_\gamma(X) = H_\gamma(X)$ was given by J. Mateu and J. Orobitg in [7]; in fact, their result is a particular case of Theorem 1 of the present paper (see also [8, Theorem 1], where the result of the paper [7] is generalized to the case of elliptic operators of an arbitrary order). Note that the proof given in this paper is essentially simpler than that in [7] and is completely constructive (in [7, 8], dual arguments based on spectral synthesis in Lizorkin–Triebel spaces are used).

In the proof of Theorem 1, we apply a generalized Vitushkin's scheme [5] of separation of singularities and approximation of functions by parts. We prove Lemma 1.1 in Secs. 2–3. In Sec. 2, we use auxiliary results to reduce Lemma 1.1 to the basic Lemma 2.7. In Sec. 3 (the main section of the paper), Lemma 2.7 is proved using a special geometric construction whose aim is to estimate and equate the Laurent coefficients at the first derivatives of the fundamental solution for localizations of the function f. In Sec. 4, we deduce Theorem 1 from Lemma 1.1 using the same reasoning as that applied in [9, §4]; here we apply some of the ideas of the paper [10].

2. Auxiliary results. Main lemma

In the proof of Lemma 1.1, we use the following Vitushkin's scheme [5]:

(1) Represent the function f as a finite sum of localizations using a proper partition of unity;

(2) expand the localizations into Laurent series and estimate Laurent coefficients;

(3) use condition (1.5) to equate a necessary number of Laurent coefficients of localizations to guarantee the inclusion $f \in H_\gamma(X)$.

A finite family of nonnegative functions $\varphi_j \in C_0^\infty(\mathbb{R}^3)$ is called a *partition of unity* on a compact set K if $\sum_j \varphi_j(x) \equiv 1$ in a neighborhood of the set K.

The function f (extended by the Whitney theorem) is representable (in the generalized sense) as a convolution $f = (\Delta f) * E$, where

$$E(x) = -\frac{1}{4\pi|x|} \tag{2.1}$$

is the fundamental solution of the Laplace equation in $\mathbb{R}^3$. Hence, a partition of unity on $\mathrm{Spt}\,(\Delta f)$ generates a representation of the function f as a finite sum of localizations,

$$f = \sum_j f_j, \quad \text{where} \quad f_j = V_{\varphi_j} f, \tag{2.2}$$

$V_\varphi f = (\varphi \Delta f) * E$ is the localization operator ([5, Chap. 2, Sec. 3] and [11, Sec. 2]), and $\varphi \in C_0^\infty(\mathbb{R}^3)$.

Further, for a multiindex $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ ($\alpha_m \in \mathbb{Z}_+$, $m = 1, 2, 3$) and for $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ we denote

$$|\alpha| = \sum_{m=1}^{3} \alpha_m, \quad \partial^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \partial x_3^{\alpha_3}},$$
$$\alpha = \alpha_1 \alpha_2 \alpha_3!, \quad x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3}.$$

Closed cubes of the form

$$Q = Q_p^{m_1, m_2, m_3} = \prod_{k=1}^{3} [m_k 2^{-p}, (m_k + 1) 2^{-p}], \tag{2.3}$$

where $(p, m_1, m_2, m_3) \in \mathbb{Z}^4$, are called *binary cubes.* Considering coverings by binary cubes, we always assume that the cubes are disjoint (i.e., they do not have common interior points).

In what follows, we denote absolute positive constants by $A, A_1, A_2, \ldots$; values of these constants may be different in different relations. We denote by $c_1(\gamma), c_2(\gamma), \ldots$ positive constants that depend only on γ in inequality (1.1).

The following statement is due to R. Harvey and J. Polking [12, Lemma 3.1].

Lemma 2.1. *Let $\{Q_j\}$ be a finite family of disjoint binary cubes. Then there exists a partition of unity $\{\varphi_j\}$ on $\bigcup_j Q_j$ such that* $\mathrm{Spt}\varphi_j \subset (3/2)Q_j$ *and the estimates $\|\nabla^n \varphi_j\|_{\mathrm{L}_\infty} \le A(n)(s(Q_j))^{-n}$ are valid for all $n \in \mathbb{Z}_+$.*

In what follows, we construct localizations using partitions of unity given by Lemma 2.1. Note that if, under the conditions of Lemma 1.1, the cube Q has the form $(3/2)Q'$ for a proper binary cube Q' and derivatives of a function ψ satisfy estimates of Lemma 2.1, then for the right-hand side of formula (1.5), the estimate $\|\nabla^3 \psi\|_{\mathrm{L}_\infty} s^3 \le A_1$ holds.

Lemma 2.2. *Let $Q = Q(a, s)$ be an open cube and let $\varphi \in C_0^\infty(\mathbb{R}^3)$ be a function with* $\mathrm{Spt}\,(\varphi) \subset Q$ *whose derivatives satisfy the estimates of Lemma* 2.1 *with respect to the cube Q. Then the localization $V_\varphi f$ has the following properties:*

(a) $\Delta(V_\varphi f) = \varphi \Delta f$; *hence,* $\mathrm{Spt}\,(\Delta(V_\varphi f)) \subset (\mathrm{Spt}\,(\varphi) \setminus X^o)$;

(b) $V_\varphi f \in C^\gamma(\mathbb{R}^3)$, *and the estimate $\|V_\varphi f\|_\gamma \le A\epsilon_{f,\gamma}(s)$ holds, where $\epsilon_{f,\gamma}$ is from* (1.2);

(c) *the estimate $\|V_\varphi f\|_{\mathrm{L}_\infty} \le A\omega_f(s)$ holds, where ω_f is the (usual) continuity modulus of f.*

The statement of Lemma 2.2 is standard. Item (a) is obvious; item (b) is proved in [7, Lemma 2.4] (see also [1, Chap. 3, Lemma 1.4]); item (c) is a particular case of [13, Lemma 14.10], this is an analog of Vitushkin's lemma (see [5, Chap. 2, Sec. 3, Lemma 2]).

Remark 2.1. Items (b) and (c) of Lemma 2.2 follow from the equality

$$V_\varphi f = \varphi f_0 + (f_0\Delta\varphi) * E - 2\sum_{|\alpha|=1}(f_0\partial^\alpha\varphi) * \partial^\alpha E,$$

where $f_0(x) = f(x) - f(a)$, and from standard estimates of a convolution.

To estimate localizations outside Q, we apply expansions in Laurent series. Everywhere for $|x-a| > 6s$, the function $V_\varphi f$ is expanded into a *Laurent series* of the form

$$V_\varphi f(x) = \sum_{|\alpha|\geq 0} c_\alpha \partial^\alpha E(x-a) \tag{2.4}$$

that converges in C^∞ (see [6, Lemma 3.2]); in (2.4), c_α are the *Laurent coefficients* given by the formulas

$$c_\alpha = c_\alpha(V_\varphi f, a) = \frac{(-1)^{|\alpha|}}{\alpha!}\int_Q f(x)\Delta(\varphi(x)(x-a)^\alpha)\,dm_x;$$

in particular, the coefficient $c_0(V_\varphi f)$ does not depend on the expansion center a.

Note that the following, more general, statement holds (see [6, Lemma 3.2], [7, Sec. 2], and [14, 1B]): Expansion (2.4) is valid for $|x-a| > 6s$ and for any convolution of the form $T * E$, where T is a distribution with $\mathrm{Spt}\,(T) \subset Q$ and

$$c_\alpha(T * E, a) = \frac{(-1)^{|\alpha|}}{\alpha!}\langle T(x)|(x-a)^\alpha\rangle; \tag{2.5}$$

in formula (2.5), the expression in angular braces is the action of the distribution T with compact support on a function from $C^\infty(\mathbb{R}^3)$.

We take $\psi(x) = \varphi(x)(x-a)^\alpha$ in inequality (1.5) to get the estimate

$$|c_\alpha(V_\varphi f, a)| \leq (\alpha!)^{-1}A_1\varepsilon(s)s^{|\alpha|}M^{1+\gamma}((9/8)Q \setminus X) \tag{2.6}$$

(here we take into account that $|x-a| \leq (\sqrt{3}/2)s$ for $x \in Q$ and apply the elementary estimate $\|\nabla^3(\varphi(x)(x-a)^\alpha)\|_{\mathrm{L}_\infty} \leq A_2\|\nabla^3\varphi\|_{\mathrm{L}_\infty}s^{|\alpha|}$).

In what follows, we consider coverings for which the lengths of cube sides do not exceed 1; without loss of generality, we assume that $\varepsilon(s) \geq \epsilon_{f,\gamma}(s)$ and $\varepsilon(2s) \leq 2\varepsilon(s)$ for $s \leq 1$. Relations (2.4)–(2.6) imply the following statement (one can find simple details concerning summation of a geometric progression, for example, in the proofs of [9, Lemmas 1.4 and 1.5]).

Lemma 2.3. *Let Q and $V_\varphi f$ be the same as in Lemma 2.2. Let $n \leq 2$ and assume that $c_\alpha(V_\varphi f, a) = 0$ for $|\alpha| < n$. Then the estimates*

$$\begin{aligned} |V_\varphi f(x)| &\leq A\varepsilon(s)s^n\frac{M^{1+\gamma}((9/8)Q\setminus X)}{|x-a|^{1+n}}, \\ |\nabla V_\varphi f(x)| &\leq A\varepsilon(s)s^n\frac{M^{1+\gamma}((9/8)Q\setminus X)}{|x-a|^{2+n}} \end{aligned} \tag{2.7}$$

are valid for $x \notin (8/7)Q$.

Now we consider the problem of approximation of localizations. We assume that for an open cube Q, the set $Q \setminus X$ is nonempty; otherwise, $\mathrm{Spt}\,(\varphi) \subset X^o$, and $V_\varphi f \equiv 0$ by Lemma 2.2.

Recall that by the Frostman theorem (see [3, Chap. 2, Theorem 1]), for any bounded open set U there exists a nonnegative measure μ such that $\mathrm{Spt}\,(\mu) \subset U$ and $\|\mu\| \geq A_1M^{1+\gamma}(U)$ (where $\|\cdot\|$ is the full variation of a measure and $A_1 > 0$ is an absolute constant); in addition, the inequality $\|\mu\|(B) \leq r^{1+\gamma}$ holds for any ball B of radius r. An easy consequence of the last inequality is the estimate $\|\mu * E\|_\gamma \leq A$, where the function E is defined in (2.1). Thus, the following statement holds.

Lemma 2.4. *For any open cube $Q = Q(a,s)$ there exists a nonnegative measure ν such that $\mathrm{Spt}(\nu) \subset ((9/8)Q\setminus X)$, $\|\nu\| = A_2M^{1+\gamma}((9/8)Q\setminus X)$, and $\|\nu * E\|_\gamma \leq 1$.*

Let $H = \varepsilon(s)(\nu * E)$. Lemma 2.4 and formula (2.5) imply the equality

$$c_0(H) = A_2\varepsilon(s)M^{1+\gamma}((9/8)Q\setminus X);$$

for $|\alpha| > 0$, the estimates

$$|c_\alpha(H,a)| \leq (\alpha!)^{-1}A\varepsilon(s)s^{|\alpha|}M^{1+\gamma}((9/8)Q\setminus X) \tag{2.8}$$

are valid.

It follows from Lemma 2.4 and estimate (2.6) that the following statement holds for $c_0(V_\varphi f)$ (where $\Psi = \lambda H$, $|\lambda| \le A_3$).

Lemma 2.5. *Assume that estimate* (1.5) *is valid, let* $Q = Q(a, s)$, *and let* $V_\varphi f$ *be the same as in Lemma* 2.2. *Then there exists a function* $\Psi \in \operatorname{Lip}^\gamma(\mathbb{R}^3)$ *such that* $\operatorname{Spt}(\Delta\Psi) \subset ((9/8)Q \setminus X)$ *and the following relations hold:*

(1) $c_0(V_\varphi f) = c_0(\Psi)$;

(2) $\|\Psi\|_\gamma \le A\varepsilon(s)$.

Lemmas 2.3 and 2.5 and estimates (2.6) and (2.8) imply the following statement.

Lemma 2.6. *Let* $V_\varphi f$ *and* Ψ *be the same as in Lemma* 2.2 *and Lemma* 2.5, *respectively. Let* $r = V_\varphi f - \Psi$. *Then*

(1) *if* $x \notin (8/7)Q$, *then*

$$\begin{aligned} &\text{(a)}\quad |r(x)| \le A\varepsilon(s)s\frac{M^{1+\gamma}((9/8)Q \setminus X)}{|x-a|^2},\\ &\text{(b)}\quad |\nabla r(x)| \le A\varepsilon(s)s\frac{M^{1+\gamma}((9/8)Q \setminus X)}{|x-a|^3}; \end{aligned} \tag{2.9}$$

(2) $|c_\alpha(r)| \le A\varepsilon(s)sM^{1+\gamma}((9/8)Q \setminus X)$ *for* $|\alpha| = 1$ *(since* $c_0(r) = 0$, *the coefficients* $c_\alpha(r)$ *with* $|\alpha| = 1$ *do not depend on the center of expansion).*

Remark 2.2. It is important to note that, in contrast with the case of approximation by analytic functions in Lipschitz norms for $0 < \gamma < 1$ (see, for example, [1, Chap. 3, Sec. 1]), Lemma 2.5 on equating the coefficient c_0 does not imply the inclusion $f \in H_\gamma(X)$ by itself; one needs a stronger statement on equality the coefficients c_α with $|\alpha| \le 1$.

Lemma 1.1 is a corollary of the following (basic) lemma which we prove in Sec. 3.

Lemma 2.7. *Let* f, $V_\varphi f$, *and* Q *be the same as in Lemma* 2.5, *and let* $s(Q) \le 1$. *Then there exists a function* $F \in \operatorname{Lip}^\gamma(\mathbb{R}^3)$ *such that* $\operatorname{Spt}(\Delta F) \subset (4Q \setminus X)$ *and the following relations hold:*

(1′) $c_\alpha(V_\varphi f, a) = c_\alpha(F, a)$ *for* $|\alpha| \le 1$;

(2′) $\|F\|_\gamma \le c_1(\gamma)\varepsilon(s)$.

One can prove the implication "Lemma 2.7 ⇒ Lemma 1.1" using [7, Lemma 2.1] (see also [14, Lemma 3.2]). We give here an essentially simpler proof of the above implication; a similar reasoning is applied in Sec. 3 in the proof of Lemma 2.7.

Proof of Lemma 1.1. Take an arbitrary number $\delta = 2^{-k}$, where $k \in \mathbb{N}$, and cover the compact set $\operatorname{Spt}(\Delta f)$ by a net of binary cubes $Q_j(a_j, \delta)$ (of the same size). Construct the corresponding partition of unity $\{\varphi_j\}$ by Lemma 2.1; let $\{f_j = V_{\varphi_j} f\}$ be the obtained family of localizations. To any function f_j assign a function $F_j = F$ by Lemma 2.7. Let $\rho_j = f_j - F_j$. It follows from Lemma 2.7 and Lemma 2.3 with $n = 2$ that if $x \notin 8Q_j$, then the following estimates are valid:

$$\begin{aligned} &\text{(a)}\quad |\rho_j(x)| \le A_1 c_1(\gamma)\varepsilon(\delta)\frac{\delta^{3+\gamma}}{|x-a_j|^3},\\ &\text{(b)}\quad |\nabla\rho_j(x)| \le A_1 c_1(\gamma)\varepsilon(\delta)\frac{\delta^{3+\gamma}}{|x-a_j|^4}. \end{aligned} \tag{2.10}$$

To prove Lemma 1.1, it is enough to establish the estimate

$$\Big\|\sum_j \rho_j\Big\|_\gamma \le c_2(\gamma)\varepsilon(\delta). \tag{2.11}$$

Fix arbitrary points $x, y \in \mathbb{R}^3$, $x \ne y$. Decompose the set of all cubes Q_j of our covering into three disjoint sets P_m, $m = 1, 2, 3$, as follows.

1. Let P_1 be the set of all cubes Q_j for which the distance to $\{x\} \cup \{y\}$ is less than 10δ. Since the number of cubes in P_1 is estimated from above by an absolute constant, Lemmas 2.2 and 2.7 imply estimate (2.11) for the sum over the corresponding indices j.

2. Let P_2 be the set of all cubes Q_j for which the distance to $\{x\}\cup\{y\}$ belongs to the segment $[10\delta, 10|x-y|]$ (if $|x-y|<\delta$, then the set P_2 is empty). It follows from estimate (2.10) (a) that, taking the sum over indices j of cubes from the set P_2, we conclude that

$$\sum_j |\rho_j(x)-\rho_j(y)| \le \sum_j (|\rho_j(x)|+|\rho_j(y)|) \le A_2 c_1(\gamma)\varepsilon(\delta)\delta^\gamma \int\limits_{10\delta\le|v|\le 10|x-y|} \frac{dm_v}{|v|^3}$$
$$\le A_3 c_1(\gamma)\varepsilon(\delta)\delta^\gamma \log\left(\frac{|x-y|}{\delta}+1\right) \le c_3(\gamma)\varepsilon(\delta)|x-y|^\gamma,$$

where $v\in\mathbb{R}^3$.

3. Let $P_3=\{Q_j\}\setminus(P_1\cup P_2)$; then for any cube $Q_j\in P_3$, the distance to $\{x\}\cup\{y\}$ is more than $\max(10\delta, 10|x-y|)$. It follows from estimate(2.10) (b) that, taking the sum over indices j of cubes from the set P_3, we conclude that

$$\sum_j |\rho_j(x)-\rho_j(y)| \le A_4|x-y| \sum_j (|\nabla\rho_j(x)|+|\nabla\rho_j(y)|)$$
$$\le A_5 c_1(\gamma)\varepsilon(\delta)\delta^\gamma|x-y| \int\limits_{|v|\ge\max(10\delta,10|x-y|)} \frac{dm_v}{|v|^4} \le A_6 c_1(\gamma)\varepsilon(\delta)|x-y|^\gamma.$$

Thus, we have obtained estimate (2.11) from Lemma 2.7. To complete the proof of Lemma 1.1, it remains to prove Lemma 2.7. □

3. Proof of Lemma 2.7

Fix a binary cube $\mathbf{Q}$; let $s(\mathbf{Q})=\delta\le 1$ and let $g=V_\varphi f$, where φ satisfies the conditions of Lemma 2.1 with respect to the cube $\mathbf{Q}$. We denote by c_m^1, $m=1,2,3$, the Laurent coefficients of functions at $(\partial/\partial x_m)E$ (i.e., the values c_α with $|\alpha|=1$). We assume that $((3/2)\mathbf{Q}\setminus X^o)\ne\varnothing$; otherwise, $g\equiv 0$. Clearly, in this case the intersection of $(13/8)\mathbf{Q}\setminus X^o$ and the complement of X contains a nonempty open set.

Lemma 3.1. *Assume that estimate* (1.5) *holds. Then there exist six functions* $G_m^p\in \mathrm{Lip}^\gamma(\mathbb{R}^3)$, $m=1,2,3$, $p=1,2$, *such that*

(1) $\mathrm{Spt}\,(\Delta G_m^p)\subset(4\mathbf{Q}\setminus X)$, $g-G_m^1(x)=O(|x|^{-2})$, $G_m^2(x)=O(|x|^{-2})$ *as* $x\to\infty$;

(2) $c_m^1(G_m^2)\ge\sum\limits_{j=1}^{3}|c_j^1(g-G_m^1)|$;

(3) $c_m^1(G_m^2)\ge 3\max_{j\ne m}|c_j^1(G_m^2)|$;

(4) $\max\left(\|g-G_m^1\|_\gamma, \|G_m^2\|_\gamma\right)\le c_4(\gamma)\varepsilon(\delta)$

(due to the relation $G_m^2(x)=O(|x|^{-2})$, the coefficients $c_m^1(G_m^2)$ do not depend on the expansion center).

It is easy to reduce Lemma 3.1 to Lemma 2.7. Indeed, let m_0 be an index for which the sum $\sum\limits_{j=1}^{3}|c_j^1(g-G_m^1)|$ is minimal over all m. Then the desired function F in Lemma 2.7 that equates for g all the coefficients c_α, $|\alpha|\le 1$, is constructed in the form $F=G_{m_0}^1+\sum\limits_{m=1}^{3}t_m G_m^2$, where t_m, $m=1,2,3$, are real coefficients. To find the coefficients t_m, one has to solve a system of three linear equations; by condition (3) of Lemma 3.1, the matrix of this system is well conditioned, while, due to the choice of m_0, the right-hand side is nor "too large" compared to elements of the main diagonal. This gives us the estimates $|t_m|\le A$, and, consequently, the estimate $\|F\|_\gamma\le c_1(\gamma)\varepsilon(\delta)$. Thus, we have reduced Lemma 1.1 to Lemma 3.1.

Proof of Lemma 3.1. Let us construct the pair of functions (G_3^1, G_3^2); the remaining two pairs (G_m^1, G_m^2) are constructed similarly.

Recall that $g=(\varphi\Delta f)*E$, where φ satisfies the conditions of Lemma 2.1 with respect to the cube $\mathbf{Q}$. In what follows, we consider a covering of the compact set $(13/8)\mathbf{Q}\setminus X^o$ by proper finite families of binary cubes Q_j and obtain the corresponding functions φ_j given by Lemma 2.1 and the localizations $g_j=V_{\varphi_j}g$. Since $\Delta g=\varphi\Delta f$, $g_j=(\varphi_j\varphi\Delta f)*E$; hence, due to inequality (2.6), the corresponding estimates

$$|c_0(g_j)|\le A\varepsilon(s(Q_j))M^{1+\gamma}((7/4)Q_j\setminus X) \tag{3.1}$$

follow from estimate (1.5) for f. We get the required covering $\{Q_j\}$ = **Cover1** using a geometric construction. We start with an auxiliary construction.

Let Q be a binary cube and take $N = 2^k$, $k \in \mathbb{N}$, such that $N \geq 8$ (in what follows, we apply Lemma 3.3 and fix N depending on γ). Decompose Q into N^3 binary cubes of the same size, with side length $s_0 = s(Q)/N$. Denote by $\mathcal{Q}_N(Q)$ the set of such cubes D for which $((7/4)D \setminus X) \neq \varnothing$. A subset of $\mathcal{Q}_N(Q)$ which consists of cubes with coinciding pairs (m_1, m_2) from (2.3) (and with different m_3) is called a *vertical row.* A vertical row is called "good" if it contains at least two cubes (denote them Q_1 and Q_2) such that $m_3(Q_2) - m_3(Q_1) \geq 7$ and, in addition,

$$\min(M^{1+\gamma}((7/4)Q_1 \setminus X), M^{1+\gamma}((7/4)Q_2 \setminus X)) \geq N^{-1}(s_0)^{1+\gamma}. \tag{3.2}$$

An ordered pair (Q_1, Q_2) is called a *compatible pair*; a cube Q is called "good" if it contains at least one "good" vertical row. Otherwise, if Q does not contain a "good" vertical row, such a cube is called "bad."

Consider a "good" vertical row; let (Q_1, Q_2) be a compatible pair, let $a_j \in (7/4)Q_j$, $j = 1, 2$, be arbitrary points, and let $h_0 = E(x - x_2) - E(x - x_1)$. Clearly, the function h_0 satisfies condition (3) of Lemma 3.1: $c_0(h_0) = 0$ and $c_3^1(h_0) > 3 \max\limits_{m \neq 3} |c_m^1(h_0)|$; in addition, $c_3^1(h_0) > 5s_0$. Hence, inequality (3.2) and Lemma 2.4 imply the following statement.

Lemma 3.2. *For any consistent pair (Q_1, Q_2) there exists a function $h_{2,1} = (\nu_2 - \nu_1) * E$ such that the ν_j, $j = 1, 2$, are (nonnegative) measures,* $\mathrm{Spt}(\nu_j) \subset ((7/4)Q_j \setminus X)$, *and the following conditions are satisfied:*

(i) $\|\nu_1\| = \|\nu_2\|$ *(hence, $c_0(h_{2,1}) = 0$, and the coefficients $c_m^1(h_{2,1})$ do not depend on the center of the Laurent expansion);*

(ii) $N^{-1}(s_0)^{1+\gamma} \leq \|\nu_1\| \leq (s_0)^{1+\gamma}$, *and, hence,*

$$c_3^1(h_{2,1}) > 5N^{-1}(s_0)^{2+\gamma}$$

(i.e., the coefficient $c_3^1(h_{2,1})$ estimates the value $(s(Q))^{2+\gamma}$ from above with a constant depending on N);

(iii) $c_3^1(h_{2,1}) > 3 \max_{m \neq 3} |c_m^1(h_{2,1})|$;

(iv) $h_{2,1} \in \mathrm{Lip}^\gamma(\mathbb{R}^3)$ *and* $\|h_{2,1}\|_\gamma \leq A$.

To any "good" cube Q we assign a function $h_{2,1} = h_{2,1}(Q)$, taking an arbitrary consistent pair.

Remark 3.1. The role of the functions $h_{2,1}$ is as follows. Let us construct for a localization $V_\varphi g$ (where φ satisfies the conditions of Lemma 2.1 with respect to Q) the function Ψ given by Lemma 2.5; let, as in Lemma 2.6, $r = V_\varphi g - \Psi$. Then, due to condition (ii) of Lemma 3.2, the value $\varepsilon(s(Q))c_3^1(h_{2,1})$ estimates from above (with a constant depending on N) the sum of $|c_m^1(r)|$, $m = 1, 2, 3$.

Remark 3.2. We construct the function G_3^2 as a finite linear combination of functions $h_{2,1}$ with positive coefficients; in this case, condition (3) of Lemma 3.1 is satisfied automatically due to condition (iii) of Lemma 3.2.

Consider an arbitrary "bad" cube Q. Condition (3.2) is not satisfied for any its vertical row; hence, for any vertical row, the sum of $M^{1+\gamma}((7/4)D \setminus X)$ over all corresponding cubes D does not exceed $A(s_0)^{1+\gamma}$. This gives us the following estimate. If Q' is a cube such that $Q' \subset Q$, then

$$\sum_{\{Q_j | Q_j \in \mathcal{Q}_N(Q),\, Q_j \subset Q'\}} (s_0)^{1-\gamma} M^{1+\gamma}((7/4)Q_j \setminus X) \leq A_1 (s(Q'))^2. \tag{3.3}$$

The geometric construction of Sec. 3 is based on the following lemma; it is essential for us in the proof of this lemma that $\gamma > 0$. Note that in the case of uniform harmonic approximations (i.e., for $\gamma = 0$), the situation is really more complicated (see, for example, [9]).

Lemma 3.3. *There exists a (large enough) number $N = N(\gamma)$ such that if Q is a "bad" cube, then the estimate*

$$\sum_{\mathcal{Q}_N(Q)} s_0 M^{1+\gamma}((7/4)Q_j \setminus X) \leq \frac{1}{2} s(Q) M^{1+\gamma}((7/4)Q \setminus X) \tag{3.4}$$

holds for summation over all the cubes Q_j from $\mathcal{Q}_N(Q)$ (recall that $N = s(Q)/s(Q_j)$, $s(Q_j) = s_0$).

Proof of Lemma 3.3. Let us show that

$$\sum_{\mathcal{Q}_N(Q)} (s_0)^{1-\gamma} M^{1+\gamma}((7/4)Q_j \setminus X) \leq A(s(Q))^{1-\gamma} M^{1+\gamma}\left(\bigcup_{\mathcal{Q}_N(Q)} (7/4)Q_j \setminus X\right); \tag{3.5}$$

estimate (3.5) immediately implies estimate (3.4) if $A/N^\gamma \le 1/2$.

Note (see [3, Chap. 2, (1.3)]) that if one replaces balls B_k by (closed) binary cubes D_k and takes $(s(D_k))^t$ instead of $(r_k)^t$ in the definition of the Hausdorff content (see (1.3)), then the obtained value $m_t(\cdot) = \inf \sum_k (s(D_k))^t$ is commensurable with $M^t(\cdot)$. Consider a countable family of binary cubes D_k that covers $\bigcup_{\mathcal{Q}_N(Q)} ((7/4)Q_j \setminus X)$ and such that

$$m_{1+\gamma}\left(\bigcup_{\mathcal{Q}_N(Q)} (7/4)Q_j \setminus X\right) \le \sum_k (s(D_k))^{1+\gamma} \le 2m_{1+\gamma}\left(\bigcup_{\mathcal{Q}_N(Q)} (7/4)Q_j \setminus X\right);$$

it is enough to prove (3.5) with the replacement of the right-hand side by

$$A_1 \sum_k (s(D_k))^{1+\gamma}(s(Q))^{1-\gamma}.$$

Decompose the set $\mathcal{Q}_N(Q)$ into two disjoint parts as follows. A cube Q^1 from $\mathcal{Q}_N(Q)$ belongs to $\mathcal{Q}_N^1(Q)$ if and only if any cube D_k of the covering that intersects $2Q^1$ satisfies the condition $s(D_k) < s_0$. Clearly, the set $\bigcup_{\mathcal{Q}_N^1(Q)} ((7/4)Q_j \setminus X)$ is covered by the union of cubes D_k such that $s(D_k) < s_0$; in addition, since $s(D_k) < s(Q_j)$ for any such cube D_k, this cube intersects not more than eight different cubes in $\mathcal{Q}_N^1(Q)$. Hence,

$$\sum_{\mathcal{Q}_N^1(Q)} m_{1+\gamma}((7/4)Q_j \setminus X) \le 8 \sum_k (s(D_k))^{1+\gamma}. \tag{3.6}$$

For any cube Q^2 from $\mathcal{Q}_N^2(Q)$, the cube $2Q^2$ intersects at least one cube D_k of the covering with $s(D_k) \ge s_0$; hence, Q^2 is contained in $5D_k$. Formula (3.3) with $Q' = D_k$ and the inequality $s(D_k) \le s(Q)$ imply that

$$\sum_{\mathcal{Q}_N^2(Q)} (s_0)^{1-\gamma} M^{1+\gamma}((7/4)Q_j \setminus X) \le A_2 \sum_k (s(D_k))^{1+\gamma}(s(Q))^{1-\gamma}. \tag{3.7}$$

Inequalities (3.6) and (3.7) prove estimate (3.5) (and, hence, estimate (3.4)). Lemma 3.3 is proved. □

In what follows, we fix N according to Lemma 3.3. Let us perform a geometric construction that consists of two parts, basic and complementary.

Construction: Basic part. We perform induction on the number of a step. Before the first step, we have a set $\mathcal{Q}^{(0)}$ which consists of 27 binary cubes with side length δ; these cubes compose the cube $3\mathbf{Q}$. In the general case, before step n, where $n \in \mathbb{N}$, we have a definite set $\mathcal{Q}^{(n-1)}$ of binary cubes with side length $N^{1-n}\delta$.

In step n, we exclude from consideration all the cubes of the set $\mathcal{Q}^{(n-1)}$ that do not intersect the compact set $(13/8)\mathbf{Q} \setminus X^o$, fix all the "good" cubes in $\mathcal{Q}^{(n-1)}$, and construct the corresponding function $h_{2,1}$. We check the two conditions of stop formulated below. If at least one of these conditions is satisfied, the basic part of the construction is completed. Otherwise, we decompose any of cubes in $\mathcal{Q}^{(n-1)}$ that has not been fixed (every such cube is a "bad" cube that intersects the set $(13/8)\mathbf{Q} \setminus X^o$) into N^3 binary cubes of the same size with side length $N^{-n}\delta$; the set of obtained cubes is $\mathcal{Q}^{(n)}$.

Stop conditions.

(1) The set $\mathcal{Q}^{(n)}$ does not contain "bad" cubes that intersect $(13/8)\mathbf{Q} \setminus X^o$.

(2) The estimate $\delta^{2+\gamma}/2^n < \sum_j (s(Q_j))^{2+\gamma}$ holds, where the sum on the right is taken over all "good" cubes fixed previously (since the intersection of the compact set $(13/8)\mathbf{Q} \setminus X^o$ and the complement of X contains a nonempty open set, for n large enough, at least one "good" cube is fixed).

Thus, our construction stops in some step. When the basic part of the construction is completed, we get a covering of the compact set $(13/8)\mathbf{Q} \setminus X^o$ by cubes of the set $\mathcal{Q}^{(n)}$, where n is the number of the last step, and by all the "good" cubes fixed previously. We denote this covering by **Cover** (after the complementary part of our construction, the covering **Cover** is transformed to the desired covering **Cover1**).

As follows from our construction, the covering **Cover** satisfies the following estimates (3.8) and (3.9). First,

$$\sum_{\{j \mid Q_j \in \mathbf{Cover}\}} s(Q_j) M^{1+\gamma}((7/4)Q_j \setminus X) \le c_5(\gamma) \sum_k c_3^1 (h_{2,1})_k, \tag{3.8}$$

where the sum on the right is taken over all the constructed functions $h_{2,1}$ that correspond to "good" cubes of the covering **Cover**.

Indeed, for the sum over "good" cubes Q_j of the covering **Cover**, estimate (3.8) follows directly from condition (ii) of Lemma 3.2 (and we can replace $M^{1+\gamma}((7/4)Q_j \setminus X)$ in the left-hand side of (3.8) by $(s(Q_j))^{1+\gamma}$). For the sum over "bad" cubes of the covering **Cover**, the estimate follows from the second stop condition and estimate (3.4) since in any step, only "bad" cubes are divided, and the corresponding sum of $s(Q_j)M^{1+\gamma}((7/4)Q_j \setminus X)$ reduces at least by one half.

Further,

$$\sum_{\{j|Q_j\in\mathbf{Cover},\,Q_j\subset D\}} s(Q_j)M^{1+\gamma}((7/4)Q_j \setminus X) \le c_5(\gamma)(s(D))^{2+\gamma} \tag{3.9}$$

for an arbitrary cube D.

Indeed, let D be a binary cube with $s(D) \le \delta$ and let n be the number of the first step in which there exists a cube $Q \in \mathcal{Q}^{(n)}$ contained in D; then $s(D)/s(Q) \le N$, and if n is the number of the last step of the basic construction, then estimate (3.9) is obvious. If n is not the number of the last step, then estimate (3.9) follows from (3.4) by induction since only "bad" cubes are divided. For an arbitrary cube D, estimate (3.9) follows from a similar estimate for a binary cube whose side length does not exceed δ.

Remark 3.3. The sense of estimate (3.8) is as follows. If we take as G_3^2 the sum of all the functions $h_{2,1}$ constructed, then (after construction of the partition of unity by the covering **Cover**) it is easy to get item (2) of Lemma 3.1. The case of item (4) of Lemma 3.1 is more complicated; it is not enough to use estimate (3.9) in this case since the multiplicity of intersection for enlarged cubes $(7/4)Q_j$ in the covering **Cover** may be arbitrary. The goal of the complementary part of our construction is to control the above multiplicity of intersection by enlarging some of the cubes of the covering **Cover** and not to "spoil" estimates (3.8) and(3.9) too much (see Lemma 3.4).

Construction: Complementary part. Let δ_0 be the minimal side length of cubes of the covering **Cover** and let D be an arbitrary cube in **Cover**; we construct for D of finite set of binary cubes, $halo(D)$. This construction is similar to that in [7, Sec. 2.3].

Set $D^{(0)} = D$. Let $D^{(1)}$ be the set consisting of D and all the binary cubes with side length $s(D)/2$ that are not contained in D but are tangent to D. Similarly, for $m > 1$, the set $D^{(m)}$ consists of all the cubes from $D^{(m-1)}$ and of all the binary cubes with side length $s(D)/2^m$ that are not contained in cubes from $D^{(m-1)}$ but are tangent to such cubes. Set $halo(D) = D^{(m)}$, where m is the maximal number for which $s(Q) \ge \delta_0$ for any cube $Q \in D^{(m)}$. Let us indicate the following obvious properties of the set $halo(D)$:

(1) if cubes Q and Q' in $halo(D)$ are tangent, then the ratio $s(Q)/s(Q')$ takes one of the values 1/2, 1, or 2;

(2) $halo(D)$ contains D (if $s(D) = \delta_0$, in particular, if D is a "bad" cube, then $halo(D) = D$); all the cubes of the family $halo(D)$ are contained in $3D$;

(3) the sum of the values $(s(Q_j))^{2+\gamma}$ over all the cubes $Q_j \in halo(D)$ does not exceed $c_6(\gamma)(s(D))^{2+\gamma}$ (here it is important that $\gamma > 0$).

Exclude from the set of all the cubes $halo(D)$ for all $D \in$ **Cover** all the cubes that are not maximal with respect to inclusion or do not intersect the compact set $(13/8)\mathbf{Q} \setminus X^o$. The set of all remaining (not excluded) cubes forms a covering **Cover1** of the compact set $(13/8)\mathbf{Q} \setminus X^o$ which we need to prove Lemma 3.1. This completes the complementary part of our construction.

Let $\mathcal{Q}_b'$ be the set of all "bad" cubes of the covering **Cover**; the side length of any such cube is minimal and equals δ_0. Let $\mathcal{Q}_g'$ be the set of all "good" cubes of the covering **Cover**. By construction, the set **Cover1** consists of the following three disjoint subsets:

(1) $\mathcal{Q}_b$ *is a subset of* $\mathcal{Q}_b'$;

(2) $\mathcal{Q}_g$ *is a subset of* $\mathcal{Q}_g'$;

(3) $\mathcal{Q}_h =$ **Cover1** $\setminus (\mathcal{Q}_b \bigcup \mathcal{Q}_g)$; *any cube in* $\mathcal{Q}_h$ *is a subset of* $\bigcup_{\{D|D\in\mathcal{Q}_g'\}} halo(D)$.

Remark 3.4. The set $\bigcup_{\{D|D\in\mathcal{Q}_g'\}} halo(D)$ is covered by the set $\bigcup_{\{D|D\in\mathcal{Q}_g\}} halo(D)$. Indeed, if a cube $D_0 \in \mathcal{Q}_g' \setminus \mathcal{Q}_g$ is not maximal with respect to inclusion into the set $\bigcup_{\{D|D\in\mathbf{Cover}\}} halo(D)$, then it follows from the definition of $halo(\cdot)$ that no cube in $halo(D_0)$ is maximal with respect to inclusion. Hence, any cube in $\mathcal{Q}_g' \setminus \mathcal{Q}_g$ is covered by a cube from the set $\bigcup_{\{D|D\in\mathcal{Q}_g\}} halo(D)$, and any cube in $\mathcal{Q}_h$ belongs to $\bigcup_{\{D|D\in\mathcal{Q}_g\}} halo(D)$.

Lemma 3.4. *The following statements are valid:*

(1)

$$\sum_{\{j \mid Q_j \in \mathbf{Cover1}\}} \chi((9/4)Q_j) \le 24,$$

where $\chi(\cdot)$ *is the indicator;*

(2)

$$\sum_{\{j \mid Q_j \in \mathcal{Q}_h\}} (s(Q_j))^{2+\gamma} \le c_7(\gamma) \sum_{\{j \mid Q_j \in \mathcal{Q}'_g\}} (s(Q_j))^{2+\gamma};$$

(3)

$$\sum_{\{j \mid Q_j \in \mathcal{Q}'_g\}} (s(Q_j))^{2+\gamma} \le c_7(\gamma) \sum_{\{j \mid Q_j \in \mathcal{Q}_g\}} (s(Q_j))^{2+\gamma};$$

(4) *if* D *is an arbitrary binary cube, then*

$$\sum_{\{j \mid Q_j \in \mathcal{Q}_h,\, Q_j \subset D\}} (s(Q_j))^{2+\gamma} \le c_7(\gamma)(s(D))^{2+\gamma};$$

(5) *for the covering* **Cover1**, *estimates* (3.8) and (3.9) *are still valid while the constant* $c_5(\gamma)$ *may be somewhat larger than that for the covering* **Cover**.

Proof of Lemma 3.4. Estimate (1) follows from property (1) of the set $halo(D)$ and from the fact that the multiplicity of intersection of binary cubes of the same size does not exceed 8. Estimate (2) follows from property (3) of the set $halo(D)$.

Before proving estimates (3) and (4), we recall that due to formula (3.2), the values $M^{1+\gamma}((7/4)Q \setminus X)$ and $(s(Q))^{1+\gamma}$ are commensurable for a cube Q from $\mathcal{Q}'_g$ (and, hence, from $\mathcal{Q}_g$). Now it follows from inequality (3.9) that for any cube $D \in \mathcal{Q}_g$, the sum of the values $(s(Q_j))^{2+\gamma}$ over all the cubes $Q_j \in \mathcal{Q}'_g$ contained in $3D$ does not exceed $c_8(\gamma)(s(D))^{2+\gamma}$.

By Remark 3.4, any cube $Q \in \mathcal{Q}'_g \setminus \mathcal{Q}_g$ is contained in a cube belonging to $halo(D)$ for a proper $D \in \mathcal{Q}_g$; hence, by property (2) of $halo(D)$, the cube Q is contained in $3D$. This proves estimate (3).

Consider estimate (4). If a cube Q_j belongs to $halo(Q)$ for a proper cube $Q \in \mathcal{Q}'_g$, $Q \subset D$, then estimate (4) for this cube follows from the commensurability of the values $M^{1+\gamma}((7/4)Q \setminus X)$ and $(s(Q))^{1+\gamma}$ and from property (3) of the set $halo(Q)$.

By construction, any remaining cube Q_j belongs to the set $halo(Q)$ for a cube Q that is tangent to the boundary of D, and $s(Q) \le 2s(D)$. Clearly, the sum of the values $(s(Q))^{2+\gamma}$ over all such cubes Q does not exceed $A(s(D))^{2+\gamma}$, and estimate (4) follows from property (3).

Statement (5) of Lemma 3.4 follows from estimates (2)–(4). Indeed, formula (3.8), condition (ii) of Lemma 3.2, and statements (2) and (3) imply that

$$\sum_{\{j \mid Q_j \in \mathcal{Q}_h\}} (s(Q_j))^{2+\gamma} \le c_9(\gamma) \sum_{\{j \mid Q_j \in \mathcal{Q}_g\}} c_3^1(h_{2,1})_j,$$

which shows that relation (3.8) is preserved. Estimate (4) shows that relation (3.9) is preserved. Lemma 3.4 is proved. □

Now we are ready to complete the proof of Lemma 3.1.

Construct a partition of unity $\{\varphi_j\}$ corresponding to the covering $\{Q_j(a_j, s_j)\} = \mathbf{Cover1}$; let $\{g_j = V_{\varphi_j} g\}$ be the corresponding family of localizations. By inequality (3.1) and Lemma 2.5, for any function g_j there exists a function $\Psi_j \in \mathrm{Lip}^\gamma(\mathbb{R}^3)$ such that $\mathrm{Spt}\,(\Delta\Psi_j) \subset ((7/4)Q_j^o \setminus X)$, $\|\Psi_j\|_\gamma \le A\varepsilon(\delta)$, and $c_0(g_j) = c_0(\Psi_j)$; in addition, $\Psi = \Psi_j$, $Q = (7/4)Q_j^o$, and the differences $r = r_j \overset{def}{=} g_j - \Psi_j$ satisfy estimates of Lemma 2.6.

Take $G_3^1 = \sum_{\{j \mid Q_j \in \mathbf{Cover1}\}} \Psi_j$ and $\widetilde{G_3^2} = \varepsilon(\delta) \sum_{\{j \mid Q_j \in \mathcal{Q}_g\}} (h_{2,1})_j$. By the definition of the functions Ψ_j, conditions (i) and (iii) of Lemma 3.2, and Remark 3.2, statements (1) and (3) of Lemma 3.1 are valid for the functions G_3^1 and $\widetilde{G_3^2}$. By condition (ii) of Lemma 3.2, estimate (3.8), statement (5) of Lemma 3.4, and Remark 3.1, for G_3^1 and $G_3^2 = c_{10}(\gamma)\widetilde{G_3^2}$ with a proper $c_{10}(\gamma)$, statement (2) of Lemma 3.1 is valid.

It remains to prove statement (4) of Lemma 3.1; for this purpose, it is necessary and sufficient to establish the following estimates:

$$\begin{aligned}&\text{(a)}\quad \Big\| \sum_{\{j \mid Q_j \in \mathbf{Cover1}\}} r_j \Big\|_\gamma \le c_{11}(\gamma)\varepsilon(\delta),\\ &\text{(b)}\quad \Big\| \sum_{\{j \mid Q_j \in \mathcal{Q}_g\}} (h_{2,1})_j \Big\|_\gamma \le c_{11}(\gamma).\end{aligned} \tag{3.10}$$

We prove estimates (3.10) using the same reasoning as that applied at the end of Sec. 2 to deduce Lemma 1.1 from Lemma 2.7. We prove estimate (3.10) (a) using Lemmas 2.5 and 2.6, estimate (3.9), and statements (1) and (5) of Lemma 3.4.

Fix arbitrary $x, y \in \mathbb{R}^3$, $x \ne y$.

(1) Let P_1 be the set of all cubes $Q_j \in \mathbf{Cover1}$ for which the cube $(9/4)Q_j$ intersects the set $\{x\} \cup \{y\}$. By statement (1) of Lemma 3.4, the number of cubes in P_1 is bounded from above by an absolute constant; thus, the sum over all cubes from P_1 satisfies estimate (3.10) (a) by Lemmas 2.2 and 2.5.

Note that, for any cube $Q_j \in \mathbf{Cover1} \setminus P_1$, the distance from any point $a_j' \in 2Q_j$ to x (to y) is commensurable with the distance from the center a_j of the cube Q_j to x (respectively, to y), and outside $2Q_j$, the functions r_j satisfy estimates (2.9).

Now (2.9), (3.9), and statement (5) of Lemma 3.4 imply the following. Let $O_k(x)$ be the set of all cubes Q_j in $\mathbf{Cover1} \setminus P_1$ such that the distance from a_j to x belongs to the segment $[2^k, 2^{k+1}]$, where $k \in \mathbb{Z}$. Then

$$\sum_{\{j \mid Q_j \in O_k(x)\}} |r_j(x)| \le A\varepsilon(\delta)2^{-2k} \sum_{\{j \mid Q_j \in O_k(x)\}} s_j M^{1+\gamma}((7/4)Q_j \setminus X) \le A_1 c_5(\gamma)\varepsilon(\delta)2^{k\gamma}, \tag{3.11}$$

and a similar estimate is valid for the sum of $|r_j(y)|$ over the set $O_k(y)$ which is defined in a similar way.

(2) Find the number $k_0 \in \mathbb{Z}$ such that $2^{k_0} \le |x-y| < 2^{k_0+1}$. Let P_2 be the set of all cubes $Q_j \in \mathbf{Cover1} \setminus P_1$ for which the estimate $\min(|a_j - x|, |a_j - y|) \le 10|x-y|$ is valid. In this case, $\max(|a_j - x|, |a_j - y|) \le 11|x-y|$, and it follows from formula (3.11) that the sum over indices j of cubes from P_2 (and, hence, over the corresponding k) can be estimated as follows:

$$\sum_j |r_j(x) - r_j(y)| \le \sum_j (|r_j(x)| + |r_j(y)|) \le A_2 c_5(\gamma)\varepsilon(\delta) \sum_{\{k \mid k \le k_0\}} 2^{k\gamma} \le c_{12}(\gamma)\varepsilon(\delta)|x-y|^\gamma.$$

(3) Let $P_3 = \mathbf{Cover1} \setminus (P_1 \cup P_2)$. The same reasoning as in the proof of inequality (3.11) shows that

$$\sum_{\{j \mid Q_j \in O_k(x)\}} |\nabla r_j(x)| \le A_2 c_5(\gamma)\varepsilon(\delta)2^{k(\gamma-1)}. \tag{3.12}$$

Taking into account the inequality $\min(|a_j - x|, |a_j - y|) > 10|x-y|$, we get the following estimate for the sum over indices j of cubes from P_3:

$$\begin{aligned}\sum_j |r_j(x) - r_j(y)| &\le A_3|x-y| \sum_j (|\nabla r_j(x)| + |\nabla r_j(y)|)\\ &\le A_4|x-y| c_5(\gamma)\varepsilon(\delta) \sum_{\{k \mid k \ge k_0\}} 2^{k(\gamma-1)} \le c_{12}(\gamma)\varepsilon(\delta)|x-y|^\gamma.\end{aligned}$$

This proves estimate (3.10) (a).

A similar (and even simpler) reasoning proves estimate (3.10) (b). We take into account that, due to conditions (i)–(iv) of Lemma 3.2 and the conditions of Lemma 2.3 with $n = 1$, the functions $(h_{2,1})_j$ satisfy the estimate $\|(h_{2,1})_j\|_\gamma \le A$, the estimates

$$|(h_{2,1})_j(x)| \le A\frac{(s_j)^{2+\gamma}}{|x-a_j|^2} \quad \text{and} \quad |\nabla(h_{2,1})_j(x)| \le A\frac{(s_j)^{2+\gamma}}{|x-a_j|^3}$$

hold outside the cubes $2Q_j$, and formula (3.9) implies the estimate

$$\sum_{\{j \mid Q_j \in \mathcal{Q}_g,\, Q_j \subset D\}} (s_j)^{2+\gamma} \le c_5(\gamma)(s(D))^{2+\gamma}$$

for an arbitrary cube D. This completes the proof of Lemma 3.1 (and of Lemma 1.1). □

4. Proof of Theorem 1

Theorem 1 follows from Lemmas 4.2 and 4.3. In Lemma 4.2, we prove the necessity of estimate (1.4) for the inclusion $f \in H_\gamma(X)$; in Lemma 4.3, we show that estimate (1.4) implies the conditions of Lemma 1.1. Since Lemma 1.1 has been proved above, this completes the proof of Theorem 1. In the proof of Theorem 1, we use some facts established in [9, Sec. 4].

First we show (see formula (4.4)) that, in fact, estimate (1.4) is a particular case of estimate (1.5) for functions ψ of simple radial structure. Let

$$\psi_1^0(x) = \frac{15}{8\pi}\chi_{B(0,1)}(x)(1-|x|^2) \text{ and } \psi_r^a(x) = r^{-3}\psi_1^0\left(\frac{x-a}{r}\right), \tag{4.1}$$

where $\chi_{(\cdot)}$ is the indicator, $a \in \mathbb{R}^3$, and $r > 0$; the constant $15/(8\pi)$ implies the norming $\int \psi_r^a(x)\, dm_x = 1$.

Note that $\psi_r^a \notin C^\infty(\mathbb{R}^3)$ (though ψ_r^a belongs to the class $\mathrm{Lip}^1(\mathbb{R}^3)$). Nevertheless, using the Green formula, one can correctly define the localizations $E * (\psi_r^a \Delta h)$ for $h \in C^\gamma(\mathbb{R}^3)$ (as well as for $h \in C(\mathbb{R}^3)$).

First we assume that $h \in C^2(\mathbb{R}^3)$. The Green formula for harmonic functions (see, for example, [15, Sec. 6, Subsec. 5, formula (29)]) implies (for $B = B(a,r)$) that

$$\int_B (\psi_r^a(x)\Delta h(x) - h(x)\Delta\psi_r^a(x))\, dm_x = \int_{\partial B}\left(\psi_r^a(x)\frac{\partial h}{\partial n}(x) - h(x)\frac{\partial \psi_r^a}{\partial n}(x)\right) ds_x. \tag{4.2}$$

Here we take into account that $\psi_r^a \equiv 0$ for $x \notin B$; the values of the normal derivative $\dfrac{\partial \psi_r^a}{\partial n}$ on ∂B are calculated from the interior of B. Since the function ψ_r^a is continuous in $\mathbb{R}^3$ and vanishes on ∂B, the term containing the normal derivative of h on ∂B vanishes as well; simple calculations show that

$$\int_B \psi_r^a(x)\Delta h(x)\, dm_x = \int_B h(x)\Delta\psi_r^a(x)\, dm_x - \int_{\partial B} h(x)\frac{\partial\psi_r^a}{\partial n}(x)\, ds_x = 15r^{-2}\left(h_{\mathrm{av}}(\partial B) - h_{\mathrm{av}}(B)\right). \tag{4.3}$$

Recall that we denote by $\langle \Psi | \varphi \rangle$ the action of a distribution Ψ with compact support on a function $\varphi \in C^\infty(\mathbb{R}^3)$; in particular, the left-hand side of inequality (1.5) (taking integration by parts into account) contains the action

$$\langle \psi\Delta f | 1\rangle = \int_Q f(x)\Delta\psi(x)\, dm_x.$$

It is natural to take the value of the integral $\int_B \psi_r^a(x)\Delta h(x)\, dm_x$ as the action $\langle \psi_r^a \Delta h | 1\rangle$ for $h \in C^2(\mathbb{R}^3)$; by equality (4.3),

$$\langle \psi_r^a \Delta h | 1\rangle = 15r^{-2}\left(h_{\mathrm{cp}}(\partial B) - h_{\mathrm{cp}}(B)\right). \tag{4.4}$$

Formula (4.4) allows us to extend the action $\langle \psi_r^a \Delta h | 1\rangle$ to functions $h \in C^\gamma(\mathbb{R}^3)$, $0 < \gamma < 1$.

We define the localizations $E * (\psi_r^a \Delta h)$ for functions $h \in C^2(\mathbb{R}^3)$ according to (4.2) and (4.3):

$$E * (\psi_r^a \Delta h)(x) = \int_B h(y)\Delta_y\left(\psi_r^a(y)E(y-x)\right) dm_y - \int_{\partial B} h(y)E(y-x)\frac{\partial\psi_r^a}{\partial n}(y)\, ds_y. \tag{4.5}$$

Since $\psi_r^a \equiv 0$ on ∂B, the integral containing the normal derivative of $E(y-x)$ on ∂B vanishes. Calculating the Laplacian $\Delta_y\left(\psi_r^a(y)E(y-x)\right)$, we take into account that E is the fundamental solution, i.e., that

$$\Delta_y\left(\psi_r^a(y)E(y-x)\right) = \psi_r^a(x) + E(y-x)\Delta_y\psi_r^a(y) + 2\nabla_y\psi_r^a(y)\nabla_y E(y-x).$$

To show that formula (4.5) preserves sense for $h \in C^\gamma(\mathbb{R}^3)$, $0 < \gamma < 1$ (and for $h \in C(\mathbb{R}^3)$), we pass to the limit and take into account that the right-hand side of (4.5) contains singular integrals with summable kernels while derivatives of h are absent.

Lemma 4.1. *Let $h \in C^\gamma(\mathbb{R}^3)$ and let $B = B(a,r)$ be an open ball. Then the following statements are valid:*

(a) *the function $E * (\psi_r^a \Delta h)$ is harmonic outside the compact set*

$$\overline{B(a,r)} \bigcap \mathrm{Spt}\,(\Delta h);$$

(b) $E * (\psi_r^a \Delta h) \in C^\gamma(\mathbb{R}^3)$ *and* $\|E * (\psi_r^a \Delta h)\|_\gamma \le A\epsilon_{h,\gamma}(r) r^{-3}$, *where* $\epsilon_{(\cdot),\gamma}$ *is from* (1.2), *and* $\lim\limits_{x\to\infty} E * (\psi_r^a \Delta h)(x) = 0$;

(c) *the Laurent decomposition of the function* $E * (\psi_r^a \Delta h)$ *(see* (2.5)*) satisfies the equality* $c_0(E * (\psi_r^a \Delta h)) = 15r^{-2}\,(h_{\mathrm{cp}}(\partial B) - h_{\mathrm{cp}}(B))$.

Proof. Statement (a) is obvious for $h \in C^2(\mathbb{R}^3)$; the passage to the case $h \in C^\gamma(\mathbb{R}^3)$ is performed by a standard regularization (by convolution of h with C_0^∞-functions of an approximative unity). Statement (b) is obtained from formula (4.5) by a direct (and simple) calculation in which we take into account the equality $\Delta h = \Delta(h - h(a))$. To prove statement (c), we compare the right-hand side of formula (4.3) and the coefficient at $E(y-x)$ in (4.5). The lemma is proved. □

The following statement is known (see, for example, [13, Theorem 5.2] and [14, Lemma 3.1]). Let $U \subset \mathbb{R}^3$ be a Borel set and let $h \in \mathrm{Lip}^\gamma(\mathbb{R}^3)$ be a function such that $\mathrm{Spt}\,(\Delta h) \subset U$. Then

$$|\langle \Delta h | 1 \rangle| \le A\|h\|_\gamma M^{1+\gamma}(U). \tag{4.6}$$

This estimate and Lemma 4.1 imply the following statement.

Lemma 4.2. *Let* $f \in H_\gamma(X)$, *then, for any open ball* B *of radius* r, *estimate* (1.4) *holds with* $k = 1$ *and* $\epsilon(r) = A\epsilon_{f,\gamma}(r)$.

Proof. Let $f \in H_\gamma(X)$. Then for any $\varepsilon_1 > 0$ we can find a function F and a neighborhood of X such that if X_1 is the closure of this neighborhood, then $\|f - F\|_\gamma(X_1) < \varepsilon_1$, and $\Delta F = 0$ on X_1. We apply the Whitney theorem (see [2, Chap. 6, Theorem 3]) to continue the difference $f - E$ to the space $\mathbb{R}^3$ by a finitary function from $C^\gamma(\mathbb{R}^3)$ that satisfies the inequality $\|f - F\|_\gamma < A_1\varepsilon_1$. Since ε_1 is arbitrary, we may assume that $\varepsilon_1 \le \epsilon_{f,\gamma}(r)$.

Lemma 4.1 implies that

$$E * (\psi_r^a \Delta F) \in C^\gamma(\mathbb{R}^3) \quad \text{and} \quad \mathrm{Spt}\Delta(E * (\psi_r^a \Delta F)) \subset (kB \setminus X)$$

for any $k > 1$, and the estimate

$$\|E * (\psi_r^a \Delta F)\|_\gamma \le A\epsilon_{f,\gamma}(r) r^{-3}$$

holds.

We apply (4.6), (2.5), and equality (c) of Lemma 4.1 to show that

$$15r^{-2}\,|F_{\mathrm{AV}}(\partial B) - F_{\mathrm{AV}}(B)| \le A\epsilon_{f,\gamma}(r) r^{-3} \inf_{k>1} M^{1+\gamma}(kB \setminus X).$$

Since $\lim\limits_{t\to k} |F_{\mathrm{AV}}(\partial(tB)) - F_{\mathrm{AV}}(tB)| = |F_{\mathrm{AV}}(\partial(kB)) - F_{\mathrm{AV}}(kB)|$,

$$15\,|F_{\mathrm{AV}}(\partial B) - F_{\mathrm{AV}}(B)| \le A\epsilon_{f,\gamma}(r) r^{-1} M^{1+\gamma}(B \setminus X).$$

Since ε_1 is arbitrary, estimate (1.4) follows from the estimate $\|f - F\|_\gamma < A_1\varepsilon_1$. Lemma 4.2 is proved. □

In what follows, we assume without loss of generality that the function ϵ in (1.4) is chosen to satisfy the inequality $\epsilon(t) \ge \epsilon_{f,\gamma}(t)$ for $t > 0$.

Lemma 4.3. *If estimate* (1.4) *holds for a function* f *and any open ball* B, *then estimate* (1.5) *holds for any cube* Q *and function* $\psi \in C^\infty(\mathbb{R}^3)$ *such that* $\mathrm{Spt}\,(\psi) \subset Q$.

Proof of Lemma 4.3. We assume that $M^{1+\gamma}((9/8)Q \setminus X) > 0$; otherwise, $\Delta f = 0$ in a neighborhood of the cube Q, and both sides of (1.5) are zero. It is also clear that we may assume that $k \ge 1$ in (1.4).

Cover the cube Q by a net of binary cubes $Q_j = Q_j(a_j, \delta)$ of the same size so that $M^{1+\gamma}((9/8)Q \setminus X) < \delta^{1+\gamma} \le 4M^{1+\gamma}((9/8)Q \setminus X)$. Let $B_j = B(a_j, 2\delta)$.

Remark 4.1. In what follows, we assume that all the balls $(4+k)B_j$ are contained in $(9/8)Q$; otherwise, the value δ is commensurable with $s = s(Q)$, and estimate (1.5) becomes a corollary of the obvious estimate

$$\left|\int_Q f(x)\Delta\psi(x)\,dm_x\right| \le \int_Q |(f(x) - f(a))\Delta\psi(x)\,dm_x| \le A\epsilon_{f,\gamma}(s)\|\nabla^2\psi\|_{\mathrm{L}_\infty} s^{3+\gamma}$$

(where a is the center of the cube Q).

Apply a construction similar to that of the paper [10] to introduce a special partition of unity $\{\varphi_{3,j}\}$ on $\bigcup_j Q_j$. Set $\psi_{3,\delta}^0 = \psi_\delta^0 * \psi_\delta^0 * \psi_\delta^0$. Clearly, $\psi_{3,\delta}^0(x) \ge 0$, $\mathrm{Spt}\,(\psi_{3,\delta}^0) \subset B(0, 3\delta)$, and $\int \psi_{3,\delta}^0(x)\,dm_x = 1$ due to the norming.

Let $\{\varphi_j\}$ be a partition of unity on $\bigcup_j Q_j$ given by Lemma 2.1. Recall that $\varphi_j \in C_0^\infty(B_j)$ (i.e., $\varphi_j \in C^\infty(\mathbb{R}^3)$ and $\mathrm{Spt}\,(\varphi_j) \subset B_j$) and $\|\nabla^n \varphi_j\|_{\mathrm{L}_\infty} \le A(n)\delta^{-n}$, $n \in \mathbb{Z}_+$. Using the functions $\psi^0_{3,\delta}$, we get a new partition of unity $\{\varphi_{3,j}\} = \{\psi^0_{3,\delta} * \varphi_j\}$; in this case, $\varphi_{3,j} \in C_0^\infty(4B_j)$ and $\|\nabla^n \varphi_{3,j}\|_{\mathrm{L}_\infty} \le A_1(n)\delta^{-n}$. We apply the new partition of unity to guarantee estimate (4.7) (see also Remark 4.2).

We refer to the following statements (i) and (ii) proved in [9, Sec. 4].

(i) Let α be a multiindex, $|\alpha| \le 2$, and let $\Psi_{\alpha,j}$ be a solution of the convolution equation

$$(x-a_j)^\alpha \varphi_{3,j} = \psi^0_\delta * \Psi_{\alpha,j}.$$

Then $\Psi_{\alpha,j} \in C_0^\infty(B(a_j, 3\delta))$, and the estimate $\|\Psi_{\alpha,j}\|_{\mathrm{L}_\infty} \le A\delta^{|\alpha|}$ holds (see [9, Lemma 4.2] and also [10, Lemma 2.5]); hence, $\|\Psi_{\alpha,j}\|_{\mathrm{L}_1} \le A_1\delta^{3+|\alpha|}$.

(ii) The equality

$$\langle \Delta f | \psi^0_\delta * \Psi_{\alpha,j}\rangle = \int \Psi_{\alpha,j}(y)\, dm_y \langle \psi^y_\delta \Delta f | 1\rangle$$

holds (see [9, Lemma 4.1]; in the formula above, we use the notation of (4.1) and (4.4)).

Statements (i) and (ii) and inequality (1.4) directly imply the following estimate (where $|\alpha| \le 2$):

$$\begin{aligned} |\langle \varphi_{3,j}\Delta f | (x-a_j)^\alpha\rangle| = \big|\langle \Delta f | \psi^0_\delta * \Psi_{\alpha,j}\rangle\big| &\le A_1\delta^{3+|\alpha|}\epsilon(\delta)\delta^{-3} \sup_{|y-a_j|\le 3\delta} M^{1+\gamma}(B(y,k\delta)\setminus X) \\ &\le A_1\epsilon(\delta)\delta^{|\alpha|} M^{1+\gamma}((4+k)B_j \setminus X). \end{aligned} \tag{4.7}$$

We apply estimate (4.7) to complete the proof of Lemma 4.3. Using the partition of unity $\{\varphi_{3,j}\}$ and the Taylor formula, we conclude that

$$\begin{aligned} \int_Q f(x)\Delta\psi(x)\,dm_x &= \langle \psi\Delta f | 1\rangle = \sum_j \langle \varphi_{3,j}\Delta f | \psi\rangle = \sum_j \psi(a_j)\langle \Delta f | \varphi_{3,j}\rangle \\ &+ \sum_j \sum_{\{\alpha | 1 \le |\alpha| \le 2\}} \frac{\partial^\alpha \psi(a_j)}{\alpha!} \langle \varphi_{3,j}\Delta f | (x-a_j)^\alpha\rangle + \sum_j \langle \Delta f | \varphi_{3,j} R_{3,j}\rangle, \end{aligned} \tag{4.8}$$

where $\partial^\alpha R_{3,j}(a_j) = 0$ for $|\alpha| \le 2$ and $\partial^\alpha R_{3,j} \equiv \partial^\alpha \psi$ for $|\alpha| = 3$. Hence, the estimates $|\partial^\alpha R_{3,j}(x)| \le A\delta^{3-|\alpha|}\|\nabla^3\psi\|_{\mathrm{L}_\infty}$ hold for $x \in \mathrm{Spt}(\varphi_{3,j}) \subset B(a_j, 4\delta)$ and $|\alpha| \le 2$. Taking into account estimates of derivatives of the functions $\varphi_{3,j}$, we conclude that $|\Delta(\varphi_{3,j}R_{3,j})(x)| \le A\delta\|\nabla^3\psi\|_{\mathrm{L}_\infty}$.

Recall that $M^{1+\gamma}((9/8)Q\setminus X) < \delta^{1+\gamma} \le 4M^{1+\gamma}((9/8)Q\setminus X)$. To complete the proof of Lemma 4.3, it remains to show that any sum on the right in formula (4.8) does not exceed

$$\varepsilon(\delta)\|\nabla^3\psi\|_{\mathrm{L}_\infty} s^3\delta^{1+\gamma},$$

where the function ε depends on ϵ, k, and $\epsilon_{f,\gamma}$.

First we consider the expression $\sum_j \langle \Delta f | \varphi_{3,j}R_{3,j}\rangle$. Integrating by parts and taking into account that the number of summation indices j does not exceed $A(s/\delta)^3$, we conclude that

$$\Big|\sum_j \langle \Delta f | \varphi_{3,j}R_{3,j}\rangle\Big| \le A_1\omega_f(\delta)\delta^3\|\nabla^3\psi\|_{\mathrm{L}_\infty}\delta\left(\frac{s}{\delta}\right)^3 \le A_1\epsilon_{f,\gamma}(\delta)\|\nabla^3\psi\|_{\mathrm{L}_\infty} s^3\delta^{1+\gamma}$$

(where ω_f is the continuity modulus of the function f).

Remark 4.2. For a similar sum for the functions $R_{2,j}$ that determine the remainder of the Taylor formula of the first order, we get an essentially weaker estimate $A_1\epsilon_{f,\gamma}(\delta)\|\nabla^2\psi\|_{\mathrm{L}_\infty} s^3\delta^\gamma$; this explains the introduction of the convolutions $\psi^0_{3,\delta}$.

Due to inequality (4.8), the remaining terms in the right-hand side of formula (4.7) satisfy the estimates

$$\Big|\sum_j \psi(a_j)\langle \Delta f | \varphi_{3,j}\rangle\Big| \le A_1\epsilon(\delta)\sum_j M^{1+\gamma}((4+k)B_j \setminus X)$$

and

$$\Big|\sum_j \sum_{\{\alpha | 1 \le |\alpha| \le 2\}} \frac{\partial^\alpha\psi(a_j)}{\alpha!}\langle \varphi_{3,j}\Delta f | (x-a_j)^\alpha\rangle\Big| \le A_2\epsilon(\delta)\|\nabla^2\psi\|_{\mathrm{L}_\infty} s^2 \sum_j M^{1+\gamma}((4+k)B_j\setminus X).$$

Thus, to complete the proof of Lemma 4.3 it remains to establish the following estimate:

$$\sum_j M^{1+\gamma}((4+k)B_j \setminus X) \le A(k)\delta^{1+\gamma}. \tag{4.9}$$

Consider a countable family of balls D_ι of radii r_ι covering the set $(9/8)Q \setminus X$ and such that $\sum_\iota (r_\iota)^{1+\gamma} \le 2M^{1+\gamma}((9/8)Q \setminus X) < 2\delta^{1+\gamma}$. It was noted in Remark 4.1 that all the balls $(4+k)B_j$ are contained in $(9/8)Q$; hence, the family $\{D_\iota\}$ covers the set $\bigcup_j((4+k)B_j \setminus X)$. Hence, the left-hand side of inequality (4.9) does not exceed $2N\delta^{1+\gamma}$, where N is the maximal number of balls $(4+k)B_j$ that can intersect a fixed ball D_ι.

Recall that $B_j = B(a_j, 2\delta)$ for disjoint binary cubes $Q_j(a_j, \delta)$. Clearly, $r(D_\iota) < 2\delta$ for all ι; hence $2N \le A(k)$. This proves estimate (4.9).

Thus, Lemma 4.3 (and Theorem 1) is proved. □

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