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J-CLOSED FINITE COLLECTIONS OF HARDY-TYPE SUBSPACES

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Several proofs of the following statement are given: If $X^0, \dots, X^n$ are BMO*-regular lattices on the circle and $x \in X^0 \cap \dots \cap X^n$, then the distances from x to the Hardy-type subspaces X^j_A are roughly attained at one and the same element of $\bigcap_j X^j_A$. Bibliography: 9 titles*

1. Introduction

Let $(X_0, X_1, \dots, X_n)$ be a compatible collection of quasi-Banach spaces. We recall that "compatible" means that each space X_j is included in one and the same topological linear space. Let $Y_0, Y_1, \dots, Y_n$ be closed subspaces of $X_0, X_1, \dots, Y_n$, respectively.

Definition 1. *The collection $(Y_0, \dots, Y_n)$ is said to be K-*closed *in $(X_0, \dots, X_n)$ if for any $y \in Y_0 + \dots + Y_n$ and any decomposition $y = x_0 + \dots + x_n$, $x_j \in X_j$, there exists another decomposition $y = y_0 + y_1 + \dots + y_n$ such that $y_j \in Y_j$ and $\|y_j\|_{Y_j} \le C\|x_j\|_{X_j}$, $j = 0, \dots, n$, where C does not depend on the vectors involved.*

Definition 2. *The collection $(Y_0, \dots, Y_n)$ is said to be J-*closed *in $(X_0, \dots, X_n)$ if for $x \in \bigcap_{j=0}^{n} X_j$ and any positive numbers $c_j > dist_{X_j}(x, Y_j)$ there exists a $y \in \bigcap_{j=0}^{n} Y_j$ such that $\|x - y\|_{X_j} < Cc_j$, $j = 0, \dots, n$, where C does not depend on x and the numbers c_j.*

The phenomenon of K-closedness has been studied in considerable detail during the last 25 years. In particular, it was shown that this phenomenon occurs in many classical situations. A principal example of K-closedness is related to Hardy-type subspaces in Banach lattices. If X is a quasinormed lattice of measurable functions on the circle that is subordinate to a certain condition of nondegeneracy (condition (*) in [1, 2] or property **B** below), we can consistently introduce an "analytic space" X_A in X by defining $X_A = X \cap N_+$, where N_+ is Smirnov's class (see [3]). Often, such spaces are called Hardy-type spaces because when $X = L^p$, X_A becomes the classical Hardy space H^p. We also need a definition of a BMO-regular lattice. We refer the reader to examples in [2] which show that classical lattices of measurable functions are BMO-regular.

Definition 3. *A quasi-Banach lattice X of measurable functions on the circle is said to be* BMO*-regular if for any $f \in X, f \ne 0$, there exists a $g \in X$ such that $g \ge |f|$, $\|g\|_X \le m\|f\|_X$, and $\|\log g\|_{BMO} \le C$ (m and C do not depend on f). The function g is called a* BMO*-majorant for f.*

A lattice X has property **B** if for any nonzero $f \in X$ there exists a $g \in X$, $g \ge |f|$, such that $\|g\| \le c\|f\|$ and $\log g \in L^1(\mathbb{T})$. It was shown in [4] that the collection $(X^0_A, \dots, X^n_A)$ is certainly K-closed if $(X^0, \dots, X^n)$ is a finite collection of BMO-regular lattices of measurable functions on the unit circle.

In the case of couples of spaces, notions of K-closedness and J-closedness are equivalent in the general situation. Basically, the proof reduces to the equivalence of the identities $a + b = u + v$ and $a - u = v - b$. However, in [5], this fact is called "the main lemma of interpolation theory" because the equivalence of K- and J-methods for couples of spaces is a consequence of it. The symbols K and J in the definition of K- and J-closedness arose because of the methods mentioned; however, we leave this apart because the notions have a straightforward geometrical meaning.

Apparently, for collections of n subspaces, where $n \ge 2$, the notions of K- and J-closedness are not equivalent any more. However, one can ask whether $(X^0_A, \dots, X^n_A)$ is J-closed if $X^0, \dots, X^n$ are BMO-regular lattices of measurable functions on the circle. The author is aware that N. Ya. Kruglyak was interested in this question.

In the present note, we give a positive answer to this question. It turns out that the situation is fairly simple provided that we use properly the information about K-closedness of Hardy-type spaces. We give three different proofs of this statement in three sections below, each of them contains additional information.

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2. Relationship between K- and J-closedness

Such a relationship still exists even though "the main lemma of interpolation theory" does not work if the number of spaces in the collection is more than 2. Let $c_j > 0$, $j = 0, \dots, n$. Along with norms in the spaces X_j, we consider the norms $c_j^{-1}\|\cdot\|_{X_j}$. Accordingly, the intersection $\bigcap\limits_{j=0}^{n} X_j$ acquires a norm related to the constants $c_0, \dots, c_n$: $|\|x\||= \max\limits_{0\le j\le n} c_j^{-1}\|x\|_{X_j}$. These normings of the spaces X_j and their "descendants" (for instance, intersections) will be called standard normings.

Lemma 1. *Let $(X_0, \dots, X_n)$ be a compatible collection of quasi-Banach spaces and let Y_j be closed subspaces of X_j for $j = 0, \dots, n$. The following statements are equivalent.*

(i) *For any standard norming of the X_j, the couple $\left(\bigcap\limits_{j\in e_1} Y_j, \bigcap\limits_{j\in e_2} Y_j\right)$ is K-closed in $\left(\bigcap\limits_{j\in e_1} X_j, \bigcap\limits_{j\in e_2} X_j\right)$ for any nonempty $e_1, e_2 \subset \{0, 1, \dots, n\}$, $e_1 \cup e_2 = \{0, 1, \dots, n\}$, and the constants in the definition of K-closedness do not depend on the norming.*

(ii) *$(Y_0, Y_1, \dots, Y_n)$ is J-closed in $(X_0, X_1, \dots, X_n)$.*

Proof. (i) $\Rightarrow$ (ii). We prove this by induction on n. As already has been mentioned, the implication is well known for $n = 2$. Assume that it is true for $n-1$ and let $x \in \bigcap\limits_{j=0}^{n} X_j$ and $\operatorname{dist}_{X_j}(x, Y_j) < c_j$, $j = 0, \dots, n$. By the inductive hypothesis, there exists a $y \in \bigcap\limits_{j=0}^{n-1} Y_j$ such that $\operatorname{dist}_{X_j}(x, y) < Cc_j$, $j = 0, \dots, n-1$. This means that if we introduce the standard norm in the intersection $X = \bigcap\limits_{j=0}^{n-1} X_j$ that corresponds to the constants $c_0, \dots, c_{n-1}$ and set $Y = \bigcap\limits_{j=0}^{n-1} Y_j$, then $\operatorname{dist}_X(x, Y) < C$ (thereby, $\operatorname{dist}_{X_n}(x, Y_n) < c_n$). However, the couple (Y, Y_n) is K-closed in (X, X_n) by assumption. Therefore, this couple is J-closed, and we are done.

(ii) $\Rightarrow$ (i). Since K-closedness and J-closedness are equivalent for couples, it suffices to prove that for any standard norming, the couple $\left(\bigcap\limits_{j\in e_1} Y_j, \bigcap\limits_{j\in e_2} Y_j\right)$ is J-closed in $\left(\bigcap\limits_{j\in e_1} X_j, \bigcap\limits_{j\in e_2} X_j\right)$ for any nonempty $e_1, e_2 \subset \{0, 1, \dots, n\}$, $e_1 \cup e_2 = \{0, 1, \dots, n\}$. Consider the standard norming that corresponds to the constants $c_0, \dots, c_n$. Let

$$f \in \left(\bigcap_{j\in e_1} X_j\right) \cap \left(\bigcap_{j\in e_2} X_j\right) = \bigcap_{j=0}^{n} X_j.$$

By our assumption, there exists a $y \in \bigcap\limits_{j=0}^{n} Y_j$ such that $\|f-y\|_{X_j} \le C \operatorname{dist}_{X_j}(f, Y_j)$. Hence,

$$\|f-y\|_{\bigcap\limits_{e_i} X_j} = \max_{j\in e_i}\left\{\frac{\|f-y\|_{X_j}}{c_j}\right\} \le C\max_{j\in e_i}\left\{\frac{\inf\limits_{y\in Y_j}\|f-y\|_{X_j}}{c_j}\right\} \le C\max_{j\in e_i}\left\{\frac{\inf\limits_{y\in \bigcap\limits_{j\in e_i} Y_j}\|f-y\|_{X_j}}{c_j}\right\}$$

$$\le C \inf_{y\in \bigcap\limits_{j\in e_i} Y_j}\max_{j\in e_i}\left\{\frac{\|f-y\|_{X_j}}{c_j}\right\} = C\operatorname{dist}_{\bigcap\limits_{e_i} X_j}\left(f, \bigcap_{e_i} Y_j\right), \quad \text{where} \quad i = 0, 1. \qquad \square$$

Now, we can easily get the first proof of this note.

Theorem 1. *Let $X^0, \dots, X^n$ be* BMO*-regular quasi-Banach lattices of measurable functions on the circle. The collection $X_A^0, \dots, X_A^n$ is J-closed in $X^0, \dots, X^n$.*

Proof. The BMO regularity condition is preserved with the same constants in the estimates if we multiply the norm in the lattice by a positive constant (we recall that a constant function has zero norm in BMO). Next, it is well known and can easily be verified that the pointwise maximum of two functions that lie in BMO also belongs to BMO. Thus, for any set $e \subset \{0,1,\dots,n\}$, the lattice $\bigcap\limits_{j\in e} X_j$ is BMO-regular in any standard norming with constants that do not depend on this norming. It is well known that the couple (X_A, Y_A) is K-closed in (X,Y) for every BMO regular lattices X and Y (see [2,6,7]). Moreover, the constant in the definition of K-closedness depends only on the constants in the property of BMO-regularity for X and Y. Thus, it remains to refer to Lemma 1. □

3. Partial retractions

In this section, we present another abstract result which allows us to prove Theorem 1 in a different way. Let $(X_0, X_1, \dots, X_n)$ be a compatible collection of Banach spaces. Let Y_j be closed subspaces in X_j, $j = 0, \dots, n$.

Definition 4. *A collection* $(Y_0, \dots, Y_n)$ *is a partial retract of* $(X_0, \dots, X_n)$ *if for any* $f \in \sum\limits_{j=0}^{n} Y_j$ *there exists a linear operator* $T : X_j \to Y_j$ *such that* $Tf = f$ *and* $\|T\|_{X_j \to Y_j} \le c$ *for any* $j = 0, \dots, n$, *where* c *does not depend on* f.

Lemma 2. *Let* $(Y_0, Y_1, \dots, Y_n)$ *be a partial retract in* $(X_0, X_1, \dots, X_n)$. *Assume that* $f_1, f_2, \dots, f_k$ *lie in* $Y_0 + Y_1 + \dots + Y_n$. *There exists an operator* $S : X_i \to Y_i$ *such that* $Sf_j = f_j$ *and* $\|S\|_{X_i \to Y_i} \le C$ *for any* $j = 1, \dots, k$, $i = 0, \dots, n$, *where* C *does not depend on* $f_1, f_2, \dots, f_k$ *for a fixed* k.

Proof. Take an operator T_1 (as in the definition of a partial retract) that fixes f_1. Then $(I - T_1)f_1 = 0$, where I is the identity operator. It is clear that $(I - T_1)f_2 \in \sum\limits_{j=0}^{n} Y_j$. Let T_2 be the operator (as in the definition of a partial retract) that fixes $(I - T_1)f_2$. Then $(I - T_2)(I - T_1)f_2 = 0$ and $(I - T_2)(I - T_1)f_1 = 0$. Continuing this procedure, we construct operators $T_1, T_2, \dots, T_k$ such that T_j maps X_i to Y_i and $\|T_j\|_{X_i \to Y_i} \le C$ for all $j = 1, \dots, k$, $i = 0, \dots, n$; moreover,

$$\prod_{j=1}^{k} (I - T_j) f_i = 0 \quad \text{for any} \quad i = 1, \dots, k.$$

Expanding the product, we obtain the equality $\prod\limits_{j=1}^{k} (I - T_j) = I - S$. It is clear that $Sf_j = f_j$ and $\|S\|_{X_i \to Y_i} \le C_1$ for any $j = 1, \dots, k$, $i = 0, \dots, n$. □

Corollary 1. *A collection* $(Y_0, \dots, Y_n)$ *is* K- *and* J-*closed in* $(X_0, \dots, X_n)$ *if the collection* $(Y_0, \dots, Y_n)$ *is a partial retract in* $(X_0, \dots, X_n)$.

Proof. The condition of K-closedness can be verified easily; therefore, we are only going to prove the J-closedness. Let $x \in \bigcap\limits_{j=0}^{n} X_j$ and let $c_j > \operatorname{dist}(x, Y_j)$, $j = 0, \dots, n$. Then there exist $y_j \in Y_j$ such that $\|x - y_j\|_{X_j} < c_j$ for every $j = 0, \dots, n$. We find S as in Lemma 2 so that it fixes each element y_j. The formula $Sx = y_j + S(x - y_j)$ shows that Sx is contained in the intersection of the spaces Y_j, $j = 0, \dots, n$. Therefore,

$$\|x - Sx\|_{X_i} \le \|x - y_j\|_{X_i} + \|S(x - y_i)\|_{X_i} \le Cc_i, \quad i = 0, \dots, n,$$

where C depends on n. □

Now we return to Theorem 1. In order to demonstrate the second proof of this theorem, we need weighted Hardy spaces. By a *weight* we mean a measurable function w on the unit circle nonnegative almost everywhere and such that $\log w$ is integrable. Set

$$L^p(w) \overset{\text{def}}{=} \left\{ f \, : \, \|f\|_{p,w} \overset{\text{def}}{=} \left(\int |f|^p w dm \right)^{1/p} < \infty \right\} \quad \text{for} \quad 0 < p < \infty.$$

In the case $p = \infty$, we define

$$L^\infty(w) \overset{\text{def}}{=} \{ f \, : \, \|f\|_{\infty,w} \overset{\text{def}}{=} \operatorname*{ess\,sup}_{\zeta \in \mathbb{T}} |f| w^{-1} < \infty \}.$$

The weighted Hardy space $H^p(w)$ is defined by the formulas $H^p(w) = W^{-1/p}H^p$ for $p < \infty$ and $H^\infty(w) = WH^\infty$, where W is the "outer" analytic function such that $|W| = w$, namely, $W = \exp(\log w + iH\log w)$, where H is the harmonic conjugate operator.

In [7], the following theorem was proved.

Theorem A. *If* $\log(w_i/w_j) \in \mathrm{BMO}$ *for every* $i, j = 0, \dots, n$, *then for every* $1 \le p \le \infty$, *the collection* $(H^p(w_0), H^p(w_1), \dots, H^p(w_n))$ *is a partial retract in* $(L^p(w_0), L^p(w_1), \dots, L^p(w_n))$.

Theorem **A** allows us to deduce Theorem 1 once again. Namely, let $x \in \bigcap_{j=0}^{n} X^j$ and let $\operatorname{dist}_{X^j}(x, X^j_A) < c_j$. Then there exist $y_j \in X^j_A$ such that $\|x - y_j\|_{X^j} \le c_j$ for every $j = 0, \dots, n$. Let w_j be appropriate BMO majorants for $x - y_j$. Note that we can ensure that $x \in L^\infty(w_j)$, $j = 0, \dots, n$, by increasing w_j and preserving the other properties. Indeed, it suffices to choose any BMO majorant w for x in $\bigcap_{j=0}^{n} X^j$ and replace w_j by $\max(\varepsilon w, w_j)$, where ε is a sufficiently small positive number. Then $y_j \in H^\infty(w_j)$ and $\|x - y_j\|_{L^\infty(w_j)} \le 1$. Using Theorem **A** and Corollary 1, we see that there exists a $y \in \bigcap_{j=0}^{n} H^\infty(w_j)$ such that $\|x - y\|_{L^\infty(w_j)} \le c$, $j = 0, \dots, n$, where $c > 1$. Then $\|x - y\|_{X_j} \le \|x - y\|_{L^\infty(w_j)} \|w_j\|_{X_j} \le C' c_j$.

4. Duality

Assume that $(X_0, X_1, \dots, X_n)$ is a compatible collection of Banach spaces and that $\bigcap_{j=0}^{n} X_j$ is dense in each X_j for $j = 0, \dots, n$. It follows that the spaces X_j^* are embedded in $\left(\bigcap_{j=0}^{n} X_j\right)^*$; therefore, the collection $(X_0^*, X_1^*, \dots, X_n^*)$ is compatible. It is well known and can easily be verified that $\left(\bigcap_{j=0}^{n} X_j\right)^* = \sum_{j=0}^{n} X_j^*$. Let Y_j be a closed subspace of X_j, $j = 0, \dots, n$. Consider the annihilators

$$Y_j^\perp \stackrel{\text{def}}{=} \left\{F \in X_j^* : F(y) = 0 \ \text{ for all } \ y \in Y_j\right\}, \quad j = 0, \dots, n.$$

As in the case of couples, the following proposition is fulfilled.

Lemma 3. *The following conditions are equivalent under the above assumptions.*

(i) *The collection* $(Y_0^\perp, Y_1^\perp, \dots, Y_n^\perp)$ *is* K*-closed in* $(X_0^*, X_1^*, \dots, X_n^*)$.
(ii) *The collection* $(Y_0, Y_1, \dots, Y_n)$ *is* J*-closed in* $(X_0, X_1, \dots, X_n)$.

The proof can easily be obtained with the use of separation theorems, and we omit it. However, there is a less immediate version of this proposition for subspaces of analytic functions in lattices.

We recall that a Banach space X of measurable functions on the circle possesses the Fatou property if the following condition is fulfilled:

$$\text{If } \ g_n \in X, \ g_n(t) \to g(t) \ \text{ a.e. and } \ \|g_n\|_X \le 1, \ \text{ then } \ g \in X \ \text{ and } \ \|g\|_X \le 1.$$

We denote by X' the order dual of X : $X' = \left\{g \ \int |fg| < \infty, f \in X\right\}$. This is a lattice of measurable functions with the norm $\|g\| = \sup\left\{\int |fg| \ : \ \|f\|_X \le 1\right\}$. The space X' always possesses the Fatou property. Moreover, the Fatou property in X is equivalent to the identity $X = X''$ (see [8]).

Lemma 4. *Let* $X^0, \dots, X^n$ *be Banach spaces of measurable functions on the circle with the Fatou property and property* ***B****. The following conditions are equivalent.*

(i) *The collection* $(X^0_A, \dots, X^n_A)$ *is* J*-closed in* $(X^0, \dots, X^n)$.
(ii) *The collection* $((X^0)'_A, \dots, (X^n)'_A)$ *is* K*-closed in* $((X^0)', \dots, (X^n)')$.

In order to relate this lemma to Lemma 3, we observe that $(X_A)^\perp \cap X' = z(X')_A$ for any lattice X of measurable functions that possesses the Fatou property and property **B**. Thus, we deal with an analog of Lemma 3, but this time annihilators are taken in order dual spaces.

Proof. First, we check that (ii) $\Rightarrow$ (i). Let $f \in \bigcap_{j=0}^{n} X^j$ and $c_j > \operatorname{dist}_{X^j}(f, X_A^j)$. We introduce standard norms $\|\cdot\|_{X^j} c_j^{-1}$ on the spaces X_j and assume that all the "descendants" of the X^j (for example, $\bigcap_j X^j$ or $(X^j)'$) are normed accordingly. We must prove that the quantity $b = \operatorname{dist}_{\cap X^j}\left(f, \cap X_A^j\right)$ is bounded from above by a constant that does not depend on f and c_j. We denote $\bigcap_j X^j$ by X; then $X_A = \bigcap_j X_A^j$. Observe also that $X' = (X^0)' + \dots + (X^n)'$ (this follows from the simple relation $((X^0)' + \dots + (X^n)')' = X$ and from the fact that a sum of spaces with the Fatou property possesses the Fatou property, see the discussion before Lemma 3 of [8]).

Now, consider the quantity

$$a = \sup\left\{\left|\int fg\right| \; : \; g \in z(X')_A, \|g\|_{X'} \le 1\right\}.$$

By Lemma 6 of [2], there exists a $\varphi \in X$ such that $\varphi - f \in X_A$ and $\|\varphi\|_X = a$, whence $b = \operatorname{dist}_X(f, X_A) \le a$. This means that if we fix an arbitrary number $\delta > 0$, we can find a $g \in z(X')_A$ such that $\|g\|_{X'} \le 1$ and $\left|\int fg\right| > b - \delta$.

Now, we refer to item (a) of Lemma 3 of [9] which says that if the collection $(Y_A^0, \dots, Y_A^n)$ is K-closed in the lattices $(Y^0, \dots, Y^n)$ that have the Fatou property, then the K-closedness is fulfilled in the following stronger form: If $h \in (Y^0, \dots, Y_A^n)_A$ and $h = u_0 + \dots + u_n$, where $u_j \in Y^j$, then there exist functions $v_j \in Y_A^j$ such that $h = v_0 + \dots + v_0$ and $\|v_j\| \le c\|u_j\|$ (it should be mentioned that in [9], this proposition was stated for couples of spaces; however, the proof is much the same in the general case). Thus, the function g constructed above can be expressed as $g = g_0 + \dots + g_n$, where $g_j \in z(X_j)'_A$ and $\|g_j\|_{(X^j)'} c_j \le C$. Therefore,

$$b - \delta < \left|\int gf\right| \le \sum \inf_{y_j \in X_A^j} \left|\int g_j(f - y_j)\right| \le C \sum \frac{\operatorname{dist}_{X^j}(f, X_A^j)}{c_j} \le (n+1)C.$$

This implies the required estimate for b.

We are going to check that (i) $\Rightarrow$ (ii). Let B_j be the ball of radius a_j with center at zero of the space $(X^j)'$, and let $h \in \sum_j (X^j)'_A$, $h \in \sum B_j$. We must prove that h is contained in the sum of the balls U_j of the spaces $(X^j)'_A$ with centers at zero and radii Ca_j, where C does not depend on f and on the numbers a_j. Assume the contrary, i.e., $h \notin U_0 + \dots + U_n$; then, arguing similarly to the proof of Lemma 7 of [2], we find an $x \in (\sum (X_j)')' = \cap X_j$ such that $\left|\int xh\right| > 1$ and

$$\sup\left\{\left|\int x(g_0 + \dots + g_n)\right| : g_j \in U_j\right\} < 1.$$

It follows that $\operatorname{dist}_{X^j}(x, zX_A^j) \le (Ca_j)^{-1}$. (We again refer to Lemma 6 of [2] as at the first stage of the proof.) Using J-closedness, we can find a $y \in \cap X_A^j$ such that $\|x - y\|_{X_j} \le C' \operatorname{dist}_{X^j}(x, X_A^j)$ for all $j = 0, \dots, n$. Therefore, representing h as $h = \sum h_j$, where $h_j \in B_j$, we can write

$$\left|\int xh\right| = \left|\int (x - y)h\right| \le \sum \left|\int (x - y)h_j\right| \le C' \sum \operatorname{dist}_{X^j}(x, X_A^j) a_j \le C'C^{-1}.$$

Choosing C sufficiently large, we arrive at a contradiction. □

In contrast to Secs. 1 and 2, duality (Lemma 4) does not require BMO-regularity. Nevertheless, if all the lattices X^j are BMO-regular, we come up with another proof of the J-closedness of the collection $(X_A^0, \dots, X_A^n)$ in $(X^0, \dots, X^n)$ (which does not include the case of quasi-Banach spaces). There exists a fairly different result saying that then the lattices $(X_j)'$ are also BMO-regular, but in this case, the K-closedness of the collection $((X^0)'_A, \dots, (X^j)'_A)$ was mentioned in [7]. In the light of simple proofs in Secs. 1 and 2, this procedure seems artificial; however, Lemma 4 is interesting in itself.

In conclusion, we note that all the results and proofs can easily be extended to the case of spaces of functions that lie in Smirnov's class with respect to the first variable in lattices of measurable functions on $(\mathbb{T} \times S, m \times \mu)$, where (S, μ) is a space with a σ-finite measure.

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