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SHARP ESTIMATES OF BEST APPROXIMATIONS BY DEVIATIONS OF WEIERSTRASS-TYPE INTEGRALS

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We establish the estimates

$$A_\sigma(f)_P \leq KP(f - f * W),$$

where W is a kernel of special type that is integrable on $\mathbb{R}$ and $A_\sigma(f)_P$ is the best approximation of a function f with respect to a seminorm P by entire functions of exponential type not greater than σ. For the uniform and integral norms, we find the least possible constant K. The estimates are obtained by linear methods of approximation. Bibliography: 7 titles.

1. Notation

In what follows, $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{Z}_+$, and $\mathbb{N}$ are the sets of reals, integers, nonnegative integers, and positive integers, respectively; $[a:b] = [a,b] \cap \mathbb{Z}$; $C(E)$ and $C^{(r)}(E)$ are the sets of continuous and r times continuously differentiable on a set E functions. The function spaces are denoted as follows: $UCB(\mathbb{R})$ is the space of bounded, uniformly continuous on $\mathbb{R}$ functions, and C is the space of 2π-periodic continuous functions, with uniform norms $\|\cdot\| = \|\cdot\|_\infty$. If $1 \leq p < \infty$, we denote by $L_p(\mathbb{R})$ the space of functions that are measurable and integrable on $\mathbb{R}$ with power p; L_p is the space of functions that are measurable, 2π-periodic, and integrable on the period with power p; the norms in these spaces are $\|f\|_p = \left(\int_E |f|^p\right)^{1/p}$, where $E = \mathbb{R}$ or $[-\pi, \pi]$, respectively. Further, $L(\mathbb{R}) = L_1(\mathbb{R})$, $L = L_1$, $L_\infty(\mathbb{R})$ is the space of measurable, essentially bounded on $\mathbb{R}$ functions with the norm

$$\|f\|_\infty = \operatorname*{vrai\,sup}_{x\in\mathbb{R}} |f(x)|,$$

and L_∞ is the subspace of 2π-periodic functions that belong to $L_\infty(\mathbb{R})$. Unless otherwise is implied by the context, function spaces may be real or complex.

Assume that $\mathfrak{M}$ is a closed subspace of $L_p(\mathbb{R})$ $(1 \leq p < \infty)$ or $UCB(\mathbb{R})$ $(p = \infty)$ and P is a seminorm defined on $\mathfrak{M}$. We say that the space $(\mathfrak{M}, P)$ is of class $\mathcal{B}$ if the following conditions are fulfilled:

(1) the space is shift-invariant, i.e., for every $f \in \mathfrak{M}$ and $h \in \mathbb{R}$, we have $f(\cdot + h) \in \mathfrak{M}$ and $P(f(\cdot + h)) = P(f)$;

(2) there exists a constant B such that $P(f) \leq B\|f\|_p$ for every $f \in \mathfrak{M}$.

As examples of spaces of class $\mathcal{B}$, we mention $\left(UCB(\mathbb{R}), \|\cdot\|_\infty\right)$, $(L_p(\mathbb{R}), \|\cdot\|_p)$, $1 \leq p < \infty$, the spaces of periodic functions $\left(C, \|\cdot\|_p\right)$, $1 \leq p \leq \infty$, and also more general spaces of uniformly continuous almost-periodic functions, with exponents belonging to a fixed set, with various norms.

Next, $\mathbf{E}_\sigma$ and $\mathbf{E}_{\sigma-0}$ are the sets of entire functions of exponential type not greater (less) than $\sigma > 0$; the best approximation of a function f with respect to a seminorm P is defined by

$$A_\sigma(f)_P = \inf_{\substack{g\in\mathbf{E}_\sigma \\ f-g\in\mathfrak{M}}} P(f-g)$$

$(\inf\varnothing = +\infty)$, the value $A_{\sigma-0}(f)_P$ is defined in the same way. Index p attached to some value means that $P(f) = \|f\|_p$. The functions are assumed to be extended to points of removable break by continuity; in other cases, the symbol $\frac{0}{0}$ is understood as 0. In addition,

$$c(f,y) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(t)e^{-iyt}\,dt$$

is the complex Fourier transform of a function $f \in L(\mathbb{R})$ and $a(f)$ and $b(f)$ are its cosine and sine Fourier transforms, i.e.,

$$a(f,y) = c(f,y) + c(f,-y) \quad\text{and}\quad b(f,y) = i\big(c(f,y) - c(f,-y)\big).$$

By analogy, if a function c is defined on a symmetric with respect to zero subset of $\mathbb{R}$, we put

$$a(y) = c(y) + c(-y) \quad\text{and}\quad b(y) = i(c(y) - c(-y)). \tag{1}$$

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We have $a = 2c$ if c is even and $b = 2i\,c$ if c is odd.

The convolution of two functions f and g is defined by

$$(f*g)(x) = \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty} f(x-t)g(t)\,dt;$$

$c(f*g) = c(f)c(g)$ with such a normalization.

2. General estimates

Assume that a convolution operator $\mathcal{W}$ with kernel $W \in L(\mathbb{R})$ is defined:

$$\mathcal{W}f = f*W.$$

We study the problem concerning the best constant in the inequality

$$A_\sigma(f)_P \le KP(f - \mathcal{W}f) \tag{2}$$

and realization of this estimate by linear approximation methods.

Let us show that this problem reduces to the classical one, which deals with approximation of classes of convolutions.

Lemma 1. *Assume that* $p \in [1,+\infty]$, $f \in L_p(\mathbb{R})$, $\varphi = (I - \mathcal{W})f$, $y_0 > 0$, *and* $c(W,y) \ne 1$ *for every* y *such that* $|y| \ge y_0$.

(1) *Then there exist functions* $Q, G \in L(\mathbb{R})$ *such that* $c(Q,y) = 0$, $c(G,y) = \frac{c(W,y)}{1-c(W,y)}$ *for* $|y| \ge y_0$, *and*

$$\mathcal{W}f = f*Q + \varphi*G. \tag{3}$$

(2) *If a function* W *is even, then* Q *and* G *can also be chosen even. If* $c(W) \in C^{(r)}\big(\mathbb{R}\setminus(-y_0,y_0)\big)$, *where* $r \in \mathbb{N}\cup\{\infty\}$, *then the function* G *can be chosen so that* $c(G) \in C^{(r)}(\mathbb{R})$.

Proof. (1) Let us take a kernel $V \in L(\mathbb{R})$ such that $c(V,y) = 0$ for $|y| \ge y_0$ and $c(W-V,y) \ne 1$ for all $y \in \mathbb{R}$. By the Wiener–Levi theorem (see [1, Sec. 75, p. 158]), the function $\frac{c(W-V)}{1-c(W-V)}$ is the Fourier transform of a function $G \in L(\mathbb{R})$. Then

$$f*(W-V) = (f - f*(W-V))*G,$$

and so equality (3) holds with $Q = V + V*G$.

(2) The statement concerning evenness is obvious. Let us prove the one concerning smoothness. We denote $w = c(W)$ and take a function $d \in C^{(r)}[-y_0,y_0]$ such that $d^{(k)}(\pm y_0 \mp) = w^{(k)}(\pm y_0 \pm)$ for every $k \in [0:r]$ and $d(y) \ne 1$ for every $y \in [-y_0,y_0]$; we put $d(y) = w(y)$ for $|y| \ge y_0$. Then $d \in C^{(r)}(\mathbb{R})$ and $\frac{d}{1-d} \in C^{(r)}(\mathbb{R})$.

We denote by L^* the set of Fourier transforms of functions of the class $L(\mathbb{R})$ [1, Sec. 75, p. 157]. Let us show that $v = w - d \in L^*$. Assume that $q \in C^{(1)}(\mathbb{R})$, q is even, $q(y) = 1$ for $|y| \le y_0$, and $q(y) = 0$ for $|y| \ge 2y_0$. Then the functions q and dq are compactly supported and belong to $C^{(1)}(\mathbb{R})$; thus, they belong to L^* (see, e.g., [2, p. 133, Remark 2]). Consequently, $wq \in L^*$, and so $v = vq = wq - dq \in L^*$ and $d = w - v \in L^*$.

It remains to denote the inverse Fourier transform of v by V and to complete the proof as in item (1). □

The term $f*Q$ in (3) belongs to $\mathbf{E}_{y_0}$ and can be included in the approximating aggregate. Thus, the problem is reduced to approximation of the convolution $\varphi*G$.

Let us recall known facts concerning approximation of classes of convolutions.

Let $y_0 > 0$ and $r \in \{1,2\}$. We denote by $CM_c^r(y_0)$ and $CM_s^r(y_0)$ (which means "completely monotone") the sets of even and odd functions c that are defined at least on $\mathbb{R}\setminus(-y_0,y_0)$, and such that for all $y \ge y_0$ (see convention (1)),

$$a(y) = \int\limits_0^{+\infty} e^{-y^r u}\,d\Phi(u) \quad\text{and}\quad b(y) = \int\limits_0^{+\infty} e^{-y^r u}\,d\Psi(u),$$

respectively, where the functions Φ and Ψ increase on $(0,+\infty)$. Next, $\widehat{CM}_c^r(y_0)$ and $\widehat{CM}_s^r(y_0)$ are the sets of even and odd functions $G \in L(\mathbb{R})$ for which $c(G) \in CM_c^r(y_0)$ or $CM_s^r(y_0)$; $\widehat{CM}^r(y_0) = \widehat{CM}_c^r(y_0) \cup \widehat{CM}_s^r(y_0)$.

Assume that a kernel G is even or odd, $G \in L(\mathbb{R}) \cap C(\mathbb{R} \setminus \{0\})$, and $G(t) = O(t^{-2})$ as $t \to \infty$; let $\varkappa$ be the parity of G, i.e., $\varkappa = 0$ if G is even and $\varkappa = 1$ if the kernel G is odd. We denote [1, Sec. 87, pp. 199–203]

$$L_\sigma(G, z) = \frac{\sin\left(\sigma z - \frac{\pi}{2}(1-\varkappa)\right)}{\sigma} \sum_{k=-\infty}^{\infty} (-1)^k \frac{G\left(\frac{(2k+1-\varkappa)\pi}{2\sigma}\right)}{z - \frac{(2k+1-\varkappa)\pi}{2\sigma}}.$$

Then $L_\sigma G$ is an even or odd function that belongs to $\mathbf{E}_\sigma \cap L(\mathbb{R})$ and interpolates G at the points $\frac{(2k+1-\varkappa)\pi}{2\sigma}$, $k \in \mathbb{Z}$, i.e., at the zeros of the functions $\sin \sigma t$ or $\cos \sigma t$ in accordance with its parity. (The value of an odd function G at zero is equal to zero, and if G is even, then its value at zero is irrelevant.) The function $L_\sigma G$ is expressed in terms of the Fourier transform as follows:

$$L_\sigma(G, t) = \int_{-\sigma}^{\sigma} c(L_\sigma, y) e^{ity}\, dy, \tag{4}$$

where

$$c(L_\sigma G, y) = \sum_{k=-\infty}^{\infty} e^{ik\pi(1-\varkappa)} c(G, y + 2k\sigma), \quad |y| \le \sigma. \tag{5}$$

These formulas are valid if we require that the series on the right in (5) converges uniformly on $[0, \sigma]$ and its sum, after multiplication by $e^{i\frac{\pi(1-\varkappa)}{2\sigma}y}$, admits an expansion in a Fourier series [1, Sec. 87, pp. 202–203].

Remark 1. It is shown in [3, Properties AM4 and AM5] that if $G \in \widehat{CM}^2(y_0)$ and $c(G) \in C^{(2)}(\mathbb{R})$, then $G \in C(\mathbb{R} \setminus \{0\})$, $G(t) = O(t^{-2})$ as $t \to \infty$, and formulas (4) and (5) are true.

Assume that

$$f = T + \varphi * G, \tag{6}$$

where $T \in \mathbf{E}_\sigma$. We denote

$$X_{\sigma,G} f = T + \varphi * L_\sigma G$$

and

$$\mathcal{K}_{\sigma,G} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} G(t) \operatorname{sign} \sin \sigma t\, dt = \frac{2}{\pi} \sum_{\nu=0}^{\infty} \frac{b(G, (2\nu+1)\sigma)}{2\nu+1}$$

if G is odd;

$$\mathcal{K}_{\sigma,G} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} G(t) \operatorname{sign} \cos \sigma t\, dt = \frac{2}{\pi} \sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{2\nu+1} a(G, (2\nu+1)\sigma)$$

if G is even.

Remark 2. Clearly, $X_{\sigma,G} f \in \mathbf{E}_\sigma$. Let $T = 0$. If a function φ has period $\frac{2\pi}{\rho}$, $\rho > 0$, then $X_{\sigma,G} f$ is a trigonometric polynomial of order less than $\frac{\sigma}{\rho}$. In particular, if $\rho \ge \sigma$, then $X_{\sigma,G} f$ is a constant. If a function φ is almost-periodic, then $X_{\sigma,G} f$ is also an almost-periodic function whose exponents belong to φ.

The following Lemmas A and B and Remark 3 are contained in [3, Lemmas 5 and 7 and Remark 4]. Lemma A includes classical results of J. Favard, N. I. Akhiezer, M. G. Krein, S. M. Nikolskii, Sun Youngshen, and other mathematicians; see, e.g., [1] and the history in [3].

Lemma A. *Assume that $G \in L(\mathbb{R}) \cap C(\mathbb{R} \setminus \{0\})$, G is even or odd, $G(t) = O(t^{-2})$ as $t \to \infty$, $\sigma > 0$, and*

$$(G(t) - L_\sigma(G, t)) \cos \sigma t \ge 0 \quad \text{or} \quad (G(t) - L_\sigma(G, t)) \sin \sigma t \ge 0,$$

respectively, for almost every $t \in \mathbb{R}$. Then

$$\mathcal{K}_{\sigma,G} = \frac{1}{2\pi} A_\sigma(G)_1 = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |G - L_\sigma G|. \tag{7}$$

If, in addition, $(\mathfrak{M}, P) \in \mathcal{B}$, $\varphi \in \mathfrak{M}$, *and the functions* f *and* φ *are related by* (6), *then*

$$P(f - X_{\sigma,G} f) \le \mathcal{K}_{\sigma,G} P(\varphi), \tag{8}$$

$$A_\sigma(f)_P \le \mathcal{K}_{\sigma,G} A_\sigma(\varphi)_P, \tag{9}$$

$$A_{\sigma-0}(f)_P \le \mathcal{K}_{\sigma,G} A_{\sigma-0}(\varphi)_P, \tag{10}$$

$$A_{\sigma-0}(f)_P \le \mathcal{K}_{\sigma,G} P(\varphi), \tag{11}$$

$$A_\sigma(f)_P \le \mathcal{K}_{\sigma,G} P(\varphi) \tag{12}$$

(*in* (10) *and* (11), $T \in \mathbf{E}_{\sigma-0}$). *In the spaces* $(UCB(\mathbb{R}), \|\cdot\|_\infty)$ *and* $(L(\mathbb{R}), \|\cdot\|_1)$, *the constant* $\mathcal{K}_{\sigma,G}$ *in* (8)–(12) *cannot be reduced. In the spaces of* $\frac{2\pi}{\sigma}$*-periodic functions with the uniform or integral norm, the constant in* (8), (10), *and* (11) *also cannot be reduced.*

Remark 3. Inequalities of type (8)–(12) can be carried over in a standard way (for instance, with the help of approximation of the function φ by its Fejér integral) from sets of continuous functions to the sets $L_\infty(\mathbb{R})$ and L_p, $1 \le p \le \infty$, with seminorms $\|\cdot\|_p$, $A_\sigma(\cdot)_p$, and $A_{\sigma-0}(\cdot)_p$. Moreover, for convolution classes of the form

$$f = T + \frac{1}{2\pi} \int_{-\infty}^{+\infty} G(x-t)\, dg(t)$$

and their periodic analogs, sharp estimates for approximations in $L_1(\mathbb{R})$ and L_1 by the variation of g are true.

In what follows, we do not dwell on the possibility of extending our estimates to wider function classes, restricting ourselves to statements for spaces of class $\mathcal{B}$.

Lemma B. *Assume that* $y_0 > 0$, G *belongs to* $\widehat{CM}^2_c(y_0)$ *or* $\widehat{CM}^2_s(y_0)$, $c(G) \in C^{(2)}(\mathbb{R})$, *and* $\sigma \ge y_0$. *Then*

$$G(t) - L_\sigma(G,t) = 2\cos\sigma t\, C_{\sigma,G}(t)$$

or

$$G(t) - L_\sigma(G,t) = 2\sin\sigma t\, S_{\sigma,G}(t),$$

respectively, where the functions $C_{\sigma,G}$ *and* $S_{\sigma,G}$ *are positive for nonconstant* Φ *and* Ψ. *The conclusion of Lemma* A *also holds.*

In the periodic case, kernels of the class $\widehat{CM}^1(1)$ were also considered by V. P. Zastavnyi and V. V. Savchuk [4].

3. Main results

If $\varphi = f - \mathcal{W}f$ and $\mathcal{W}f$ is represented by (3), we put

$$Y_{\sigma,W} f = f * Q + \varphi * L_\sigma G.$$

In other words, $Y_{\sigma,W} = X_{\sigma,G}\mathcal{W}$. Then

$$\mathcal{W}f - Y_{\sigma,W} f = \varphi * (G - L_\sigma G).$$

Since the difference $G - L_\sigma G$ does not change if one adds a function from $\mathbf{E}_\sigma$ to G, the operator $Y_{\sigma,W}$ does not depend on the choice of Q and G in (3).

An analog of Remark 2 holds for the operator $Y_{\sigma,W}$.

Remark 4. As follows from the definition, $Y_{\sigma,W}$ is a convolution operator with kernel

$$D_\sigma = Q + (L_\sigma G - W * L_\sigma G).$$

Let us express the Fourier transform of D_σ in terms of the Fourier transform of the initial kernel W under the assumption that formula (4) holds. Denoting $Q = V + V * G$, as in Lemma 1, we get the equality

$$c(D_\sigma, y) = c(V,y)(1 + c(G,y)) + (1 - c(W,y)) \sum_{l\in\mathbb{Z}} (-1)^l c(G, y + 2l\sigma).$$

Isolating the term with $l = 0$ and taking into account that $c(G) = \frac{c(W-V)}{1-c(W-V)}$ and $c(V,z) = 0$ for $|z| \ge \sigma$, we conclude that the equality

$$c(D_\sigma, y) = c(W,y) - (1 - c(W,y)) \sum_{l\in\mathbb{Z}\setminus\{0\}} (-1)^{l-1} \frac{c(W, y+2l\sigma)}{1 - c(W, y+2l\sigma)} \tag{13}$$

holds for $|y| \le \sigma$.

Theorem 1. *Assume that* $(\mathfrak{M}, P) \in \mathcal{B}$, $f \in \mathfrak{M}$, $y_0 > 0$, $W \in L(\mathbb{R})$, *the function* W *is even,* $c(W, y) \ne 1$ *for every* $y \ge y_0$, $G \in L(\mathbb{R})$, *the function* G *is even,* $G(t) = O(t^{-2})$ *as* $t \to \infty$, $c(G, y) = \frac{c(W,y)}{1-c(W,y)}$ *for every* $y \ge y_0$, $\sigma \ge y_0$,

$$(G(t) - L_\sigma(G, t)) \cos \sigma t \ge 0 \quad \textit{for almost every } t \in \mathbb{R}, \tag{14}$$

and $\varphi = (I - \mathcal{W})f$. *Then*

$$P(f - Y_{\sigma,W} f) \le (1 + \mathcal{K}_{\sigma,G}) P(\varphi), \tag{15}$$

$$A_\sigma(f)_P \le (1 + \mathcal{K}_{\sigma,G}) A_\sigma(\varphi)_P, \tag{16}$$

$$A_{\sigma-0}(f)_P \le (1 + \mathcal{K}_{\sigma,G}) A_{\sigma-0}(\varphi)_P, \tag{17}$$

$$A_{\sigma-0}(f)_P \le (1 + \mathcal{K}_{\sigma,G}) P(\varphi), \tag{18}$$

$$A_\sigma(f)_P \le (1 + \mathcal{K}_{\sigma,G}) P(\varphi). \tag{19}$$

In the spaces $(UCB(\mathbb{R}), \|\cdot\|_\infty)$ *and* $(L(\mathbb{R}), \|\cdot\|_1)$, *the constant in* (15)–(19) *cannot be reduced, even for functions orthogonal to* $\mathbf{E}_\sigma$. *In the spaces of* $\frac{2\pi}{\sigma}$*-periodic functions with the uniform or integral norm, the constant in* (15), (17), *and* (18) *cannot be reduced, even for functions with zero mean value.*

Proof. (1) By Lemma 1,

$$f = \varphi + f * Q + \varphi * G,$$

where $f * Q \in \mathbf{E}_{y_0} \cap L(\mathbb{R})$. Thus, all the inequalities follow from Lemma A. To prove (17), one has to take into account that (see [1, Sec. 99, p. 232])

$$A_{\sigma-0}(f * Q)_P \le A_{\sigma-0}(f)_P A_{\sigma-0}(Q)_1 = A_{\sigma-0}(f)_P A_\sigma(Q)_1 = 0.$$

Let us prove the sharpness of the inequalities.

(2) First, we consider the space $L_\infty(\mathbb{R})$ and notice that, by (7), the inequality

$$\|\varphi + \varphi * (G - L_\sigma G)\|_\infty \le (1 + \mathcal{K}_{\sigma,G}) \|\varphi\|_\infty$$

turns into equality for the function $\varphi_\sigma(t) = \operatorname{sign} \cos \sigma t$. Consequently, inequality (15) turns into equality if $f - \mathcal{W} f = \varphi_\sigma$. One may take

$$\psi_\sigma = \varphi_\sigma + \varphi_\sigma * G$$

as f. The function ψ_σ has period $\frac{2\pi}{\sigma}$ and is expanded in the Fourier series,

$$\psi_\sigma(t) = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)\bigl(1 - c(W, (2k+1)\sigma)\bigr)} \cos{(2k+1)\sigma t}.$$

Thus, $Y_{\sigma,W} \psi_\sigma = 0$. Moreover (see [1, Sec. 96, p. 227]), $A_{\sigma-0}(\psi_\sigma)_\infty = \|\psi_\sigma\|_\infty$. This implies that (18) and, consequently, the previous inequalities (15) and (17) also turn into equalities for the function ψ_σ.

(3) Let us prove the sharpness of (19) in the space $L_\infty(\mathbb{R})$ on the set of functions that are orthogonal to $\mathbf{E}_\sigma$. We take $\rho > \sigma$ and put

$$\psi_\rho = \varphi_\rho + \varphi_\rho * G.$$

As earlier,

$$Y_{\sigma,W} \psi_\rho = 0, \quad (I - \mathcal{W})\psi_\rho = \varphi_\rho, \quad \|(I - \mathcal{W})\psi_\rho\|_\infty = \|\varphi_\rho\|_\infty = \|\varphi_\sigma\|_\infty,$$

and

$$A_\sigma(\psi_\rho)_\infty = \|\psi_\rho\|_\infty.$$

By the Lebesgue dominated convergence theorem, $\psi_\rho \underset{\rho \to \sigma+}{\longrightarrow} \psi_\sigma$ almost everywhere, and so,

$$\varliminf_{\rho \to \sigma+} \|\psi_\rho\|_\infty \ge \|\psi_\sigma\|_\infty.$$

That proves the sharpness of (19).

The sharpness of the inequalities on the sets of continuous functions is deduced from their sharpness on the sets of functions that belong to $L_\infty(\mathbb{R})$ in a standard way (e.g., with the help of approximation of the functions φ_ρ by their Fejér integrals).

(4) Let us prove the sharpness in the case of the integral norm. The operator $I-\mathcal{W}$ is invertible on the sets of functions that are orthogonal to $\mathbf{E}_\sigma$. Thus,

$$\sup_{\substack{f\perp\mathbf{E}_\sigma\\ \|(I-\mathcal{W})f\|_p\le 1}} A_\sigma(f)_p = \sup_{\substack{\varphi\perp\mathbf{E}_\sigma\\ \|\varphi\|_p\le 1}} A_\sigma(\varphi+\varphi*G)_p$$

for every $p\in[1,\infty]$. By the duality relations (see [5, §1.4]),

$$\sup_{\substack{\varphi\perp\mathbf{E}_\sigma\\ \|\varphi\|_1\le 1}} A_\sigma(\varphi+\varphi*G)_1 = \sup_{\substack{\varphi\perp\mathbf{E}_\sigma\\ \|\varphi\|_1\le 1}} \sup_{\substack{g\perp\mathbf{E}_\sigma\\ \|g\|_\infty\le 1}} \left|\int_{-\infty}^{+\infty}(\varphi+\varphi*G)\,g\right|$$

$$= \sup_{\substack{g\perp\mathbf{E}_\sigma\\ \|g\|_\infty\le 1}} \sup_{\substack{\varphi\perp\mathbf{E}_\sigma\\ \|\varphi\|_1\le 1}} \left|\int_{-\infty}^{+\infty}(g+g*G)\,\varphi\right| = \sup_{\substack{g\perp\mathbf{E}_\sigma\\ \|g\|_\infty\le 1}} A_\sigma(g+g*G)_\infty.$$

The sharpness in $L_\infty(\mathbb{R})$, which is already proved, implies that the last upper bound is equal to $1+\mathcal{K}_{\sigma,G}$. Thus, inequality (19) is sharp in $L(\mathbb{R})$ (and, consequently, the previous inequalities (15)–(18) are also sharp).

The sharpness of (18) (and, consequently, of the previous inequalities (15) and (17)) in the space L of $\frac{2\pi}{\sigma}$-periodic functions is shown similarly:

$$\sup_{\substack{\varphi\perp 1\\ \|\varphi\|_1\le 1}} A_{\sigma-0}(\varphi+\varphi*G)_1 = \sup_{\substack{\varphi\perp 1\\ \|\varphi\|_1\le 1}} \sup_{\substack{g\perp 1\\ \|g\|_\infty\le 1}} \left|\int_{-\frac{\pi}{\sigma}}^{\frac{\pi}{\sigma}}(\varphi+\varphi*G)\,g\right|$$

$$= \sup_{\substack{g\perp 1\\ \|g\|_\infty\le 1}} \sup_{\substack{\varphi\perp 1\\ \|\varphi\|_1\le 1}} \left|\int_{-\frac{\pi}{\sigma}}^{\frac{\pi}{\sigma}}(g+g*G)\,\varphi\right| = \sup_{\substack{g\perp 1\\ \|g\|_\infty\le 1}} A_{\sigma-0}(g+g*G)_\infty.$$

Here we use the fact that, for such functions, $A_{\sigma-0}$ coincides with the best approximation by constants. □

By Lemma B, condition (14) is fulfilled if we require that the kernel G belongs to $\widehat{CM}_c^2(y_0)$. In the next lemma, we show that the property to belong to the class $\widehat{CM}_c^2(y_0)$ is preserved under the passage from the kernel W to G.

Lemma 2. *If $y_0>0$, $w\in CM_c^2(y_0)$, and $w(y)<1$ for every $y\ge y_0$, then $\frac{w}{1-w}\in CM_c^2(y_0)$.*

Proof. We denote $g=\frac{w}{1-w}$. For every $y\ge y_0$,

$$g(y)=\sum_{k=1}^{\infty} w^k(y).$$

Changing variables in the double integral, we can ascertain that the product of two functions from $CM_c^2(y_0)$ also belongs to $CM_c^2(y_0)$ (see [6, p. 88, Theorem 11.5]). Consequently, by induction, $w^k\in CM_c^2(y_0)$ for every $k\in\mathbb{N}$, i.e.,

$$w^k(y)=\int_0^{+\infty} e^{-y^2u}\,d\Phi_k(u), \quad y\ge y_0,$$

where the function Φ_k increases. Without loss of generality, we may assume that $\Phi_k(0)=\Phi_k(0+)=0$ and integrate over the closed ray $[\,0,+\infty)$. We put

$$F_n=\sum_{k=1}^{n}\Phi_k, \quad F=\sum_{k=1}^{\infty}\Phi_k, \quad \text{and} \quad g_n(y)=\sum_{k=1}^{n} w^k(y)=\int_0^{+\infty} e^{-y^2u}\,dF_n(u).$$

Clearly, F increases. Let us prove that all the values of F are finite. If $u^*>0$ and $F(u^*)=+\infty$, then

$$g_n(y_0)\ge \int_{[0,u^*)} e^{-y_0^2u}\,dF_n(u)\ge e^{-y_0^2u^*}F_n\Big|_0^{u^*} = e^{-y_0^2u^*}F_n(u^*) \underset{n\to\infty}{\longrightarrow} +\infty,$$

which contradicts the relation $g_n(y_0)\underset{n\to\infty}{\longrightarrow} g(y_0)$. Thus, all the values of F are finite.

Consequently ([7, §7.2, p. 177]), we can pass to the limit under the sign of the Stieltjes integral, i.e.,

$$g(y) = \int_0^{+\infty} e^{-y^2 u}\, dF(u). \qquad \square$$

Theorem 2. *Assume that* $(\mathfrak{M}, P) \in \mathcal{B}$, $f \in \mathfrak{M}$, $y_0 > 0$, $W \in \widehat{CM}^2_c(y_0)$, $c(W,y) < 1$ *for every* $y \ge y_0$, $c(W) \in C^{(2)}(\mathbb{R} \setminus (-y_0, y_0))$, $c(G,y) = \frac{c(W,y)}{1-c(W,y)}$ *for every* $y \ge y_0$, $\varphi = (I - \mathcal{W})f$, *and* $\sigma \ge y_0$. *Then the conclusion of Theorem* 1 *holds.*

To prove Theorem 2, one has to choose, by Lemma 1, an even function G such that $c(G) \in C^{(2)}(\mathbb{R})$, and then combine Remark 1, Theorem 1, Lemma B, and Lemma 2.

Remark 5. In the statement of Lemma 2, the class $CM^2(y_0)$ can be replaced by $CM^1(y_0)$. Since functions that belong to $CM^1(y_0)$ are multiply monotone, for kernels from $\widehat{CM}^1_c(y_0)$, Theorem 2 can be simply obtained by applying the B. Nagy theorem [1, Sec. 88, pp. 203–207].

In the next two remarks, we assume the conditions of Theorem 2 to be fulfilled.

Remark 6. The constant $\mathcal{K}_{\sigma,G}$ can be converted as follows:

$$\mathcal{K}_{\sigma,G} = \frac{4}{\pi}\sum_{\nu=0}^{\infty}\frac{(-1)^\nu}{2\nu+1}\,\frac{c(W,(2\nu+1)\sigma)}{1-c(W,(2\nu+1)\sigma)} = \frac{4}{\pi}\sum_{\nu=0}^{\infty}\frac{(-1)^\nu}{2\nu+1}\sum_{k=1}^{\infty} c^k(W,(2\nu+1)\sigma)$$
$$= \sum_{k=1}^{\infty}\frac{4}{\pi}\sum_{\nu=0}^{\infty}\frac{(-1)^\nu}{2\nu+1}\, c^k(W,(2\nu+1)\sigma) = \sum_{k=1}^{\infty}\mathcal{K}_{\sigma,W_k}.$$

Here W_k is the $(k-1)$-fold convolution of a kernel W with itself. To prove the validity of changing the summation order, one can group the terms with numbers $\nu = 2s$ and $\nu = 2s+1$ and take into account that $c(W,y) \in (0,1)$ and $c(W,y)$ decreases for $y \ge y_0$.

Remark 7. By Remark 6, the series $\sum_{k=1}^{\infty}(W_k - L_\sigma W_k)$ converges in the space $L(\mathbb{R})$ because $\frac{1}{2\pi}\|W_k - L_\sigma W_k\|_1 = \mathcal{K}_{\sigma,W_k}$ by (7). According to (4), for $|y| \le \sigma$, the sum on the right in (13) equals $c(G - L_\sigma G, y)$. Expanding each fraction as a geometric progression and changing the order of summation as in Remark 6, we obtain the equality

$$c(G - L_\sigma G, y) = \sum_{k=1}^{\infty} c(W_k - L_\sigma W_k, y).$$

For $|y| > \sigma$, the last inequality is obvious in view of the relation between G and W.

Consequently, the sum of the series $\sum_{k=1}^{\infty}(W_k - L_\sigma W_k)$ in the space $L(\mathbb{R})$ is equal to $G - L_\sigma G$, and the expansion

$$\mathcal{W}f - Y_{\sigma,W}f = \varphi * (G - L_\sigma G) = \varphi * \sum_{k=1}^{\infty}(W_k - L_\sigma W_k)$$

holds.

4. Estimates of approximations by linear combinations of seminorms of differences

In [3], the author generalized and strengthened the estimate from above of Lemma B in the form mentioned in the title of this section. To strengthen Theorem 2 in a similar way, we introduce new notation and formulate previous results from [3].

Central differences and moduli of continuity of a function f are defined by

$$\delta_t^r(f,x) = \sum_{k=0}^{r}(-1)^k C_r^k f\left(x + \frac{rt}{2} - kt\right) \quad \text{and} \quad \omega_r(f,h)_P = \sup_{0\le t\le h} P(\delta_t^r f),$$

respectively.

Assume that $y_0 > 0$, $G \in \widehat{CM}^2(y_0)$, $c(G) \in C^{(2)}(\mathbb{R})$, $c_0(y) = c(G, y)$ for every y such that $|y| \geq y_0$, and $0 < h < \frac{2\pi}{y_0}$. Let $c_{h0}(y) = c_0(y)$ $(|y| \geq y_0)$,

$$c_{h\nu}(y) = \frac{1}{2i} \sum_{s=-\infty}^{\infty} \left(c_{h,\nu-1}(y) - c_{h,\nu-1}\left(\frac{2\pi s}{h}\right) \right) \frac{(-1)^{s-1}}{\pi s - hy/2}, \quad |y| \geq y_0, \quad \nu \in \mathbb{N},$$

$$c_{h\nu}(0) = \sum_{\substack{s=-\infty \\ s\neq 0}}^{\infty} (-1)^{s-1} c_{h\nu}\left(\frac{2\pi s}{h}\right) \quad \nu \in \mathbb{Z}_+,$$

and

$$K_{h\nu}(t) = \begin{cases} \frac{2\pi}{h} \sum\limits_{s=-\infty}^{\infty} c_{h\nu}\left(\frac{2\pi s}{h}\right) e^{i\frac{2\pi s}{h}t}, & |t| \leq \frac{h}{2}, \\ 0, & |t| > \frac{h}{2}, \end{cases} \quad (\nu \in \mathbb{Z}_+).$$

By the general convention (1), we define the functions $a_{h\nu}$ and $b_{h\nu}$.

If $c_0 \in CM_c^r(y_0)$, then $c_{h1} \in CM_s^r(y_0)$, and if $c_0 \in CM_s^r(y_0)$, then $-c_{h1} \in CM_c^r(y_0)$. The functions $c_{h\nu}$ and $K_{h\nu}$ have the same parity as c_0 for ν even and the opposite parity to that of c_0 for ν odd. For each $m \in \mathbb{Z}_+$ there exists a function $G_{hm} \in L(\mathbb{R})$ of the same parity as c_{hm} such that $c(G_{hm}) \in C^{(2)}(\mathbb{R})$ and $c(G_{hm}, y) = c_{hm}(y)$ for every y such that $|y| \geq y_0$. The functions $K_{h\nu}$ do not change their sign on $\left(0, \frac{h}{2}\right)$ and are integrable. For every $m \in \mathbb{N}$, the expansions

$$G = \sum_{\nu=0}^{m-1} \delta_h^\nu K_{h\nu} + \delta_h^m G_{hm} + M_{hm}$$

and

$$f = T + \varphi * M_{hm} + \delta_h^m \varphi * G_{hm} + \sum_{\nu=0}^{m-1} \delta_h^\nu \varphi * K_{h\nu}$$

hold, where $M_{hm} \in \mathbf{E}_{y_0} \cap L(\mathbb{R})$ and the functions f and φ are related by (6).

Let $\sigma \geq y_0$. We put

$$U_{\sigma hm} f = U_{\sigma hm,G} f = T + \varphi * M_{hm} + \delta_h^m \varphi * L_\sigma G_{hm},$$

$$A_{h\nu} = \begin{cases} \frac{2}{\pi} \left| \sum\limits_{s=0}^{\infty} \frac{1}{2s+1} b_{h\nu}\left(\frac{2\pi(2s+1)}{h}\right) \right| & \text{if the parity of } \nu \text{ is opposite to that of } G, \\ \left| \sum\limits_{s=1}^{\infty} (-1)^{s-1} a_{h\nu}\left(\frac{2\pi s}{h}\right) \right| & \text{if the parity of } \nu \text{ is equal to that of } G, \end{cases}$$

and

$$B_{\sigma hm} = \begin{cases} \frac{2}{\pi} \left| \sum\limits_{s=0}^{\infty} \frac{1}{2s+1} b_{hm}((2s+1)\sigma) \right| & \text{if the parity of } m \text{ is opposite to that of } G, \\ \frac{2}{\pi} \left| \sum\limits_{s=0}^{\infty} \frac{(-1)^s}{2s+1} a_{hm}((2s+1)\sigma) \right| & \text{if the parity of } m \text{ is equal to that of } G. \end{cases}$$

We do not indicate the dependence of the constants on the kernel G in the notation: $A_{h\nu} = A_{h\nu,G}$ and $B_{\sigma hm} = B_{\sigma hm,G}$. A remark analogous to Remark 2 is valid for the operators $U_{\sigma hm,G}$.

The following theorem is established in [3, Theorem 1].

Theorem C. *Assume that* $(\mathfrak{M}, P) \in \mathcal{B}$, $\varphi \in \mathfrak{M}$, $y_0 > 0$, $G \in \widehat{CM}^2(y_0)$, $c(G) \in C^{(2)}(\mathbb{R})$, *functions* f *and* φ *are connected by* (6), $m \in \mathbb{N}$, $0 < h < \frac{2\pi}{y_0}$, *and* $\sigma \geq y_0$. *Then*

$$P(f - U_{\sigma hm} f) \leq A_{h0} P(\varphi) + \sum_{\nu=1}^{m-1} A_{h\nu} P\left(\delta_h^\nu \varphi\right) + B_{\sigma hm} P\left(\delta_h^m \varphi\right).$$

If, in addition, the kernel G *is odd, then* $A_{h0} P(\varphi)$ *can be replaced by* $A_{h0} \frac{\omega_1(\varphi,h)_P}{2}$.

We use Theorem C for even kernels G.

We put $Y_{\sigma hm} = Y_{\sigma hm,W} = U_{\sigma hm,G} \mathcal{W}$. A remark analogous to Remark 2 is also valid for the operators $Y_{\sigma hm,W}$. Applying Theorem C, we immediately come to the following statement.

Theorem 3. *Assume that* $(\mathfrak{M}, P) \in \mathcal{B}$, $f \in \mathfrak{M}$, $y_0 > 0$, $W \in \widehat{CM}_c^2(y_0)$, $c(W,y) < 1$ *for every* $y \geq y_0$, $c(W) \in C^{(2)}(\mathbb{R} \setminus (-y_0, y_0))$, $c(G,y) = \frac{c(W,y)}{1-c(W,y)}$ *for every* $y \geq y_0$, $\varphi = (I - \mathcal{W})f$, $m \in \mathbb{N}$, $0 < h < \frac{2\pi}{y_0}$, *and* $\sigma \geq y_0$. *Then*

$$P(f - Y_{\sigma h m} f) \leq (1 + A_{h0}) P(\varphi) + \sum_{\nu=1}^{m-1} A_{h\nu} P(\delta_h^\nu \varphi) + B_{\sigma h m} P(\delta_h^m \varphi).$$

In [3, Lemma 13 and Corollary 5], the equalities

$$B_{\sigma, \frac{\pi}{\sigma}, m} = 2^{-m} \left(\mathcal{K}_{\sigma,G} - \sum_{\nu=0}^{m-1} 2^\nu A_{\frac{\pi}{\sigma}, \nu} \right) \quad \text{and} \quad U_{\sigma, \frac{\pi}{\sigma}, m, G} = X_{\sigma,G}$$

are established. Consequently,

$$Y_{\sigma, \frac{\pi}{\sigma}, m, W} = Y_{\sigma, W}.$$

Thus, for $h = \frac{\pi}{\sigma}$, Theorem 3 takes the following form.

Corollary 1. *Assume that* $(\mathfrak{M}, P) \in \mathcal{B}$, $f \in \mathfrak{M}$, $y_0 > 0$, $W \in \widehat{CM}_c^2(y_0)$, $c(W,y) < 1$ *for every* $y \geq y_0$, $c(W) \in C^{(2)}(\mathbb{R} \setminus (-y_0, y_0))$, $c(G,y) = \frac{c(W,y)}{1-c(W,y)}$ *for every* $y \geq y_0$, $\varphi = (I - \mathcal{W})f$, $m \in \mathbb{N}$, *and* $\sigma \geq y_0$. *Then*

$$P(f - Y_{\sigma,W} f) \leq (1 + A_{\frac{\pi}{\sigma},0}) P(\varphi) + \sum_{\nu=1}^{m-1} A_{\frac{\pi}{\sigma},\nu} P(\delta_{\frac{\pi}{\sigma}}^\nu \varphi) + \left(\mathcal{K}_{\sigma,G} - \sum_{\nu=0}^{m-1} 2^\nu A_{\frac{\pi}{\sigma},\nu} \right) 2^{-m} P(\delta_{\frac{\pi}{\sigma}}^m \varphi). \tag{20}$$

Let $\eta_\nu(\varphi, h)_P = 2^{-\nu} P(\delta_h^\nu \varphi)$. Since the sequence $\{\eta_\nu(\varphi, h)_P\}_{\nu=0}^\infty$ decreases, there exists the limit

$$\eta_\infty(\varphi, h)_P = \lim_{\nu \to \infty} \eta_\nu(\varphi, h)_P.$$

The decreasing of the sequence $\{\eta_\nu(\varphi, h)_P\}_{\nu=0}^\infty$ implies that the right-hand side of (20) decreases with respect to m. For this reason, the best estimate is achieved at the limit as $m \to \infty$.

Corollary 2. *Assume that* $(\mathfrak{M}, P) \in \mathcal{B}$, $f \in \mathfrak{M}$, $y_0 > 0$, $W \in \widehat{CM}_c^2(y_0)$, $c(W,y) < 1$ *for every* $y \geq y_0$, $c(W) \in C^{(2)}(\mathbb{R} \setminus (-y_0, y_0))$, $c(G,y) = \frac{c(W,y)}{1-c(W,y)}$ *for every* $y \geq y_0$, $\varphi = (I - \mathcal{W})f$, *and* $\sigma \geq y_0$. *Then*

$$P(f - Y_{\sigma,W} f) \leq (1 + A_{\frac{\pi}{\sigma},0}) P(\varphi) + \sum_{\nu=1}^{\infty} 2^\nu A_{\frac{\pi}{\sigma},\nu} \eta_\nu \left(\varphi, \frac{\pi}{\sigma} \right)_P + \left(\mathcal{K}_{\sigma,G} - \sum_{\nu=0}^{\infty} 2^\nu A_{\frac{\pi}{\sigma},\nu} \right) \eta_\infty \left(\varphi, \frac{\pi}{\sigma} \right)_P. \tag{21}$$

Inequalities (20) and (21) strengthen (15) and are sharp in the cases pointed in Theorems 1 and 2.

5. Applications

We restrict ourselves to explicit form of the inequalities of Corollary 1 only for $m = 0$ (i.e., of Theorems 1 and 2) and $m = 1$. Recall that

$$A_{\frac{\pi}{\sigma},0} = 2 \sum_{s=1}^{\infty} (-1)^{s-1} \frac{c(W, 2s\sigma)}{1 - c(W, 2s\sigma)}$$

and

$$\mathcal{K}_{\sigma,G} = \frac{4}{\pi} \sum_{s=0}^{\infty} \frac{(-1)^s}{(2s+1)} \frac{c(W, (2s+1)\sigma)}{(1 - c(W, (2s+1)\sigma))}.$$

5.1. Assume that $\lambda > 0$ and let

$$W_\lambda(t) = \int_{-\infty}^{+\infty} e^{-\lambda y^2} e^{ity} \, dy = \frac{\sqrt{\pi}}{\sqrt{\lambda}} e^{-\frac{t^2}{4\lambda}}$$

be the Weierstrass kernel (the heat conduction kernel). Clearly, $W_\lambda \in \widehat{CM}_c^2(y_0)$ for every $y_0 > 0$. The convolution $f * W_\lambda$ is the Weierstrass integral of a function f. Since y_0 is arbitrary, the estimates are valid for all $\sigma > 0$.

Corollary 3. *Assume that* $(\mathfrak{M}, P) \in \mathcal{B}$, $f \in \mathfrak{M}$, $\lambda, \sigma > 0$, *and* $\varphi = f - f * W_\lambda$. *Then*

$$P(f - Y_{\sigma,W_\lambda} f) \le \left(1 + 2\sum_{s=1}^{\infty}(-1)^{s-1}\frac{e^{-\lambda(2s\sigma)^2}}{1 - e^{-\lambda(2s\sigma)^2}}\right)P(\varphi)$$
$$+ \left(\frac{2}{\pi}\sum_{s=0}^{\infty}\frac{(-1)^s}{2s+1}\frac{e^{-\lambda((2s+1)\sigma)^2}}{1 - e^{-\lambda((2s+1)\sigma)^2}} - \sum_{s=1}^{\infty}(-1)^{s-1}\frac{e^{-\lambda(2s\sigma)^2}}{1 - e^{-\lambda(2s\sigma)^2}}\right)P\big(\delta^1_{\frac{\pi}{\sigma}}\varphi\big)$$

and

$$P(f - Y_{\sigma,W_\lambda} f) \le \left(1 + \frac{4}{\pi}\sum_{s=0}^{\infty}\frac{(-1)^s}{2s+1}\frac{e^{-\lambda((2s+1)\sigma)^2}}{1 - e^{-\lambda((2s+1)\sigma)^2}}\right)P(\varphi).$$

In [2, p. 285, Corollary 2], for spaces of periodic functions and $\sigma \in \mathbb{N}$, the following inequality is established:

$$A_{\sigma-0}(f)_P \le \left(1 + \frac{\pi^2}{4\lambda\sigma^2}\right)P(f - f * W_\lambda).$$

5.2. Assume that $\lambda > 0$ and let

$$P_\lambda(t) = \int_{-\infty}^{+\infty} e^{-\lambda|y|}e^{ity}\,dy = \frac{2\lambda}{\lambda^2 + t^2}$$

be the Poisson kernel. Clearly, $P_\lambda \in \widehat{CM}^1_c(y_0)$ for every $y_0 > 0$. The convolution $f * P_\lambda$ is the Poisson integral of a function f. Since y_0 is arbitrary, the estimates are valid for all $\sigma > 0$.

Corollary 4. *Assume that* $(\mathfrak{M}, P) \in \mathcal{B}$, $f \in \mathfrak{M}$, $\lambda, \sigma > 0$, *and* $\varphi = f - f * P_\lambda$. *Then*

$$P(f - Y_{\sigma,P_\lambda} f) \le \left(1 + 2\sum_{s=1}^{\infty}(-1)^{s-1}\frac{e^{-2\lambda\sigma s}}{1 - e^{-2\lambda\sigma s}}\right)P(\varphi)$$
$$+ \left(\frac{2}{\pi}\sum_{s=0}^{\infty}\frac{(-1)^s}{2s+1}\frac{e^{-(2s+1)\lambda\sigma}}{1 - e^{-(2s+1)\lambda\sigma}} - \sum_{s=1}^{\infty}(-1)^{s-1}\frac{e^{-2\lambda\sigma s}}{1 - e^{-2\lambda\sigma s}}\right)P\big(\delta^1_{\frac{\pi}{\sigma}}\varphi\big)$$

and

$$P(f - Y_{\sigma,P_\lambda} f) \le \left(1 + \frac{4}{\pi}\sum_{s=0}^{\infty}\frac{(-1)^s}{2s+1}\frac{e^{-\lambda(2s+1)\sigma}}{1 - e^{-\lambda(2s+1)\sigma}}\right)P(\varphi).$$

In [2, c. 276, Corollary 1], for spaces of periodic functions and $\sigma \in \mathbb{N}$, the following inequality is established:

$$A_{\sigma-0}(f)_P \le \left(1 + \frac{4}{\pi\lambda\sigma}\sum_{j=0}^{\infty}\frac{(-1)^j}{(2j+1)^2}\right)P(f - f * P_\lambda).$$

As a rule, estimates of type (2) are studied if, as for the Weierstrass and Poisson kernels, there is a family of kernels $\{W_\lambda\}$ that is an approximate identity. One more example is provided by the family of kernels

$$\Theta_\lambda(t) = \int_{-\infty}^{+\infty}\frac{e^{ity}}{\operatorname{ch}\lambda y}\,dy = \frac{\pi}{\lambda\operatorname{ch}\frac{\pi t}{2\lambda}}, \quad \lambda > 0,$$

that belong to $\widehat{CM}^2_c(y_0)$ for every $y_0 > 0$. The kernel Θ_λ arises in the description of classes of functions that are analytic in the strip $\{z : |\operatorname{Im} z| < \lambda\}$ ([1, Sec. 110, pp. 267–268]). However, the results of this paper are valid, for example, for the Bernoulli kernels or kernels of the type AP_λ, $A > 0$, while deviations of convolutions with such kernels do not have a clear approximative sense.

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