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OPERATOR LIPSCHITZ FUNCTIONS AND LINEAR FRACTIONAL TRANSFORMATIONS

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It is known that the function $t^2 \sin \frac{1}{t}$ is an operator Lipschitz function on the real line $\mathbb{R}$. We prove that the function sin *can be replaced by any operator Lipschitz function f with $f(0) = 0$. In other words, for every operator Lipschitz function f, the function $t^2 f(\frac{1}{t})$ is also operator Lipschitz if $f(0) = 0$. The function f can be defined on an arbitrary closed subset of the complex plane $\mathbb{C}$. Moreover, the linear fractional transformation $\frac{1}{t}$ can be replaced by every linear fractional transformation φ. In this case, we assert that the function $\dfrac{f \circ \varphi}{\varphi'}$ is operator Lipschitz for every operator Lipschitz function f provided that $f(\varphi(\infty)) = 0$. Bibliography: 12 titles.*

1. Introduction

It is well known that every operator Lipschitz function on a nondegenerate closed interval is differentiable at each point of the interval. E. V. Kissin and V. S. Shulman [8] proved that there exists an operator Lipschitz function with discontinuous derivative. They proved that the function $t^2 \sin \frac{1}{t}$ is operator Lipschitz,[1] at least locally. Then other methods of construction of operator Lipschitz functions with discontinuous derivatives were discovered in [10].

A starting point of the present paper is the following question.

Is it true that the function $t^2 f(\frac{1}{t})$ is locally operator Lipschitz for every operator Lipschitz function f?

We give a positive answer to this question. Moreover, we exhibit, using absolutely elementary methods, that for every operator Lipschitz function f defined on a closed set of the real numbers, the function $t^2 f(\frac{1}{t})$ is also operator Lipschitz under an additional assumption that $f(0) = 0$. Of course, this additional assumption is required only in the case where it makes sense, i.e., where 0 belongs to the domain of f. Let us observe that $t^2 f(\frac{1}{t}) = t^2\big(f(\frac{1}{t}) - f(0)\big) + f(0)\,t^2$, which implies that the function $t^2 f(\frac{1}{t})$ is locally operator Lipschitz even if $f(0) \neq 0$.

It should be noted that it is not necessary to consider only functions f of a real variable; the function f can also be defined on a closed set of the complex numbers. Moreover, a linear change of a variable exhibits that the linear fractional transformation $\frac{1}{z}$ can be replaced by practically every linear fractional transformation φ. In this case, we can assert that if f is operator Lipschitz, then $T_\varphi f \stackrel{\text{def}}{=} \dfrac{f \circ \varphi}{\varphi'}$ is also operator Lipschitz under the condition that $f(\varphi(\infty)) = 0$.

In Sec. 2, we give the definition of operator Lipschitz functions. In this section, we also collect simplest results concerning operator Lipschitz functions which are used in the following sections.

In Sec. 3, we deal with operator Lipschitz functions defined on the real line. We start this section with the simplest version of the theorem concerning the operator Lipschitzness of the function $t^2 f(\frac{1}{t})$ (Theorem 3.1). The proof is absolutely elementary and founded in essence only on Theorem 2.4.

It should be noted that an important role in the study of operator Lipschitz functions is played by Schur multipliers which are not used at all in this paper. Nevertheless, our proof of Theorem 3.1 as well as of a series of other theorems of this paper can easily be interpreted in terms of Schur multipliers.

In Sec. 3, we also specify a theorem concerning the operator Lipschitzness of the function $t^2 f(\frac{1}{t})$. Let us observe that the function $t^2 f(\frac{1}{t})$ is the difference of the antiderivatives of the functions $2tf(\frac{1}{t})$ and $f'(\frac{1}{t})$. Theorem 3.6 asserts that under the assumption of Theorem 3.1, together with the function $t^2 f(\frac{1}{t})$, the antiderivatives of the functions $tf(\frac{1}{t})$ and $f'(\frac{1}{t})$ are also operator Lipschitz.

Dilations, rotations, and translations allow us to replace the linear fractional transformation $\frac{1}{t}$ by an arbitrary real linear fractional transformation φ. In particular, we prove that the set of all derivatives of operator Lipschitz functions is invariant with respect to a linear fractional change of a variable (Theorem 3.10).

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[1]It should be noted that in [8], operator Lipschitz functions were considered only on compact sets.

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In Sec. 4, we state some auxiliary results. We consider there spaces of multipliers of spaces of operator Lipschitz functions.

Some results of Sec. 3 can be carried over to the case of functions of a complex variable. Section 5 is devoted to these extensions. Moreover, in this section, together with transformations $T_\varphi f \stackrel{\text{def}}{=} \dfrac{f\circ\varphi}{\varphi'}$, where φ is a complex linear fractional transformation, we also consider the transformations $T_{\varphi,n} f \stackrel{\text{def}}{=} \dfrac{f\circ\varphi}{|\varphi'|\,\mathrm{sgn}^n(\varphi')}$, where n is an integer. Clearly, $T_\varphi = T_{\varphi,1}$. We note that in the case of operator Lipschitz functions defined on a nondegenerate closed set $\mathfrak{F}$, $\mathfrak{F}\subset\mathbb{C}$, we manage to compute explicitly the norms of the operators $T_{\varphi,n}$ for $n\neq 0$, see Theorem 5.4 and Remark 1 after Corollary 5.5.

In the case where φ is the Cayley transformation, the operators $T_{\varphi,n}$ allow us to construct (see Theorem 5.6) linear bijections[2] of the space of operator Lipschitz functions on the unit circle $\mathbb{T}$ vanishing at -1 onto the space of operator Lipschitz functions on $\mathbb{R}$.

The same operators yield similar results concerning operator Lipschitz functions on the disk and half-plane.

In addition, in Sec. 5 we consider the operator T_φ for $\varphi(t) = \tan t$. Corollary 5.9 says in essence that this operator is a linear bijection of the space of operator Lipschitz functions on $\mathbb{R}$ onto the space of π-periodic operator Lipschitz functions on $\mathbb{R}$ vanishing at $\frac{\pi}{2}$ and so at all points of the form $\frac{\pi}{2}+\pi n$, where n is an integer.

In Sec. 6, we make an attempt to carry over Theorem 3.10 mentioned above to the case of functions defined on $\mathbb{C}$. This attempt is not completely successful: In the complex case, the corresponding space of functions is invariant with respect to linear fractional transformations φ only in the case where $\varphi(z) = az$ or $\varphi(z) = \frac{a}{z}$, where $a\in\mathbb{C}$, $a\neq 0$.

In Sec. 7, we first give the definition and describe simplest properties of commutator Lipschitz functions. Then we study the transformation T_φ on spaces of commutator Lipschitz functions.

As an example, we give a new proof of the commutator Lipschitzness of the function $(z-1)^2\exp\big((z-1)^{-1}\big)$ on the closed unit disk. This result was obtained by another method in [7].

In addition, in Sec. 7 we exhibit that the methods of our paper allow us to carry over some results of the paper [9] from the disk to the half-plane.

The paper is finished by Theorem 7.12 which we can consider as a generalization of Theorem 3.10 to the case of commutator Lipschitz functions.

Now we list some notation used throughout this paper.

Let $\mathrm{Aut}(\widehat{\mathbb{C}})$ denote the Möbius group of linear fractional transformations of the extended complex plane $\widehat{\mathbb{C}} \stackrel{\text{def}}{=} \mathbb{C}\cup\{\infty\}$.

We denote by $\mathrm{Aut}(\mathbb{C})$ the set of all linear transformations of the complex plane, i.e., $\mathrm{Aut}(\mathbb{C}) = \{\varphi\in\mathrm{Aut}(\widehat{\mathbb{C}}) : \varphi(\infty)=\infty\}$.

Let $\mathfrak{F}\subset\mathbb{C}$. The symbols $\mathrm{Lip}(\mathfrak{F})$, $\mathrm{OL}(\mathfrak{F})$, and $\mathrm{CL}(\mathfrak{F})$ denote the space of Lipschitz functions, the space of operator Lipschitz functions (see Sec. 2), and the space of commutator Lipschitz functions (see Sec. 7) on $\mathfrak{F}$, respectively.

Let $a\in\mathfrak{F}\subset\mathbb{C}$. Put $\mathrm{OL}_a(\mathfrak{F}) \stackrel{\text{def}}{=} \{f\in\mathrm{OL}(\mathfrak{F}) : f(a)=0\}$ and $\mathrm{CL}_a(\mathfrak{F}) \stackrel{\text{def}}{=} \{f\in\mathrm{CL}(\mathfrak{F}) : f(a)=0\}$.

The spaces of derivatives (in the corresponding senses) of operator Lipschitz functions and commutator Lipschitz functions, $(\mathrm{OL})'(\widehat{\mathbb{R}})$, $(\mathrm{OL})'_a(\mathbb{C})$, and $(\mathrm{CL})'(\mathfrak{F})$, are defined in Secs. 3, 6, and 7, respectively.

In this paper, we consider Hilbert spaces only over the complex field $\mathbb{C}$. Unless the contrary is explicitly stated, the word "operator" means a bounded operator acting in a Hilbert space (or sometimes from a Hilbert space into another Hilbert space). In particular, a normal operator means a bounded normal operator. In principle, one can assume that all operators act in a fixed infinite-dimensional separable or nonseparable Hilbert space for which we use no notation. If we need to consider one more Hilbert space, then we introduce a notation for this space. In essence, for each statement of this paper, one can say that if it is true for an infinite-dimensional Hilbert space, then it is true for every Hilbert space including finite-dimensional Hilbert spaces. Moreover, some statements remain correct and true even in the case where some operators act from a Hilbert space to another Hilbert space.

[2]Here and in what follows, we avoid to use the term "isomorphism." In the given case, this is connected with the fact that in accordance with our definitions, the first space is normed but the second space is not.

Example. Let us consider the following statement. The inequality

$$\|f(M)R - Rf(N)\| \le \|MR - RN\|$$

is fulfilled for every operator R and for every normal operators M and N. This statement describes a property of a function $f : \mathbb{C} \to \mathbb{C}$. It makes sense if M acts in a Hilbert space $\mathcal{H}_1$ and N acts in a Hilbert space $\mathcal{H}_2$. Then R must act from $\mathcal{H}_2$ into $\mathcal{H}_1$. If we prove this statement in the case where both spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ coincide with our fixed infinite-dimensional space, then of course it is true for every Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$.

2. Operator Lipschitz functions

Let $\mathfrak{F}$ be an arbitrary subset of the complex plane $\mathbb{C}$. We denote by $\operatorname{Lip}(\mathfrak{F})$ the space of functions $f : \mathfrak{F} \to \mathbb{C}$ satisfying *the Lipschitz condition*:

$$|f(z) - f(w)| \le C|z - w|, \quad z, w \in \mathbb{C}. \tag{2.1}$$

We denote by $\|f\|_{\operatorname{Lip}(\mathfrak{F})}$ the minimal constant C satisfying (2.1). Put $\|f\|_{\operatorname{Lip}(\mathfrak{F})} \stackrel{\text{def}}{=} +\infty$ if $f \notin \operatorname{Lip}(\mathfrak{F})$.

Usually we require that $\mathfrak{F}$ be closed but at first, by formal reasons, it is convenient for us to consider the case of an arbitrary set $\mathfrak{F} \subset \mathbb{C}$. Clearly, each function $f \in \operatorname{Lip}(\mathfrak{F})$ admits a unique continuous extension $\widetilde{f}$ to the closure $\operatorname{clos}\mathfrak{F}$ of $\mathfrak{F}$, $\widetilde{f} \in \operatorname{Lip}(\operatorname{clos}\mathfrak{F})$, and $\|\widetilde{f}\|_{\operatorname{Lip}(\operatorname{clos}\mathfrak{F})} = \|f\|_{\operatorname{Lip}(\mathfrak{F})}$.

It follows easily from the spectral theorem for a pair of commuting normal operators that the inequality

$$\|f(M) - f(N)\| \le \|f\|_{\operatorname{Lip}(\mathfrak{F})}\|M - N\| \tag{2.2}$$

is fulfilled for every commuting normal operators M and N with spectra in $\mathfrak{F}$.

We need the following well-known statement which allows us to approximate an arbitrary normal operator by normal operators with finite spectra.

Lemma 2.1. *Let N be a normal operator. Assume that a set Λ, $\Lambda \subset \mathbb{C}$, is an ε-net for the spectrum $\sigma(N)$ of the operator N, i.e., for every $\zeta \in \sigma(N)$ there exists a point $\lambda \in \Lambda$ such that $|\lambda - \zeta| < \varepsilon$. Then there exists a normal operator N_0 such that $NN_0 = N_0N$, $\|N - N_0\| < \varepsilon$, and $\sigma(N_0)$ is a finite subset of Λ.*

Proof. By virtue of compactness of the spectrum of N, there exists a finite ε-net Λ_0 of $\sigma(N)$ such that $\Lambda_0 \subset \Lambda$. Then we can find a Borel function $\eta : \sigma(N) \to \Lambda_0$ such that $\sup\{|z - \eta(z)| : z \in \sigma(N)\} < \varepsilon$. It remains to put $N_0 \stackrel{\text{def}}{=} \eta(N)$. □

We say that a complex-valued function f on $\mathfrak{F}$ is *operator Lipschitz* if there exists a constant C such that

$$\|f(M) - f(N)\| \le C\|M - N\| \tag{2.3}$$

for every normal operators M and N with finite spectra $\sigma(M)$ and $\sigma(N)$ lying in $\mathfrak{F}$. We denote by $\operatorname{OL}(\mathfrak{F})$ the set of operator Lipschitz functions defined on $\mathfrak{F}$. We denote by $\|f\|_{\operatorname{OL}(\mathfrak{F})}$ the minimal constant C satisfying (2.3). Put $\|f\|_{\operatorname{OL}(\mathfrak{F})} = +\infty$ if $f \notin \operatorname{OL}(\mathfrak{F})$.

If a function f is defined on a larger set $\Lambda \supset \mathfrak{F}$, then for the sake of brevity we write $f \in \operatorname{OL}(\mathfrak{F})$ and $\|f\|_{\operatorname{OL}(\mathfrak{F})}$ instead of $f|\mathfrak{F} \in \operatorname{OL}(\mathfrak{F})$ and $\|f|\mathfrak{F}\|_{\operatorname{OL}(\mathfrak{F})}$. We also apply this agreement for other function spaces.

It is easily seen that $\operatorname{OL}(\mathfrak{F}) \subset \operatorname{Lip}(\mathfrak{F})$ and $\|f\|_{\operatorname{Lip}(\mathfrak{F})} \le \|f\|_{\operatorname{OL}(\mathfrak{F})}$ for every $f \in \operatorname{OL}(\mathfrak{F})$. In particular, each function $f \in \operatorname{OL}(\mathfrak{F})$ is continuous. Thus, the operator $f(M)$ is well defined for every normal operator M with $\sigma(M) \subset \mathfrak{F}$. It follows from Lemma 2.1 and inequality (2.2) that the inequality

$$\|f(M) - f(N)\| \le \|f\|_{\operatorname{OL}(\mathfrak{F})}\|M - N\| \tag{2.4}$$

holds for every normal[3] operators M and N with $\sigma(M), \sigma(N) \subset \mathfrak{F}$.

Let us observe also that Lemma 2.1 and inequality (2.2) imply the equality

$$\|f\|_{\operatorname{OL}(\mathfrak{F})} = \sup\big\{\|f\|_{\operatorname{OL}(\Lambda)} : \Lambda \subset \mathfrak{F},\ \Lambda \text{ is finite}\big\} \tag{2.5}$$

and even a stronger statement:

$$\|f\|_{\operatorname{OL}(\mathfrak{F})} = \sup\big\{\|f\|_{\operatorname{OL}(\Lambda)} : \Lambda \subset \mathfrak{F}_0,\ \Lambda \text{ is finite}\big\}, \tag{2.6}$$

where $\mathfrak{F}_0$ is a dense subset of $\mathfrak{F}$.

We need the following elementary statement.

[3] In this paper, we consider only bounded operators although standard methods exhibit that inequality (2.4) also holds for unbounded M and N (e.g., see [3]).

Lemma 2.2. *Let $f \in \mathrm{OL}(\mathfrak{F})$. Then there exists a unique extension of f to a continuous function $\widetilde{f}$ on $\operatorname{clos}\mathfrak{F}$, and $\|\widetilde{f}\|_{\mathrm{OL}(\operatorname{clos}\mathfrak{F})} = \|f\|_{\mathrm{OL}(\mathfrak{F})}$.*

Proof. The existence and uniqueness of a continuous extension follow from the fact that $\mathrm{OL}(\mathfrak{F}) \subset \mathrm{Lip}(\mathfrak{F})$. The equality $\|\widetilde{f}\|_{\mathrm{OL}(\operatorname{clos}\mathfrak{F})} = \|f\|_{\mathrm{OL}(\mathfrak{F})}$ follows from (2.6). □

It is well known that if $\mathfrak{F} \subset \mathbb{R}$, then every function $f \in \mathrm{OL}(\mathfrak{F})$ is differentiable at each nonisolated point in $\mathfrak{F}$ (see [5] and [8]). Similarly, for unbounded sets $\mathfrak{F} \subset \mathbb{R}$, one can assert that every function $f \in \mathrm{OL}(\mathfrak{F})$ has a finite derivative at infinity if we put $f'(\infty) \stackrel{\text{def}}{=} \lim\limits_{|t|\to+\infty} t^{-1}f(t)$, see also Theorem 4.16 of [1].

The result concerning the differentiability of an operator Lipschitz function of a real variable yields the following statement for an operator Lipschitz function of a complex variable. If $f \in \mathrm{OL}(\mathfrak{F})$, where $\mathfrak{F} \subset \mathbb{C}$, then, for every straight line $l \subset \mathbb{C}$, the restriction $f|(l \cap \mathfrak{F})$ has a finite derivative at each nonisolated point in $l \cap \mathfrak{F}$ and at infinity provided that $l \cap \mathfrak{F}$ is unbounded. In fact, as follows from results by Kissin and Shulman [8], the straight line l can be replaced by a simple closed curve of class C^2 on the Riemann sphere $\widehat{\mathbb{C}}$.

In particular, a function $f \in \mathrm{OL}(\mathfrak{F})$ must have finite direction derivatives along all directions at each interior point of $\mathfrak{F}$. Nevertheless, Theorem 3.5 in [3] exhibits that an operator Lipschitz function f does not have to be differentiable as a function of two real variables at each (interior) point of its domain, see also Corollary 4.3 below.

The following statement is contained in Theorem 3.1 of [3] but in essence it can be extracted from [8], where it is considered not only for operator norms but also for arbitrary symmetric norms.

Theorem 2.3. *Let f be a function defined on a closed subset $\mathfrak{F}$ of $\mathbb{C}$. Then the following three statements are equivalent:*

(i) $\|f(M) - f(N)\| \le \|M - N\|$ *for every normal operators M and N such that $\sigma(M), \sigma(N) \subset \mathfrak{F}$;*

(ii) $\|f(N)R - Rf(N)\| \le \max\big(\|NR - RN\|, \|N^*R - RN^*\|\big)$ *for every operator R and every normal operator N such that $\sigma(N) \subset \mathfrak{F}$;*

(iii) $\|f(M)R - Rf(N)\| \le \max\big(\|MR - RN\|, \|M^*R - RN^*\|\big)$ *for every operator R and every normal operators M and N such that $\sigma(M), \sigma(N) \subset \mathfrak{F}$.*

Remark. As was noted above, it suffices to verify statement (i) only for normal operators M and N with finite spectra lying in $\mathfrak{F}$. Moreover, equality (2.6) implies that we can require in addition that the spectra of M and N be contained in a fixed dense subset of $\mathfrak{F}$. The same can be said concerning statements (ii) and (iii) with the only difference in the case of statement (ii) where we have one normal operator N, so we should require the corresponding additional conditions only for N.

It also follows from this remark that we can remove the assumption of closedness of $\mathfrak{F}$ in Theorem 2.3.

Assuming in Theorem 2.3 that $\mathfrak{F}$ is contained in $\mathbb{R}$, we obtain the following statement concerning self-adjoint operators.

Theorem 2.4. *Let f be a continuous function defined on a closed subset $\mathfrak{F}$ of $\mathbb{R}$. Then the following three statements are equivalent:*

(i) $\|f(A) - f(B)\| \le \|A - B\|$ *for every self-adjoint operators A and B such that $\sigma(A), \sigma(B) \subset \mathfrak{F}$;*

(ii) $\|f(A)R - Rf(A)\| \le \|AR - RA\|$ *for every operator R and every self-adjoint operator A such that $\sigma(A) \subset \mathfrak{F}$;*

(iii) $\|f(A)R - Rf(B)\| \le \|AR - RB\|$ *for every operator R and every self-adjoint operators A and B such that $\sigma(A), \sigma(B) \subset \mathfrak{F}$.*

Let us also state a version of Theorem 2.3 for unitary operators.

Theorem 2.5. *Let f be a continuous function defined on a closed subset $\mathfrak{F}$ of the unit circle $\mathbb{T} \stackrel{\text{def}}{=} \{z \in \mathbb{C} : |z| = 1\}$. Then the following three statements are equivalent:*

(i) $\|f(U) - f(V)\| \le \|U - V\|$ *for every unitary operators U and V such that $\sigma(U), \sigma(V) \subset \mathfrak{F}$;*

(ii) $\|f(U)R - Rf(U)\| \le \|UR - RU\|$ *for every operator R and every unitary operators U such that $\sigma(U) \subset \mathfrak{F}$;*

(iii) $\|f(U)R - Rf(V)\| \le \|UR - RV\|$ *for every operator R and every unitary operators U and V such that $\sigma(U), \sigma(V) \subset \mathfrak{F}$.*

The Bochner integral (see, e.g., [4]) allows us to integrate[4] in $\mathrm{OL}(\mathfrak{F})$ strongly measurable functions $\omega \mapsto \Phi_\omega$ such that the function $\omega \mapsto \|\Phi_\omega\|$ is summable. Recall that a strongly measurable function is almost separably-valued. On the other hand, the space $\mathrm{OL}(\mathfrak{F})$ can be separable only for degenerate sets $\mathfrak{F}$. Therefore, the Bochner integral is not enough convenient to integrate $\mathrm{OL}(\mathfrak{F})$-valued functions. For example, it is easily seen that the function $\Phi(\omega) = \sin \omega x$, where $\omega \in [0,1]$, is not Bochner integrable in $\mathrm{OL}(\mathbb{R})$ because it is not almost separably-valued.

We need the following lemma concerning integration in $\mathrm{OL}(\mathfrak{F})$ of weakly measurable functions.

Lemma 2.6. *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. Assume that a mapping $\omega \mapsto \Phi_\omega$ defined almost everywhere on Ω acts into $\mathrm{OL}(\mathfrak{F})$ and possesses the following properties:*

(a) *the function $\omega \mapsto \Phi_\omega(z)$ is measurable for all $z \in \mathfrak{F}$;*

(b) *the function $\omega \mapsto \Phi_\omega(z_0)$ is summable for some $z_0 \in \mathfrak{F}$;*

(c) *the function $\omega \mapsto \|\Phi_\omega\|_{\mathrm{OL}(\mathfrak{F})}$ is summable.*

Then the function $\omega \mapsto \Phi_\omega(z)$ is summable for all $z \in \mathfrak{F}$, the function $f(z) \overset{\text{def}}{=} \int_\Omega \Phi_\omega(z)\, d\mu(\omega)$ belongs to $\mathrm{OL}(\mathfrak{F})$, and

$$\|f\|_{\mathrm{OL}(\mathfrak{F})} \le \int_\Omega \|\Phi_\omega\|_{\mathrm{OL}(\mathfrak{F})}\, d\mu(\omega).$$

Proof. To prove the summability of the function $\omega \mapsto \Phi_\omega(z)$, it suffices to note that

$$|\Phi_\omega(z)| \le |\Phi_\omega(z_0)| + |\Phi_\omega(z) - \Phi_\omega(z_0)| \le |\Phi_\omega(z_0)| + |z - z_0| \cdot \|\Phi_\omega\|_{\mathrm{OL}(\mathfrak{F})}.$$

It remains to estimate $\|f\|_{\mathrm{OL}(\mathfrak{F})}$. The required estimate is evident in the case where $\mathfrak{F}$ is finite because in this case, the space $\mathrm{OL}(\mathfrak{F})$ is finite-dimensional, and so weak measurability coincides with strong measurability. The case of an arbitrary set $\mathfrak{F}$ reduces to the case of a finite set $\mathfrak{F}$ with the help of equality (2.5). □

We need one more elementary statement.

Lemma 2.7. *Let f be a function defined on a set $\mathfrak{F} \cup \{z_0\}$, where $\mathfrak{F}$ is a closed subset of the complex plane $\mathbb{C}$, $z_0 \in \mathbb{C} \setminus \mathfrak{F}$. Then*

$$\|f\|_{\mathrm{OL}(\mathfrak{F} \cup \{z_0\})} \le \|f\|_{\mathrm{OL}(\mathfrak{F})} + \sup_{z \in \mathfrak{F}} \frac{|f(z) - f(z_0)|}{|z - z_0|}.$$

Proof. It suffices to consider the case where $z_0 = 0$ and $f(0) = 0$. Put

$$g(z) \overset{\text{def}}{=} \begin{cases} z^{-1} f(z) & \text{if } z \in \mathfrak{F}, \\ 0 & \text{if } z = 0. \end{cases}$$

By Theorem 2.3, we have to prove the following inequality:

$$\|f(N)R - Rf(N)\| \le \bigl(\|f\|_{\mathrm{OL}(\mathfrak{F})} + \sup |g|\bigr) \max\bigl(\|NR - RN\|, \|N^*R - RN^*\|\bigr)$$

for every operator R and every normal operator N with finite spectrum lying in $\mathfrak{F} \cup \{0\}$. Let $\mathcal{H}_0$ denote the range of N. Note that $\mathcal{H}_0$ is a closed subspace since N is a normal operator with finite spectrum. It is clear that $\mathcal{H}_0$ is a reducing subspace for N. Hence, the operator $N_0 : \mathcal{H}_0 \to \mathcal{H}_0$, $N_0 u \overset{\text{def}}{=} Nu$, is normal. It is easily seen that $\sigma(N_0) \subset \mathfrak{F}$. Let P denote the orthogonal projection onto $\mathcal{H}_0$, and let Q denote the orthogonal projection onto the kernel of N. Then $P + Q = I$, where I denotes the identity operator. Clearly,

$$f(N)R - Rf(N) = f(N)RP + f(N)RQ - PRf(N) - QRf(N),$$

whence

$$\|f(N)R - Rf(N)\| \le \|f(N)RP - PRf(N)\| + \|f(N)RQ - QRf(N)\|.$$

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[4]Rigorously speaking, $\mathrm{OL}(\mathfrak{F})$ is not a Banach space because $\|\mathrm{const}\|_{\mathrm{OL}(\mathfrak{F})} = 0$, but it can be changed into a Banach space replacing the seminorm $\|f\|_{\mathrm{OL}(\mathfrak{F})}$ by the norm $\|f\|_{\mathrm{OL}(\mathfrak{F})} + \varepsilon |f(z_0)|$, where z_0 is a point in the set $\mathfrak{F}$, $\varepsilon > 0$. Thus, we have to require not only the summability of $\omega \mapsto \|\Phi_\omega\|$ but also the summability of $\omega \mapsto \Phi_\omega(z_0)$ for some $z_0 \in \mathfrak{F}$.

Let us observe that

$$\begin{aligned}\|f(N)RP-PRf(N)\| &= \|f(N_0)PRP-PRPf(N_0)\|\\ &\le \|f\|_{\mathrm{OL}(\mathfrak{F})}\max\big(\|N_0PRP-PRPN_0\|,\|N_0^*PRP-PRPN_0^*\|\big)\\ &= \|f\|_{\mathrm{OL}(\mathfrak{F})}\max\big(\|P(NR-RN)P\|,\|P(N^*R-RN^*)P\|\big)\\ &\le \|f\|_{\mathrm{OL}(\mathfrak{F})}\max\big(\|NR-RN\|,\|N^*R-RN^*\|\big).\end{aligned}$$

Now we prove that

$$\|f(N)RQ-QRf(N)\|\le\big(\sup|g|\big)\|NR-RN\|.$$

To do this, we use the following elementary identity:

$$\|PAQ+QBP\|=\max(\|PAQ\|,\|QBP\|)$$

for every operators A and B. We have

$$\begin{aligned}\|f(N)RQ-QRf(N)\| &= \|Pf(N)RQ-QRf(N)P\|=\max(\|f(N)RQ\|,\|QRf(N)\|)\\ &\le \|g(N)\|\max(\|NRQ\|,\|QRN\|)\\ &= \|g(N)\|\max(\|P(NR-RN)Q\|,\|Q(RN-NR)P\|)\le\big(\sup|g|\big)\|NR-RN\|.\end{aligned}$$ □

3. Operator Lipschitz functions on the real line and linear fractional transformations

Theorem 3.1. *Let $f\in\mathrm{OL}(\mathbb{R})$. Put*

$$g(t)\stackrel{\mathrm{def}}{=}\begin{cases} t^2(f(t^{-1})-f(0)) & \text{if } t\ne0,\\ 0 & \text{if } t=0.\end{cases}\tag{3.1}$$

Then $g\in\mathrm{OL}(\mathbb{R})$ and $\|g\|_{\mathrm{OL}(\mathbb{R})}\le3\|f\|_{\mathrm{OL}(\mathbb{R})}$.

Proof. We can assume that $\|f\|_{\mathrm{OL}(\mathbb{R})}=1$ and $f(0)=0$. By Lemma 2.2, it suffices to prove that

$$\|A^2f(A^{-1})-B^2f(B^{-1})\|\le3\|A-B\|$$

for every invertible self-adjoint operators A and B. We have

$$f(A^{-1})A^2-B^2f(B^{-1})=f(A^{-1})A(A-B)+f(A^{-1})AB-ABf(B^{-1})+(A-B)Bf(B^{-1}).$$

Clearly,

$$\|Af(A^{-1})\|\le\sup_{t\ne0}|t^{-1}f(t)|\le\|f\|_{\mathrm{Lip}(\mathbb{R})}\le\|f\|_{\mathrm{OL}(\mathbb{R})}=1.$$

Hence, $\|f(A^{-1})A(A-B)\|\le\|A-B\|$, and the same reasoning shows that $\|(A-B)Bf(B^{-1})\|\le\|A-B\|$. Finally, applying Theorem 2.4, we conclude that

$$\|f(A^{-1})AB-ABf(B^{-1})\|\le\|A^{-1}AB-ABB^{-1}\|=\|A-B\|.$$ □

To state a generalization of Theorem 3.1, we define the transformation T_φ,

$$(T_\varphi f)(t)\stackrel{\mathrm{def}}{=}\begin{cases}\dfrac{f(\varphi(t))}{\varphi'(t)} & \text{if } t\in\mathbb{R} \text{ and } \varphi(t)\ne\infty,\\ 0 & \text{if } t\in\mathbb{R} \text{ and } \varphi(t)=\infty.\end{cases}$$

Here f denotes a function defined on $\mathbb{R}$ and φ is a real linear fractional transformation which is not linear.

Recall that $\mathrm{OL}_a(\mathbb{R})\stackrel{\mathrm{def}}{=}\{f\in\mathrm{OL}(\mathbb{R}):f(a)=0\}$, where $a\in\mathbb{R}$.

Theorem 3.2. *Let φ be a (nonlinear) real linear fractional transformation. Then*

$$\|T_\varphi f\|_{\mathrm{OL}(\mathbb{R})}\le3\|f\|_{\mathrm{OL}(\mathbb{R})}$$

for all $f\in\mathrm{OL}_a(\mathbb{R})$, where $a=\varphi(\infty)$.

Proof. Put $\phi_c(t)\stackrel{\mathrm{def}}{=}ct^{-1}$, where $c\in\mathbb{R}$, $c\ne0$. Note that the case where $\varphi=\phi_c$ reduces to Theorem 3.1 (i.e., to the case $c=1$) with the help of a homothety with center at zero. In the general case, the transformation φ can be represented in the form $\varphi(t)=a+\dfrac{c}{t-b}$, where $b=\varphi^{-1}(\infty)$, $c\in\mathbb{R}$, $c\ne0$. Put $\tau_h(t)\stackrel{\mathrm{def}}{=}t-h$. The evident identity $T_\varphi f=(T_{\phi_c}(f\circ\tau_a^{-1}))\circ\tau_b$ allows us to reduce the general case to the case $\varphi=\phi_c$. □

Corollary 3.3. *Let the assumptions of Theorem* 3.2 *be satisfied. Then*

$$3\|T_\varphi f\|_{\mathrm{OL}(\mathbb{R})} \ge \|f\|_{\mathrm{OL}(\mathbb{R})}$$

for all $f \in \mathrm{OL}_a(\mathbb{R})$, *where* $a = \varphi(\infty)$.

Proof. Let ψ be the linear fractional transformation which is the inverse of φ. It is easily seen that $T_\psi(T_\varphi f) = f$ and $(T_\varphi f)(\psi(\infty)) = 0$. Hence,

$$\|f\|_{\mathrm{OL}(\mathbb{R})} \le 3\|T_\varphi f\|_{\mathrm{OL}(\mathbb{R})}$$

by Theorem 3.2. □

Theorem 3.4. *Let f be a continuous function on the real line $\mathbb{R}$ such that $(x-a)f(x) \in \mathrm{OL}(\mathbb{R})$ for some $a \in \mathbb{R}$. Put $F(x) \overset{\mathrm{def}}{=} \int_0^x f(t)\,dt$. Then $F \in \mathrm{OL}(\mathbb{R})$ and $\|F\|_{\mathrm{OL}(\mathbb{R})} \le \|(x-a)f(x)\|_{\mathrm{OL}(\mathbb{R})}$.*

Proof. We can assume that $a = 0$. Then

$$F(x) = \int_0^1 xf(tx)\,dt,$$

and we can apply Lemma 2.6 because

$$\|xf(tx)\|_{\mathrm{OL}(\mathbb{R})} = \|xf(x)\|_{\mathrm{OL}(\mathbb{R})}$$

for all $t \in (0,1]$. □

Corollary 3.5. *Let $(\Theta_1 f) \overset{\mathrm{def}}{=} tf'(t)$. Then $\Theta_1(\mathrm{OL}(\mathbb{R})) \supset \mathrm{OL}_0(\mathbb{R})$.*

Let f and g denote the same as in Theorem 3.1. Then

$$g(x) = \int_0^x g'(t)\,dt = 2\int_0^x t\big(f(t^{-1}) - f(0)\big)\,dt - \int_0^x f'(t^{-1})\,dt \overset{\mathrm{def}}{=} 2g_1(x) - g_2(x). \tag{3.2}$$

The following theorem exhibits that not only $g \in \mathrm{OL}(\mathbb{R})$ but the functions g_1 and g_2 also must belong to $\mathrm{OL}(\mathbb{R})$.

Theorem 3.6. *Let $f \in \mathrm{OL}(\mathbb{R})$ and let functions g_1 and g_2 be defined by formula* (3.2). *Then $g_1, g_2 \in \mathrm{OL}(\mathbb{R})$, $\|g_1\|_{\mathrm{OL}(\mathbb{R})} \le 3\|f\|_{\mathrm{OL}(\mathbb{R})}$, and $\|g_2\|_{\mathrm{OL}(\mathbb{R})} \le 9\|f\|_{\mathrm{OL}(\mathbb{R})}$.*

Proof. Note that $\|g\|_{\mathrm{OL}(\mathbb{R})} \le 3\|f\|_{\mathrm{OL}(\mathbb{R})}$ by Theorem 3.1. Now applying Theorem 3.4 to the function g, we get the inequalities $\|g_1\|_{\mathrm{OL}(\mathbb{R})} \le \|g\|_{\mathrm{OL}(\mathbb{R})} \le 3\|f\|_{\mathrm{OL}(\mathbb{R})}$. To estimate the $\mathrm{OL}(\mathbb{R})$-seminorm of g_2, it suffices to use the equality $g_2 = 2g_1 - g$. □

Corollary 3.7. *Let $f \in \Theta_1(\mathrm{OL}(\mathbb{R}))$, where Θ_1 denotes the same as in Corollary* 3.5. *Let us define g by formula* (3.1). *Then $g \in \Theta_1(\mathrm{OL}(\mathbb{R}))$.*

Put

$$(L_\varphi f)(x) \overset{\mathrm{def}}{=} \int_0^x f'(\varphi(t))\,dt,$$

where f is a Lipschitz function on $\mathbb{R}$ and φ is a real linear fractional transformation. It is clear that $L_{\varphi\circ\psi} = L_\psi \circ L_\varphi$ for every real linear fractional transformations φ and ψ. Moreover, $L_\varphi f = f - f(0)$ if $\varphi(t) = t$.

Theorem 3.8. *Let φ be a real linear fractional transformation. Then*

$$\frac{1}{9}\|f\|_{\mathrm{OL}(\mathbb{R})} \le \|L_\varphi f\|_{\mathrm{OL}(\mathbb{R})} \le 9\|f\|_{\mathrm{OL}(\mathbb{R})}$$

for all $f \in \mathrm{OL}(\mathbb{R})$.

Proof. Note that $\|L_\varphi f\|_{\mathrm{OL}(\mathbb{R})} = \|f\|_{\mathrm{OL}(\mathbb{R})}$ if φ is a nonconstant real-valued linear function. Moreover, by Theorem 3.6, $\|L_\varphi f\|_{\mathrm{OL}(\mathbb{R})} \le 9\|f\|_{\mathrm{OL}(\mathbb{R})}$ in the case where $\varphi(t) = t^{-1}$. Every (nonlinear) real linear fractional transformation φ can be represented in the form $\varphi = \varphi_1 \circ \varphi_2 \circ \varphi_3$, where φ_1 and φ_3 are real-valued linear functions and $\varphi_2(t) = t^{-1}$. Now the inequality $\|L_\varphi f\|_{\mathrm{OL}(\mathbb{R})} \le 9\|f\|_{\mathrm{OL}(\mathbb{R})}$ is evident. The inequality $\|f\|_{\mathrm{OL}(\mathbb{R})} \le 9\|L_\varphi f\|_{\mathrm{OL}(\mathbb{R})}$ reduces to the inequality which had just been proved with the help of consideration of the inverse transformation of φ in the same way as in the proof of Corollary 3.3. □

We recall that $\widehat{\mathbb{R}}$ denotes the one-point compactification of the real line $\mathbb{R}$. Put $f'(\infty) \stackrel{\text{def}}{=} \lim\limits_{t\to\infty} t^{-1}f(t)$. As was noted in Sec. 2, every function $f \in \mathrm{OL}(\mathbb{R})$ has a finite derivative at each point $t \in \widehat{\mathbb{R}}$. Denote by $(\mathrm{OL})'(\widehat{\mathbb{R}})$ the space of all bounded functions h defined everywhere on $\widehat{\mathbb{R}}$ and such that $f'(t) = h(t)$ for all $t \in \widehat{\mathbb{R}}$ for some function $f \in \mathrm{OL}(\mathbb{R})$. Put $\|h\|_{(\mathrm{OL})'(\widehat{\mathbb{R}})} \stackrel{\text{def}}{=} \|f\|_{\mathrm{OL}(\mathbb{R})}$. It is clear that $\|h\|_{(\mathrm{OL})'(\widehat{\mathbb{R}})}$ is well defined. The space $(\mathrm{OL})'(\widehat{\mathbb{R}})$ with the norm $\|\cdot\|_{(\mathrm{OL})'(\widehat{\mathbb{R}})}$ is a Banach space.

Theorem 3.4 readily implies the following theorem.

Theorem 3.9. *Let $f \in \mathrm{OL}(\mathbb{R})$. Put*

$$g(t) \stackrel{\text{def}}{=} \begin{cases} \dfrac{f(t)-f(t_0)}{t-t_0} & \text{if } t \in \mathbb{R},\ t \neq t_0, \\ f'(t_0) & \text{if } t = t_0, \\ f'(\infty) & \text{if } t = \infty, \end{cases}$$

where $t_0 \in \mathbb{R}$. Then $g \in (\mathrm{OL})'(\mathbb{R})$ and $\|g\|_{(\mathrm{OL})'(\mathbb{R})} \le \|f\|_{\mathrm{OL}(\mathbb{R})}$.

Proof. Put $F(x) \stackrel{\text{def}}{=} \int_0^x g(t)\,dt$. Then $F'(x) = g(x)$ for all $x \in \widehat{\mathbb{R}}$ and $\|F\|_{\mathrm{OL}(\mathbb{R})} \le \|(x-t_0)g(x)\|_{\mathrm{OL}(\mathbb{R})} = \|f\|_{\mathrm{OL}(\mathbb{R})}$ by Theorem 3.4. ☐

Now we restate Theorem 3.8.

Theorem 3.10. *Let φ be a real linear fractional transformation. Then*

$$\frac{1}{9}\|h\|_{(\mathrm{OL})'(\widehat{\mathbb{R}})} \le \|h\circ\varphi\|_{(\mathrm{OL})'(\widehat{\mathbb{R}})} \le 9\|h\|_{(\mathrm{OL})'(\widehat{\mathbb{R}})}$$

for every function h defined on $\widehat{\mathbb{R}}$.

Proof. Put $f \stackrel{\text{def}}{=} \int_0^x h(t)\,dt$. It is clear that $f(0) = 0$ and $f'(x) = h(x)$ for all $x \in \widehat{\mathbb{R}}$. Applying Theorem 3.8 to the function f, we get the inequalities

$$\frac{1}{9}\|h\|_{(\mathrm{OL})'(\widehat{\mathbb{R}})} \le \|(L_\varphi f)'\|_{(\mathrm{OL})'(\widehat{\mathbb{R}})} \le 9\|h\|_{(\mathrm{OL})'(\widehat{\mathbb{R}})}.$$

It is easily seen that $(L_\varphi f)' = h\circ\varphi$ almost everywhere on $\mathbb{R}$, but we need to verify that this equality is fulfilled everywhere on $\widehat{\mathbb{R}}$. Again it suffices to restrict ourselves to the case $\varphi(t) = t^{-1}$. We rewrite equality (3.2) as follows:

$$(L_\varphi f)(x) = \int_0^x f'(t^{-1})\,dt = -\int_0^x t^2\big(f(t^{-1})\big)'\,dt = -g(x) + 2\int_0^x tf(t^{-1})\,dt.$$

It is clear that

$$g'(x) = \begin{cases} 2xf(x^{-1}) - f'(x^{-1}) & \text{if } x \in \mathbb{R},\ \ x \neq 0, \\ f'(\infty) & \text{if } x = 0, \\ f'(0) & \text{if } x = \infty. \end{cases}$$

Note that the function $tf(t^{-1})$ extends to a continuous function on $\widehat{\mathbb{R}}$. Hence,

$$\left(\int_0^x tf(t^{-1})\,dt\right)' = \begin{cases} xf(x^{-1}) & \text{if } x \in \mathbb{R}, \quad x \neq 0, \\ f'(\infty) & \text{if } x = 0, \\ f'(0) & \text{if } x = \infty. \end{cases}$$

Now it is clear that the equality $(L_\varphi f)' = h\circ\varphi$ is fulfilled everywhere on $\widehat{\mathbb{R}}$. ☐

Example. Let us consider the function

$$u(t) \stackrel{\text{def}}{=} \begin{cases} \cos t & \text{if } t \in \mathbb{R}, \\ 0 & \text{if } t = \infty. \end{cases}$$

It is clear that $u \in (\mathrm{OL})'(\widehat{\mathbb{R}})$. Hence, the function $\dfrac{1+u(2t)}{2}$ also belongs to $(\mathrm{OL})'(\widehat{\mathbb{R}})$. Note that $u^2(t) = \dfrac{1+u(2t)}{2}$ for all $t \in \mathbb{R}$ but $u^2(t) \neq \dfrac{1+u(2t)}{2}$ for $t = \infty$. Hence, $u^2 \notin (\mathrm{OL})'(\widehat{\mathbb{R}})$. Of course, using a linear fractional change of a variable, we can construct a function $h \in (\mathrm{OL})'(\widehat{\mathbb{R}})$ such that h^2 does not coincide identically on $\mathbb{R}$ with any function in $(\mathrm{OL})'(\widehat{\mathbb{R}})$. Thus, the space $(\mathrm{OL})'(\widehat{\mathbb{R}})$ is not an algebra with respect to the point-to-point multiplication even if we consider functions in $(\mathrm{OL})'(\widehat{\mathbb{R}})$ as functions defined on $\mathbb{R}$.

The author does not know whether $(\mathrm{OL})'(\widehat{\mathbb{R}})$ is an algebra if we identify functions that coincide almost everywhere, i.e., if we consider $(\mathrm{OL})'(\widehat{\mathbb{R}})$ as a subspace of $L^\infty(\mathbb{R})$.

4. Multipliers of operator Lipschitz functions

Let $\mathfrak{F}$ be a closed subset of the complex plane $\mathbb{C}$. Denote by $\mathfrak{M}(\mathfrak{F})$ the set of functions $w : \mathfrak{F} \to \mathbb{C}$ such that $f \in \mathrm{OL}(\mathfrak{F}) \Longrightarrow wf \in \mathrm{OL}(\mathfrak{F})$ for every $f \in \mathrm{OL}(\mathfrak{F})$.

We recall that $\mathrm{OL}_a(\mathfrak{F}) \stackrel{\text{def}}{=} \{f \in \mathrm{OL}(\mathfrak{F}) : f(a) = 0\}$, where $a \in \mathfrak{F}$. Denote by $\mathfrak{M}_a(\mathfrak{F})$ the set of functions $w : \mathfrak{F} \to \mathbb{C}$ such that $f \in \mathrm{OL}_a(\mathfrak{F}) \Longrightarrow wf \in \mathrm{OL}_a(\mathfrak{F})$ for every $f \in \mathrm{OL}_a(\mathfrak{F})$. Put

$$\|w\|_{\mathfrak{M}_a(\mathfrak{F})} \stackrel{\text{def}}{=} \sup\bigl\{\|wf\|_{\mathrm{OL}(\mathfrak{F})} : f \in \mathrm{OL}_a(\mathfrak{F}), \|f\|_{\mathrm{OL}(\mathfrak{F})} \le 1\bigr\}.$$

Remark. It follows from Lemma 2.7 that $\mathfrak{M}(\mathfrak{F}) = \mathfrak{M}_a(\mathfrak{F})$ if a is an isolated point of $\mathfrak{F}$.

It is easily seen that $\|w\|_{\mathfrak{M}_a(\mathfrak{F})} = 0$ if and only if $w(z) = 0$ for all $z \in \mathfrak{F} \setminus \{a\}$. Thus, the quantity $\|w\|_{\mathfrak{M}_a(\mathfrak{F})}$ depends only on the restriction $w|(\mathfrak{F} \setminus \{a\})$, and so it does not depend on the value of w at a. Taking this into account, sometimes we consider the space $\mathfrak{M}_a(\mathfrak{F})$ as a space of functions defined on $\mathfrak{F} \setminus \{a\}$.

The following theorem gives us a description of the space $\mathfrak{M}_a(\mathfrak{F})$.

Theorem 4.1. *Let w be a function defined on a closed set $\mathfrak{F}$, $\mathfrak{F} \subset \mathbb{C}$. Assume that $a \in \mathfrak{F}$. Then the following statements are equivalent:*

(i) $w \in \mathfrak{M}_a(\mathfrak{F})$;

(ii) *the function $(z-a)w(z)$ is operator Lipschitz on $\mathfrak{F}$;*

(iii) *the function $(\overline{z}-\overline{a})w(z)$ is operator Lipschitz on $\mathfrak{F}$.*

Proof. It suffices to consider the case where $a = 0$. The implications (i) $\Longrightarrow$ (ii) and (i) $\Longrightarrow$ (iii) are trivial. Let us prove that (ii) $\Longrightarrow$ (i). It is clear that

$$\sup\{|w(z)| : z \in \mathfrak{F}, z \neq 0\} \le \|zw(z)\|_{\mathrm{Lip}(\mathfrak{F})} \le \|zw(z)\|_{\mathrm{OL}(\mathfrak{F})}.$$

We can assume that $w(0) = 0$. Then $\sup|w| \le \|zw(z)\|_{\mathrm{OL}(\mathfrak{F})}$.

Let $f \in \mathrm{OL}_0(\mathfrak{F})$. Put

$$g(z) \stackrel{\text{def}}{=} \begin{cases} z^{-1}f(z) & \text{if } z \in \mathfrak{F} \text{ and } z \neq 0, \\ 0 & \text{if } z = 0. \end{cases}$$

Then $\sup|g| \le \|f\|_{\mathrm{Lip}(\mathfrak{F})} \le \|f\|_{\mathrm{OL}(\mathfrak{F})}$. Using the identity

$$\begin{aligned} w(M)f(M) - w(N)f(N) &= w(M)(f(M)-f(N)) + (w(M)-w(N))f(N) \\ &= w(M)(f(M)-f(N)) - w(M)(M-N)g(N) + (Mw(M)-Nw(N))g(N), \end{aligned}$$

we get the inequalities

$$\begin{aligned} \|w(M)f(M) - w(N)f(N)\| \le \bigl(\sup|w|\bigr)\|f(M)-f(N)\| + \bigl(\sup|w|\bigr)\bigl(\sup|g|\bigr)\|M-N\| \\ + \bigl(\sup|g|\bigr)\|Mw(M)-Nw(N)\| \le 3\|zw(z)\|_{\mathrm{OL}(\mathfrak{F})}\|f\|_{\mathrm{OL}(\mathfrak{F})}\|M-N\|. \end{aligned}$$

The implication (iii) $\Longrightarrow$ (i) can be proved in a similar way. Let $f \in \mathrm{OL}_0(\mathfrak{F})$. Put

$$g(z) \stackrel{\text{def}}{=} \begin{cases} \overline{z}^{-1} f(z) & \text{if } z \in \mathfrak{F} \quad \text{and} \quad z \neq 0, \\ 0 & \text{if } z = 0. \end{cases}$$

We have

$$\begin{aligned} w(M)f(M) - w(N)f(N) &= w(M)(f(M) - f(N)) + (w(M) - w(N))f(N) \\ &= w(M)(f(M) - f(N)) - w(M)(M^* - N^*)g(N) + (M^*w(M) - N^*w(N))g(N), \end{aligned}$$

whence

$$\begin{aligned} \|w(M)f(M) - w(N)f(N)\| &\le \big(\sup|w|\big)\|f(M) - f(N)\| + \big(\sup|w|\big)\big(\sup|g|\big)\|M - N\| \\ &+ \big(\sup|g|\big)\|M^*w(M) - N^*w(N)\| \le 3\|\overline{z}w(z)\|_{\mathrm{OL}(\mathfrak{F})}\|f\|_{\mathrm{OL}(\mathfrak{F})}\|M - N\|. \end{aligned} \qquad \square$$

Remark 1. If the space $\mathfrak{M}_a(\mathfrak{F})$ is considered as a space of functions defined on $\mathfrak{F} \setminus \{a\}$, then Theorem 4.1 can be restated as follows:

$$\mathfrak{M}_a(\mathfrak{F}) = (z - a)^{-1}\,\mathrm{OL}_a(\mathfrak{F}) = (\overline{z} - \overline{a})^{-1}\,\mathrm{OL}_a(\mathfrak{F}).$$

Remark 2. It is clear that

$$\|w\|_{\mathfrak{M}_a(\mathfrak{F})} \ge \max\big(\|(z - a)w(z)\|_{\mathrm{OL}(\mathfrak{F})}, \|(\overline{z} - \overline{a})w(z)\|_{\mathrm{OL}(\mathfrak{F})}\big).$$

It can be seen from the proof of the theorem that the following inequality holds:

$$\|w\|_{\mathfrak{M}_a(\mathfrak{F})} \le 3\min\big(\|(z - a)w(z)\|_{\mathrm{OL}(\mathfrak{F})}, \|(\overline{z} - \overline{a})w(z)\|_{\mathrm{OL}(\mathfrak{F})}\big). \tag{4.1}$$

Put

$$\operatorname{sgn} z \stackrel{\text{def}}{=} \begin{cases} \dfrac{z}{|z|} & \text{if } z \neq 0, \\ 0 & \text{if } z = 0, \end{cases}$$

Remark 3. Let n be a positive integer. The equality $\|\overline{z}\operatorname{sgn}^{2n} z\|_{\mathrm{OL}(\mathbb{C})} = 2n-1$ proved in [3] (see also Corollary 4.3 below) and estimate (4.1) imply the following inequality:

$$\|\operatorname{sgn}^{2n}\|_{\mathfrak{M}_0(\mathfrak{F})} \le 6n - 3, \tag{4.2}$$

where $\mathfrak{F}$ is a closed set of the complex numbers containing 0.

The following theorem exhibits that estimate (4.2) can be improved for $n \ge 2$.

Theorem 4.2. *Let $\mathfrak{F}$ be a closed subset of the complex plane $\mathbb{C}$, $a \in \mathfrak{F}$. Then the function $\operatorname{sgn}^{2n}(z - a)$ belongs*[5] *to the space $\mathfrak{M}_a(\mathfrak{F})$ for all integers n, and*

$$\|\operatorname{sgn}^{2n}(z - a)\|_{\mathfrak{M}_a(\mathfrak{F})} \le 2|n| + 1.$$

Before proving this theorem, let us note that it yields Theorem 3.5 of [3] as a corollary.

Corollary 4.3. *Let $n \in \mathbb{Z}$. Put $g_n(z) \stackrel{\text{def}}{=} z\operatorname{sgn}^{2n} z$ for $z \neq 0$ and $g_n(0) \stackrel{\text{def}}{=} 0$. Them $\|g_n\|_{\mathrm{OL}(\mathbb{C})} = |2n + 1|$.*

Proof. The equality $\overline{g}_n = g_{-n-1}$ allows us to restrict ourselves to the case $n \ge 0$. The result is evident for $n = 0$. Clearly, $\|g_n\|_{\mathrm{OL}(\mathbb{C})} \le \|\operatorname{sgn}^{2n} z\|_{\mathfrak{M}_0(\mathfrak{F})}\|g_0\|_{\mathrm{OL}(\mathbb{C})} = 2n + 1$. It remains to note that $\|g_n\|_{\mathrm{OL}(\mathbb{C})} \ge \|g_n\|_{\mathrm{Lip}(\mathbb{T})} = 2n + 1$. $\square$

To prove Theorem 4.2, we need the following lemma.

Lemma 4.4. *Let M and N be normal operators. Then*

$$\|(\operatorname{sgn} M)^{2n}N - (\operatorname{sgn} N)^{2n}N\| \le 2n\|M - N\|$$

for all positive integers n.

[5] Here, at least for $n \le 0$, the space $\mathfrak{M}_a(\mathfrak{F})$ should be considered as a space of functions defined on $\mathfrak{F} \setminus \{a\}$ because the function $\operatorname{sgn}^{2n}(z - a)$ is not defined for $z = a$ if $n \le 0$.

Proof. In the case $n = 1$, the result follows from the equality

$$(\operatorname{sgn} M)^2 N - (\operatorname{sgn} N)^2 N = -\operatorname{sgn}^2 M (M^* - N^*) \operatorname{sgn}^2 N + (M - N) \operatorname{sgn}^2 N.$$

The identity

$$\begin{aligned}(\operatorname{sgn} M)^{2n} N &- (\operatorname{sgn} N)^{2n} N \\ &= (\operatorname{sgn} M)^2 \big((\operatorname{sgn} M)^{2n-2} N - (\operatorname{sgn} N)^{2n-2} N\big) + \big((\operatorname{sgn} M)^2 N - (\operatorname{sgn} N)^2 N\big)(\operatorname{sgn} N)^{2n-2}\end{aligned}$$

allows us to make the induction step from $n-1$ to n. □

Proof of Theorem 4.2. We can assume that $a = 0$. The case $n = 0$ is trivial. Note that if $n < 0$, then

$$\operatorname{sgn}^{2n} z = \operatorname{sgn}^{-2n} \overline{z} = \overline{\operatorname{sgn}^{-2n} z}$$

for $z \neq 0$. Hence, $\| \operatorname{sgn}^{2n} \|_{\mathfrak{M}_0(\mathfrak{F})} = \| \operatorname{sgn}^{-2n} \|_{\mathfrak{M}_0(\mathfrak{F})}$. Thus, the case $n < 0$ is reduced to the case $n > 0$. Let $n > 0$. We need to prove that

$$\|f(z) \operatorname{sgn}^{2n} z\|_{\mathrm{OL}(\mathfrak{F})} \le (2n+1)\|f\|_{\mathrm{OL}(\mathfrak{F})}$$

for all $f \in \mathrm{OL}_0(\mathfrak{F})$. Put

$$g(z) \stackrel{\text{def}}{=} \begin{cases} z^{-1} f(z) & \text{if } z \in \mathfrak{F} \text{ and } z \neq 0, \\ 0 & \text{if } z = 0. \end{cases}$$

It is clear that

$$f(M)(\operatorname{sgn} M)^{2n} - f(N)(\operatorname{sgn} N)^{2n} = (\operatorname{sgn} M)^{2n}\big(f(M) - f(N)\big) + \big((\operatorname{sgn} M)^{2n} N - (\operatorname{sgn} N)^{2n} N\big) g(N)$$

for every normal operators M and N. Now Lemma 4.4 implies that

$$\begin{aligned}\|f(M)(\operatorname{sgn} M)^{2n} - f(N)(\operatorname{sgn} N)^{2n}\| &\le \|f(M) - f(N)\| + (\sup |g|)\|(\operatorname{sgn} M)^{2n} N - (\operatorname{sgn} N)^{2n} N\| \\ &\le (2n+1)\|f\|_{\mathrm{OL}(\mathfrak{F})}\|M - N\|.\end{aligned}$$

□

Remark 1. Let $n > 0$. It is clear that $\|z\|_{\mathrm{OL}(\mathfrak{F})} = 1$ if $\mathfrak{F}$ contains at least two points. From the proof of Corollary 4.3 it is not difficult to see that $\|z \operatorname{sgn}^{2n} z\|_{\mathrm{OL}(\mathfrak{F})} \ge 2n+1$ if, for example, $\mathfrak{F}$ contains infinitely many points of a circle centered at the origin. Hence, the constant $2|n|+1$ in Theorem 4.2 cannot be diminished for such sets $\mathfrak{F}$.

Remark 2. The function $\operatorname{sgn}^{2n+1} z$ does not belong to $\mathfrak{M}_0(\mathfrak{F})$ if, for example, $\mathfrak{F}$ contains a straight line or at least a union of two disjoint rays lying on a straight line. This follows from the fact that the restriction of the function $z \operatorname{sgn}^{2n+1} z$ on every straight line does not have a derivative at infinity. Note that the function $z \operatorname{sgn}^{2n+1} z$ does not have a derivative at zero along any direction. Hence, the function $\operatorname{sgn}^{2n+1} z$ cannot belong to $\mathfrak{M}_0(\mathfrak{F})$ if $\mathfrak{F}$ contains an open interval passing through the origin.

Now we pass to a description of $\mathfrak{M}(\mathfrak{F})$. It is clear that the product of two bounded functions in $\mathrm{OL}(\mathfrak{F})$ belongs to $\mathrm{OL}(\mathfrak{F})$. This readily implies the following assertion.

Theorem 4.5. *Let $\mathfrak{F}$ be a compact subset of $\mathbb{C}$. Then $\mathrm{OL}(\mathfrak{F})$ is an algebra. In other words, $\mathfrak{M}(\mathfrak{F}) = \mathrm{OL}(\mathfrak{F})$.*

In the case of an arbitrary closed set $\mathfrak{F}$, $\mathfrak{F} \subset \mathbb{C}$, the following statement holds.

Theorem 4.6. *Let w be a function defined on a closed set $\mathfrak{F}$, $\mathfrak{F} \subset \mathbb{C}$. Then the following statements are equivalent:*
(i) *$w \in \mathfrak{M}(\mathfrak{F})$;*
(ii) *the functions w and $zw(z)$ are operator Lipschitz on $\mathfrak{F}$;*
(iii) *the functions w and $\overline{z}w(z)$ are operator Lipschitz on $\mathfrak{F}$.*

Proof. If $0 \in \mathfrak{F}$, then the result follows from Theorem 4.1 because $\mathfrak{M}(\mathfrak{F}) = \mathrm{OL}(\mathfrak{F}) \cap \mathfrak{M}_0(\mathfrak{F})$. But if $0 \notin \mathfrak{F}$, then the case of the set $\mathfrak{F}$ is reduced to the investigated case of the set $\mathfrak{F} \cup \{0\}$. □

Theorem 4.7. *Let w be a function defined on a closed set $\mathfrak{F}$, $\mathfrak{F}\subset\mathbb{C}$. Assume that $a\in\mathbb{C}\setminus\mathfrak{F}$. Then the following statements are equivalent:*

(i) $w\in\mathfrak{M}(\mathfrak{F})$;

(ii) *the functions w and $zw(z)$ are operator Lipschitz on $\mathfrak{F}$;*

(iii) *the function $(z-a)w(z)$ is operator Lipschitz on $\mathfrak{F}$;*

(iv) *the functions w and $\overline{z}w(z)$ are operator Lipschitz on $\mathfrak{F}$;*

(v) *the function $(\overline{z}-\overline{a})w(z)$ is operator Lipschitz on $\mathfrak{F}$.*

Proof. It suffices to consider the case where $a=0$. The equivalence of statements (i), (ii), and (iv) follows from Theorem 4.6. Let us prove the equivalence of statements (i), (iii), and (v). For this, we take an arbitrary extension of w to the set $\mathfrak{F}\cup\{0\}$. Now the desired equivalence follows from Theorem 4.1 and the equality $\mathfrak{M}(\mathfrak{F}\cup\{0\})=\mathfrak{M}_0(\mathfrak{F}\cup\{0\})$, see the remark at the beginning of this section. ☐

5. Operator Lipschitz functions of a complex variable and linear fractional transformations

Theorem 5.1. *Let $f\in\mathrm{OL}(\mathbb{C})$. Put*

$$g_n(z)\stackrel{\mathrm{def}}{=}\begin{cases} z^{1+n}\overline{z}^{1-n}(f(z^{-1})-f(0)) & \text{if } z\ne 0,\\ 0 & \text{if } z=0,\end{cases}$$

where $n\in\mathbb{Z}$. Then $g_n\in\mathrm{OL}(\mathbb{C})$, and

$$c_n^{-1}\|f\|_{\mathrm{OL}(\mathbb{C})}\le\|g_n\|_{\mathrm{OL}(\mathbb{C})}\le c_n\|f\|_{\mathrm{OL}(\mathbb{C})},\tag{5.1}$$

where $c_n=\max(3,2|n|+1)$.

Proof. Denote by T_n the transformation that maps f to g_n. Note that $T_n(T_nf)=f-f(0)$. Thus, we can restrict ourselves to the case of upper estimates of the seminorm of g_n. In addition, $T_{-n}f=\overline{T_n\overline{f}}$. Hence, it suffices to consider the case where $n\ge0$. We can assume that $\|f\|_{\mathrm{OL}(\mathbb{C})}=1$ and $f(0)=0$. Let us consider the case $n=0$. By Lemma 2.2, it suffices to prove that

$$\|M^*Mf(M^{-1})-N^*Nf(N^{-1})\|\le 3\|M-N\|$$

for every invertible normal operators M and N. We have

$$M^*Mf(M^{-1})-N^*Nf(N^{-1})=f(M^{-1})M(M^*-N^*)+f(M^{-1})MN^*-MN^*f(N^{-1})+(M-N)N^*f(N^{-1}).\tag{5.2}$$

Hence,

$$\begin{aligned}\|M^*Mf(M^{-1})-N^*Nf(N^{-1})\|&\le\|f(M^{-1})M\|\|M^*-N^*\|+\|M-N\|\|N^*f(N^{-1})\|\\&\quad+\max(\|M^{-1}MN^*-MN^*N^{-1}\|,\|(M^*)^{-1}MN^*-MN^*(N^*)^{-1}\|)\\&\le 2\|M-N\|+\max(\|N^*-MN^*N^{-1}\|,\|(M^*)^{-1}MN^*-M\|).\end{aligned}$$

It remains to estimate $\|N^*-MN^*N^{-1}\|$ and $\|(M^*)^{-1}MN^*-M\|$. We have

$$\|N^*-MN^*N^{-1}\|=\|(N-M)N^{-1}N^*\|\le\|M-N\|$$

and

$$\|(M^*)^{-1}MN^*-M\|=\|M(M^*)^{-1}(N^*-M^*)\|\le\|M^*-N^*\|.$$

The case $n=1$ can be considered in a similar way but instead of (5.2) we must use the following identity:

$$f(M^{-1})M^2-N^2f(N^{-1})=f(M^{-1})M(M-N)+f(M^{-1})MN-MNf(N^{-1})+(M-N)Nf(N^{-1}).$$

Then it remains to note that

$$\|(M^{-1})^*MN-MN(N^{-1})^*\|=\|\big((M^{-1})^*M\big)(N^*-M^*)\big(N(N^{-1})^*\big)\|\le\|M-N\|.$$

Finally, let $n\ge2$. Then

$$\begin{aligned}g_n(M)-g_n(N)&=(\operatorname{sgn}M)^{2n-2}g_1(M)-(\operatorname{sgn}N)^{2n-2}g_1(N)\\&=(\operatorname{sgn}M)^{2n-2}\big(g_1(M)-g_1(N)\big)+\big((\operatorname{sgn}M)^{2n-2}N-(\operatorname{sgn}N)^{2n-2}N\big)Nf(N^{-1}).\end{aligned}$$

Now it is clear that

$$\|g_n(M)-g_n(N)\|\le(2n+1)\|M-N\|$$

in view of Lemma 4.4 and the case $n=1$ just considered. ☐

Let $\mathfrak{F}$ be a closed subset of the complex plane $\mathbb{C}$. Put $\mathfrak{F}_{-1} \stackrel{\text{def}}{=} \{0\} \cup \{z \in \mathbb{C} : z^{-1} \in \mathfrak{F}\}$. It is clear that $\mathfrak{F}_{-1}$ is also closed. In addition, $(\mathfrak{F}_{-1})_{-1} = \{0\} \cup \mathfrak{F}$.

Now we state the following generalization of Theorem 5.1.

Theorem 5.2. *Let $f \in \mathrm{OL}(\mathfrak{F})$, where $\mathfrak{F}$ is a closed subset of $\mathbb{C}$ with $\mathfrak{F} \ni 0$. Put*

$$g_n(z) \stackrel{\text{def}}{=} \begin{cases} z^{1+n}\overline{z}^{1-n}\big(f(z^{-1}) - f(0)\big) & \text{if } z \in \mathfrak{F}_{-1} \text{ and } z \neq 0, \\ 0 & \text{if } z = 0, \end{cases}$$

where $n \in \mathbb{Z}$. Then $g_n \in \mathrm{OL}(\mathfrak{F}_{-1})$ and

$$c_n^{-1}\|f\|_{\mathrm{OL}(\mathfrak{F})} \le \|g_n\|_{\mathrm{OL}(\mathfrak{F}_{-1})} \le c_n\|f\|_{\mathrm{OL}(\mathfrak{F})}, \tag{5.3}$$

where $c_n = \max(3, 2|n| + 1)$.

Proof. As in the proof of Theorem 5.1, it suffices to get only upper estimates of the seminorm of g_n, and we can assume that $n \ge 0$. Observe that if $\mathfrak{F}$ is not bounded, then we can repeat almost word-by-word the proof of Theorem 5.1. Indeed, in this case, 0 is a limit point of $\mathfrak{F}_{-1}$, and we can rewrite the estimates of the proof of Theorem 5.1 for normal operators M and N with spectra containing in $\mathfrak{F}_{-1} \setminus \{0\}$. Then it is enough to apply Lemma 2.2.

We are going to modify a little the proof of Theorem 5.1 so that it would work in the general case. We can assume that $\|f\|_{\mathrm{OL}(\mathfrak{F})} = 1$ and $f(0) = 0$. First we consider the case $n = 0$. By Theorem 2.3, it suffices to prove that

$$\|g_0(N)R - Rg_0(N)\| \le 3\max\big(\|NR - RN\|, \|N^*R - RN^*\|\big)$$

for every operators N and R such that N is a normal operator with finite spectrum lying in $\mathfrak{F}_{-1}$. Put

$$h_0(z) \stackrel{\text{def}}{=} \begin{cases} \overline{z}f(z^{-1}) & \text{if } z \in \mathfrak{F}_{-1} \text{ and } z \neq 0, \\ 0 & \text{if } z = 0, \end{cases}$$

and $h_1(z) \stackrel{\text{def}}{=} h_0(z)\,\mathrm{sgn}^2 z$. Clearly, $g_0(z) = zh_0(z) = \overline{z}h_1(z)$ for all $z \in \mathfrak{F}_{-1}$, and

$$\sup_{\mathfrak{F}_{-1}} |h_0| = \sup_{\mathfrak{F}_{-1}} |h_1| \le \|f\|_{\mathrm{Lip}(\mathfrak{F})} \le \|f\|_{\mathrm{OL}(\mathfrak{F})} = 1.$$

We have

$$g_0(N)R - Rg_0(N) = Nh_0(N)R - RN^*h_1(N) = h_0(N)(NR - RN) + h_0(N)RN - N^*Rh_1(N) + (N^*R - RN^*)h_1(N),$$

whence

$$\|g_0(N)R - Rg_0(N)\| \le 2\max\big(\|NR - RN\|, \|N^*R - RN^*\|\big) + \|h_0(N)RN - N^*Rh_1(N)\|.$$

Thus, we must prove the following inequality:

$$\|h_0(N)RN - N^*Rh_1(N)\| \le \max\big(\|NR - RN\|, \|N^*R - RN^*\|\big). \tag{5.4}$$

If N is invertible, then, applying Theorem 2.3, we get the inequalities

$$\begin{aligned} \|h_0(N)RN - N^*Rh_1(N)\| &= \|f(N^{-1})N^*RN - N^*RNf(N^{-1})\| \\ &\le \max\big(\|N^{-1}N^*RN - N^*R)\|, \big\|RN - N^*RN(N^*)^{-1}\|\big) \\ &= \max\big(\|N^{-1}N^*(RN - NR)\|, \big\|(RN^* - N^*R)N(N^*)^{-1}\|\big) \\ &\le \max\big(\|RN - NR\|, \big\|RN^* - N^*R\|\big). \end{aligned}$$

Now we prove this inequality for every normal operator with finite spectrum contained in $\mathfrak{F}_{-1}$. Denote by $\mathcal{H}_0$ the range of the operator N. Note that $\mathcal{H}_0$ is closed because the spectrum of N is finite. Let us consider the operator N_0 on the space $\mathcal{H}_0$ such that $N_0u = Nu$ for all $u \in \mathcal{H}_0$. The operator N_0 is normal since $\mathcal{H}_0$ is a reducing subspace for N. Moreover, N_0 is invertible because $\sigma(N_0) = \sigma(N) \setminus \{0\}$. Thus, inequality (5.4) for N_0 is already proved. Let P denote the orthogonal projection onto $\mathcal{H}_0$. Clearly, the ranges of $h_0(N)$ and $h_1(N)$ are contained in $\mathcal{H}_0$, $\ker h_0(N) \supset \ker N$, and $\ker h_1(N) \supset \ker N$. Hence,

$$\begin{aligned} \|h_0(N)RN - N^*Rh_1(N)\| &= \|h_0(N_0)PRPN_0 - N_0^*PRPh_1(N_0)\| = \|f(N_0^{-1})N_0^*PRPN_0 - N_0^*PRPN_0f(N_0^{-1})\| \\ &\le \max\big(\|N_0^{-1}N_0^*PRPN_0 - N_0^*PRP\|, \|PRPN_0 - N_0^*PRPN_0(N_0^{-1})^*\|\big) \end{aligned}$$

by Theorem 2.3. It remains to note that

$$\|N_0^{-1}N_0^*PRPN_0 - N_0^*PRP\| = \|PRPN_0 - N_0PRP\| = \|P(RN-NR)P\| \le \|NR-RN\|$$

and

$$\|PRPN_0 - N_0^*PRPN_0(N_0^{-1})^*\| = \|PRPN_0^* - N_0^*PRP\| = \|P(RN^*-N^*R)P\| \le \|N^*R-RN^*\|.$$

Now we pass to the case $n=1$. The proof repeats to a great extent the proof for $n=0$, and we use the notation introduced in the case $n=0$. We have

$$g_1(N)R - Rg_1(N) = Nh_1(N)R - RNh_1(N) = h_1(N)(NR-RN) + h_1(N)RN - NRh_1(N) + (NR-RN)h_1(N),$$

whence

$$\|g_1(N)R - Rg_1(N)\| \le 2\|NR-RN\| + \|h_1(N)RN - NRh_1(N)\|.$$

We must prove that

$$\|h_1(N)RN - NRh_1(N)\| \le \max\big(\|NR-RN\|, \|N^*R-RN^*\|\big).$$

If N is invertible, then

$$\begin{aligned}\|h_1(N)RN - NRh_1(N)\| &= \|f(N^{-1})NRN - NRNf(N^{-1})\| \\ &\le \max\big(\|RN-NR\|, \|(N^{-1})^*NRN - NRN(N^{-1})^*\|\big) \\ &= \max\big(\|RN-NR\|, \|(N^{-1})^*N(RN^*-N^*R)N(N^{-1})^*\|\big) \\ &= \max\big(\|NR-RN\|, \|N^*R-RN^*\|\big).\end{aligned}$$

Now we prove this inequality for every normal operator with finite spectrum contained in $\mathfrak{F}_{-1}$. We have

$$\begin{aligned}\|h_1(N)RN - NRh_1(N)\| = \|h_1(N_0)PRPN_0 - N_0PRPh_1(N_0)\| = \|f(N_0^{-1})N_0PRPN_0 - N_0PRPN_0f(N_0^{-1})\| \\ \le \max\big(\|PRPN_0 - N_0PRP\|, \|(N_0^{-1})^*N_0PRPN_0 - N_0PRPN_0(N_0^{-1})^*\|\big)\end{aligned}$$

This proves the desired inequality because

$$\|PRPN_0 - N_0PRP\| = \|P(RN-NR)P\| \le \|NR-RN\|$$

and

$$\begin{aligned}\|(N_0^{-1})^*N_0PRPN_0 - N_0PRPN_0(N_0^{-1})^*\| &= \|(N_0^{-1})^*N_0(PRPN_0^* - N_0^*PRP)N_0(N_0^{-1})^*\| \\ &= \|PRPN_0^* - N_0^*PRP\| = \|P(RN^*-N^*R)P\| \le \|N^*R-RN^*\|.\end{aligned}$$

Let us pass to the case $n \ge 2$. Clearly,

$$\begin{aligned}g_n(M) - g_n(N) &= (\operatorname{sgn} M)^{2n-2}g_1(M) - (\operatorname{sgn} N)^{2n-2}g_1(N) \\ &= (\operatorname{sgn} M)^{2n-2}\big(g_1(M) - g_1(N)\big) + \big((\operatorname{sgn} M)^{2n-2}N - (\operatorname{sgn} N)^{2n-2}N\big)\big)h_1(N)\end{aligned}$$

for every normal operators M and N such that $\sigma(M), \sigma(N) \subset \mathfrak{F}_{-1}$. It remains to use Lemma 4.4 and the case $n=1$ considered above. ☐

Corollary 5.3. *Let $f \in \mathrm{OL}(\mathfrak{F})$, where $\mathfrak{F}$ is a closed subset of $\mathbb{C}$ with $\mathfrak{F} \not\ni 0$. Put*

$$g_n(z) \stackrel{\text{def}}{=} \begin{cases} z^{1+n}\overline{z}^{1-n}f(z^{-1}) & \text{if } z \in \mathfrak{F}_{-1} \text{ and } z \ne 0, \\ 0 & \text{if } z = 0, \end{cases}$$

where $n \in \mathbb{Z}$. Then $g_n \in \mathrm{OL}(\mathfrak{F}_{-1})$ and

$$c_n^{-1}\max\left(\|f\|_{\mathrm{OL}(\mathfrak{F})}, \sup_{z\in\mathfrak{F}}\frac{|f(z)|}{|z|}\right) \le \|g_n\|_{\mathrm{OL}(\mathfrak{F}_{-1})} \le c_n\left(\|f\|_{\mathrm{OL}(\mathfrak{F})} + \sup_{z\in\mathfrak{F}}\frac{|f(z)|}{|z|}\right),$$

where $c_n = \max(3, 2|n|+1)$.

Proof. We can consider the extension of f to f_0 on the set $\mathfrak{F}_0 \stackrel{\text{def}}{=} \mathfrak{F} \cup \{0\}$ such that $f_0(0) = 0$. Observe that

$$\max\left(\|f\|_{\mathrm{OL}(\mathfrak{F})}, \sup_{z\in\mathfrak{F}}\frac{|f(z)|}{|z|}\right) \le \|f_0\|_{\mathrm{OL}(\mathfrak{F}_0)} \le \|f\|_{\mathrm{OL}(\mathfrak{F})} + \sup_{z\in\mathfrak{F}}\frac{|f(z)|}{|z|},$$

where the first inequality is evident and the second inequality follows from Lemma 2.7. It remains to apply Theorem 5.2 to the function f_0. ☐

We recall that $\mathrm{Aut}(\widehat{\mathbb{C}})$ denotes the Möbius group of linear fractional transformations of the extended complex plane $\widehat{\mathbb{C}} \stackrel{\text{def}}{=} \mathbb{C} \cup \{\infty\}$ and $\mathrm{Aut}(\mathbb{C}) \stackrel{\text{def}}{=} \{f \in \mathrm{Aut}(\widehat{\mathbb{C}}) : f(\infty) = \infty\}$.

Let $\varphi \in \mathrm{Aut}(\widehat{\mathbb{C}})$. With every closed subset $\mathfrak{F}$ of $\mathbb{C}$ we associate the set $\mathfrak{F}_\varphi \stackrel{\text{def}}{=} \mathbb{C} \cap \varphi^{-1}(\mathfrak{F} \cup \{\infty\})$. It is easily seen that the closedness of $\mathfrak{F}$ implies the closedness of $\mathfrak{F}_\varphi$. Clearly, $\mathfrak{F}_\varphi \stackrel{\text{def}}{=} \varphi^{-1}(\mathfrak{F})$ if $\varphi \in \mathrm{Aut}(\mathbb{C})$. If $\varphi \notin \mathrm{Aut}(\mathbb{C})$, then $\varphi^{-1}(\infty) \in \mathfrak{F}_\varphi$.

Note also that $(\mathfrak{F}_\varphi)_{\varphi^{-1}} = \mathbb{C} \cap (\mathfrak{F} \cup \varphi(\infty))$. In other words, $(\mathfrak{F}_\varphi)_{\varphi^{-1}} = \mathfrak{F} \cup \varphi(\infty)$ if $\varphi \notin \mathrm{Aut}(\mathbb{C})$ and $(\mathfrak{F}_\varphi)_{\varphi^{-1}} = \mathfrak{F}$ if $\varphi \in \mathrm{Aut}(\mathbb{C})$. With every function f on $\mathfrak{F}$ we associate the function $T_{\varphi,n}f$ defined on the set $\mathfrak{F}_\varphi$ as follows:

$$(T_{\varphi,n}f)(z) \stackrel{\text{def}}{=} \begin{cases} \dfrac{f(\varphi(z))}{|\varphi'(z)| \operatorname{sgn}^n(\varphi'(z))} & \text{if } z \in \mathfrak{F}_\varphi \text{ and } \varphi(z) \neq \infty, \\ 0, & \text{if } \varphi(z) = \infty \text{ and } z \in \mathbb{C}, \end{cases}$$

where $n \in \mathbb{Z}$.

If $\varphi \in \mathrm{Aut}(\mathbb{C})$, then $(T_{\varphi,n}f)(z) = \dfrac{f(\varphi(z))}{|\varphi'(z)| \operatorname{sgn}^n(\varphi'(z))}$ for all $z \in \mathfrak{F}_\varphi$. In this case, $\varphi' = \text{const}$. However, we are mostly interested in the case where $\varphi \notin \mathrm{Aut}(\mathbb{C})$.

Theorem 5.4. *Let $\varphi \in \mathrm{Aut}(\widehat{\mathbb{C}})$ and $f \in \mathrm{OL}(\mathfrak{F})$, where $\mathfrak{F}$ is a closed subspace of $\mathbb{C}$. Assume that $\varphi(\infty) \in \mathfrak{F}$ and $f(\varphi(\infty)) = 0$. Then the function $T_{\varphi,n}f$ is operator Lipschitz, and*

$$c_n^{-1} \|f\|_{\mathrm{OL}(\mathfrak{F})} \le \|T_{\varphi,n}f\|_{\mathrm{OL}(\mathfrak{F}_\varphi)} \le c_n \|f\|_{\mathrm{OL}(\mathfrak{F})} \tag{5.5}$$

for all $n \in \mathbb{Z}$, where $c_n = \max(3, 2|n|+1)$. Moreover, $T_{\varphi,n}(\mathrm{OL}_{\varphi(\infty)}(\mathfrak{F})) = \mathrm{OL}_{\varphi^{-1}(\infty)}(\mathfrak{F}_\varphi)$.

Proof. First we prove (5.5). Put $\phi_c(z) \stackrel{\text{def}}{=} cz^{-1}$, where $c \in \mathbb{C}$, $c \neq 0$. Note that $T_{\phi_1,n}f = (-1)^n g_n$, where g_n denotes the same as in Theorem 5.2. Thus, the case $\varphi = \phi_1$ is reduced to Theorem 5.2. The case $\varphi = \phi_c$ is reduced to the case $c = 1$ with the help of dilations and rotations. Now we reduce the general case to the case $\varphi = \phi_c$. Put $a \stackrel{\text{def}}{=} \varphi(\infty)$ and $b \stackrel{\text{def}}{=} \varphi^{-1}(\infty)$. By the assumptions, $a \in \mathfrak{F} \subset \mathbb{C}$. Hence, $b \in \mathbb{C}$, and φ can be represented in the form $\varphi(z) = a + \dfrac{c}{z-b}$, where $c \neq 0$. Put $\tau_h(z) \stackrel{\text{def}}{=} z - h$, where $h \in \mathbb{C}$. Then $T_{\varphi,n}f = (T_{\phi_c,n}(f \circ \tau_a^{-1})) \circ \tau_b$. This proves (5.5) and the inclusion

$$T_{\varphi,n}(\mathrm{OL}_{\varphi(\infty)}(\mathfrak{F})) \subset \mathrm{OL}_{\varphi^{-1}(\infty)}(\mathfrak{F}_\varphi). \tag{5.6}$$

To prove the opposite inclusion, it suffices to note that $f = T_{\varphi,n}(T_{\varphi^{-1},n}f)$ and to apply inclusion (5.6) to the linear fractional transformation φ^{-1} and the set $\mathfrak{F}_\varphi$. □

Corollary 5.5. *Let $\varphi \in \mathrm{Aut}(\widehat{\mathbb{C}})$ and $f \in \mathrm{OL}(\mathfrak{F})$, where $\mathfrak{F}$ is a closed subspace of $\mathbb{C}$. Assume that $\varphi(\infty) \notin \mathfrak{F}$. Then $T_{\varphi,n}f \in \mathrm{OL}(\mathfrak{F}_\varphi)$ and*

$$c_n^{-1} \max\left(\|f\|_{\mathrm{OL}(\mathfrak{F})}, \sup_{z \in \mathfrak{F}} \frac{|f(z)|}{|z - \varphi(\infty)|} \right) \le \|T_{\varphi,n}f\|_{\mathrm{OL}(\mathfrak{F}_\varphi)} \le c_n \left(\|f\|_{\mathrm{OL}(\mathfrak{F})} + \sup_{z \in \mathfrak{F}} \frac{|f(z)|}{|z - \varphi(\infty)|} \right)$$

for all $n \in \mathbb{Z}$, where $c_n = \max(3, 2|n|+1)$. Moreover,[6] *$T_{\varphi,n}(\mathrm{OL}(\mathfrak{F})) = \mathrm{OL}_{\varphi^{-1}(\infty)}(\mathfrak{F}_\varphi)$.*

Proof. If $\varphi(\infty) = \infty$, then $\varphi \in \mathrm{Aut}(\mathbb{C})$. Then it is clear that $T_{\varphi,n}(\mathrm{OL}(\mathfrak{F})) = \mathrm{OL}(\mathfrak{F}_\varphi)$ and $\|T_{\varphi,n}f\|_{\mathrm{OL}(\mathfrak{F}_\varphi)} = \|f\|_{\mathrm{OL}(\mathfrak{F})}$ for all $f \in \mathrm{OL}(\mathfrak{F})$ and $n \in \mathbb{Z}$. Now let $\varphi(\infty) \in \mathbb{C}$. To reduce in this case the corollary to Theorem 5.4, we can extend each function $f \in \mathrm{OL}(\mathfrak{F})$ to a function $f_0 \in \mathrm{OL}(\mathfrak{F} \cup \{\varphi(\infty)\})$ by putting $f_0(\varphi(\infty)) = 0$. Then $T_\varphi f_0 = T_\varphi f$. Hence,

$$T_{\varphi,n}(\mathrm{OL}(\mathfrak{F})) = T_{\varphi,n}\big(\mathrm{OL}_{\varphi(\infty)}(\mathfrak{F} \cup \{\varphi(\infty)\})\big) = \mathrm{OL}_{\varphi^{-1}(\infty)}(\mathfrak{F}_\varphi).$$

To complete the proof, it suffices to apply Lemma 2.7 in the same way as is done in the proof of Corollary 5.3. □

Remark 1. If $f(z) = \overline{z}$, then $g_n(z) \stackrel{\text{def}}{=} (T_n f)(z) = z \operatorname{sgn}^{2n} z = \overline{z} \operatorname{sgn}^{2n+2} z$. Recall that $\|g_n\|_{\mathrm{OL}(\mathbb{C})} = 2n+1$ for $n \ge 0$ by Corollary 4.3, see also Theorem 3.5 of [3]. From this it is clear that for $n \neq 0$, the constant c_n in inequality (5.1) cannot be improved. The same one can say about the constant c_n^{-1}. Taking into account Remark 1 after the proof of Theorem 4.2, we see that for $n \neq 0$, the constants c_n^{-1} and c_n cannot be improved in inequality (5.3) if, for example, infinitely many points of $\mathfrak{F}$ lie on a circle centered at zero. From this it is

[6] Here we put $\mathrm{OL}_\infty(\mathfrak{F}) \stackrel{\text{def}}{=} \mathrm{OL}(\mathfrak{F})$.

clear that for $n \neq 0$, the constant c_n (respectively, c_n^{-1}) cannot be improved in inequality (5.5) if, for example, infinitely many points of $\mathfrak{F}$ lie on a circle centered at $\varphi(\infty)$ (respectively, $\varphi^{-1}(\infty)$). The author does not know whether the constants $c_0^{-1} = \frac{1}{3}$ and $c_0 = 3$ in inequality (5.1) can be improved.

Remark 2. Almost all results of this paper concerning operator Lipschitz functions have natural analogs for usual Lipschitz functions. Moreover, in most cases the proofs also work for the spaces of Lipschitz functions $\mathrm{Lip}(\mathfrak{F})$ if we replace there operators by numbers or by operators acting in a one-dimensional Hilbert space; in such cases, we get estimates with the same constants. In particular, we can rewrite Remark 1 for the spaces $\mathrm{Lip}(\mathfrak{F})$ with the only difference: now the sharpness of the constants c_n and c_n^{-1} for nondegenerate $\mathfrak{F}$ can easily be verified even for $n = 0$. To exhibit this, it suffices to prove that the constant $c_0 = 3$ in the corresponding analog of (5.1) cannot be improved. For this purpose, we put $f(z) \stackrel{\text{def}}{=} |z-1|$. Then $g_0(z) = |z^2 - z| - |z|^2$. It remains to note that $\|f\|_{\mathrm{Lip}(\mathbb{C})} = 1$ and $\|g_0\|_{\mathrm{Lip}(\mathbb{C})} \geq \|g_0\|_{\mathrm{Lip}([0,1])} = 3$.

Remark 3. Inequalities (5.1) readily imply the following inequalities:

$$\|g_n\|_{\mathrm{Lip}(\mathbb{C})} \leq c_n \|f\|_{\mathrm{OL}(\mathbb{C})} \quad \text{and} \quad \|f\|_{\mathrm{Lip}(\mathbb{C})} \leq c_n \|g_n\|_{\mathrm{OL}(\mathbb{C})}. \tag{5.7}$$

Arguments considered in Remark 1 exhibit that the constant c_n cannot be diminished even in (5.7) provided that $n \neq 0$. On the other hand, one can prove that for $n = 0$, the constant $c_0 = 3$ in (5.7) can be improved.

Now we consider some special cases of results obtained in this section. Let κ denote the Cayley transformation, $\kappa(z) \stackrel{\text{def}}{=} \frac{i-z}{i+z}$. Applying Theorem 5.4 to $\varphi = \kappa$ and $\mathfrak{F} = \mathbb{T}$, we conclude that $T_{\kappa,n}(\mathrm{OL}_{-1}(\mathbb{T})) \subset \mathrm{OL}_{-i}(\mathbb{R} \cup \{-i\})$ for all $n \in \mathbb{Z}$. Applying once more Theorem 5.4, now to $\varphi = \kappa^{-1}$ and $\mathfrak{F} = \mathbb{R} \cup \{\infty\}$, we conclude that $T_{\kappa,n}(\mathrm{OL}_{-1}(\mathbb{T})) = \mathrm{OL}_{-i}(\mathbb{R} \cup \{-i\})$ for all $n \in \mathbb{Z}$. Put $X_n f \stackrel{\text{def}}{=} (T_{\kappa,n} f)|\mathbb{R}$, where $n \in \mathbb{Z}$. Then $X_n(\mathrm{OL}_{-1}(\mathbb{T})) = \mathrm{OL}(\mathbb{R})$.

Theorem 5.6. *Let $n \in \mathbb{Z}$. Then X_n is a linear bijection from $\mathrm{OL}_{-1}(\mathbb{T})$ onto $\mathrm{OL}(\mathbb{R})$, and*

$$(2c_n)^{-1}\|f\|_{\mathrm{OL}(\mathbb{T})} \leq \|X_n f\|_{\mathrm{OL}(\mathbb{R})} + |(X_n f)(0)| \leq 2c_n \|f\|_{\mathrm{OL}(\mathbb{T})}$$

for all $f \in \mathrm{OL}_{-1}(\mathbb{T})$, where $c_n = \max(3, 2|n| + 1)$.

Proof. We need to prove the inequalities only because the rest has been explained in essence above. It follows from Theorem 5.4 that

$$c_n^{-1}\|f\|_{\mathrm{OL}(\mathbb{T})} \leq \|X_n f\|_{\mathrm{OL}(\mathbb{R}\cup\{-i\})} \leq c_n \|f\|_{\mathrm{OL}(\mathbb{T})}.$$

It remains to prove that

$$\frac{1}{2}\|X_n f\|_{\mathrm{OL}(\mathbb{R}\cup\{-i\})} \leq \|X_n f\|_{\mathrm{OL}(\mathbb{R})} + |(X_n f)(0)| \leq 2\|X_n f\|_{\mathrm{OL}(\mathbb{R}\cup\{-i\})}. \tag{5.8}$$

Note that $\|X_n f\|_{\mathrm{OL}(\mathbb{R})} \leq \|X_n f\|_{\mathrm{OL}(\mathbb{R}\cup\{-i\})}$ and

$$|(X_n f)(0)| = |(X_n f)(0) - (X_n f)(-i)| \leq \|X_n f\|_{\mathrm{OL}(\mathbb{R}\cup\{-i\})},$$

whence we get the second inequality in (5.8). Applying Lemma 2.7, we get the inequalities

$$\begin{aligned}\|X_n f\|_{\mathrm{OL}(\mathbb{R}\cup\{-i\})} &\leq \|X_n f\|_{\mathrm{OL}(\mathbb{R})} + \sup_{t\in\mathbb{R}} \frac{|(X_n f)(t)|}{|t+i|} \\ &\leq \|X_n f\|_{\mathrm{OL}(\mathbb{R})} + \sup_{t\in\mathbb{R}} \frac{|(X_n f)(t) - (X_n f)(0)|}{|t+i|} + \sup_{t\in\mathbb{R}} \frac{|(X_n f)(0)|}{|t+i|} \leq 2\|X_n f\|_{\mathrm{OL}(\mathbb{R})} + |(X_n f)(0)|,\end{aligned}$$

which readily implies the first inequality in (5.8). □

Remark 1. In the same way, one can construct a linear bijection X_n from $\mathrm{OL}_{-1}(\overline{\mathbb{D}})$ onto $\mathrm{OL}(\overline{\mathbb{C}}_+)$ such that

$$(2c_n)^{-1}\|f\|_{\mathrm{OL}(\overline{\mathbb{D}})} \leq \|X_n f\|_{\mathrm{OL}(\overline{\mathbb{C}}_+)} + |(X_n f)(0)| \leq 2c_n \|f\|_{\mathrm{OL}(\overline{\mathbb{D}})}$$

for all $f \in \mathrm{OL}_{-1}(\overline{\mathbb{D}})$, where

$$\overline{\mathbb{D}} \stackrel{\text{def}}{=} \{z \in \mathbb{C} : |z| \leq 1\} \text{ and } \overline{\mathbb{C}}_+ \stackrel{\text{def}}{=} \{z \in \mathbb{C} : \operatorname{Im} z \geq 0\}.$$

Remark 2. Applying Theorem 5.2 to $\mathfrak{F} = \overline{\mathbb{D}}$ and Corollary 5.3 to $\mathfrak{F} = \mathbb{C} \setminus \mathbb{D}$, we can obtain linear bijections from $\mathrm{OL}_0(\overline{\mathbb{D}})$ onto $\mathrm{OL}(\mathbb{C} \setminus \mathbb{D})$.

Let $a \in (0, +\infty)$. Denote by $\mathrm{OL}(\mathbb{R}/a\mathbb{Z})$ the set of functions in $\mathrm{OL}(\mathbb{R})$ with period a. Put

$$\mathrm{OL}_c(\mathbb{R}/a\mathbb{Z}) \stackrel{\text{def}}{=} \{f \in \mathrm{OL}(\mathbb{R}/a\mathbb{Z}) : f(c) = 0\},$$

where $c \in \mathbb{R}$.

We need the following statement.

Lemma 5.7. *The transformation that maps a function $f(\zeta)$ on the unit circle $\mathbb{T}$ to the function $f(e^{it})$ is an isomorphism from $\mathrm{OL}(\mathbb{T})$ onto $\mathrm{OL}(\mathbb{R}/2\pi\mathbb{Z})$, and*

$$\frac{1}{3\sqrt{2}\pi}\|f\|_{\mathrm{OL}(\mathbb{T})} \le \|f(e^{it})\|_{\mathrm{OL}(\mathbb{R})} \le \|f\|_{\mathrm{OL}(\mathbb{T})}$$

for every $f \in \mathrm{OL}(\mathbb{T})$.

Here we do not prove this lemma. A proof can be deduced from the paper [2]. Note that the upper estimate of $\|f(e^{it})\|_{\mathrm{OL}(\mathbb{R})}$ follows from the fact that $\|e^{it}\|_{\mathrm{OL}(\mathbb{R})} = 1$. The lower estimate of $\|f(e^{it})\|_{\mathrm{OL}(\mathbb{R})}$ can be derived from Lemma 9.8 of [2], at least for smooth f. One can avoid the assumption of smoothness with the help of an approximate identity.

Let us consider the operator Y that maps a function $f : \mathbb{R} \to \mathbb{C}$ to $Yf : \mathbb{R} \to \mathbb{C}$, $(Yf)(t) \stackrel{\text{def}}{=} (1+t^2)f(\arctan t)$.

Theorem 5.8. *The mapping Y is a linear bijection from $\mathrm{OL}_{\frac{\pi}{2}}(\mathbb{R}/\pi\mathbb{Z})$ onto $\mathrm{OL}(\mathbb{R})$, and*

$$\frac{1}{6}\|f\|_{\mathrm{OL}(\mathbb{R})} \le \|Yf\|_{\mathrm{OL}(\mathbb{R})} + |(Yf)(0)| \le 18\sqrt{2}\pi\|f\|_{\mathrm{OL}(\mathbb{R})}$$

for every $f \in \mathrm{OL}_{\frac{\pi}{2}}(\mathbb{R}/\pi\mathbb{Z})$.

Proof. Let $g \in \mathrm{OL}_{\frac{\pi}{2}}(\mathbb{R}/\pi\mathbb{Z})$. By Lemma 5.7, there exists a unique function $f \in \mathrm{OL}_{-1}(\mathbb{T})$ such that $g(t/2) = f(e^{it})$. Moreover, it follows from Lemma 5.7 that

$$\frac{2}{3\sqrt{2}\pi}\|f\|_{\mathrm{OL}(\mathbb{T})} \le \|g\|_{\mathrm{OL}(\mathbb{R})} \le 2\|f\|_{\mathrm{OL}(\mathbb{T})}. \tag{5.9}$$

We have

$$(Yg)(t) = (1+t^2)g(\arctan t) = (1+t^2)f(e^{2i\arctan t}) = (1+t^2)f\left(\frac{i-t}{i+t}\right) = 2(X_0 f)(t).$$

Now Theorem 5.6 implies that Y is a linear bijection. Moreover, it follows from Theorem 5.6 that

$$\frac{1}{3}\|f\|_{\mathrm{OL}(\mathbb{T})} \le \|Yg\|_{\mathrm{OL}(\mathbb{R})} + |(Yg)(0)| \le 12\|f\|_{\mathrm{OL}(\mathbb{T})}.$$

Together with inequalities (5.9), this implies the required estimates. ☐

Let Z denote the operator that maps a function $g : \mathbb{R} \to \mathbb{C}$ to the function

$$(Zg)(x) \stackrel{\text{def}}{=} \begin{cases} (\cos^2 x)\, g(\tan x) & \text{if } \cos x \ne 0, \\ 0 & \text{if } \cos x = 0. \end{cases}$$

Corollary 5.9. *The mapping Z is a linear bijection from $\mathrm{OL}(\mathbb{R})$ onto $\mathrm{OL}_{\frac{\pi}{2}}(\mathbb{R}/\pi\mathbb{Z})$, and*

$$\frac{1}{18\sqrt{2}\pi}(\|g\|_{\mathrm{OL}(\mathbb{R})} + |g(0)|) \le \|Zg\|_{\mathrm{OL}(\mathbb{R})} \le 6(\|g\|_{\mathrm{OL}(\mathbb{R})} + |g(0)|)$$

for every $g \in \mathrm{OL}(\mathbb{R})$.

6. Operator Lipschitz functions and differential operators

In Sec. 3, we have proved that for every $f \in \mathrm{OL}(\mathbb{R})$ there exists a function $g \in \mathrm{OL}(\mathbb{R})$ such that $g'(t) = f'(t^{-1})$. Our next aim is to obtain an analog of this statement for the space $\mathrm{OL}(\mathbb{C})$. We are going to construct a differential operator $\mathfrak{D}$ possessing the following property: For every function $f \in \mathrm{OL}(\mathbb{C})$ there exists a function $g \in \mathrm{OL}(\mathbb{C})$ such that $(\mathfrak{D}g)(z) = (\mathfrak{D}f)(z^{-1})$. Unfortunately, this differential operator is not invariant with respect to translations. This circumstance does not allow us to obtain in full measure analogs of the one-dimensional result; in contrast to the one-dimensional case, we cannot replace the linear fractional transformation z^{-1} by an arbitrary linear fractional transformation.

Put $Df \stackrel{\text{def}}{=} \frac{\partial f}{\partial z}$, $\overline{D}f \stackrel{\text{def}}{=} \frac{\partial f}{\partial \overline{z}}$, $\mathfrak{D}f \stackrel{\text{def}}{=} Df + \frac{\overline{z}}{z}\overline{D}f$, and $\Theta_2 f \stackrel{\text{def}}{=} z\mathfrak{D}f = zDf + \overline{z}\overline{D}f = x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}$. The operators D, $\overline{D}$, and Θ_2 are defined on the space of distributions $\mathscr{D}'(\mathbb{C})$, and the operator $\mathfrak{D}$ is defined on the space of distributions $\mathscr{D}'(\mathbb{C}\setminus\{0\})$. It is easily seen that the equality $\mathfrak{D}f = 0$ (or $\Theta_2 f = 0$, which is the same) holds in the space $\mathscr{D}'(\mathbb{C}\setminus\{0\})$ if and only if f is a homogeneous distribution of degree 0. Note also that $\mathfrak{D}f \in L^\infty(\mathbb{C})$ for every function $f \in \mathrm{Lip}(\mathbb{C})$. Moreover, if $\mathfrak{D}f = 0$ for a function $f \in \mathrm{Lip}(\mathbb{C})$, then $f = \mathrm{const}$ because f should be continuous and homogeneous of degree 0 at the same time.

We define the space $(\mathrm{OL})'_0(\mathbb{C})$ as the set of functions $g \in L^\infty(\mathbb{C})$ such that $g = \mathfrak{D}f$ for some function $f \in \mathrm{OL}(\mathbb{C})$, $\|g\|_{(\mathrm{OL})'_0(\mathbb{C})} \stackrel{\text{def}}{=} \|f\|_{\mathrm{OL}(\mathbb{C})}$. Clearly, $\|g\|_{(\mathrm{OL})'_0(\mathbb{C})}$ is a well defined norm on the space $(\mathrm{OL})'_0(\mathbb{C})$. Note that, unlike the class $(\mathrm{OL})'(\mathbb{R})$, functions of the class $(\mathrm{OL})'_0(\mathbb{C})$ are defined only up to a set of measure zero. Moreover, unlike the class $(\mathrm{OL})'(\mathbb{R})$, the space $(\mathrm{OL})'_0(\mathbb{C})$ is not invariant with respect to translations. Thus, if we have a statement concerning the space $(\mathrm{OL})'_0(\mathbb{C})$ and we want to transfer the origin to a point $a \in \mathbb{C}$, then the space $(\mathrm{OL})'_0(\mathbb{C})$ must be replaced by $(\mathrm{OL})'_a(\mathbb{C})$,

$$(\mathrm{OL})'_a(\mathbb{C}) \stackrel{\text{def}}{=} \left\{(Df)(z) + \frac{\overline{z}-\overline{a}}{z-a}(\overline{D}f)(z) : f \in \mathrm{OL}(\mathbb{C})\right\}.$$

Theorem 6.1. *Let $g \in (\mathrm{OL})'_0(\mathbb{C})$. Then $g(z)\,\mathrm{sgn}^{2n} z \in (\mathrm{OL})'_0(\mathbb{C})$ for all $n \in \mathbb{Z}$, and*

$$\|g(z)\,\mathrm{sgn}^{2n} z\|_{(\mathrm{OL})'_0(\mathbb{C})} \le (2|n|+1)\|g\|_{(\mathrm{OL})'_0(\mathbb{C})}.$$

Proof. By the definition of the space $(\mathrm{OL})'_0(\mathbb{C})$, there exists a function $f \in \mathrm{OL}(\mathbb{C})$ such that $\mathfrak{D}f = g$. Then $\mathfrak{D}(f(z)\,\mathrm{sgn}^{2n} z) = g(z)\,\mathrm{sgn}^{2n} z$. Hence, $g(z)\,\mathrm{sgn}^{2n} z \in (\mathrm{OL})'_0(\mathbb{C})$, and

$$\|g(z)\,\mathrm{sgn}^{2n} z\|_{(\mathrm{OL})'_0(\mathbb{C})} = \|f(z)\,\mathrm{sgn}^{2n} z\|_{\mathrm{OL}(\mathbb{C})} \le (2|n|+1)\|f\|_{\mathrm{OL}(\mathbb{C})} = \|g\|_{(\mathrm{OL})'_0(\mathbb{C})}$$

by Theorem 4.2. □

Theorem 6.2. *Let $f \in \mathrm{OL}(\mathbb{C})$. Then the function $\frac{f(z)-f(0)}{z}$ belongs to the space $(\mathrm{OL})'_0(\mathbb{C})$, and*

$$\left\|\frac{f(z)-f(0)}{z}\right\|_{(\mathrm{OL})'_0(\mathbb{C})} \le \|f\|_{\mathrm{OL}(\mathbb{C})}.$$

Proof. We can assume that $f(0) = 0$. The function f is a Lipschitz function; therefore, it is differentiable almost everywhere on $\mathbb{C}$ as a function of two real variables, see, e.g., [11, Chap. VIII, Theorem 1]. Hence, the equality

$$f(z) = \int_0^1 \frac{(\Theta_2 f)(tz)\,dt}{t} = z\int_0^1 (\mathfrak{D}f)(tz)\,dt$$

holds for almost all $z \in \mathbb{C}$. Put

$$h(z) \stackrel{\text{def}}{=} \int_0^1 \frac{f(tz)\,dt}{t}.$$

Then $h \in \mathrm{OL}(\mathbb{C})$, and $\|h\|_{\mathrm{OL}(\mathbb{C})} \le \|f\|_{\mathrm{OL}(\mathbb{C})}$ by Lemma 2.6. Note that $\mathfrak{D}\left(\frac{f(tz)}{t}\right) = (\mathfrak{D}f)(tz)$ for almost all $t \in [0,1]$ and almost all $z \in \mathbb{C}$. Hence,

$$(\mathfrak{D}h)(z) = \int_0^1 (\mathfrak{D}f)(tz)\,dt = \frac{f(z)}{z}$$

for almost all $z \in \mathbb{C}$. □

Corollary 6.3. *Let $f \in \mathrm{OL}(\mathbb{C})$. Then the function $\frac{f(z)-f(0)}{z}\,\mathrm{sgn}^{2n} z$ belongs to the space $(\mathrm{OL})'_0(\mathbb{C})$, and*

$$\left\|\frac{f(z)-f(0)}{z}\,\mathrm{sgn}^{2n} z\right\|_{(\mathrm{OL})'_0(\mathbb{C})} \le (2|n|+1)\|f\|_{\mathrm{OL}(\mathbb{C})}$$

for all $n \in \mathbb{Z}$. In particular, $\left\|\frac{f(z)-f(0)}{\overline{z}}\right\|_{(\mathrm{OL})'_0(\mathbb{C})} \le 3\|f\|_{\mathrm{OL}(\mathbb{C})}$.

Theorem 6.4. *Let $g \in (\mathrm{OL})_0'(\mathbb{C})$. Then $g(az^{-1}) \in (\mathrm{OL})_0'(\mathbb{C})$ for all $a \in \mathbb{C}$, $a \neq 0$, and*

$$\frac{1}{9}\|g(az^{-1})\|_{(\mathrm{OL})_0'(\mathbb{C})} \le \|g\|_{(\mathrm{OL})_0'(\mathbb{C})} \le 9\|g(az^{-1})\|_{(\mathrm{OL})_0'(\mathbb{C})}.$$

Proof. By virtue of the equality $\mathfrak{D}(f(az)) = a(\mathfrak{D}f)(az)$, it suffices to consider the case $a = 1$. We represent the function g in the form $g = \mathfrak{D}f$, where $f \in \mathrm{OL}_0(\mathbb{C})$. Put $h \stackrel{\mathrm{def}}{=} T_1 f$, where T_1 denotes the same as in the proof of Theorem 5.1. If follows from Theorem 5.1 that $\|h\|_{\mathrm{OL}(\mathbb{C})} \le 3\|f\|_{\mathrm{OL}(\mathbb{C})} = 3\|g\|_{(\mathrm{OL})_0'(\mathbb{C})}$. Note that

$$(\mathfrak{D}h)(z) = 2z^{-1}h(z) - g(z^{-1}).$$

Hence,

$$\|g(z^{-1})\|_{(\mathrm{OL})_0'(\mathbb{C})} \le 2\|z^{-1}h(z)\|_{(\mathrm{OL})_0'(\mathbb{C})} + \|\mathfrak{D}h\|_{(\mathrm{OL})_0'(\mathbb{C})} \le 3\|h\|_{\mathrm{OL}(\mathbb{C})} \le 9\|g\|_{(\mathrm{OL})_0'(\mathbb{C})}.$$ □

7. Commutator Lipschitz functions and linear fractional transformations

A function f defined on a set $\mathfrak{F}$, $\mathfrak{F} \subset \mathbb{C}$, is said to be[7] *commutator Lipschitz* if there exists a constant C such that

$$\|f(M)R - Rf(N)\| \le C\|MR - RN\| \tag{7.1}$$

for every normal operators M and N with finite spectra lying in $\mathfrak{F}$ and every operator R.

Denote by $\mathrm{CL}(\mathfrak{F})$ the set of commutator Lipschitz functions on $\mathfrak{F}$. Denote by $\|f\|_{\mathrm{OL}(\mathfrak{F})}$ the minimal constant C satisfying (7.1). Put $\|f\|_{\mathrm{OL}(\mathfrak{F})} = +\infty$ if $f \notin \mathrm{OL}(\mathfrak{F})$. Clearly, $\mathrm{CL}(\mathfrak{F}) \subset \mathrm{OL}(\mathfrak{F})$ and $\|f\|_{\mathrm{OL}(\mathfrak{F})} \le \|f\|_{\mathrm{CL}(\mathfrak{F})}$. In the same way as in the case of operator Lipschitz functions, it is easy to verify that

$$\|f(M)R - Rf(N)\| \le \|f\|_{\mathrm{CL}(\mathfrak{F})}\|MR - RN\|$$

for every operator R and every normal operators M and N such that $\sigma(M), \sigma(N) \subset \mathfrak{F}$. Similarly, one can prove the following analog of Lemma 2.2.

Lemma 7.1. *Let $f \in \mathrm{CL}(\mathfrak{F})$. Then there exists a unique extension of f to a continuous function $\widetilde{f}$ on $\operatorname{clos}\mathfrak{F}$, and $\|\widetilde{f}\|_{\mathrm{CL}(\operatorname{clos}\mathfrak{F})} = \|f\|_{\mathrm{CL}(\mathfrak{F})}$.*

Results of the paper [5] imply that every function $f \in \mathrm{CL}(\mathfrak{F})$ is differentiable as a function of a complex variable at each nonisolated point of $\mathfrak{F}$. Moreover, if $\mathfrak{F}$ is not bounded, then there exists the finite limit $\lim\limits_{|z|\to+\infty} z^{-1}f(z) \stackrel{\mathrm{def}}{=} f'(\infty)$ for every $f \in \mathrm{CL}(\mathfrak{F})$. Thus, a function $f \in \mathrm{CL}(\mathfrak{F})$ is analytic at each interior point of $\mathfrak{F}$. From this it readily follows that $\mathrm{CL}(\mathbb{C}) = \{az + b : a, b \in \mathbb{C}\}$.

A closed set $\mathfrak{F}$ is called a Fuglede set if $\overline{z} \in \mathrm{CL}(\mathfrak{F})$ (see [8], where this notion was introduced for compact $\mathfrak{F}$). Thus, if $\mathfrak{F}$ is a Fuglede set, then the function $\overline{z}$ is differentiable as a function of a complex variable at each nonisolated point of $\mathfrak{F}$. Moreover, there exists the limit $\lim\limits_{|z|\to+\infty} z^{-1}\overline{z}$ if $\mathfrak{F}$ is an unbounded Fuglede set. Clearly, $\|f\|_{\mathrm{OL}(\mathfrak{F})} \le \|f\|_{\mathrm{CL}(\mathfrak{F})} \le \|\overline{z}\|_{\mathrm{CL}(\mathfrak{F})}\|f\|_{\mathrm{OL}(\mathfrak{F})}$ for every function defined on $\mathfrak{F}$. Hence, the equality $\mathrm{OL}(\mathfrak{F}) = \mathrm{CL}(\mathfrak{F})$ holds if and only if $\mathfrak{F}$ is a Fuglede set, see [8]. Note that $\|\overline{z}\|_{\mathrm{CL}(\mathfrak{F})} \ge 1$ if $\mathfrak{F}$ contains at least two points. Results of the paper [6] imply that $\|\overline{z}\|_{\mathrm{CL}(\mathfrak{F})} \le 1$ if and only if $\mathfrak{F}$ is contained in a circle or straight line. Thus, $\|f\|_{\mathrm{OL}(\mathfrak{F})} = \|f\|_{\mathrm{CL}(\mathfrak{F})}$ for every function f on a set $\mathfrak{F}$ if and only if $\mathfrak{F}$ lies on a circle or straight line.

Now we state an analogue of Theorem 2.3 for commutator Lipschitz functions.

Theorem 7.2. *Let f be a continuous function defined on a closed subset $\mathfrak{F}$ of $\mathbb{C}$. Then the following two statements are equivalent:*

(i) *$\|f(N)R - Rf(N)\| \le \|NR - RN\|$ for all operators R and all normal operators N such that $\sigma(N) \subset \mathfrak{F}$;*

(ii) *$\|f(M)R - Rf(N)\| \le \|MR - RN\|$ for all operators R and all normal operators M and N such that $\sigma(M), \sigma(N) \subset \mathfrak{F}$.*

This theorem was proved in [3], see also [8], where a similar statement is proved for symmetric norms.

It should be noted that Lemmas 2.6 and 2.7 have natural analogs for commutator Lipschitz functions. We state the analogs without proofs because they are similar to the proofs of Lemmas 2.6 and 2.7.

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[7] In the papers [8] and [10], such functions are called *commutator bounded* or rather *commutator $\mathfrak{S}^b$-bounded*.

Lemma 7.3. *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. Assume that a mapping $\omega \mapsto \Phi_\omega$ defined almost everywhere on Ω acts into $\mathrm{CL}(\mathfrak{F})$ and possesses the following properties:*

(a) *the function $\omega \mapsto \Phi_\omega(z)$ is measurable for all $z \in \mathfrak{F}$;*

(b) *the function $\omega \mapsto \Phi_\omega(z_0)$ is summable for some $z_0 \in \mathfrak{F}$;*

(c) *the function $\omega \mapsto \|\Phi_\omega\|_{\mathrm{CL}(\mathfrak{F})}$ is summable.*

Then the function $\omega \mapsto \Phi_\omega(z)$ is summable for all $z \in \mathfrak{F}$, the function $f(z) \stackrel{\text{def}}{=} \int\limits_\Omega \Phi_\omega(z)\,d\mu(\omega)$ belongs to $\mathrm{CL}(\mathfrak{F})$, and

$$\|f\|_{\mathrm{CL}(\mathfrak{F})} \le \int\limits_\Omega \|\Phi_\omega\|_{\mathrm{CL}(\mathfrak{F})}\,d\mu(\omega).$$

Lemma 7.4. *Let f be a function on $\mathfrak{F} \cup \{z_0\}$, where $\mathfrak{F}$ is a closed subset of $\mathbb{C}$, $z_0 \in \mathbb{C} \setminus \mathfrak{F}$. Then*

$$\|f\|_{\mathrm{CL}(\mathfrak{F}\cup\{z_0\})} \le \|f\|_{\mathrm{CL}(\mathfrak{F})} + \sup_{z\in\mathfrak{F}} \frac{|f(z) - f(z_0)|}{|z - z_0|}.$$

We need one more lemma.

Lemma 7.5. *Let $f \in \mathrm{CL}(\mathfrak{F})$, where $\mathfrak{F}$ is a closed subset of $\mathbb{C}$. Then the function $\dfrac{f(z)}{z-a}$ belongs to $\mathrm{CL}(\mathfrak{F})$ for all $a \in \mathbb{C} \setminus \mathfrak{F}$, and*

$$\left\|\frac{f(z)}{z-a}\right\|_{\mathrm{CL}(\mathfrak{F})} \le \left(\|f\|_{\mathrm{CL}(\mathfrak{F})} + \sup_{z\in\mathfrak{F}} \frac{|f(z)|}{|z-a|}\right)(\mathrm{dist}(a, \mathfrak{F}))^{-1}.$$

Proof. It suffices to consider the case where $a = 0$. By Theorem 7.2, we have to prove that for

$$C = (\mathrm{dist}(0, \mathfrak{F}))^{-1}\|f\|_{\mathrm{CL}(\mathfrak{F})} + (\mathrm{dist}(0, \mathfrak{F}))^{-1} \sup_{z\in\mathfrak{F}} \frac{|f(z)|}{|z|},$$

the inequality

$$\|N^{-1}f(N)R - RN^{-1}f(N)\| \le C\|NR - RN\|$$

holds for all operators R and all normal operators N such that $\sigma(N) \subset \mathfrak{F}$. To prove this inequality, it suffices to note that

$$N^{-1}f(N)R - RN^{-1}f(N) = N^{-1}(f(N)R - Rf(N)) + N^{-1}(RN - NR)N^{-1}f(N). \qquad \square$$

Now we are going to state an analogue of Theorem 5.4 for commutator Lipschitz functions. As was noted above, commutator Lipschitz functions are differentiable as functions of a complex variable at each nonisolated point of their domains. Thus, we are interested in the transformations $T_{\varphi,n}$ only for $n = 1$ because for $n \ne 1$, the function $|\varphi'(z)|\,\mathrm{sgn}^n(\varphi')$ is not differentiable at any point as a function of a complex variable if, of course, we exclude the trivial case where $\varphi \in \mathrm{Aut}(\mathbb{C})$. Put $T_\varphi \stackrel{\text{def}}{=} T_{\varphi,1}$. Recall that $\mathrm{CL}_a(\mathfrak{F}) \stackrel{\text{def}}{=} \{f \in \mathrm{CL}(\mathfrak{F}) : f(a) = 0\}$, where $a \in \mathfrak{F}$.

Theorem 7.6. *Let $\varphi \in \mathrm{Aut}(\widehat{\mathbb{C}})$ and $f \in \mathrm{CL}(\mathfrak{F})$, where $\mathfrak{F}$ is a closed subset of $\mathbb{C}$. Assume that $\varphi(\infty) \in \mathfrak{F}$ and $f(\varphi(\infty)) = 0$. Then $T_\varphi f \in \mathrm{CL}(\mathfrak{F}_\varphi)$, and*

$$\frac{1}{3}\|f\|_{\mathrm{CL}(\mathfrak{F})} \le \|T_\varphi f\|_{\mathrm{CL}(\mathfrak{F}_\varphi)} \le 3\|f\|_{\mathrm{CL}(\mathfrak{F})}.$$

Moreover, $T_\varphi(\mathrm{CL}_{\varphi(\infty)}(\mathfrak{F})) = \mathrm{CL}_{\varphi^{-1}(\infty)}(\mathfrak{F}_\varphi)$.

Proof. The equality $T_\varphi(\mathrm{CL}_{\varphi(\infty)}(\mathfrak{F})) = \mathrm{CL}_{\varphi^{-1}(\infty)}(\mathfrak{F}_\varphi)$ can be deduced from the inclusion $T_\varphi(\mathrm{CL}_{\varphi(\infty)}(\mathfrak{F})) \subset \mathrm{CL}_{\varphi^{-1}(\infty)}(\mathfrak{F}_\varphi)$ in the same way as in the case of operator Lipschitz functions, see the proof of Theorem 5.4. Thus, it suffices to prove only the inequalities. In the same way as in the case of operator Lipschitz functions, it suffices to consider the case where $\varphi(z) = z^{-1}$. We can assume that $\|f\|_{\mathrm{OL}(\mathfrak{F})} = 1$ and $f(0) = 0$. Put $g \stackrel{\text{def}}{=} T_\varphi f$. It follows from the equality $f = T_\varphi g$ that it suffices to prove the upper estimate for $\|g\|_{\mathrm{OL}(\mathfrak{F})}$, i.e., by virtue of Theorem 7.2, the following inequality:

$$\|g(N)R - Rg(N)\| \le 3\|NR - RN\|$$

for every operator R and every normal operator N with spectrum in $\mathfrak{F}_{-1}$. Moreover, by Lemma 2.1, we can assume, in addition, that the spectrum of N is finite. Put

$$h(z) \stackrel{\text{def}}{=} \begin{cases} -zf(z^{-1}) & \text{if } z \in \mathfrak{F}_{-1} \text{ and } z \neq 0, \\ 0 & \text{if } z = 0. \end{cases}$$

Clearly, $g(z) = zh(z)$ for all $z \in \mathfrak{F}_{-1}$ and

$$\sup_{\mathfrak{F}_{-1}} |h| \le \|f\|_{\mathrm{Lip}(\mathfrak{F})} \le \|f\|_{\mathrm{OL}(\mathfrak{F})} = 1.$$

We have

$$g(N)R - Rg(N) = h(N)(NR - RN) + h(N)RN - NRh(N) + (NR - RN)h(N),$$

whence

$$\|g(N)R - Rg(N)\| \le 2\|NR - RN\| + \|h(N)RN - NRh(N)\|.$$

Thus, it remains to prove the following inequality:

$$\|h(N)RN - NRh(N)\| \le \|NR - RN\| \tag{7.2}$$

for every operator R and every normal operator N with finite spectrum lying in $\mathfrak{F}_{-1}$. If N is invertible, then

$$\|h(N)RN - NRh(N)\| = \|f(N^{-1})NRN - NRNf(N^{-1})\| \le \|N^{-1}NRN - NRNN^{-1}\| = \|NR - RN\|.$$

Denote by $\mathcal{H}_0$ the range of the operator N. Note that $\mathcal{H}_0$ is closed since the spectrum of N is finite. Let us consider the operator N_0 on the space $\mathcal{H}_0$ such that $N_0u = Nu$ for all $u \in \mathcal{H}_0$. The operator N_0 is normal because $\mathcal{H}_0$ is a reducing subspace for N. Moreover, N_0 is invertible because $\sigma(N_0) = \sigma(N) \setminus \{0\}$. Thus, inequality (7.2) for the operator N_0 is already proved. Let P denote the orthogonal projection onto $\mathcal{H}_0$. Clearly, the range of $h(N)$ is contained in $\mathcal{H}_0$ and $\ker h(N) \supset \ker N$. Hence,

$$\begin{aligned} \|h(N)RN - NRh(N)\| &= \|h(N_0)PRPN_0 - N_0PRPh(N_0)\| \\ &= \|f(N_0^{-1})N_0PRPN_0 - N_0PRPN_0f(N_0^{-1})\| \\ &\le \|N_0^{-1}N_0PRPN_0 - N_0PRPN_0N_0^{-1}\| \\ &= \|P(RN - NR)P\| \le \|RN - NR\|. \end{aligned}$$

□

Corollary 7.7. *Let $\varphi \in \mathrm{Aut}(\widehat{\mathbb{C}})$ and $f \in \mathrm{CL}(\mathfrak{F})$, where $\mathfrak{F}$ is a closed subset of $\mathbb{C}$. Assume that $\varphi(\infty) \notin \mathfrak{F}$. Then $T_\varphi f \in \mathrm{OL}(\mathfrak{F}_\varphi)$, and*

$$\frac{1}{3}\max\left(\|f\|_{\mathrm{CL}(\mathfrak{F})}, \sup_{z\in\mathfrak{F}} \frac{|f(z)|}{|z - \varphi(\infty)|}\right) \le \|T_\varphi f\|_{\mathrm{CL}(\mathfrak{F}_\varphi)} \le 3\left(\|f\|_{\mathrm{CL}(\mathfrak{F})} + \sup_{z\in\mathfrak{F}} \frac{|f(z)|}{|z - \varphi(\infty)|}\right).$$

Moreover,[8] $T_\varphi(\mathrm{CL}(\mathfrak{F})) = \mathrm{CL}_{\varphi^{-1}(\infty)}(\mathfrak{F}_\varphi)$.

Proof. If $\varphi(\infty) = \infty$, then $\varphi \in \mathrm{Aut}(\mathbb{C})$. Then, clearly, $T_\varphi(\mathrm{CL}(\mathfrak{F})) = \mathrm{CL}(\mathfrak{F}_\varphi)$ and $\|T_\varphi f\|_{\mathrm{CL}(\mathfrak{F}_\varphi)} = \|f\|_{\mathrm{CL}(\mathfrak{F})}$ for all $f \in \mathrm{CL}(\mathfrak{F})$. Now let $\varphi(\infty) \in \mathbb{C}$. To reduce in this case the corollary to Theorem 7.6, we can extend each function $f \in \mathrm{CL}(\mathfrak{F})$ to a function $f_0 \in \mathrm{CL}(\mathfrak{F} \cup \{\varphi(\infty)\})$ on the set $\mathfrak{F} \cup \{\varphi(\infty)\}$ by putting $f_0(\varphi(\infty)) = 0$. Then $T_\varphi f_0 = T_\varphi f$. Hence,

$$T_\varphi(\mathrm{CL}(\mathfrak{F})) = T_\varphi\big(\mathrm{CL}_{\varphi(\infty)}(\mathfrak{F} \cup \{\varphi(\infty)\})\big) = \mathrm{CL}_{\varphi^{-1}(\infty)}(\mathfrak{F}_\varphi).$$

To prove the inequalities, we note that

$$\max\left(\|f\|_{\mathrm{CL}(\mathfrak{F})}, \sup_{z\in\mathfrak{F}} \frac{|f(z)|}{|z - \varphi(\infty)|}\right) \le \|f_0\|_{\mathrm{CL}(\mathfrak{F}\cup\{\varphi(\infty)\})} \le \|f\|_{\mathrm{CL}(\mathfrak{F})} + \sup_{z\in\mathfrak{F}} \frac{|f(z)|}{|z - \varphi(\infty)|},$$

where the first inequality is evident and the second inequality follows from Lemma 7.4. It remains to apply Theorem 7.6 one more time. □

[8] Here we put $\mathrm{CL}_\infty(\mathfrak{F}) \stackrel{\text{def}}{=} \mathrm{CL}(\mathfrak{F})$.

Example 1. It is well known that the exponent belongs to the space $\mathrm{CL}(\mathfrak{F})$ if the projection of $\mathfrak{F}$ on the real line is bounded from above. Indeed,

$$\exp(M)R - R\exp(N) = -\exp(M)\int_0^1 d(\exp(-tM)R\exp(tN)) = \int_0^1 \exp((1-t)M)(MR - RN)\exp(tN)\,dt,$$

whence

$$\|\exp(M)R - R\exp(N)\| \le e^{\sup \operatorname{Re}\mathfrak{F}}\|MR - RN\|$$

for every operator R and normal operators M and N with spectra in $\mathfrak{F}$. Applying Theorem 7.6 to the commutator Lipschitz function $f = -\exp|\mathfrak{F}$ with $\mathfrak{F} = \left\{\operatorname{Re} z \le -\frac{1}{2}\right\}$ and the linear fractional function $\varphi(z) = (z-1)^{-1}$, we see that the function

$$h(z) \stackrel{\text{def}}{=} \begin{cases} (z-1)^2\exp\left((z-1)^{-1}\right) & \text{if } z \in \mathbb{C} \text{ and } z \ne 1, \\ 0 & \text{if } z = 1, \end{cases}$$

is commutator Lipschitz in the closed unit disk.

This was proved (in a different way) by Kissin and Shulman [7] to answer a question posed by J. P. Williams [12].

Note that if we take as $\mathfrak{F}$ the half-plane $\{\operatorname{Re} z \le 0\}$, then we conclude that h is commutator Lipschitz in the half-plane $\{\operatorname{Re} z \le 1\}$. If we take as $\mathfrak{F}$ the half-plane $\{\operatorname{Re} z \le a\}$ with $a > 0$, then we conclude that h is commutator Lipschitz in each annulus of the form $\{|z - 1 - \varepsilon| \ge \varepsilon\}$, where $\varepsilon > 0$.

Example 2. Theorem 1.1 in Kissin and Shulman's paper [9] contains the following equality:

$$\mathrm{CL}(\overline{\mathbb{D}}) = \{f \in C_A : f|\mathbb{T} \in \mathrm{OL}(\mathbb{T})\}, \tag{7.3}$$

where C_A denotes the disk algebra. Results of our work allow us to carry over this equality from the disk to the half-plane:

$$\mathrm{CL}(\overline{\mathbb{C}}_+) = \{f \in \mathcal{A}_1 : f|\mathbb{R} \in \mathrm{OL}(\mathbb{R})\}, \tag{7.4}$$

where $\mathcal{A}_1$ is the set of all functions f defined and continuous in the closed upper half-plane $\overline{\mathbb{C}}_+$, analytic in the open upper half-plane $\mathbb{C}_+$, and such that $\sup\limits_{z\in\mathbb{C}_+}(1+|z|)^{-1}|f(z)| < +\infty$.

To prove equality (7.4), it suffices to verify the inclusion $\{f \in \mathcal{A}_1 : f|\mathbb{R} \in \mathrm{OL}(\mathbb{R})\} \subset \mathrm{CL}(\overline{\mathbb{C}}_+)$ because the opposite inclusion is evident. Let $g \in \mathcal{A}_1$ and let $g|\mathbb{R} \in \mathrm{OL}(\mathbb{R})$. Put

$$f(z) \stackrel{\text{def}}{=} \begin{cases} \dfrac{(g\circ\varphi)(z)}{\varphi'} & \text{if } |z| \le 1, \quad z \ne -1, \\ 0 & \text{if } z = -1, \end{cases}$$

where $\varphi(z) = \kappa^{-1}(z) = i\dfrac{1-z}{1+z}$. Let us prove that $f \in C_A$. To do this, it suffices to verify that f is continuous at -1. We have

$$\lim_{z\to -1}|f(z)| = \frac{1}{2}\lim_{z\to -1}\left(|z+1|^2\left|g\left(i\frac{1-z}{1+z}\right)\right|\right) = 0$$

because $|g(w)| \le c + c|w|$ for some $c > 0$. In the notation of Theorem 5.6, $X_1(f|\mathbb{T}) = g|\mathbb{R}$. Hence, $f \in \mathrm{OL}_{-1}(\mathbb{T})$ by Theorem 5.6. Thus, $f \in \mathrm{CL}(\overline{\mathbb{D}})$ by equality (7.3). Now applying Theorem 7.6 to $\mathfrak{F} = \overline{\mathbb{D}}$ and $\varphi = \kappa$, we conclude that $g \in \mathrm{CL}(\overline{\mathbb{C}}_+)$.

Now let $\mathfrak{F}$ denote a perfect subset of $\widehat{\mathbb{C}}$. Then we can define the space $(\mathrm{CL})'(\mathfrak{F})$ consisting of the derivatives (in the complex sense) of functions in $\mathrm{CL}(\mathfrak{F}\cap\mathbb{C})$, i.e., $(\mathrm{CL})'(\mathfrak{F}) \stackrel{\text{def}}{=} \{f' : f \in \mathrm{CL}(\mathfrak{F}\cap\mathbb{C})\}$. Put

$$\|h\|_{(\mathrm{CL})'(\mathfrak{F})} \stackrel{\text{def}}{=} \inf\left\{\|f\|_{\mathrm{CL}(\mathfrak{F}\cap\mathbb{C})} : f'(z) = h(z) \text{ for any } z \in \mathfrak{F}\right\}.$$

Clearly, $(\mathrm{CL})'(\mathfrak{F})$ is a Banach space.

Let φ be a linear fractional transformation. We are going to consider the question about the equality

$$\left\{h\circ\varphi : h \in (\mathrm{CL})'(\mathfrak{F})\right\} = (\mathrm{CL})'(\varphi^{-1}(\mathfrak{F})). \tag{7.5}$$

This equality is trivial if $\varphi \in \mathrm{Aut}(\mathbb{C})$. However, in general, it is not true. For example, if $\mathfrak{F} = \overline{\mathbb{D}}$ and $\varphi(z) = \frac{1}{z}$, then φ belongs to $\left\{h\circ\varphi : h \in (\mathrm{CL})'(\mathfrak{F})\right\}$ but does not belong to $(\mathrm{CL})'(\varphi^{-1}(\mathfrak{F}))$. This is related to the fact that the set $\varphi^{-1}(\mathfrak{F})\cap\mathbb{C}$ is not simply connected in the given case.

The author does not know whether equality (7.5) is true under the condition that both sets $\mathfrak{F} \cap \mathbb{C}$ and $\varphi^{-1}(\mathfrak{F}) \cap \mathbb{C}$ are simply connected. We are going to prove this equality under the condition that both sets $\mathfrak{F} \cap \mathbb{C}$ and $\varphi^{-1}(\mathfrak{F}) \cap \mathbb{C}$ are convex.

First we prove the following generalization of Theorem 3.4.

Theorem 7.8. *Let f be a continuous function on a perfect subset $\mathfrak{F}$ of $\widehat{\mathbb{C}}$. Assume that $\mathfrak{F} \cap \mathbb{C}$ is a starlike set with respect to a point a. Then if the function $(z-a)f(z)$ belongs to $\mathrm{CL}(\mathfrak{F} \cap \mathbb{C})$, then $f \in (\mathrm{CL})'(\mathfrak{F})$ and $\|f\|_{(\mathrm{CL})'(\mathfrak{F})} \le \|(z-a)f(z)\|_{\mathrm{CL}(\mathfrak{F}\cap\mathbb{C})}$.*

Proof. We can assume that $a = 0$. Let $[0, z]$ denote the line segment connecting the point 0 with a point $z \in \mathfrak{F} \cap \mathbb{C}$. Put

$$F(z) = \int\limits_{[0,z]} f(\zeta)\, d\zeta = \int\limits_0^1 z f(tz)\, dt.$$

Note that $zf(tz) \in \mathrm{CL}(t^{-1}(\mathfrak{F} \cap \mathbb{C}))$ and

$$\|zf(tz)\|_{\mathrm{CL}(\mathfrak{F}\cap\mathbb{C})} \le \|zf(tz)\|_{\mathrm{CL}(t^{-1}(\mathfrak{F}\cap\mathbb{C}))} = \|zf(z)\|_{\mathrm{CL}(\mathfrak{F}\cap\mathbb{C})}$$

for all $t \in (0, 1]$. Thus, Lemma 7.3 implies that $F \in \mathrm{CL}(\mathfrak{F} \cap \mathbb{C})$ and $\|F\|_{\mathrm{CL}(\mathfrak{F}\cap\mathbb{C})} \le \|zf(z)\|_{\mathrm{CL}(\mathfrak{F}\cap\mathbb{C})}$. It remains to verify that $F'(z) = f(z)$ for all $z \in \mathfrak{F}$. This is evident for $z = 0$ and for $z = \infty$ if $\infty \in \mathfrak{F}$. Now let $z = z_0$, $z_0 \in \mathfrak{F}$, $z_0 \in \mathbb{C} \setminus \{0\}$. The function F is differentiable as a function of a complex variable everywhere on $\mathfrak{F} \cap \mathbb{C}$ because $F \in \mathrm{CL}(\mathfrak{F})$ and all the points of $\mathfrak{F}$ are limit. Thus, to prove the equality $F'(z_0) = f(z_0)$, it suffices to note that

$$F'(z_0) = \lim_{s\to 0} \frac{F((1+s)z_0) - F(z_0)}{sz_0} = \lim_{s\to 0} \frac{1}{s} \int\limits_1^{1+s} f(tz_0)\, dt = f(z_0). \qquad \square$$

Theorem 7.9. *Let $f \in \mathrm{CL}(\mathfrak{F} \cap \mathbb{C})$, where $\mathfrak{F}$ is a perfect subset of $\widehat{\mathbb{C}}$. Assume that $\mathfrak{F} \cap \mathbb{C}$ is a starlike set with respect to a point a. Then the function $\dfrac{f(z) - f(a)}{z - a}$ belongs to $(\mathrm{CL})'(\mathfrak{F})$, and*

$$\left\| \frac{f(z) - f(a)}{z - a} \right\|_{(\mathrm{CL})'(\mathfrak{F})} \le \|f\|_{\mathrm{CL}(\mathfrak{F}\cap\mathbb{C})}.$$

Proof. The result readily follows from Theorem 7.8. $\square$

Theorem 7.10. *Let $f \in \mathrm{CL}(\mathfrak{F} \cap \mathbb{C})$, where $\mathfrak{F}$ is a perfect subset of $\widehat{\mathbb{C}}$. Assume that the set $\mathfrak{F} \cap \mathbb{C}$ is convex. Then the function $\dfrac{f(z)}{z-a}$ belongs to $(\mathrm{CL})'(\mathfrak{F})$, and*

$$\left\| \frac{f(z)}{z-a} \right\|_{(\mathrm{CL})'(\mathfrak{F})} \le 2\|f\|_{\mathrm{CL}(\mathfrak{F}\cap\mathbb{C})} + \sup_{z\in\mathfrak{F}\cap\mathbb{C}} \frac{|f(z)|}{|z-a|}$$

for all $a \in \mathbb{C} \setminus \mathfrak{F}$.

Proof. Let b is the point in $\mathfrak{F}$ nearest to a. Note that

$$\frac{f(z)(z-b)}{z-a} = f(z) + (a-b)\frac{f(z)}{z-a}.$$

Using Theorem 7.8 and Lemma 7.5, we get the relations

$$\begin{aligned} \left\| \frac{f(z)}{z-a} \right\|_{(\mathrm{CL})'(\mathfrak{F})} &\le \|f\|_{\mathrm{CL}(\mathfrak{F}\cap\mathbb{C})} + |a-b| \left(\|f\|_{\mathrm{CL}(\mathfrak{F}\cap\mathbb{C})} + \sup_{\zeta\in\mathfrak{F}\cap\mathbb{C}} \frac{|f(z)|}{|z-a|} \right) (\mathrm{dist}(a, \mathfrak{F}))^{-1} \\ &= 2\|f\|_{\mathrm{CL}(\mathfrak{F}\cap\mathbb{C})} + \sup_{\zeta\in\mathfrak{F}\cap\mathbb{C}} \frac{|f(z)|}{|z-a|}. \end{aligned} \qquad \square$$

Theorem 7.11. *Let $\varphi \in \mathrm{Aut}(\widehat{\mathbb{C}}) \setminus \mathrm{Aut}(\mathbb{C})$ and let $h \in (\mathrm{CL})'(\mathfrak{F})$, where $\mathfrak{F}$ is a perfect subset of $\widehat{\mathbb{C}}$. Assume that the set $\varphi^{-1}(\mathfrak{F}) \cap \mathbb{C}$ is convex. Then $h \circ \varphi \in (\mathrm{CL})'(\varphi^{-1}(\mathfrak{F}))$ and $\|h \circ \varphi\|_{(\mathrm{CL})'(\varphi^{-1}(\mathfrak{F}))} \le 49\|h\|_{(\mathrm{CL})'(\mathfrak{F})}$.*

Proof. Let a be the nearest to $\varphi(\infty)$ point of $\mathfrak{F}$. We can take a function $f \in \mathrm{CL}(\mathfrak{F})$ such that $f' = h$ and $f(a) = 0$. We have

$$h \circ \varphi = \left(\frac{f \circ \varphi}{\varphi'}\right)' + \frac{f \circ \varphi}{\varphi'} \cdot \frac{\varphi''}{\varphi'} = (T_\varphi f)' - \frac{2}{z - \varphi^{-1}(\infty)} T_\varphi f. \tag{7.6}$$

Let us prove that

$$\|T_\varphi f\|_{\mathrm{CL}(\varphi^{-1}(\mathfrak{F})\cap\mathbb{C})} \le \begin{cases} 3\|f\|_{\mathrm{CL}(\mathfrak{F}\cap\mathbb{C})} & \text{if } \ \varphi(\infty) \in \mathfrak{F}, \\ 9\|f\|_{\mathrm{CL}(\mathfrak{F}\cap\mathbb{C})} & \text{if } \ \varphi(\infty) \notin \mathfrak{F}. \end{cases} \tag{7.7}$$

If $\varphi(\infty) \in \mathfrak{F}$, then $\|T_\varphi f\|_{\mathrm{CL}(\varphi^{-1}(\mathfrak{F})\cap\mathbb{C})} \le 3\|f\|_{\mathrm{CL}(\mathfrak{F}\cap\mathbb{C})}$ by Theorem 7.6. If $\varphi(\infty) \notin \mathfrak{F}$, then, applying Corollary 7.7, we get the inequalities

$$\begin{aligned} \|T_\varphi f\|_{\mathrm{CL}(\varphi^{-1}(\mathfrak{F})\cap\mathbb{C})} &\le 3\left(\|f\|_{\mathrm{CL}(\mathfrak{F}\cap\mathbb{C})} + \sup_{z\in\mathfrak{F}} \frac{|f(z) - f(a)|}{|z - \varphi(\infty)|}\right) \\ &\le \|f\|_{\mathrm{CL}(\mathfrak{F}\cap\mathbb{C})}\left(3 + 3\sup_{z\in\mathfrak{F}} \frac{|z-a|}{|z - \varphi(\infty)|}\right) \le 9\|f\|_{\mathrm{CL}(\mathfrak{F}\cap\mathbb{C})}. \end{aligned}$$

First we consider the case where $\infty \in \mathfrak{F}$. Then identity (7.6), inequality (7.7) and Theorem 7.9 imply that

$$\|h \circ \varphi\|_{(\mathrm{CL})'(\varphi^{-1}(\mathfrak{F}))} \le \begin{cases} 9\|f\|_{\mathrm{CL}(\mathfrak{F}\cap\mathbb{C})} & \text{if } \ \varphi(\infty) \in \mathfrak{F} \ \text{ and } \ \infty \in \mathfrak{F}, \\ 27\|f\|_{\mathrm{CL}(\mathfrak{F}\cap\mathbb{C})} & \text{if } \ \varphi(\infty) \notin \mathfrak{F} \text{ and } \ \infty \in \mathfrak{F}. \end{cases}$$

To examine the case where $\infty \notin \mathfrak{F}$, we need, in addition, the following inequality:

$$\sup_{z\in\varphi^{-1}(\mathfrak{F})\cap\mathbb{C}} \frac{|(T_\varphi f)(z)|}{|z - \varphi^{-1}(\infty)|} \le \begin{cases} \|f\|_{\mathrm{CL}(\mathfrak{F}\cap\mathbb{C})} & \text{if } \ \ \varphi(\infty) \in \mathfrak{F}, \\ 2\|f\|_{\mathrm{CL}(\mathfrak{F}\cap\mathbb{C})} & \text{if } \ \ \varphi(\infty) \notin \mathfrak{F}. \end{cases} \tag{7.8}$$

We have

$$\frac{|(T_\varphi f)(z)|}{|z - \varphi^{-1}(\infty)|} = \frac{|f(\varphi(z)) - f(a)|}{|\varphi'(z)| \cdot |z - \varphi^{-1}(\infty)|} \le \frac{|\varphi(z) - a|}{|\varphi'(z)| \cdot |z - \varphi^{-1}(\infty)|}\|f\|_{\mathrm{Lip}(\mathfrak{F}\cap\mathbb{C})}.$$

A straightforward calculation shows that

$$\frac{|\varphi(z) - \varphi(\infty)|}{|\varphi'(z)| \cdot |z - \varphi^{-1}(\infty)|} = 1$$

for all $z \in \mathbb{C}$. This proves the required inequality for $a = \varphi(\infty)$, i.e., in the case where $\varphi(\infty) \in \mathfrak{F}$. If $\varphi(\infty) \notin \mathfrak{F}$, then

$$\frac{|\varphi(z) - a|}{|\varphi'(z)| \cdot |z - \varphi^{-1}(\infty)|} \le \frac{|\varphi(z) - \varphi(\infty)| + |\varphi(\infty) - a|}{|\varphi'(z)| \cdot |z - \varphi^{-1}(\infty)|} \le 2\frac{|\varphi(z) - \varphi(\infty)|}{|\varphi'(z)| \cdot |z - \varphi^{-1}(\infty)|} = 2$$

for $z \in \varphi^{-1}(\mathfrak{F})\cap\mathbb{C}$. Thus, in the case where $\infty \notin \mathfrak{F}$, identity (7.6), inequality (7.7), Theorem 7.10, and inequality (7.8) imply that

$$\|h \circ \varphi\|_{(\mathrm{CL})'(\varphi^{-1}(\mathfrak{F}))} \le \begin{cases} 17\|f\|_{\mathrm{CL}(\mathfrak{F}\cap\mathbb{C})} & \text{if } \ \varphi(\infty) \in \mathfrak{F} \ \text{ and } \ \infty \notin \mathfrak{F}, \\ 49\|f\|_{\mathrm{CL}(\mathfrak{F}\cap\mathbb{C})} & \text{if } \ \varphi(\infty) \notin \mathfrak{F} \ \text{ and } \ \infty \notin \mathfrak{F}. \end{cases}$$ □

Theorem 7.11 readily implies the following result.

Theorem 7.12. *Let $\varphi \in \mathrm{Aut}(\widehat{\mathbb{C}})$, where $\mathfrak{F}$ is a perfect subset of $\widehat{\mathbb{C}}$. Assume that the sets $\mathfrak{F}\cap\mathbb{C}$ and $\varphi^{-1}(\mathfrak{F})\cap\mathbb{C}$ are convex. Then for every function h on $\mathfrak{F}$, the inclusion $h \circ \varphi \in (\mathrm{CL})'(\varphi^{-1}(\mathfrak{F}))$ holds if and only if $h \in (\mathrm{CL})'(\mathfrak{F})$. Moreover, $\frac{1}{49}\|h\|_{(\mathrm{CL})'(\mathfrak{F})} \le \|h \circ \varphi\|_{(\mathrm{CL})'(\varphi^{-1}(\mathfrak{F}))} \le 49\|h\|_{(\mathrm{CL})'(\mathfrak{F})}$.*

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