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COMPARISON OF THE DISCRETE AND UNIFORM NORMS OF POLYNOMIALS ON AN INTERVAL AND A CIRCULAR ARC

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The paper proves inequalities involving the discrete and uniform norms of polynomials on an interval and a circular arc. The proofs are based on Bernstein type inequalities. The inequalities obtained supplement the corresponding results by Rakhmanov, Shekhtman, Sheil-Small, and Dubinin. Bibliography: 8 titles.

Introduction

In approximation theory, the question on comparing the discrete and uniform norms of polynomials on various subsets of the complex plane naturally arises.

Introduce the following notation:

$$\mathcal{P}_n^r := \left\{ p(z) : p(z) = \sum_{k=0}^{n} c_k z^k,\ c_k \in \mathbb{R},\, c_n \neq 0 \right\},$$

$$\mathcal{P}_n^c := \left\{ p(z) : p(z) = \sum_{k=0}^{n} c_k z^k,\ c_k \in \mathbb{C},\, c_n \neq 0 \right\},$$

$$\Gamma_\alpha = \{z = e^{i\varphi} : -\alpha \le \varphi \le \alpha\},\ \ 0 < \alpha < \pi.$$

$$H = \max_{x\in[-1,1]} P(x),\ \ L = \min_{x\in[-1,1]} P(x),\ \ \overline{H} = \max_{x\in[-1,1]} |P(x)|,\ \ P(x) \in \mathcal{P}_n^r.$$

$$M = \max_{z\in\Gamma_\alpha} |P(z)|,\ \ m = \min_{x\in\Gamma_\alpha} |P(z)|,\ \ P(x) \in \mathcal{P}_n^c.$$

Rakhmanov and Shekhtman in [1] established the inequality

$$\max_{w^N=1} |P(w)| \ge \left(1 + C\log\frac{N}{N-n}\right)^{-1} \max_{|z|=1} |P(z)| \tag{1}$$

for algebraic polynomials $P(z) \in \mathcal{P}_n^c$ and positive integers $N > n$, where the absolute constant C can be bounded by the number 16.

Somewhat later, Sheil-Small [2] proved the bound

$$\max_{w^N=1} |P(w)| \ge \sqrt{\frac{N-n}{N}} \max_{|z|=1} |P(z)|. \tag{2}$$

Dubinin (see [3,4]) established the inequality

$$\max_{w^N=1} |P(w)| \ge \cos\frac{\pi n}{2N} \max_{|z|=1} |P(z)|, \tag{3}$$

in which equality is attained in the case where $P(z) = (z\exp(i\pi/N))^n + 1$ and N is an arbitrary multiple of n.

Inequality (1) is sharper than (2) for n/N close to unity but worse than the Sheil–Small bound for small values of n/N. For $n < N < 2n$, the bound (3) is inferior to (2). For $N = 2n$ the bounds (2) and (3) coincide, whereas for $N > 2n$ inequality (3) strengthens inequalities (1) and (2) (see [3,4]).

Among papers considering bounds involving the discrete norm on the interval $(-1, 1)$, we mention the work by Coppersmith and Rivlin [5], and also the recent paper by Rakhmanov [6].

The present paper aims at deriving inequalities involving the uniform and discrete norms of polynomials on the interval $[-1, 1]$ and an arc Γ_α by applying the methods suggested in [3,4]. The proofs are based on inequalities

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for derivatives of polynomials on appropriate sets. For instance, for a polynomial $P(z) \in \mathcal{P}_n^r$ and $x \in [-1, 1]$, the following strengthened version of the Bernstein–Szegö inequality, which stems from Theorem 4.4 in [7], is valid:

$$|P'(x)|\sqrt{1-x^2} \le \gamma_n \sqrt{\overline{H}^2 - P^2(x)}, \tag{4}$$

where

$$\gamma_n = \left(n - 1 + \frac{2^{2-n}|c_n|}{H - L}\right) \le n. \tag{5}$$

Equality in (4) for arbitrary $x \in [-1, 1]$ is attained for $P(z) = aT_n(z)$, where a is a real and $T_n(z)$ is the Chebyshev polynomial of the first order of degree n.

The Bernstein type inequality for polynomials on a circular arc contains the result below, established in [8].

Lemma 1. *For a polynomial $P \in \mathcal{P}_n^c$ and $z \in \Gamma_\alpha$, the following inequality holds:*

$$|(|P(z)|^2)'_\varphi| \le \frac{n|z+1|\sqrt{(M^2 - |P(z)|^2)(|P(z)|^2 - m^2)}}{|\sqrt{z^2 - 2z\cos\alpha + 1}|}. \tag{6}$$

Equality in (6) *is attained for the polynomial*

$$P_\alpha(z) = \begin{cases} \prod\limits_{k=1}^{n/2} (z^2 - 2a_k z + 1) & \text{for } n \text{ odd}, \\ (z-1)\prod\limits_{k=1}^{(n-1)/2} (z^2 - 2a_k z + 1) & \text{for } n \text{ even}, \end{cases}$$

where $a_k = \cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2}\cos\frac{\pi(2k-1)}{n}$.

Main results

Assertion 1. *For a polynomial $P(z) \in \mathcal{P}_n^r$ and arbitrary $-1 \le x_1, x_2 \le 1$,*

$$\left|\arcsin\frac{|P(x)|}{\overline{H}}\Big|_{x_1}^{x_2}\right| \le \gamma_n|\arcsin x_1 - \arcsin x_2|, \tag{7}$$

where γ_n is defined in (5).

Proof. For definiteness, let $x_1 < x_2$. On integrating inequality (4) written as

$$\frac{|P'(x)|}{\sqrt{\overline{H}^2 - P^2(x)}} \le \frac{\gamma_n}{\sqrt{1-x^2}}$$

over the interval (x_1, x_2), performing the change of variable $u = u(x) = P(x)$, and using the fact that the modulus of an integral does not exceed the integral of the modulus, we obtain

$$\gamma_n(\arcsin x_2 - \arcsin x_1) \ge \pm\int\limits_{u_1}^{u_2} \frac{du}{\sqrt{\overline{H}^2 - u^2}} = \pm\arcsin\frac{u}{\overline{H}}\Big|_{u_1}^{u_2},$$

where $u_k = P(x_k)$, $k = 1, 2$. It remains to apply the elementary inequality $|a - b| \ge ||a| - |b||$. This completes the proof. □

Assertion 1 implies the following two theorems.

Theorem 1. *For any polynomial $P(z) \in \mathcal{P}_n^r$ and for any positive integer $N \ge \dfrac{1}{2\sin^2(\pi/4n)}$, it holds that*

$$\max_{k=0,\dots,N}|P(x_k)| \ge \cos\left(\gamma_n \arccos\left(1 - \frac{1}{N}\right)\right)\max_{x\in[-1,1]}|P(x)| \ge T_n\left(1 - \frac{1}{N}\right)\max_{x\in[-1,1]}|P(x)|,$$

where $x_k = -1 + \dfrac{2k}{N}$, $k = 0, \dots, N$.

Proof. Let x_0 be one of the points at which the maximum $\overline{H}$ is attained. Among the points x_k, $k = 0, \dots, N$, there is at least one point, say, x_m such that $|x_m - x_0| \le \frac{1}{N}$. Inequality (7) yields

$$\arcsin \frac{|P(x_m)|}{\overline{H}} \ge \frac{\pi}{2} - \gamma_n |\arcsin x_m - \arcsin x_0| \ge \frac{\pi}{2} - \gamma_n \left(\frac{\pi}{2} - \arcsin \left(1 - \frac{1}{N}\right)\right)$$
$$= \frac{\pi}{2} - \gamma_n \arccos \left(1 - \frac{1}{N}\right) \ge 0.$$

The second inequality stems from the oddness of the function $\arcsin x$ and also from its properties of being increasing and convex downword on the interval $[0, 1]$. Then

$$\frac{|P(x_m)|}{\overline{H}} \ge \cos \left(\gamma_n \arccos \left(1 - \frac{1}{N}\right)\right),$$

implying that

$$|P(x_m)| \ge \overline{H} \cos \left(\gamma_n \arccos \left(1 - \frac{1}{N}\right)\right) \ge \overline{H} \cos \left(n \arccos \left(1 - \frac{1}{N}\right)\right) = \overline{H}\, T_n \left(1 - \frac{1}{N}\right).$$

It remains to observe that $|P(x_m)| \le \max\limits_{k=0,\dots,N} |P(x_i)|$. This completes the proof of the theorem. □

A drawback of the above bound is that the quantity $\frac{1}{2\sin^2(\pi/4n)}$ grows too rapidly as n increases. In the theorem below, this drawback is alleviated by passing to an appropriate nonuniform partition of the interval $[-1, 1]$.

Theorem 2. *For any polynomial $P(z) \in \mathcal{P}_n^r$ and for any positive integer $N \ge n$, the following inequality holds*:

$$\max_{k=0,\dots,N} |P(x_k)| \ge \cos \frac{\pi \gamma_n}{2N} \max_{x \in [-1,1]} |P(x)|;$$

here, $x_k = \sin\left(-\frac{\pi}{2} + \frac{\pi k}{N}\right)$, $k = 0, \dots, N$.

Proof. Let x_0 be one of the points at which the maximum $\overline{H}$ is attained. Among the points x_k, $k = 0, \dots, N$, there is at least one point, say, x_m such that $|\arcsin x_m - \arcsin x_0| \le \frac{\pi}{2N}$. Inequality (7) yields

$$\arcsin \frac{|P(x_m)|}{\overline{H}} \ge \frac{\pi}{2} - \gamma_n |\arcsin x_m - \arcsin x_0| \ge \frac{\pi}{2} - \frac{\pi \gamma_n}{2N} \ge 0.$$

Therefore,

$$\frac{|P(x_m)|}{\overline{H}} \ge \cos \frac{\pi \gamma_n}{2N},$$

whence

$$|P(x_m)| \ge \overline{H} \cos \frac{\pi \gamma_n}{2N}.$$

It remains to observe that $|P(x_m)| \le \max\limits_{k=0,\dots,N} |P(x_i)|$. The theorem is proved completely. □

Assertion 2. *For a polynomial $P(z) \in \mathcal{P}_n^c$ with $m \ne M$ and for arbitrary reals φ_1 and φ_2 such that $e^{i\varphi_1}, e^{i\varphi_2} \in \Gamma_\alpha$, it holds that*

$$\left| \arcsin \sqrt{\frac{|P(e^{i\varphi})|^2 - m^2}{M^2 - m^2}} \,\Bigg|_{\varphi_1}^{\varphi_2} \right| \le n \left| \arcsin \frac{\sin \frac{\varphi}{2}}{\sin \frac{\alpha}{2}} \,\Bigg|_{\varphi_1}^{\varphi_2} \right|. \tag{8}$$

Proof. For definiteness, let $\varphi_1 < \varphi_2$. Write (6) in the form

$$\frac{|(|P(z)|^2)'_\varphi|}{\sqrt{(M^2 - |P(z)|^2)(|P(z)|^2 - m^2)}} \le \frac{n|z+1|}{|\sqrt{z^2 - 2z\cos\alpha + 1}|}, \quad z = e^{i\varphi},$$

and integrate it over the interval (φ_1, φ_2). For the left-hand side, by successively performing the changes $u = u(\varphi) = |P(\varphi)|^2$ and $t = \sqrt{u - m^2}$, we obtain

$$\int\limits_{\varphi_1}^{\varphi_2} \frac{|(|P(e^{i\varphi})|^2)'_\varphi|}{\sqrt{(M^2 - |P(e^{i\varphi})|^2)(|P(e^{i\varphi})|^2 - m^2)}} d\varphi \geq \pm \int\limits_{\varphi_1}^{\varphi_2} \frac{u'_\varphi d\varphi}{\sqrt{(u - m^2)(M^2 - u)}}$$

$$= \pm \int\limits_{u(\varphi_1)}^{u(\varphi_2)} \frac{du}{\sqrt{(u - m^2)(M^2 - u)}} = \pm 2 \int\limits_{t(u_1)}^{t(u_2)} \frac{dt}{\sqrt{M^2 - m^2 - t^2}}$$

$$= \pm 2 \arcsin \frac{t}{\sqrt{M^2 - m^2}} \Bigg|_{t_1}^{t_2} = \pm 2 \arcsin \sqrt{\frac{|P^2(e^{i\varphi})|^2 - m^2}{M^2 - m^2}} \Bigg|_{\varphi_1}^{\varphi_2};$$

here, $u_k = u(\varphi_k)$, $t_k = t(u_k)$, $k = 1, 2$.

Consider the right-hand side:

$$\int\limits_{\varphi_1}^{\varphi_2} \frac{n|e^{i\varphi} + 1| d\varphi}{|\sqrt{e^{i2\varphi} - 2e^{i\varphi}\cos\alpha + 1}|} = \int\limits_{\varphi_1}^{\varphi_2} \frac{n\left|\left(e^{i\varphi/2} + \dfrac{1}{e^{i\varphi/2}}\right)\right| d\varphi}{\left|\sqrt{e^{i\varphi} + \dfrac{1}{e^{i\varphi}} - 2\cos\alpha}\right|}$$

$$= \int\limits_{\varphi_1}^{\varphi_2} \frac{n\left|\dfrac{1}{2}\left(e^{i\varphi/2} + \dfrac{1}{e^{i\varphi/2}}\right)\right| d\varphi}{\left|\sqrt{\left(\dfrac{1}{2}\left(e^{i\varphi/2} + \dfrac{1}{e^{i\varphi/2}}\right)\right)^2 - \dfrac{1 + \cos\alpha}{2}}\right|} = \int\limits_{\varphi_1}^{\varphi_2} \frac{n\,|\cos(\varphi/2)|\, d\varphi}{\left|\sqrt{\cos^2(\varphi/2) - \cos^2(\alpha/2)}\right|}$$

$$= \int\limits_{\varphi_1}^{\varphi_2} \frac{n\cos(\varphi/2) d\varphi}{\sqrt{\sin^2(\alpha/2) - \sin^2(\varphi/2)}} = 2n \arcsin \frac{\sin(\varphi/2)}{\sin(\alpha/2)} \Bigg|_{\varphi_1}^{\varphi_2}.$$

This completes the proof. □

Remark. Letting α tend to π, we arrive at Theorem 2 in [3].

Theorem 3. *For any polynomial $P(z) \in \mathcal{P}_n^c$ and an arbitrary positive integer*

$$N \geq \frac{\alpha}{\alpha - 2\arcsin(\sin(\alpha/2)\cos(\pi/2n))},$$

the following inequality is valid:

$$\max_{k=0,\dots,N} |P(e^{i\varphi_k})| \geq \cos\left(n \arccos \frac{\sin\left(\dfrac{\alpha}{2} - \dfrac{\alpha}{2N}\right)}{\sin\dfrac{\alpha}{2}}\right) \max_{\varphi\in[-\alpha,\alpha]} |P(e^{i\varphi})|$$

$$= T_n\left(\frac{\sin\left(\dfrac{\alpha}{2} - \dfrac{\alpha}{2N}\right)}{\sin\dfrac{\alpha}{2}}\right) \max_{\varphi\in[-\alpha,\alpha]} |P(e^{i\varphi})|;$$

here, $\varphi_k = -\alpha + \dfrac{2\alpha k}{N}$, $k = 0, \dots, N$.

Proof. Let $z_0 = e^{i\varphi_0}$ be one of the points at which the maximum M is attained. Among the points φ_k, $k = 0, \dots, N$, there is at least one point, say, φ_m such that $|\varphi_m - \varphi_0| \le \frac{\alpha}{N}$. Inequality (8) yields

$$\arcsin\sqrt{\frac{|P(e^{i\varphi_m})|^2 - m^2}{M^2 - m^2}} \ge \frac{\pi}{2} - n\left|\arcsin\frac{\sin\frac{\varphi_m}{2}}{\sin\frac{\alpha}{2}} - \arcsin\frac{\sin\frac{\varphi_0}{2}}{\sin\frac{\alpha}{2}}\right|$$

$$\ge \frac{\pi}{2} - n\left(\frac{\pi}{2} - \arcsin\frac{\sin\left(\frac{\alpha}{2} - \frac{\alpha}{2N}\right)}{\sin\frac{\alpha}{2}}\right) = \frac{\pi}{2} - n\arccos\frac{\sin\left(\frac{\alpha}{2} - \frac{\alpha}{2N}\right)}{\sin\frac{\alpha}{2}} \ge 0.$$

The second inequality stems from the oddness of the function

$$y(x) = \arcsin\frac{\sin\frac{x}{2}}{\sin\frac{\alpha}{2}}$$

and also from its properties of being increasing and convex downward on the interval $[0, \alpha]$. Thus,

$$\sqrt{\frac{|P(e^{i\varphi_m})|^2 - m^2}{M^2 - m^2}} \ge \cos\left(n\arccos\frac{\sin\left(\frac{\alpha}{2} - \frac{\alpha}{2N}\right)}{\sin\frac{\alpha}{2}}\right),$$

implying that

$$|P(e^{i\varphi_m})| \ge M\cos\left(n\arccos\frac{\sin\left(\frac{\alpha}{2} - \frac{\alpha}{2N}\right)}{\sin\frac{\alpha}{2}}\right) = M\,T_n\left(\frac{\sin\left(\frac{\alpha}{2} - \frac{\alpha}{2N}\right)}{\sin\frac{\alpha}{2}}\right).$$

It remains to observe that $|P(e^{i\varphi_m})| \le \max_{k=0,\dots,N}|P(e^{i\varphi_k})|$. This completes the proof. □

Theorem 4. *For any polynomial $P(z) \in \mathcal{P}_n^c$ and an arbitrary positive integer $N \ge n$, it holds that*

$$\max_{k=0,\dots,N}|P(e^{i\varphi_k})| \ge \cos\frac{\pi n}{2N}\max_{\varphi\in[-\alpha,\alpha]}|P(e^{i\varphi})|,$$

where $\varphi_k = 2\arcsin\left(\sin\frac{\alpha}{2}\sin\left(-\frac{\pi}{2} + \frac{\pi k}{N}\right)\right)$, $k = 0, \dots, N$.

Proof. Let $z_0 = e^{i\varphi_0}$ be one of the points at which the maximum M is attained. Among the points φ_k, $k = 0, \dots, N$, there is at least one point, say, φ_m such that

$$\left|\arcsin\frac{\sin\frac{\varphi_m}{2}}{\sin\frac{\alpha}{2}} - \arcsin\frac{\sin\frac{\varphi_0}{2}}{\sin\frac{\alpha}{2}}\right| \le \frac{\pi}{2N}.$$

Inequality (8) yields

$$\arcsin\sqrt{\frac{|P(e^{i\varphi_m})|^2 - m^2}{M^2 - m^2}} \ge \frac{\pi}{2} - n\left|\arcsin\frac{\sin\frac{\varphi_m}{2}}{\sin\frac{\alpha}{2}} - \arcsin\frac{\sin\frac{\varphi_0}{2}}{\sin\frac{\alpha}{2}}\right| \ge \frac{\pi}{2} - \frac{\pi n}{2N} \ge 0.$$

Therefore,

$$\sqrt{\frac{|P(e^{i\varphi_m})|^2 - m^2}{M^2 - m^2}} \ge \cos\frac{\pi n}{2N},$$

implying that

$$|P(e^{i\varphi_k})| \ge M\cos\frac{\pi n}{2N}.$$

It remains to observe that $|P(e^{i\varphi_m})| \le \max_{k=0,\dots,N}|P(e^{i\varphi_k})|$. The theorem is proved completely. □

Letting α tend to π in Theorems 3 and 4, we arrive at the Dubinin bound (3) up to rotation of the unit circle.

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