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ON AN ESTIMATE IN THE CLASS OF TYPICALLY REAL FUNCTIONS

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Let $T(c_2, c_3)$ be the class of functions $f(z) = z + \sum_{n=2}^{\infty} c_n z^n$ regular and typically real in the disk $|z| < 1$ with fixed values of the coefficients c_2 and c_3. The boundary functions of the region of values of $f(z_0)$ $(0 < |z_0| < 1)$ and sharp estimates for $f(r)$, $0 < r < 1$, in the class $T(c_2, c_3)$ are determined. Bibliography: 6 titles.

Introduction

Let T be the class of functions

$$f(z) = z + \sum_{n=2}^{\infty} c_n z^n$$

regular and typically real in the disk $U = \{z \in \mathbb{C} : |z| < 1\}$, i.e., the class of functions satisfying in U the condition

$$\operatorname{Im} z \cdot \operatorname{Im} f(z) > 0 \quad \text{for} \quad \operatorname{Im} z \neq 0.$$

Jenkins [1] obtained the following sharp estimate for $f(r)$ in the class T with a prescribed value of c_2 $(-2 \leq c_2 \leq 2)$:

$$\frac{r}{1 - c_2 r + r^2} \leq f(r) \leq \frac{r(1 + c_2 r + r^2)}{(1 - r^2)^2}, \quad 0 < r < 1. \tag{1}$$

Equality on the left-hand side of (1) occurs only for the function $z(1 - c_2 z + z^2)^{-1}$, and equality on its right-hand side occurs only for the function $z(1 + c_2 z + z^2)(1 - z^2)^{-2}$.

In [2], the region of values of $f(z)$ and, in particular, a sharp estimate for $f(r)$ were found in the class T in the case where the values of c_2 and c_3 are prescribed.

In the present paper, some of the results obtained in [2] are supplemented.

1.

Let D be the region of values of the system $\{c_2, c_3, f(r)\}$, $0 < r < 1$, on the class T. Set

$$x_1 = c_2, \quad x_2 = c_3, \quad x_3 = f(r), \quad \rho = r + \frac{1}{r}.$$

Theorem 1. 1. *The set of values D is the set of all points*

$$X = X(x_1, x_2, x_3) \in \mathbb{R}^3$$

defined by the inequalities

$$\varepsilon + (2 - \varepsilon\rho)x_3 \geq 0, \quad \varepsilon(x_2 + 1) - (2 + \varepsilon\rho)x_1 + 2\rho \geq 0, \quad \Delta_\varepsilon \geq 0, \tag{2}$$

where

$$\Delta_\varepsilon = \begin{vmatrix} \varepsilon + (2 - \varepsilon\rho)x_3 & \varepsilon x_1 - 2 \\ \varepsilon x_1 - 2 & \varepsilon(x_2 + 1) - x_1(\varepsilon\rho + 2) + 2\rho \end{vmatrix}, \quad \varepsilon = \pm 1.$$

2. $\operatorname{Int} D = \{X \in \mathbb{R}^3 : \varepsilon + (2 - \varepsilon\rho)x_3 > 0,\ \Delta_\varepsilon > 0,\ \varepsilon = \pm 1\}$.

3. *To the boundary ∂D only the functions*

$$f_\varepsilon(z) = \frac{\lambda_1 z}{1 - 2\varepsilon z + z^2} + \frac{\lambda_2 z}{1 - 2tz + z^2}, \quad \varepsilon = \pm 1, \tag{3}$$

where $\lambda_1 + \lambda_2 = 1$, $\lambda_1, \lambda_2 \geq 0$, $0 < t < 1$, correspond.

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Theorem 2. *Let $f(z) = z + c_2 z^2 + c_3 z^3 + \cdots \in T$. Then the following sharp estimates hold:*

$$f(r) \le \frac{1}{\rho - 2}\left[1 - \frac{(2-c_2)^2}{c_3 + 1 - c_2(\rho+2) + 2\rho}\right],$$
$$f(r) \ge \frac{1}{\rho + 2}\left[1 - \frac{(2+c_2)^2}{c_3 + 1 - c_2(\rho-2) - 2\rho}\right],$$

and equalities occur only for the functions

$$f_\varepsilon(z) = \frac{z}{1 - 2\varepsilon z + z^2}\left[1 + \frac{(c_2 - 2\varepsilon)^2 z}{c_2 - 2\varepsilon - z(c_3 + 1 - 2\varepsilon c_2) + z^2(c_2 - 2\varepsilon)}\right]$$

with $\varepsilon = \pm 1$.

2.

Proof of Theorem 1. As is known (see [3, 4]), a function $f(z)$, $f(0) = 0$, $f'(0) = 1$, regular in U belongs to the class T if and only if this function admits the integral representation

$$f(z) = \int_{-1}^{1} z(1 - 2tz + z^2)^{-1} d\alpha(t), \quad \alpha \in M_1, \quad z \in U, \tag{4}$$

where M_1 is the class of functions $\alpha(t)$ nondecreasing on the interval $[-1, 1]$ and satisfying the condition

$$\int_{-1}^{1} d\alpha(t) = 1.$$

By using (4), for the system $\{c_2, c_3, f(r)\}$ we obtain the integral representation

$$c_2 = \int_{-1}^{1} 2t\, d\alpha(t), \quad c_3 = \int_{-1}^{1} (4t^2 - 1)\, d\alpha(t), \quad f(r) = \int_{-1}^{1} (\rho - 2t)^{-1} d\alpha(t), \quad \alpha \in M_1. \tag{5}$$

Set $y_1 = \frac{c_2}{2}$, $y_2 = \frac{c_3+1}{4}$, $y_3 = f(r)$.

As in known (see [5, pp. 84–85]), representation (5) holds if and only if from the validity of the inequality

$$\varphi(t) = \delta_0 + \delta_1 t + \delta_2 t^2 + \delta_3(\rho - 2t)^{-1} \ge 0 \tag{6}$$

for all $t, -1 \le t \le 1$, it follows that

$$\delta_0 + \delta_1 y_1 + \delta_2 y_2 + \delta_3 y_3 \ge 0.$$

Write (6) in the form

$$\varphi(t) = \frac{A_3(t)}{\rho - 2t} \ge 0, \quad -1 \le t \le 1, \tag{7}$$

where $A_3(t)$ is an algebraic polynomial in t of degree 3.

By using the Markov–Lukacs theorem (see [5, pp. 84–85]), represent (7) in the form

$$\begin{aligned}
\varphi(t) &= \frac{1+t}{\rho - 2t}(\beta_0 + \beta_1 t)^2 + \frac{1-t}{\rho - 2t}(\gamma_0 + \gamma_1 t)^2 = \beta_0^2\left[-\frac{1}{2} + \frac{2+\rho}{2(\rho - 2t)}\right] \\
&+ \beta_1^2\left[-\frac{\rho(\rho+2)}{8} - \frac{(2+\rho)t}{4} - \frac{t^2}{2} + \frac{\rho^2(2+\rho)}{8(\rho - 2t)}\right] \\
&+ 2\beta_0\beta_1\left[-\frac{2+\rho}{4} - \frac{t}{2} + \frac{\rho(\rho+2)}{4(\rho - 2t)}\right] \\
&+ \gamma_0^2\left[\frac{1}{2} + \frac{2-\rho}{2(\rho - 2t)}\right] + \gamma_1^2\left[\frac{\rho(\rho-2)}{8} + \frac{(\rho-2)t}{4} + \frac{t^2}{2} + \frac{\rho^2(2-\rho)}{8(\rho - 2t)}\right] \\
&+ 2\gamma_0\gamma_1\left[\frac{\rho - 2}{4} + \frac{t}{2} + \frac{\rho(2-\rho)}{4(\rho - 2t)}\right] \ge 0,
\end{aligned}$$

where β_0, β_1 and γ_0, γ_1 are real numbers.

The validity of the inequality

$$H(X) = \int_{-1}^{1} \varphi(t)\, d\alpha(t) \geq 0, \quad \alpha(t) \in M_1,$$

implies the nonnegativity of all principal minors of the matrices $A_1(X)$ and $A_{-1}(X)$ (see [6, p. 270]). Here,

$$A_\varepsilon(X) = \begin{pmatrix} \varepsilon + x_3(2-\varepsilon\rho) & \varepsilon\rho - 2 + \varepsilon x_1 + (2-\varepsilon\rho)\rho x_3 \\ \varepsilon\rho - 2 + \varepsilon x_1 + (2-\varepsilon\rho)\rho x_3 & (\varepsilon\rho-2)\rho + (\varepsilon\rho-2)x_1 + \varepsilon(x_2+1) + \rho^2(2-\varepsilon\rho)x_3 \end{pmatrix}.$$

The latter assertion amounts to the nonnegativity of all principal minors of the matrices

$$B_\varepsilon(X) = \begin{pmatrix} \varepsilon + x_3(2-\varepsilon\rho) & \varepsilon x_1 - 2 \\ \varepsilon x_1 - 2 & 2\rho + \varepsilon(1+x_2) - x_1(2+\varepsilon\rho), \end{pmatrix}, \quad \varepsilon = \pm 1.$$

The first assertion of Theorem 1 is proved. The second and third assertions follow from Lemmas 1, 3 and from Lemma 2 in [2], respectively. □

Using the integral representation of the system $\{c_2, c_3\}$ on the class T

$$c_2 = \int_{-1}^{1} 2t\, d\alpha(t), \quad c_3 = \int_{-1}^{1} (4t^2-1)\, d\alpha(t), \quad \alpha(t) \in M_1,$$

we find the region of values of the system $\{c_2, c_3\}$ defined by the inequalities $c_3 + 1 \geq c_2^2$, $c_3 \leq 3$.

Let $T(c_2, c_3)$ be the class of functions $f(z) \in T$ with fixed values of the coefficients c_2 and c_3 satisfying the conditions

$$c_3 + 1 > c_2^2, \quad c_3 < 3.$$

By $D(c_2, c_3)$ denote the region of values of $f(r)$, $0 < r < 1$, where $f \in T(c_2, c_3)$.

Proof of Theorem 2. By the second assertion of Theorem 1, the set $\operatorname{Int} D(c_2, c_3)$ is defined by the inequalities

$$\begin{gathered} x_3(\rho-2) < 1, \qquad x_3(\rho+2) > 1, \\ [1-(\rho-2)x_3][c_3+1-c_2(\rho+2)+2\rho] > (c_2-2)^2, \\ [-1+x_3(\rho+2)x_3][2\rho-1+c_3-c_2(2-\rho)] > (c_2+2)^2. \end{gathered} \tag{8}$$

From inequalities (8) we obtain the estimates for $f(r)$. The set $D(c_2, c_3)$ is the interval of the real axis, and to its endpoints there correspond the functions (3) satisfying the conditions

$$\lambda_1 + \lambda_2 = 1, \quad 2\lambda_1\varepsilon + 2t\lambda_2 = c_2, \quad 3\lambda_1 + (4t^2-1)\lambda_2 = c_3. \tag{9}$$

By (9),

$$\lambda_2 = \frac{c_3 - 3}{4(t^2-1)} = \frac{c_2 - 2\varepsilon}{2(t-\varepsilon)},$$

implying that

$$2t = \frac{c_3-3}{c_2-2\varepsilon} - 2\varepsilon.$$

Theorem 2 is proved. □

By $\widetilde{D}(c_2, c_3)$ denote the region of values of $f(z_1)$ on the class $T(c_2, c_3)$.

The following result is valid.

Theorem 3. *Let* $\operatorname{Im} z_1 \neq 0$ *and let* $f \in T\ (c_2, c_3)$. *Then the following assertions hold*:

1. $\operatorname{Int} \widetilde{D} = \{w = f(z_1) : \Delta_1 \cap \Delta_2\}$, *where*

$$\Delta_1 = \left\{ w \in \mathbb{C} : \left| w - \frac{1}{\zeta_1 - \overline{\zeta}_1} \left[1 - \frac{(c_2 - \overline{\zeta}_1)^2}{\Delta} \right] \right| < \frac{1}{|\zeta_1 - \overline{\zeta}_1|} \left[1 - \frac{|c_2 - \zeta_1|^2}{\Delta} \right] \right\},$$

with

$$\Delta = |\zeta_1 - c_2|^2 + c_3 + 1 - c_2^2,$$

and

$$\Delta_2 = \left\{ w \in \mathbb{C} : \left| w - \frac{1}{\zeta_1^2 - 4} \left(\zeta_1 + c_2 + \frac{c_3 - 3}{\zeta_1 - \overline{\zeta}_1} \right) \right| < \frac{3 - c_3}{|\zeta_1 - \overline{\zeta}_1|\, |\zeta_1^2 - 4|} \right\}, \quad \zeta_1 = z_1 + \frac{1}{z_1}.$$

2. *To the points on* $\partial\Delta_1$ *there correspond only the functions*

$$f_1(z) \equiv f_1(z;t_1) = \frac{z}{1-2t_1z+z^2}\left[1+\frac{(c_2-2t_1)^2z}{(c_2-2t_1)(1+z^2)-(c_3+1-2t_1c_2)z}\right], \quad t_1\in[-1,1],$$

whereas to the points on $\partial\Delta_2$ *there correspond only the functions*

$$f_2(z) \equiv f_2(z;t) = \frac{z^2}{(1-z^2)^2}\left[c_2+\frac{1+z^2}{z}+\frac{(c_3-3)z}{1-2tz+z^2}\right], \quad t\in\left[\frac{2c_2-c_3-1}{2(2-c_2)},\frac{2c_2+c_3+1}{2(2+c_2)}\right]\equiv[t_1,t_2].$$

To the points of the intersection of the boundary arcs $\partial\Delta_1$ *and* $\partial\Delta_2$ *there correspond the functions*

$$f(z) = \frac{z}{(1\pm z)^2}\left[1+\frac{(c_2\pm 2)^2z}{(c_2\pm 2)(1+z^2)-(c_3+1\pm 2c_2)^z}\right].$$

Proof of Theorem 3. The first assertion of Theorem 3 was proved in [2]. Prove the second assertion.

By Lemma 2 in [2], the boundary functions corresponding to the points on $\partial\Delta_1$ and $\partial\Delta_2$ are of the form

$$f_1(z) = \frac{\lambda_1}{\zeta-2t_1}+\frac{\lambda_2}{\zeta-2t_2}, \quad t_1\neq t_2, \quad \lambda_1,\lambda_2\geq 0,$$

and

$$f_2(z) = \frac{\mu_1}{\zeta+2}+\frac{\mu_2}{\zeta-2}+\frac{\mu_3}{\zeta-2t}, \quad \mu_j\geq 0, \quad j=1,2,3, \quad \zeta = z+\frac{1}{z},$$

where

$$\lambda_1+\lambda_2=1,\ 2t_1\lambda_1+2t_2\lambda_2=c_2, \quad (4t_1^2-1)\lambda_1+(4t_2^2-1)\lambda_2=c_3, \tag{10}$$

and

$$\mu_1+\mu_2+\mu_3=1, \quad -2\mu_1+2\mu_2+2t\mu_3=c_2, \quad 3\mu_1+3\mu_2+(4t^2-1)\mu_3=c_3. \tag{11}$$

In view of (10), we have

$$\lambda_1=1-\lambda_2, \quad 2t_1+2\lambda_2(t_2-t_1)=c_2, \quad 4t_1^2-1+4\lambda_2(t_2^2-t_1^2)=c_3, \tag{12}$$

whence

$$2\lambda_2(t_2-t_1)=c_2-2t_1, \quad 2(c_2-2t_1)(t_1+t_2)=c_3+1-4t_1^2, \tag{13}$$

and we find

$$2t_2=\frac{c_3+1-2c_2t_1}{c_2-2t_1}. \tag{14}$$

In view of (11), we have

$$\mu_1=1-\mu_2-\mu_3, \quad 2\mu_3(1+t)=c_2+2-4\mu_2, \quad 4\mu_3(1-t^2)=3-c_3, \quad 8\mu_2(1-t)=c_3-3+2(1-t)(c_2+2). \tag{15}$$

From (12)–(14) we obtain

$$f_1(z)=\frac{1}{\zeta-2t_1}+\frac{2\lambda_2(t_2-t_1)}{(\zeta-2t_1)(\zeta-2t_2)}=\frac{1}{\zeta-2t_1}\left[1+\frac{c_2-2t_1}{\zeta-\frac{c_3+1-2c_2t_1}{c_2-2t_1}}\right]$$

and, using (15), we derive

$$\begin{aligned} f_2(z) &= \frac{1}{\zeta+2}+\frac{4\mu_2}{\zeta^2-4}+\frac{2\mu_3(1+t)}{(\zeta+2)(\zeta-2t)} \\ &= \frac{1}{\zeta+2}+\frac{4\mu_2}{\zeta^2-4}+\frac{c_2+2-4\mu_2}{(\zeta+2)(\zeta-2t)} \\ &= \frac{1}{\zeta+2}+\frac{c_2+2}{(\zeta+2)(\zeta-2t)}+\frac{(c_2+2)2(1-t)+c_3-3}{(\zeta^2-4)(\zeta-2t)} \\ &= \frac{1}{\zeta^2-4}\left(c_2+\zeta+\frac{c_3-3}{\zeta-2t}\right). \end{aligned}$$

Observe that $f_1(z;-1)=f_2(z;t_2)$ and $f_1(z;1)=f_2(z;t_1)$.

Theorem 3 is proved. □

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