Journal of Mathematical Sciences, Vol. 193, No. 1, August, 2013

SHARP ESTIMATES OF BEST APPROXIMATIONS IN TERMS OF HOLOMORPHIC FUNCTIONS OF WEIERSTRASS-TYPE OPERATORS

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Estimates of the form

$$A_\sigma(f)_P \le KP(\Phi(\mathcal{W})f),$$

where W is a kernel of a special type summable on $\mathbb{R}$, a function Φ is holomorphic in a neighborhood of the spectrum of W, and $A_\sigma(f)_P$ is the best approximation of a function f by entire functions of exponential type not greater than σ with respect to a seminorm P, are established. In some cases, for the uniform and integral norms the least possible constant K is found. The estimates are obtained by linear approximation methods. Bibliography: 13 *titles.*

1. Preliminary facts

1.1. Notation. In what follows, $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{Z}_+$, and $\mathbb{N}$ are the sets of reals, integers, nonnegative integers, and positive integers, respectively; $[a:b]=[a,b]\cap\mathbb{Z}$; $C(E)$ and $C^{(r)}(E)$ are the sets of functions continuous and r times continuously differentiable on a set E. The function spaces considered are as follows: $UCB(\mathbb{R})$ is the space of functions bounded and uniformly continuous on $\mathbb{R}$, and C is the space of 2π-periodic continuous functions, with the uniform norm $\|\cdot\|=\|\cdot\|_\infty$; for $1\le p<\infty$, $L_p(\mathbb{R})$ and L_p are the spaces of functions measurable and p-integrable on $\mathbb{R}$ or on the period, respectively, with the norm $\|f\|_p=\left(\int_E |f|^p\right)^{1/p}$, where $E=\mathbb{R}$ or $[-\pi,\pi]$; $L(\mathbb{R})=L_1(\mathbb{R})$; $L=L_1$; $L_\infty(\mathbb{R})$ is the space of measurable, essentially bounded on $\mathbb{R}$ functions f with the norm

$$\|f\|_\infty=\operatorname*{vrai\,sup}_{x\in\mathbb{R}}|f(x)|;$$

L_∞ is the subspace of the space of 2π-periodic functions that belong to $L_\infty(\mathbb{R})$. Unless the context implies otherwise, function spaces may be real or complex.

Assume that $\mathfrak{M}$ is a closed subspace of $L_p(\mathbb{R})$ ($1\le p<\infty$) or $UCB(\mathbb{R})$ ($p=\infty$), P is a seminorm defined on $\mathfrak{M}$. We say that the space $(\mathfrak{M},P)$ is in class $\mathcal{B}$ if the following conditions are fulfilled:

(1) the space is shift-invariant, i.e., for every $f\in\mathfrak{M}$ and $h\in\mathbb{R}$, we have $f(\cdot+h)\in\mathfrak{M}$ and $P(f(\cdot+h))=P(f)$;

(2) there exists a constant B such that $P(f)\le B\|f\|_p$ for every $f\in\mathfrak{M}$.

As examples of spaces of class $\mathcal{B}$, we mention the spaces $\big(UCB(\mathbb{R}),\|\cdot\|_\infty\big)$, $(L_p(\mathbb{R}),\|\cdot\|_p)$ ($1\le p<\infty$), the spaces of periodic functions $\big(C,\|\cdot\|_p\big)$ ($1\le p\le\infty$), and also more general spaces of uniformly continuous almost-periodic functions with exponents belonging to a fixed set and with various norms.

Further, $\mathbf{E}_\sigma$ and $\mathbf{E}_{\sigma-0}$ are the sets of entire functions of exponential type not greater (less) than $\sigma>0$; the best approximation of a function f with respect to a seminorm P is defined by

$$A_\sigma(f)_P=\inf_{\substack{g\in\mathbf{E}_\sigma\\ f-g\in\mathfrak{M}}}P(f-g)$$

($\inf\varnothing=+\infty$); the value $A_{\sigma-0}(f)_P$ is defined similarly. The subscript p attached to a value means that $P(f)=\|f\|_p$. Functions are assumed to be extended to points of removable discontinuities by continuity; in other cases, the symbol $\frac{0}{0}$ is understood as 0. In addition,

$$c(f,y)=\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}f(t)e^{-iyt}\,dt$$

is the complex Fourier transform of a function $f\in L(\mathbb{R})$; $a(f)$ and $b(f)$ are its cosine and sine Fourier transforms, i.e.,

$$a(f,y)=c(f,y)+c(f,-y),\quad b(f,y)=i\big(c(f,y)-c(f,-y)\big).$$

By analogy, if a function c is defined on a subset of $\mathbb{R}$ symmetric with respect to zero, then we set

$$a(y)=c(y)+c(-y),\quad b(y)=i(c(y)-c(-y)).\tag{1.1}$$

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Translated from *Zapiski Nauchnykh Seminarov POMI*, Vol. 404, 2012, pp. 18–60. Original article submitted May 5, 2012.

 1072-3374/13/1931-0008

We have $a = 2c$ if c is even, and $b = 2ic$ if c is odd. The set of Fourier transforms of functions belonging to $L(\mathbb{R})$ is denoted by L^*.

The convolution of two functions f and g is defined by

$$(f * g)(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(x-t)g(t)\,dt;$$

with such a normalization, $c(f*g) = c(f)c(g)$. The $(k-1)$-fold convolution of a function W with itself is denoted by $W^{[k]}$. By definition,

$$\sum_{k\in\mathbb{Z}} a_k = \sum_{k=-\infty}^{\infty} a_k = \lim_{N\to\infty} \sum_{k=-N}^{N} a_k.$$

1.2. Approximation of classes of convolutions. Assume that a convolution operator $\mathcal{W}$ with a kernel $W \in L(\mathbb{R})$ is defined,

$$\mathcal{W} f = f * W,$$

$w = c(W)$, and a function Φ satisfies the following condition:

$\Phi1$. Φ is defined at least on the set $w(\mathbb{R}) \cup \{0\}$, $\Phi(0) = 1$, $(\Phi - 1) \circ w \in L^*$.

Here and below, the number 1 and the constant function identically equal to 1 are denoted by the same symbol. The function whose Fourier transform is $(\Phi-1)\circ w$ is denoted by $(\Phi - 1)^\circ(W)$. Define the operator $\Phi(\mathcal{W})$ by

$$\Phi(\mathcal{W})f = f + (\Phi-1)^\circ(W) * f.$$

We study the problem on the best constant in the inequality

$$A_\sigma(f)_P \le KP(\Phi(\mathcal{W})f) \tag{1.2}$$

and on realization of this estimate by linear approximation methods.

By the Wiener–Levy theorem [1, Sec. 75, p. 158], for the condition $(\Phi-1)\circ w \in L^*$ to be fulfilled, it is sufficient that the function Φ be holomorphic in a neighborhood of the set $w(\mathbb{R}) \cup \{0\}$ and that $\Phi(0) = 1$. The assumption that Φ is holomorphic in a neighborhood of the set $w(\mathbb{R}) \cup \{0\}$ does not cover some important applications, so we weaken it. We fix $y_0 > 0$ and assume that, in addition to $\Phi1$, the function Φ satisfies the following condition:

$\Phi2$. Φ is holomorphic in a neighborhood of the set $w\big(\mathbb{R}\setminus(-y_0, y_0)\big) \cup \{0\}$.

We show that problem (1.2) reduces to the classical problem on approximation of classes of convolutions.

Lemma 1. *Let $p \in [1, +\infty]$, $f \in L_p(\mathbb{R})$, $W \in L(\mathbb{R})$, $w = c(W)$, $y_0 > 0$, a function Φ satisfy conditions $\Phi1$ and $\Phi2$, $\varphi = \Phi(\mathcal{W})f$, $\Phi(w(y)) \ne 0$ for every y such that $|y| \ge y_0$. Then the following assertions hold:*

1. *There exists a function $G \in L(\mathbb{R})$ such that $c(G, y) = \frac{1-\Phi(w(y))}{\Phi(w(y))}$ for $|y| \ge y_0$.*
2. *If G is a function from item 1, then there exists a function $Q \in \mathbf{E}_{y_0} \cap L(\mathbb{R})$ such that*

$$(I - \Phi(\mathcal{W}))f = f * Q + \varphi * G. \tag{1.3}$$

3. *If a function W is even, then G and Q can also be chosen even.*

Proof. 1. Take a kernel $V \in L(\mathbb{R})$ whose Fourier transform $v = c(V)$ satisfies the following conditions: $v(y) = 0$ for $|y| \ge y_0$; $(w - v)(\mathbb{R})$ is contained in the domain of holomorphy of Φ, and $\Phi((w-v)(y)) \ne 0$ for every $y \in \mathbb{R}$. By the Wiener–Levy theorem, the function $\frac{1-\Phi\circ(w-v)}{\Phi\circ(w-v)}$ is the Fourier transform of a function $G \in L(\mathbb{R})$.

2. Since

$$(1-\Phi)^\circ(W - V) = G - (1-\Phi)^\circ(W-V) * G,$$

equality (1.3) holds with

$$Q = (1-\Phi)^\circ(W) - (1-\Phi)^\circ(W-V) + \big((1-\Phi)^\circ(W) - (1-\Phi)^\circ(W-V)\big) * G. \tag{1.4}$$

This implies that $c(Q, y) = 0$ for $|y| \ge y_0$.

3. If a function W is even, then V can also be chosen even, and the functions Q and G are even as well. □

The term $f * Q$ belongs to $\mathbf{E}_{y_0}$, and for $\alpha \ge y_0$ it can be included in the approximating aggregate. Thus, the problem is reduced to approximating the convolution $\varphi * G$.

Recall some known facts concerning approximation of classes of convolutions.

Assume that a function $G \in L(\mathbb{R})$ is either even or odd, $\sigma > 0$. We set

$$\mathcal{K}_{\sigma,G} = \left| \frac{1}{2\pi} \int\limits_{-\infty}^{+\infty} G(t) \operatorname{sign} \sin \sigma t \, dt \right| = \left| \frac{2}{\pi} \sum_{s=0}^{\infty} \frac{b(G,(2s+1)\sigma)}{2s+1} \right|$$

if G is odd, and

$$\mathcal{K}_{\sigma,G} = \left| \frac{1}{2\pi} \int\limits_{-\infty}^{+\infty} G(t) \operatorname{sign} \cos \sigma t \, dt \right| = \left| \frac{2}{\pi} \sum_{s=0}^{\infty} \frac{(-1)^s}{2s+1} a(G,(2s+1)\sigma) \right|$$

if G is even.

The following condition plays an important role in the problem of approximation of classes of convolutions.

N1. There exist a function $G_\sigma \in \mathbf{E}_\sigma \cap L(\mathbb{R})$ and a number $\varepsilon \in \{-1, 1\}$ such that in accordance with the parity of G,

$$\text{either} \quad \varepsilon\big(G(t) - G_\sigma(t)\big) \cos \sigma t \ge 0 \quad \text{or} \quad \varepsilon\big(G(t) - G_\sigma(t)\big) \sin \sigma t \ge 0$$

for almost every $t \in \mathbb{R}$.

Assume that

$$f = T + \varphi * G, \tag{1.5}$$

where $T \in \mathbf{E}_\sigma$, and let condition N1 be fulfilled. We define the operator $\mathcal{X}_{\sigma,G}$ by the relation

$$\mathcal{X}_{\sigma,G} f = T + \varphi * G_\sigma.$$

Remark 1. Clearly, $\mathcal{X}_{\sigma,G} f \in \mathbf{E}_\sigma$. Let $T = 0$. If a function φ has period $\frac{2\pi}{\rho}$ ($\rho > 0$), then $\mathcal{X}_{\sigma,G} f$ is a trigonometric polynomial of order less than $\frac{\sigma}{\rho}$. In particular, if $\rho \ge \sigma$, then $\mathcal{X}_{\sigma,G} f$ is a constant. If a function φ is almost-periodic, then $\mathcal{X}_{\sigma,G} f$ also is an almost-periodic function whose exponents belong to φ.

Lemma A. *Let a function $G \in L(\mathbb{R})$ be even or odd, $\sigma > 0$, and let condition* N1 *be fulfilled. Then*

$$\mathcal{K}_{\sigma,G} = \frac{1}{2\pi} A_\sigma(G)_1 = \frac{1}{2\pi} \int\limits_{-\infty}^{+\infty} |G - G_\sigma|. \tag{1.6}$$

If, in addition, $(\mathfrak{M}, P) \in \mathcal{B}$, $\varphi \in \mathfrak{M}$, and the functions f and φ are interrelated via (1.5), *then*

$$P(f - \mathcal{X}_{\sigma,G} f) \le \mathcal{K}_{\sigma,G} P(\varphi), \tag{1.7}$$

$$A_\sigma(f)_P \le \mathcal{K}_{\sigma,G} A_\sigma(\varphi)_P, \tag{1.8}$$

$$A_{\sigma-0}(f)_P \le \mathcal{K}_{\sigma,G} A_{\sigma-0}(\varphi)_P, \tag{1.9}$$

$$A_{\sigma-0}(f)_P \le \mathcal{K}_{\sigma,G} P(\varphi), \tag{1.10}$$

$$A_\sigma(f)_P \le \mathcal{K}_{\sigma,G} P(\varphi) \tag{1.11}$$

(*in* (1.9) *and* (1.10), $T \in \mathbf{E}_{\sigma-0}$). *In the spaces $(UCB(\mathbb{R}), \|\cdot\|_\infty)$ and $(L(\mathbb{R}), \|\cdot\|_1)$, the constant $\mathcal{K}_{\sigma,G}$ in inequalities* (1.7)–(1.11) *cannot be reduced. In the spaces of $\frac{2\pi}{\sigma}$-periodic functions with the uniform or integral norm, the constant in* (1.7), (1.9), *and* (1.10) *neither can be reduced.*

Lemma A includes the classical results of J. Favard, N. I. Akhiezer, M. G. Krein, S. M. Nikolskii, Sun Youngshen and other mathematicians, and it is essentially contained in [1, Sec. 87, p. 202, Sec. 100, p. 235]. The above statement is taken from [2, Lemma 5], with a difference. In [1, 2], the kernel G satisfied additional restrictions. It allowed one to take the function $L_\sigma G$, interpolating G, as G_σ. The proof of Lemma A in the above, formally more general, form does not differ from the one presented in [1, 2], because the latter proof did not use the explicit expression of G_σ.

Remark 2 [2, Remark 4]. Inequalities of the types (1.7)–(1.11) can be carried over in a standard way (for instance, with the use of approximation of the function φ by its Fejér integral) from sets of continuous functions to the sets $L_\infty(\mathbb{R})$ and L_p ($1 \le p \le \infty$) with the seminorms $\|\cdot\|_p$, $A_\sigma(\cdot)_p$, and $A_{\sigma-0}(\cdot)_p$. Moreover, for convolution classes of the form

$$f = T + \frac{1}{2\pi} \int\limits_{-\infty}^{+\infty} G(x - t) \, dg(t)$$

and for their periodic analogs similar sharp estimates for approximations in $L_1(\mathbb{R})$ and L_1 by the variation of g are valid.

In what follows, we will not dwell on the possibility of extending our estimates to wider function classes, restricting ourselves to spaces of class $\mathcal{B}$.

We also mention that the sign of the expression under the modulus sign in the definition of the constant $\mathcal{K}_{\sigma,G}$ is equal to the number ε from condition N1.

Let us describe the situation where a kernel can be interpolated in more detail and present the interpolating function in explicit form.

Assume that a kernel G is either even or odd, $G \in L(\mathbb{R}) \cap C(\mathbb{R} \setminus \{0\})$, $G(t) = O(t^{-2})$ as $t \to \infty$, $\varkappa$ is the parity of G, i.e., $\varkappa = 0$ if G is even and $\varkappa = 1$ if G is odd. We denote [1, Sec. 87, pp. 199–203]

$$L_\sigma G(z) = \frac{\sin\left(\sigma z - \frac{\pi}{2}(1-\varkappa)\right)}{\sigma} \sum_{q=-\infty}^{\infty} (-1)^q \frac{G\left(\frac{(2q+1-\varkappa)\pi}{2\sigma}\right)}{z - \frac{(2q+1-\varkappa)\pi}{2\sigma}}. \tag{1.12}$$

Then $L_\sigma G$ is an even or an odd function that belongs to $\mathbf{E}_\sigma \cap L(\mathbb{R})$ and interpolates G at the points $\frac{(2k+1-\varkappa)\pi}{2\sigma}$ ($k \in \mathbb{Z}$), i.e., at the zeros of the functions $\sin\sigma t$ or $\cos\sigma t$ in accordance with its parity. (The value of an odd function G at the zero point is equal to zero, and if G is even, then its value at zero is irrelevant.) In terms of the Fourier transform, the function $L_\sigma G$ is as follows:

$$L_\sigma G(t) = \int_{-\sigma}^{\sigma} c(L_\sigma, y) e^{ity}\, dy, \tag{1.13}$$

$$c(L_\sigma G, y) = \sum_{s=-\infty}^{\infty} (-1)^{s(1-\varkappa)} c(G, y + 2s\sigma), \quad |y| \le \sigma. \tag{1.14}$$

For these formulas to be valid, it is sufficient that the series on the right-hand side of (1.14) converge uniformly on $[0,\sigma]$ and its sum, upon multiplying by $e^{i\frac{\pi(1-\varkappa)}{2\sigma}y}$, admit a Fourier series expansion [1, Sec. 87, pp. 202–203].

In [1], a more general situation was considered, where the evenness or oddness of G was not needed.

Now we state two lemmas which provide conditions sufficient for condition N1 to be fulfilled.

Lemma B [1, Sec. 88]**.** *Assume that $G \in L(\mathbb{R})$, $\sigma > 0$.*

1. *If G is even and if $a(G)$ is three times differentiable on $\mathbb{R}$ and three times monotone on $[\sigma, +\infty)$ (i.e., $(-1)^r a^{(r)}(G,y) \ge 0$ for $r \in [0:3]$ and $y \ge \sigma$), then*

$$\big(G(t) - L_\sigma G(t)\big)\cos\sigma t \ge 0$$

for every $t \in \mathbb{R}$.

2. *If G is odd, $b(G) \in C^{(2)}(\mathbb{R})$, $b(G)$ is twice monotone on $[\sigma, +\infty)$ (i.e., $(-1)^r b^{(r)}(G,y) \ge 0$ for $r \in [0:2]$ and $y \ge \sigma$), then*

$$\big(G(t) - L_\sigma G(t)\big)\sin\sigma t \ge 0$$

for every $t \in \mathbb{R}$.

In both cases, the conclusion of Lemma A *holds.*

The assumptions of Lemma B are called the B. Nagy conditions.

In [2], the function classes $\widehat{CM}^r(y_0)$ were introduced, and Lemma C presented below was established.

Let $y_0 > 0$, $r \in \{1, 2\}$. By $CM_c^r(y_0)$ and $CM_s^r(y_0)$ (which means "completely monotone") we denote the sets of even and odd functions c defined at least on $\mathbb{R} \setminus (-y_0, y_0)$ and such that for all $y \ge y_0$ the function a or b, respectively (see the convention (1.1)), is of the form

$$y \mapsto \int_0^{+\infty} e^{-y^r u} d\Psi(u), \quad \text{the function } \Psi \text{ increases on } (0, +\infty). \tag{1.15}$$

Let, in addition, $\widehat{CM}_c^r(y_0)$ and $\widehat{CM}_s^r(y_0)$ be the sets of even and odd functions $G \in L(\mathbb{R})$ for which $c(G) \in CM_c^r(y_0)$ or $CM_s^r(y_0)$; $\widehat{CM}^r(y_0) = \widehat{CM}_c^r(y_0) \cup \widehat{CM}_s^r(y_0)$.

As is known [2, Property AM1], $\widehat{CM}^1(y_0) \subset \widehat{CM}^2(y_0)$. Functions from $\widehat{CM}^r(y_0)$ satisfy the Nagy conditions for $r = 1$, but not necessarily for $r = 2$.

Remark 3. As was shown in [1, Sec. 88, pp. 204–205], if a kernel G satisfies the Nagy conditions, then $G \in C(\mathbb{R} \setminus \{0\})$, $G(t) = O(t^{-2})$ as $t \to \infty$, and the formulas (1.13) and (1.14) are true. As was shown in [2, Properties AM4 and AM5], if $G \in \widehat{CM}^2(y_0)$, $c(G) \in C^{(2)}(\mathbb{R})$, and $\sigma \geq y_0$, then the same conclusion holds.

Remark 4. The analysis of the proof in [1] shows that the condition of the threefold (twofold) differentiability in the Nagy conditions is redundant. Instead of the condition mentioned, one may assume that $c(G) \in C^{(2)}[-\sigma, \sigma]$ and understand the multiple monotonicity in the following sense: A function g is called r times monotone on $[\sigma, +\infty)$ if $(-1)^k \Delta_t^k g \geq 0$ on $[\sigma, +\infty)$ for all $t > 0$ and $k \in [0 : r]$, where $\Delta_t^k g$ is the kth forward difference of the function g with step t. Then the conclusions of Remark 3 and Lemma B hold. We will use this remark in the example of Sec. 5.4.

Remark 5. It is easy to show [3, Lemma 1] that if $g \in L^*$, $h \in C^{(1)}[-y_0, y_0]$, and $h = g$ on $\mathbb{R} \setminus (-y_0, y_0)$, then $h \in L^*$. Therefore, every function $G \in L(\mathbb{R})$ can be represented in the form $G = H + V$, where $H, V \in L(\mathbb{R})$, $V \in \mathbf{E}_{y_0}$, $c(H) \in C^{(\infty)}[-y_0, y_0]$. Moreover, by the Borel theorem [4, p. 124, Problem VII.2.15], we may assume that at the points $\pm y_0$ the function $c(H)$ has prescribed one-sided derivatives of all orders. In addition, if G is even or odd and if the boundary conditions have the corresponding parity (namely, at the opposite points, an even function has equal derivatives of even order and opposite derivatives of odd order, and vice versa for an odd function), then the functions H and V can be chosen of the same parity. This remark will be used in the proof of Theorem 3 in Sec. 3.

Lemma C [2, Lemma 7]**.** *Assume that $y_0 > 0$, G belongs to either $\widehat{CM}_c^2(y_0)$ or $\widehat{CM}_s^2(y_0)$, $c(G) \in C^{(2)}(\mathbb{R})$, $\sigma \geq y_0$. Then, for all $t \in \mathbb{R}$,*

$$\textit{either} \quad \big(G(t) - L_\sigma G(t)\big) \cos \sigma t \geq 0 \quad \textit{or} \quad \big(G(t) - L_\sigma G(t)\big) \sin \sigma t \geq 0,$$

respectively. The assertion of Lemma A *also holds.*

To obtain sharp estimates of the type (1.2) with the use of Lemma A, it is necessary to understand which properties of the functions W and Φ provide for the validity of condition N1 for the function G.

In [3, Lemma 2], the author proved that if $y_0 > 0$, $w \in CM_c^2(y_0)$, and $w(y) < 1$ for all $y \geq y_0$, then $\frac{w}{1-w} \in CM_c^2(y_0)$. This allowed him to obtain sharp estimates in the case where $W \in \widehat{CM}_c^2(y_0)$ and $\Phi(z) = 1 - z$, i.e., estimates for approximations of Weierstrass-type integrals by deviations. Babenko and Kryakin [5] obtained a sharp estimate of the best approximation in terms of the deviation of the first-order Steklov function. In [6], estimates of the type (1.2) were proved in the case where the operators $\mathcal{W}$ are linear combinations of Steklov averages and $\Phi(z) = (1 - z)^q$ $(q \in \mathbb{N})$, i.e., estimates of approximations in terms of powers of deviations of such combinations. In [7], the sharpness of one of the estimates mentioned above was established. In more detail, these results are discussed in Sec. 5.

Let us mention that in the general case, G may be not continuous, and the condition $G(t) = O(t^{-2})$ as $t \to \infty$ may be not fulfilled. So, it is unclear whether we can apply formula (1.12) to define the function $L_\sigma G$.

Now we prove the following lemma, which is useful for further constructions.

Lemma 2. *Let $G_1, G_2 \in L(\mathbb{R}) \cap C(\mathbb{R} \setminus \{0\})$ and let $G_1(t), G_2(t) = O(t^{-2})$ as $t \to \infty$. Then the function $G_1 * G_2$ has the same properties.*

Proof. 1. The summability of convolution is well known. Let us prove its continuity on $\mathbb{R} \setminus \{0\}$. Fix $t \neq 0$ and put $E_t = \left[-\frac{|t|}{2}, \frac{|t|}{2}\right]$. We have

$$\begin{aligned} 2\pi\big(G_1 * G_2(t+h) - G_1 * G_2(t)\big) &= \int_{\mathbb{R}} G_1(u)\big(G_2(t+h-u) - G_2(t-u)\big)\,du \\ &= \int_{\mathbb{R}} G_1(t-v)\big(G_2(v+h) - G_2(v)\big)\,dv. \end{aligned}$$

Split the integral into the integrals $I_1(h)$ and $I_2(h)$ over the segment E_t and its complement, respectively. If $v \in E_t$, then $|t - v| \geq |t|/2$. Hence

$$|I_1(h)| = \left| \int_{E_t} G_1(t-v)\big(G_2(v+h) - G_2(v)\big)\,dv \right| \leq \sup_{|u| \geq |t|/2} |G_1(u)| \cdot \int_{\mathbb{R}} \big|G_2(v+h) - G_2(v)\big|\,dv.$$

The upper bound is finite by the assumptions on G_1, and the integral tends to zero as $h \to 0$ by the continuity of the shift. Consequently, $I_1(h) \to 0$. The integral $I_2(h)$ also tends to zero by virtue of the estimate

$$|I_2(h)| = \left| \int\limits_{\mathbb{R}\setminus E_t} G_1(t-v)\big(G_2(v+h) - G_2(v)\big)\, dv \right| \le \int\limits_{\mathbb{R}} |G_1| \cdot \sup_{|v|\ge|t|/2} |G_2(v+h) - G_2(v)|.$$

2. Prove the relation $G_1 * G_2(t) = O(t^{-2})$ as $t \to \infty$. We have

$$2\pi t^2\, G_1 * G_2(t) = \int\limits_{\mathbb{R}} G_1(u) t^2 G_2(t-u)\, du.$$

Split the integral into the integrals $J_1(t)$ and $J_2(t)$ over the segment $E_t = \left[-\frac{|t|}{2}, \frac{|t|}{2}\right]$ and its complement. If $u \in E_t$, then $t^2 = 4\left(\frac{t}{2}\right)^2 \le 4(t-u)^2$. Therefore,

$$|J_1(t)| \le \int\limits_{E_t} \big|G_1(u) 4(t-u)^2 G_2(t-u)\big|\, du \le 4 \int\limits_{\mathbb{R}} |G_1| \cdot \sup_{|v|\ge|t|/2} v^2 |G_2(v)|.$$

If $u \notin E_t$, then $t^2 = 4\left(\frac{t}{2}\right)^2 \le 4u^2$. Hence

$$|J_2(t)| \le \int\limits_{\mathbb{R}\setminus E_t} 4u^2 \big|G_1(u) G_2(t-u)\big|\, du \le 4 \int\limits_{\mathbb{R}} |G_2| \cdot \sup_{|u|\ge|t|/2} u^2 |G_1(u)|.$$

These estimates imply the boundedness of both integrals on the set $\mathbb{R} \setminus [-\Delta, \Delta]$ for every $\Delta > 0$. □

Remark 6. Since the convolution of two functions of the same parity is even and the convolution of two functions of different parities is odd, under the assumptions of Lemma 2, the function $L_\sigma(G_1 * G_2)$ is well defined. In particular, if an even or an odd function Λ satisfies the assumptions of Lemma 2, then the function $L_\sigma \Lambda^{[k]}$ is well defined for all $k \in \mathbb{N}$.

2. Construction of operators and general estimates

Let us outline the plan of construction of approximating operators, assuming that all the series under consideration converge in the appropriate sense.

Conditions Φ1 and Φ2 imply that in a neighborhood of zero the function $\frac{1}{\Phi}$ has a power series expansion of the form

$$\frac{1}{\Phi(z)} = 1 + \sum_{k=1}^{\infty} c_k z^k, \quad |z| < R. \tag{2.1}$$

Consequently, if $|w(y)| < R$ for all $y : |y| \ge y_0 > 0$, then the function G from Lemma 1 satisfies the relation

$$c(G, y) = \sum_{k=1}^{\infty} c_k w^k(y), \quad |y| \ge y_0.$$

Thus,

$$G = \sum_{k=1}^{\infty} c_k W^{[k]} + V,$$

where $V \in \mathbf{E}_{y_0}$. If $\sigma \ge y_0$ and we know how to construct approximating functions $T_\sigma W^{[k]}$ for the kernels $W^{[k]}$, then we can set

$$T_\sigma G = \sum_{k=1}^{\infty} c_k\, T_\sigma W^{[k]} + V.$$

In this case,

$$\varphi * G - \varphi * T_\sigma G = \sum_{k=1}^{\infty} c_k\, \varphi * (W^{[k]} - T_\sigma W^{[k]}),$$

$$P(\varphi * G - \varphi * T_\sigma G) \le \left(\sum_{k=1}^{\infty} |c_k| \big\| W^{[k]} - T_\sigma W^{[k]} \big\|_1 \right) P(\varphi).$$

In addition, it turns out that if all the coefficients c_k are nonnegative and the differences $W^{[k]} - T_\sigma W^{[k]}$ have equal signs, as in condition N1, then the difference $G - T_\sigma G$ has the same property, and the estimate becomes sharp in the cases indicated in Lemma A.

This plan is carried out with some modifications in this and the next sections. We must impose some restrictions on functions. These restrictions provide for the possibility of approximating terms, for the series convergence, and for the possibility of changing the order of operations.

In what follows, we assume that the function W has the following form:

W1. A function Λ is even or odd, $\Lambda \in L(\mathbb{R}) \cap C(\mathbb{R} \setminus \{0\})$, $\Lambda = O(t^{-2})$ as $t \to \infty$; $\varkappa$ is the parity of Λ; $\theta > 0$; $\{\alpha_\nu\}_{\nu\in\mathbb{Z}}$ is a two-sided number sequence, $\alpha_{-\nu-\varkappa} = \alpha_\nu$ for all $\nu \in \mathbb{Z}$; the summator operator $\mathcal{S}$ is defined by

$$\mathcal{S}f(x) = \sum_{\nu\in\mathbb{Z}} (-1)^{\nu+\varkappa} \alpha_\nu f\left(x - \left(\nu + \frac{\varkappa}{2}\right)\theta\right), \quad \sum_{\nu\in\mathbb{Z}} |\alpha_\nu| < +\infty; \tag{2.2}$$

$V \in \mathbf{E}_{y_0} \cap L(\mathbb{R})$, V is even,

$$W = \mathcal{S}\Lambda + V.$$

Then

$$w = \gamma\lambda + v, \tag{2.3}$$

where $v = c(V)$, the function $\lambda = c(\Lambda)$ is either even or odd,

$$\gamma(y) = \sum_{\nu\in\mathbb{Z}} (-1)^{\nu+\varkappa} \alpha_\nu e^{i(\nu+\frac{\varkappa}{2})\theta y}, \tag{2.4}$$

and the parities of γ and λ coincide. The presence of the multipliers $(-1)^{\nu+\varkappa}$ is convenient in formulating the conditions on the sign of the coefficients.

If $(\mathfrak{M}, P) \in \mathcal{B}$, then the operator $\mathcal{S}$ acts from $\mathfrak{M}$ to $\mathfrak{M}$. We denote the norm of S as an operator from $UCB(\mathbb{R})$ to $UCB(\mathbb{R})$ by $\|\mathcal{S}\|$, and its seminorm

$$N_P(\mathcal{S}) = \sup_{f\in\mathfrak{M}} \frac{P(\mathcal{S}f)}{P(f)}$$

is denoted by $N_P(\mathcal{S})$. Obviously,

$$N_P(\mathcal{S}) \le \sum_{\nu\in\mathbb{Z}} |\alpha_\nu| = \|\mathcal{S}\|,$$

and the norms of $\mathcal{S}$ in the spaces $UCB(\mathbb{R})$, $L_\infty(\mathbb{R})$, and $L_1(\mathbb{R})$ are equal. It is also obvious that $N_P(\mathcal{S}^k) \le N_P^k(\mathcal{S})$.

Write an explicit expression for the powers of the operator $\mathcal{S}$:

$$\begin{aligned} \mathcal{S}^k f(x) &= \sum_{\nu\in\mathbb{Z}^k} (-1)^{\nu_1+\ldots+\nu_k+k\varkappa} \alpha_{\nu_1} \ldots \alpha_{\nu_k} f\left(x - \left(\nu_1 + \ldots + \nu_k + \tfrac{k\varkappa}{2}\right)\theta\right) \\ &= \sum_{\mu\in\mathbb{Z}} (-1)^{\mu+k\varkappa} \left(\sum_{\substack{\nu\in\mathbb{Z}^k \\ \nu_1+\ldots+\nu_k=\mu}} \alpha_{\nu_1} \ldots \alpha_{\nu_k} \right) f\left(x - \left(\mu + \tfrac{k\varkappa}{2}\right)\theta\right). \end{aligned} \tag{2.5}$$

The finite-difference operator of rth order, δ_θ^r,

$$\delta_\theta^r f(x) = \sum_{k=0}^{r} (-1)^k C_r^k f\left(x + \frac{r\theta}{2} - k\theta\right),$$

is an important example of an operator $\mathcal{S}$ of such a kind. For this operator, $\gamma(y) = \left(2i \sin \frac{y\theta}{2}\right)^r$. For an even $r = 2m$ and for an odd $r = 2m+1$, the differences are written as

$$\delta_\theta^{2m} f(x) = \sum_{\nu=-m}^{m} (-1)^{m+\nu} C_{2m}^{m-\nu} f(x - \nu\theta),$$

$$\delta_\theta^{2m+1} f(x) = \sum_{\nu=-m-1}^{m} (-1)^{m+\nu+1} C_{2m+1}^{m-\nu} f\left(x - \left(\nu + \frac{1}{2}\right)\theta\right).$$

For $\sigma \ge y_0$, we put

$$T_\sigma W = \mathcal{S} L_\sigma \Lambda + V,$$

where $L_\sigma\Lambda$ is expressed by (1.12).

From the convergence of the series in (2.2) it follows that $T_\sigma W \in \mathbf{E}_\sigma \cap L(\mathbb{R})$ and $c(T_\sigma W) = \gamma c(L_\sigma \Lambda) + v$.

Denote $\widetilde{W} = W - V = \mathcal{S}\Lambda$. Obviously, the operator $\mathcal{S}$ commutes with the convolution operator. If

$$G_n = \sum_{k=1}^{n} c_k \widetilde{W}^{[k]} + V = \sum_{k=1}^{n} c_k \mathcal{S}^k \Lambda^{[k]} + V,$$

then we set

$$T_\sigma G_n = \sum_{k=1}^{n} c_k T_\sigma \widetilde{W}^{[k]} + V = \sum_{k=1}^{n} c_k \mathcal{S}^k L_\sigma \Lambda^{[k]} + V. \tag{2.6}$$

By Remark 6, the functions $L_\sigma \Lambda^{[k]}$ are well defined. By linearity, $T_\sigma W \in \mathbf{E}_\sigma \cap L(\mathbb{R})$.

For the next construction step, we assume that one more condition is fulfilled:

W2. The series

$$\sum_{k=1}^{\infty} |c_k| \|\mathcal{S}^k\| \left\|\Lambda^{[k]} - L_\sigma \Lambda^{[k]}\right\|_1 \tag{2.7}$$

is convergent.

The convergence of the series (2.7) implies that the series

$$\sum_{k=1}^{\infty} c_k \big(\widetilde{W}^{[k]} - T_\sigma \widetilde{W}^{[k]}\big) \tag{2.8}$$

absolutely converges in $L(\mathbb{R})$. Consequently, the series

$$\sum_{k=1}^{\infty} c_k \big(\widetilde{w}^k - c\big(T_\sigma \widetilde{W}^{[k]}\big)\big)$$

converges absolutely, and the series whose terms are the absolute values of its terms converges uniformly on $\mathbb{R}$. In particular, condition W2 implies the following assertion:

W3. For every $y : |y| \geq y_0$, the series $\sum\limits_{k=1}^{\infty} c_k w^k(y)$ converges absolutely.

Indeed, for $|y| \geq y_0$, the relations $\widetilde{w}(y) = w(y)$ and $c\big(T_\sigma \widetilde{W}^{[k]}, y\big) = 0$ hold because $T_\sigma \widetilde{W}^{[k]} \in \mathbf{E}_\sigma$, $\sigma \geq y_0$.

Remark 7. If the series (2.1) absolutely converges for $|z| = R$, then condition W3 means that $|w(y)| \leq R$ for all y such that $|y| \geq y_0$. Otherwise it means that $|w(y)| < R$ for all y such that $|y| \geq y_0$.

Changing, if necessary, $c(\Lambda)$ on $(-y_0, y_0)$ and adding a correction term to V, we may assume that $|\widetilde{w}(y)| < R$ or, respectively, that $|\widetilde{w}(y)| \leq R$ for all $y \in \mathbb{R}$, whence the series $\sum\limits_{k=1}^{\infty} c_k \widetilde{w}^k(y)$ absolutely converges for all $y \in \mathbb{R}$. In addition, we may assume that the range of $\widetilde{w}$ is contained in the domain of holomorhy of Φ. Under this agreement,

$$c(G_n) = \sum_{k=1}^{n} c_k\, \widetilde{w}^k + v \to \sum_{k=1}^{\infty} c_k\, \widetilde{w}^k + v = \frac{1}{\Phi \circ (w - v)} - 1 + v \tag{2.9}$$

as $n \to \infty$ uniformly on $\mathbb{R}$. By the Wiener–Levy theorem, the right-hand side of (2.9) is the Fourier transform, $c(G)$, of a certain function $G \in L(\mathbb{R})$. This function G is one of those mentioned in the first assertion of Lemma 1.

Let

$$T_\sigma G = G - \sum_{k=1}^{\infty} c_k \big(\widetilde{W}^{[k]} - T_\sigma \widetilde{W}^{[k]}\big). \tag{2.10}$$

Obviously, $T_\sigma G \in L(\mathbb{R})$.

From the convergence of the series (2.8) in $L(\mathbb{R})$, from the inclusions $T_\sigma \widetilde{W}^{[k]} \in \mathbf{E}_\sigma$, and from relation (2.9) it follows that

$$c(T_\sigma G) = c(G) - \sum_{k=1}^{\infty} c_k \big(\widetilde{w}^k - c(T_\sigma \widetilde{W}^{[k]})\big) = \sum_{k=1}^{\infty} c_k\, \gamma^k\, c\big(L_\sigma \Lambda^{[k]}\big). \tag{2.11}$$

Consequently, $c(T_\sigma G, y) = 0$ for all $y : |y| > \sigma$, whence $T_\sigma G \in \mathbf{E}_\sigma$.

If $\varphi = \Phi(\mathcal{W}) f$ and $f - \Phi(\mathcal{W}) f$ is represented by (1.3) we put

$$\mathcal{Z}_{\sigma, W, \Phi} f = f * Q + \varphi * T_\sigma G.$$

Then

$$f - \mathcal{Z}_{\sigma,W,\Phi} f = \varphi + \varphi * (G - T_\sigma G).$$

Remark 8. One can imagine the situation where the function W admits different representations of the form $W = \mathcal{S}\Lambda + V$, and the operators $\mathcal{Z}_{\sigma,W,\Phi}$ constructed may be different. It leads to no misunderstanding if we assume that for a fixed representation from condition W1, the operators are constructed on the basis of the latter representation. If no representation is fixed, then we may assume that $\mathcal{Z}_{\sigma,W,\Phi}$ denotes any of the operators constructed.

A remark similar to Remark 1 is also valid for the operators $\mathcal{Z}_{\sigma,W,\Phi}$.

Remark 9. The absolute convergence of the series (2.8) in $L(\mathbb{R})$ implies [7, Lemma 4] that for any space $(\mathfrak{M}, P) \in \mathcal{B}$, the series $\sum\limits_{k=1}^{\infty} c_k \big(\widetilde{\mathcal{W}}^k - \mathcal{T}_\sigma \widetilde{\mathcal{W}}^k\big)$ converges absolutely with respect to the seminorm in the space of operators acting from $(\mathfrak{M}, P)$ to $(\mathfrak{M}, P)$. Here, $\widetilde{\mathcal{W}}^k$ and $\mathcal{T}_\sigma \widetilde{\mathcal{W}}^k$ are the convolution operators with the kernels $\widetilde{W}^{[k]}$ and $T_\sigma \widetilde{W}^{[k]}$.

Remark 10. The definition implies that $\mathcal{Z}_{\sigma,W,\Phi}$ is a convolution operator with the kernel

$$Z_\sigma = Q + T_\sigma G + (\Phi - 1)^\circ(W) * T_\sigma G.$$

Let us express the Fourier transform of the kernel Z_σ. By formula (1.4) for the kernel Q and by (2.11), we have

$$c(Z_\sigma) = (\Phi \circ (w - v) - \Phi \circ w)(1 + c(G)) + (\Phi \circ w) \sum_{k=1}^{\infty} c_k \, \gamma^k \, c\big(L_\sigma \Lambda^{[k]}\big).$$

By taking into account that $c(G) = \frac{1 - \Phi \circ (w - v)}{\Phi \circ (w - v)}$, we obtain

$$c(Z_\sigma) = \left(1 + \frac{\Phi \circ w}{\Phi \circ (w - v)}\right) + (\Phi \circ w) \sum_{k=1}^{\infty} c_k \, \gamma^k \, c\big(L_\sigma \Lambda^{[k]}\big).$$

Theorem 1. *Assume that* $(\mathfrak{M}, P) \in \mathcal{B}$, $f \in \mathfrak{M}$, $y_0 > 0$, $\sigma \geq y_0$, *a function* Φ *satisfies conditions* $\Phi 1$ *and* $\Phi 2$, *a function* W *satisfies conditions* W1 *and* W2; *for every* $k \in \mathbb{N}$ *for which* $c_k \neq 0$, *there exists an* $\varepsilon_k \in \{-1, 1\}$ *such that*

$$\varepsilon_k \big(\Lambda^{[k]}(t) - L_\sigma \Lambda^{[k]}(t)\big) \cos\left(\sigma t - \frac{k \varkappa \pi}{2}\right) \geq 0$$

for almost every $t \in \mathbb{R}$; $\varphi = \Phi(\mathcal{W}) f$. *Then*

$$P(f - \mathcal{Z}_{\sigma,W,\Phi} f) \leq P(\varphi) + \sum_{k=1}^{\infty} |c_k| \, \mathcal{K}_{\sigma,\Lambda^{[k]}} P(\mathcal{S}^k \varphi), \tag{2.12}$$

$$P(f - \mathcal{Z}_{\sigma,W,\Phi} f) \leq \left(1 + \sum_{k=1}^{\infty} |c_k| \, \|\mathcal{S}^k\| \mathcal{K}_{\sigma,\Lambda^{[k]}}\right) P(\varphi), \tag{2.13}$$

$$A_\sigma(f)_P \leq \left(1 + \sum_{k=1}^{\infty} |c_k| \, \|\mathcal{S}^k\| \mathcal{K}_{\sigma,\Lambda^{[k]}}\right) A_\sigma(\varphi)_P, \tag{2.14}$$

$$A_{\sigma-0}(f)_P \leq \left(1 + \sum_{k=1}^{\infty} |c_k| \, \|\mathcal{S}^k\| \mathcal{K}_{\sigma,\Lambda^{[k]}}\right) A_{\sigma-0}(\varphi)_P, \tag{2.15}$$

$$A_{\sigma-0}(f)_P \leq \left(1 + \sum_{k=1}^{\infty} |c_k| \, \|\mathcal{S}^k\| \mathcal{K}_{\sigma,\Lambda^{[k]}}\right) P(\varphi), \tag{2.16}$$

$$A_\sigma(f)_P \leq \left(1 + \sum_{k=1}^{\infty} |c_k| \, \|\mathcal{S}^k\| \mathcal{K}_{\sigma,\Lambda^{[k]}}\right) P(\varphi). \tag{2.17}$$

Proof. As the series (2.8) absolutely converges in $L(\mathbb{R})$, the equality

$$f - \mathcal{Z}_{\sigma,W,\Phi} f = \varphi + \sum_{k=1}^{\infty} c_k \, \varphi * \big(\widetilde{W}^{[k]} - T_\sigma \widetilde{W}^{[k]}\big)$$

holds, and the series on its right-hand side absolutely converges with respect to the seminorm P. By using the definition of $T_\sigma \widetilde{W}^{[k]}$ and the fact that $\mathcal{S}$ and the convolution operator commute, we obtain

$$f - \mathcal{Z}_{\sigma,W,\Phi} f = \varphi + \sum_{k=1}^{\infty} c_k\, \mathcal{S}^k \varphi * \bigl(\Lambda^{[k]} - L_\sigma \Lambda^{[k]}\bigr).$$

Applying the triangle inequality and (1.6), we arrive at (2.12). Inequality (2.13) follows from (2.12).

The simplest way to derive (2.14) from (2.13) is to take $A_\sigma(\cdot)_P$ as the seminorm in (2.13) and to use the relation $A_\sigma(f - \mathcal{Z}_{\sigma,W,\Phi}(f))_P = A_\sigma(f)_P$. Now we give another proof, which is common for inequalities (2.14) and (2.15). By Lemma 1,

$$f = \varphi + f * Q + \varphi * G,$$

where $f * Q \in \mathbf{E}_{y_0} \cap L(\mathbb{R})$. Moving the seminorm under the integral sign [2, Corollary 2], we have

$$A_\sigma(f)_P \le \bigl(1 + A_\sigma(G)_1\bigr) A_\sigma(\varphi)_P.$$

It remains to apply the inequality

$$A_\sigma(G)_1 \le \|G - T_\sigma G\|_1 \le \sum_{k=1}^{\infty} |c_k|\, \|\mathcal{S}^k\| \mathcal{K}_{\sigma,\Lambda^{[k]}}.$$

Inequality (2.15) is proved in a similar way. In addition, one should take into account that [1, Sec. 99, p. 232] $A_{\sigma-0}(G)_1 = A_\sigma(G)_1$ and

$$A_{\sigma-0}(f * Q)_P \le A_{\sigma-0}(f)_P A_{\sigma-0}(Q)_1 = A_{\sigma-0}(f)_P A_\sigma(Q)_1 = 0.$$

Inequalities (2.16) and (2.17) trivially follow from (2.15). □

3. Sharp inequalities

In this section, for $\theta = \frac{\pi}{\sigma}$, we establish additional properties of approximating operators and show that the inequalities of Theorem 1 are sharp in some cases.

Remark 11. If $\theta = \frac{\pi}{\sigma}$, then the values of $T_\sigma G_n$ and G_n at the points $z_q = \bigl(q + \frac{1}{2}\bigr)\frac{\pi}{\sigma}$ $(q \in \mathbb{Z})$ coincide.

Proof. Denoting $\tau_\nu = (-1)^{\nu+\varkappa}\alpha_\nu$, where $\varkappa$ is the parity of Λ, we have

$$T_\sigma \mathcal{S}\Lambda(z_q) = \sum_{\nu\in\mathbb{Z}} \tau_\nu L_\sigma \Lambda\Bigl(z_q - \Bigl(\nu + \frac{\varkappa}{2}\Bigr)\frac{\pi}{\sigma}\Bigr).$$

In both cases, the points $z_q - \bigl(\nu + \frac{\varkappa}{2}\bigr)\frac{\pi}{\sigma}$ are nodes of interpolation, whence

$$T_\sigma \mathcal{S}\Lambda(z_q) = \sum_{\nu\in\mathbb{Z}} \tau_\nu \Lambda\Bigl(z_q - \Bigl(\nu + \frac{\varkappa}{2}\Bigr)\frac{\pi}{\sigma}\Bigr) = \mathcal{S}\Lambda(z_q).$$

Thus, the relation $T_\sigma \widetilde{W}^{[k]}(z_q) = \widetilde{W}^{[k]}(z_q)$ is proved for $k = 1$. In order to prove it for an arbitrary k, we apply the relation established to the function $\Lambda^{[k]}$ and operator $\mathcal{S}^k$, which is of the same form as $\mathcal{S}$. By linearity, $T_\sigma G_n(z_q) = G_n(z_q)$. □

Remark 11 implies that if $G_n \in C(\mathbb{R} \setminus \{0\})$ and $G_n(t) = O(t^{-2})$ as $t \to \infty$, then $T_\sigma G_n = L_\sigma G_n$. This is true because a summable interpolating function is unique [8, Sec. 4.3.1, p. 195]. Therefore, for $\theta = \frac{\pi}{\sigma}$ we will use the notation $L_\sigma G_n$ rather than $T_\sigma G_n$.

Inequalities (2.13)–(2.17) are senseless in the case where the series (2.7) is divergent because the constants on their right-hand sides are infinite. The construction of approximating operators was also based on the convergence of (2.7). However, straightforward verification of the convergence of this series is not always convenient.

From now on, as above, we assume that condition W1 is fulfilled, but we do not require that condition W2 be fulfilled, i.e., the series (2.7) does not necessarily converge. Instead, we assume that the function W satisfies the weaker condition W3 and the following two additional conditions:

W4. The function $|\lambda|$ decreases on $[y_0, +\infty)$.

W5. For every $k \in \mathbb{N}$ such that $c_k \ne 0$,

$$L_\sigma \Lambda^{[k]}(t) = \int_{-\sigma}^{\sigma} c\bigl(L_\sigma \Lambda^{[k]}, y\bigr) e^{ity}\, dy$$

and

$$c\big(L_\sigma \Lambda^{[k]}, y\big) = \sum_{s=-\infty}^{\infty} (-1)^{s(1-k\varkappa)} c\big(\Lambda^{[k]}, y + 2s\sigma\big), \qquad |y| \le \sigma.$$

Here, c_k are the coefficients of the series (2.1); the kernel Λ is defined in W1; $\lambda = c(\Lambda)$, and $\varkappa$ is the parity of Λ.

We will also use the agreement from Remark 7, which is valid under condition W3.

Later (see Remark 14) we will show that under some additional assumptions, which provide for the sharpness of the inequalities, condition W2 is fulfilled automatically.

Lemma 3. *Assume that* $y_0 > 0$, $\sigma \ge y_0$, $\theta = \frac{\pi}{\sigma}$, *and a function* W *satisfies conditions* W1, W3, W4, *and* W5. *Then the sequence* $\{L_\sigma G_n\}$ *converges on* $\mathbb{R}$ *pointwisely.*

Proof. Without loss of generality, we may assume that $V = 0$, i.e., $\widetilde{W} = W$.

1. Consider the case where the functions γ and λ from formula (2.3) are even. For definiteness, we assume that $\lambda \ge 0$ on $[y_0, +\infty)$. Note that the function γ has period 2σ. We denote $\xi_{n\sigma} = c(L_\sigma G_n)$. By condition W5 and equality (2.6),

$$L_\sigma G_n(t) = \int_{-\sigma}^{\sigma} \xi_{n\sigma}(y) e^{ity}\, dy \tag{3.1}$$

and

$$\xi_{n\sigma}(y) = \sum_{k=1}^{n} c_k \sum_{s=-\infty}^{\infty} (-1)^s \gamma^k(y) \lambda^k(y + 2s\sigma), \qquad |y| \le \sigma. \tag{3.2}$$

Prove that the sequence $\{\xi_{n\sigma}(y)\}$ converges and that the limit passage under the integral sign on the right-hand side of (3.1) is possible. Consider the iterated series

$$\sum_{k=1}^{\infty} c_k \sum_{s=1}^{\infty} (-1)^{s-1} \gamma^k(y) \lambda^k(y + 2s\sigma). \tag{3.3}$$

Since the function λ decays to zero on $[\sigma, +\infty)$,

$$0 \le \sum_{q=1}^{\infty} \big(\lambda^k(y + (4q-2)\sigma) - \lambda^k(y + 4q\sigma)\big) \le \lambda^k(y + 2\sigma).$$

The series $\sum\limits_{k=1}^{\infty} |c_k||w^k(y + 2\sigma)|$ being convergent, the double series

$$\sum_{k,q=1}^{\infty} c_k \gamma^k(y) \big(\lambda^k(y + (4q-2)\sigma) - \lambda^k(y + 4q\sigma)\big)$$

converges absolutely. Consequently, the sums of the iterated series are equal, i.e.,

$$\sum_{k=1}^{\infty} c_k \gamma^k(y) \sum_{q=1}^{\infty} \big(\lambda^k(y + (4q-2)\sigma) - \lambda^k(y + 4q\sigma)\big) = \sum_{q=1}^{\infty} \sum_{k=1}^{\infty} c_k \gamma^k(y) \big(\lambda^k(y + (4q-2)\sigma) - \lambda^k(y + 4q\sigma)\big). \tag{3.4}$$

We use the following assertion: If $a_s \to 0$ and the series $\sum\limits_{q=1}^{\infty} (a_{2q-1} - a_{2q})$ converges, then the series $\sum\limits_{s=1}^{\infty} (-1)^{s-1} a_s$ converges to the same sum. Application of this assertion to the inner sum on the left-hand side of equality (3.4), where $a_s = \lambda^k(y + 2s\sigma)$, is obvious. On the right-hand side, $a_s = \sum\limits_{k=1}^{\infty} c_k \gamma^k(y) \lambda^k(y + 2s\sigma)$. This series converges uniformly with respect to $s \in \mathbb{N}$, and its terms tend to zero as $s \to \infty$. Consequently, the condition $a_s \to 0$ is fulfilled. Thus, the series (3.3) converges, and the order of summation can be changed. In addition, the partial sums of the series (3.3) satisfy the inequality

$$\left| \sum_{k=1}^{n} c_k \sum_{s=1}^{\infty} (-1)^{s-1} \gamma^k(y) \lambda^k(y + 2s\sigma) \right| \le \sum_{k=1}^{\infty} |c_k| |\gamma^k(y)| \lambda^k(y + 2\sigma) = M(y),$$

and $M \in L[-\sigma, \sigma]$.

The sum over the negative indices s in the series (3.2), by the evenness of λ, reduces to the situation just considered, and the term with $s=0$ is treated separately.

Denote $\xi_\sigma(y)=\lim\limits_{n\to\infty}\xi_{n\sigma}(y)$. By the Lebesgue dominated convergence theorem, $\xi_\sigma\in L[-\sigma,\sigma]$, and we may pass to the limit under the integral sign on the right-hand side of (3.1).

2. The case where the functions γ and λ in (2.3) are odd is considered analogously. Here, we must take into account that by (2.4), $\gamma(y+2\sigma)=-\gamma(y)$ and, for k odd, group together the terms with opposite numbers in the sum over s. □

Thus, under the assumptions of Lemma 3, the formula

$$\xi_\sigma(y)=\sum_{s=-\infty}^{\infty}(-1)^s\sum_{k=1}^{\infty}c_k\gamma^k(y+2s\sigma)\lambda^k(y+2s\sigma) \tag{3.5}$$

is valid, and we can define the function $L_\sigma G$ by the relation

$$L_\sigma G(t)=\lim_{n\to\infty}L_\sigma G_n(t)=\int\limits_{-\sigma}^{\sigma}\xi_\sigma(y)e^{ity}\,dy; \tag{3.6}$$

here, the convergence is uniform with respect to t; G is the function defined in Remark 7. Thus, $L_\sigma G\in\mathbf{E}_\sigma$. We will not study the question whether the right-hand sides of (3.6) and (1.12) coincide in the general case, and by $L_\sigma G$ we will mean the function defined by (3.6).

If $\varphi=\Phi(\mathcal{W})f$ and $f-\Phi(\mathcal{W})f$ is represented by (1.3) we put

$$\mathcal{Y}_{\sigma,W,\Phi}f=f*Q+\varphi*L_\sigma G.$$

In other terms, $\mathcal{Y}_{\sigma,W,\Phi}=\mathcal{X}_{\sigma,G}\big(I-\Phi(\mathcal{W})\big)$. Then

$$f-\mathcal{Y}_{\sigma,W,\Phi}f=\varphi+\varphi*(G-L_\sigma G).$$

The remark analogous to Remark 1 is valid for the operators $\mathcal{Y}_{\sigma,W,\Phi}$ as well.

Remark 12. The definition implies that $\mathcal{Y}_{\sigma,W,\Phi}$ is a convolution operator with the kernel

$$Y_\sigma=Q+L_\sigma G+(\Phi-1)^\circ(W)*L_\sigma G.$$

Let us express the Fourier transform of the kernel Y_σ in terms of the Fourier transform of the original kernel W. By applying, as in Remark 10, formula (1.4) for the kernel Q, we obtain

$$c(Y_\sigma)=(\Phi\circ(w-v)-\Phi\circ w)(1+c(G))+(\Phi\circ w)c(L_\sigma G).$$

Writing $c(L_\sigma G)$ as in (3.5), separating the term with $s=0$, and taking into account that $c(G)=\frac{1-\Phi\circ(w-v)}{\Phi\circ(w-v)}$, whereas $v(y)=0$ for $|y|\ge\sigma$, we conclude that for $|y|\ge\sigma$,

$$\begin{aligned}c(Y_\sigma,y)&=1-(\Phi\circ w)(y)-(\Phi\circ w)(y)\sum_{s\in\mathbb{Z}\setminus\{0\}}(-1)^{s-1}\frac{1-(\Phi\circ w)(y+2s\sigma)}{(\Phi\circ w)(y+2s\sigma)}\\&=1-(\Phi\circ w)(y)-(\Phi\circ w)(y)\sum_{k=1}^{\infty}c_k\sum_{s\in\mathbb{Z}\setminus\{0\}}(-1)^{s-1}w^k(y+2s\sigma).\end{aligned} \tag{3.7}$$

This formula implies that the operator $\mathcal{Y}_{\sigma,W,\Phi}$ does not depend on the choice of the functions Λ and V and of the operator $\mathcal{S}$ in representation (2.2).

Remark 13. Let $\theta=\frac{\pi}{\sigma}$ and let conditions W1–W5 be fulfilled, i.e., the functions $T_\sigma G$ and $L_\sigma G$ be defined by (2.10) and (3.6), respectively. Then the equality of their Fourier transforms implies that $T_\sigma G=L_\sigma G$. Consequently, in this case, $\mathcal{Y}_{\sigma,W,\Phi}=\mathcal{Z}_{\sigma,W,\Phi}$.

Recall that c_k are the coefficients of the series (2.1); the function Λ, the coefficients α_ν, and the operator $\mathcal{S}$ are defined in condition W1.

Theorem 2. *Assume that $(\mathfrak{M},P)\in\mathcal{B}$, $f\in\mathfrak{M}$, $y_0>0$, $\sigma\ge y_0$, $\theta=\frac{\pi}{\sigma}$; a function Φ satisfies conditions* Φ1 *and* Φ2*; a function W satisfies conditions* W1, W3, W4, *and* W5; *$c_k\ge0$ for every $k\in\mathbb{N}$; $\alpha_\nu\ge0$ for every $\nu\in\mathbb{Z}$; for every k such that $c_k\ne0$ and for almost every $t\in\mathbb{R}$,*

$$\big(\Lambda^{[k]}(t)-L_\sigma\Lambda^{[k]}(t)\big)\cos\left(\sigma t-\frac{k\varkappa\pi}{2}\right)\ge0, \tag{3.8}$$

$\varphi = \Phi(\mathcal{W})f$. *Then*

$$P(f - \mathcal{Y}_{\sigma,W,\Phi}f) \le P(\varphi) + \sum_{k=1}^{\infty} c_k \mathcal{K}_{\sigma,\Lambda^{[k]}} P(\mathcal{S}^k\varphi), \tag{3.9}$$

$$P(f - \mathcal{Y}_{\sigma,W,\Phi}f) \le (1 + \mathcal{K}_{\sigma,G})P(\varphi), \tag{3.10}$$

$$A_\sigma(f)_P \le (1 + \mathcal{K}_{\sigma,G})A_\sigma(\varphi)_P, \tag{3.11}$$

$$A_{\sigma-0}(f)_P \le (1 + \mathcal{K}_{\sigma,G})A_{\sigma-0}(\varphi)_P, \tag{3.12}$$

$$A_{\sigma-0}(f)_P \le (1 + \mathcal{K}_{\sigma,G})P(\varphi), \tag{3.13}$$

$$A_\sigma(f)_P \le (1 + \mathcal{K}_{\sigma,G})P(\varphi). \tag{3.14}$$

In the spaces $(UCB(\mathbb{R}), \|\cdot\|_\infty)$ *and* $(L(\mathbb{R}), \|\cdot\|_1)$, *the constant occurring in inequalities* (3.10)–(3.14) *cannot be reduced, even if one restricts himself to the functions orthogonal to* $\mathbf{E}_\sigma$. *In the spaces of* $\frac{2\pi}{\sigma}$*- periodic functions with the uniform or integral norm, the constant in* (3.10), (3.12), (3.13) *neither can be reduced, even if one restricts himself to the functions with the zero mean value.*

Proof. 1. First, we prove the inequalities themselves. Verify that for every $k \in \mathbb{N}$ such that $c_k \ne 0$ and for almost every $t \in \mathbb{R}$,

$$\big(\widetilde{W}^{[k]}(t) - L_\sigma \widetilde{W}^{[k]}(t)\big)\cos\sigma t \ge 0.$$

For brevity, we denote

$$f_k = \Lambda^{[k]} - L_\sigma\Lambda^{[k]}, \qquad \beta_\mu = \sum_{\substack{\nu\in\mathbb{Z}^k \\ \nu_1+\ldots+\nu_k=\mu}} \alpha_{\nu_1}\ldots\alpha_{\nu_k}.$$

By (2.5),

$$\big(\widetilde{W}^{[k]}(t) - L_\sigma \widetilde{W}^{[k]}(t)\big)\cos\sigma t = \mathcal{S}^k f_k(t)\cos\sigma t = \sum_{\mu\in\mathbb{Z}} (-1)^{\mu+k\varkappa}\beta_\mu f_k\left(t - \left(\mu + \frac{k\varkappa}{2}\right)\frac{\pi}{\sigma}\right)\cos\sigma t$$

$$= \sum_{\mu\in\mathbb{Z}} \beta_\mu f_k\left(t - \frac{\mu\pi}{\sigma} - \frac{k\varkappa\pi}{2\sigma}\right)\cos\left(\sigma\left(t - \frac{\mu\pi}{\sigma} - \frac{k\varkappa\pi}{2\sigma}\right) - \frac{k\varkappa\pi}{2}\right).$$

By the nonnegativity of the coefficients α_ν and by (3.8), the right-hand side is nonnegative.

The nonnegativity of the coefficients c_k implies that for every $n \in \mathbb{N}$ and almost every $t \in \mathbb{R}$,

$$\big(G_n(t) - L_\sigma G_n(t)\big)\cos\sigma t \ge 0.$$

Denote

$$H_n(t) = \big(G_n(t) - L_\sigma G_n(t)\big)\cos\sigma t, \qquad H(t) = \big(G(t) - L_\sigma G(t)\big)\cos\sigma t.$$

By the positivity of the Fejér operators σ_N, the inequality $\sigma_N H_n(t) \ge 0$ holds for all $N > 0$ and $t \in \mathbb{R}$. By Remark 7, the series $\sum\limits_{k=1}^{\infty} c_k\widetilde{w}^k$ uniformly converges on $\mathbb{R}$, i.e., $c(G_n)$ uniformly converges to $c(G)$ on $\mathbb{R}$. In addition, $L_\sigma G_n \to L_\sigma G$ as $n \to \infty$ uniformly on $\mathbb{R}$. Multiplication by $\cos\sigma t$ spoils none of the two convergences. Consequently, for every $N > 0$,

$$\sigma_N H_n \to \sigma_N H$$

uniformly on $\mathbb{R}$ as $n \to \infty$. Hence $\sigma_N H(t) \ge 0$ for all $N > 0$ and $t \in \mathbb{R}$. By letting N tend to ∞, we obtain the inequality

$$(G(t) - L_\sigma G(t))\cos\sigma t \ge 0$$

for almost every t.

The sequence $\{|G_n(t) - L_\sigma G_n(t)|\}$ increases for almost every t. By the Levy theorem,

$$\|G_n - L_\sigma G_n\|_1 \to \|G - L_\sigma G\|_1. \tag{3.15}$$

Using the expression for the constant in (1.6), we can write relation (3.15) as

$$\sum_{k=1}^{\infty} c_k \mathcal{K}_{\sigma,\widetilde{W}^{[k]}} = \mathcal{K}_{\sigma,G}.$$

Thus, condition W2 is fulfilled, whence, by Theorem 1, all inequalities (3.9)–(3.14) hold as well.

For the subsequent reasoning, we observe that inequality (3.9) strengthens (3.10) because

$$\mathcal{K}_{\sigma,\Lambda^{[k]}} P(\mathcal{S}^k \varphi) \le \mathcal{K}_{\sigma,\Lambda^{[k]}} \left(\sum_{\nu\in\mathbb{Z}} \alpha_\nu\right)^k P(\varphi) = \mathcal{K}_{\sigma,\widetilde{W}^{[k]}} P(\varphi).$$

Now we prove that the inequalities are sharp.

2. First we note that by (1.6), in the space $L_\infty(\mathbb{R})$ the inequality

$$\|\varphi + \varphi * (G - L_\sigma G)\|_\infty \le (1 + \mathcal{K}_{\sigma,G})\|\varphi\|_\infty$$

turns into equality for the function $\varphi_\sigma(t) = \operatorname{sign}\cos\sigma t$. Therefore, if $\Phi(\mathcal{W})f = \varphi_\sigma$, then inequality (3.10) in the space $L_\infty(\mathbb{R})$ turns into equality. As such a function we may take

$$\psi_\sigma = \varphi_\sigma + \varphi_\sigma * G.$$

The function ψ_σ is periodic with period $\frac{2\pi}{\sigma}$, and it has the Fourier series expansion

$$\psi_\sigma(t) = \frac{4}{\pi}\sum_{s=0}^{\infty} \frac{(-1)^s}{(2s+1)(\Phi\circ w)((2s+1)\sigma)} \cos{(2s+1)\sigma t}.$$

Hence $\mathcal{Y}_{\sigma,W,\Phi}\psi_\sigma = 0$. In addition [1, Sec. 96, p. 227], $A_{\sigma-0}(\psi_\sigma)_\infty = \|\psi_\sigma\|_\infty$. This implies that inequality (3.13) and, consequently, the previous inequalities (3.10) and (3.12) as well turn into equalities on the function ψ_σ.

3. Prove that inequality (3.14) is sharp in the space $L_\infty(\mathbb{R})$ on the set of functions orthogonal to $\mathbf{E}_\sigma$. Take $\rho > \sigma$ and put

$$\psi_\rho = \varphi_\rho + \varphi_\rho * G.$$

Then, as above,

$$\mathcal{Y}_{\sigma,W,\Phi}\psi_\rho = 0, \quad \Phi(\mathcal{W})\psi_\rho = \varphi_\rho, \quad \|\Phi(\mathcal{W})\psi_\rho\|_\infty = \|\varphi_\rho\|_\infty = \|\varphi_\sigma\|_\infty, \quad A_\sigma(\psi_\rho)_\infty = \|\psi_\rho\|_\infty.$$

By the Lebesgue dominated convergence theorem, $\psi_\rho \underset{\rho\to\sigma+}{\longrightarrow} \psi_\sigma$ almost everywhere. For this reason,

$$\varliminf_{\rho\to\sigma+} \|\psi_\rho\|_\infty \ge \|\psi_\sigma\|_\infty,$$

which proves that inequality (3.14) is sharp.

The unimprovability of the inequalities on the sets of continuous functions is deduced (e.g., with the use of approximation of functions φ_ρ by their Fejér integrals) from their unimprovability on the sets of functions belonging to $L_\infty(\mathbb{R})$.

4. Prove the sharpness of the inequalities for the integral norm. The operator $\Phi(\mathcal{W})$ is invertible on sets of functions orthogonal to $\mathbf{E}_\sigma$. Therefore, for all $p \in [1,\infty]$, we have

$$\sup_{\substack{f\perp\mathbf{E}_\sigma\\ \|\Phi(\mathcal{W})f\|_p\le 1}} A_\sigma(f)_p = \sup_{\substack{\varphi\perp\mathbf{E}_\sigma\\ \|\varphi\|_p\le 1}} A_\sigma(\varphi + \varphi * G)_p.$$

Now we use the duality relations (see [9, §1.4])

$$\begin{aligned}\sup_{\substack{\varphi\perp\mathbf{E}_\sigma\\ \|\varphi\|_1\le 1}} A_\sigma(\varphi+\varphi*G)_1 &= \sup_{\substack{\varphi\perp\mathbf{E}_\sigma\\ \|\varphi\|_1\le 1}} \sup_{\substack{g\perp\mathbf{E}_\sigma\\ \|g\|_\infty\le 1}} \left|\int_{-\infty}^{+\infty} (\varphi+\varphi*G)\, g\right| \\ &= \sup_{\substack{g\perp\mathbf{E}_\sigma\\ \|g\|_\infty\le 1}} \sup_{\substack{\varphi\perp\mathbf{E}_\sigma\\ \|\varphi\|_1\le 1}} \left|\int_{-\infty}^{+\infty} (g+g*G)\,\varphi\right| = \sup_{\substack{g\perp\mathbf{E}_\sigma\\ \|g\|_\infty\le 1}} A_\sigma(g+g*G)_\infty.\end{aligned}$$

The sharpness of the inequalities proved for $L_\infty(\mathbb{R})$ implies that the latter upper bound is equal to $1 + \mathcal{K}_{\sigma,G}$. Thus, inequality (3.14) (and, consequently, the previous inequalities (3.10)–(3.13) as well) is sharp in $L(\mathbb{R})$.

The sharpness of inequality (3.13) (and, consequently, of the previous inequalities (3.10) and (3.12)) in the space L of $\frac{2\pi}{\sigma}$-periodic functions is proved similarly:

$$\begin{aligned}\sup_{\substack{\varphi\perp 1\\ \|\varphi\|_1\le 1}} A_{\sigma-0}(\varphi+\varphi*G)_1 &= \sup_{\substack{\varphi\perp 1\\ \|\varphi\|_1\le 1}} \sup_{\substack{g\perp 1\\ \|g\|_\infty\le 1}} \left|\int_{-\frac{\pi}{\sigma}}^{\frac{\pi}{\sigma}} (\varphi+\varphi*G)\, g\right| \\ &= \sup_{\substack{g\perp 1\\ \|g\|_\infty\le 1}} \sup_{\substack{\varphi\perp 1\\ \|\varphi\|_1\le 1}} \left|\int_{-\frac{\pi}{\sigma}}^{\frac{\pi}{\sigma}} (g+g*G)\,\varphi\right| = \sup_{\substack{g\perp 1\\ \|g\|_\infty\le 1}} A_{\sigma-0}(g+g*G)_\infty.\end{aligned}$$

Here, we have used the fact that for such functions, $A_{\sigma-0}$ coincides with the best approximation by constants. $\square$

Remark 14. In proving Theorem 2, we have shown that under the assumptions of this theorem, condition W2 is fulfilled and

$$\mathcal{K}_{\sigma,G}=\sum_{k=1}^{\infty}c_k\mathcal{K}_{\sigma,\widetilde{W}^{[k]}}=\sum_{k=1}^{\infty}c_k\Bigl(\sum_{\nu\in\mathbb{Z}}\alpha_\nu\Bigr)^k\mathcal{K}_{\sigma,\Lambda^{[k]}}.$$

The constants are expressed in terms of the Fourier transform as follows:

$$\mathcal{K}_{\sigma,G}=\frac{4}{\pi}\sum_{s=0}^{\infty}\frac{(-1)^s}{2s+1}\,\frac{1-(\Phi\circ w)((2s+1)\sigma)}{(\Phi\circ w)((2s+1)\sigma)}=\frac{4}{\pi}\sum_{s=0}^{\infty}\frac{(-1)^s}{2s+1}\sum_{k=1}^{\infty}c_kw^k((2s+1)\sigma),$$

$$\mathcal{K}_{\sigma,\widetilde{W}^{[k]}}=\frac{4}{\pi}\sum_{s=0}^{\infty}\frac{(-1)^s}{2s+1}w^k((2s+1)\sigma).$$

By Remark 13, under the assumptions of Theorem 2, the equality $\mathcal{Y}_{\sigma,W,\Phi}=\mathcal{Z}_{\sigma,W,\Phi}$ holds true.

Theorem 3. *Assume that* $(\mathfrak{M},P)\in\mathcal{B}$, $f\in\mathfrak{M}$, $y_0>0$, $\sigma\ge y_0$, $\theta=\frac{\pi}{\sigma}$, *a function* Φ *satisfies conditions* $\Phi1$ *and* $\Phi2$, *a function* W *satisfies conditions* W1 *and* W3, $c_k\ge 0$ *for every* $k\in\mathbb{N}$, $\alpha_\nu\ge 0$ *for every* $\nu\in\mathbb{Z}$, $\varphi=\Phi(\mathcal{W})f$. *Also assume that a function* Λ *satisfies one of the following two conditions:*

(1) *in accordance with the parity of* Λ, *one of the functions* $a(\Lambda)$ *or* $b(\Lambda)$ *is three times monotone on* $[y_0,+\infty)$;

(2) $\Lambda\in\widehat{CM}^2(y_0)$, $c(\Lambda)\in C^{(2)}(\mathbb{R}\setminus(-y_0,y_0))$.

Then the assertion of Theorem 2 *holds.*

The threefold monotonicity is understood in the sense of Remark 4.

Proof. By Remark 5, without loss of generality we may assume that the function $\lambda=c(\Lambda)$ is twice continuously differentiable on $[-y_0,y_0]$ (in the second case, on $\mathbb{R}$). Show that the assumptions of Theorem 2 are satisfied. The monotonicity property W4 is obvious. The parity of the function $\Lambda^{[k]}$ coincides with that of $k\varkappa$, where $\varkappa$ is the parity of Λ. We apply formulas (1.1). If the function Λ is even, then, for every $k\in\mathbb{N}$,

$$a\bigl(\Lambda^{[k]}\bigr)=2c\bigl(\Lambda^{[k]}\bigr)=2c^k(\Lambda)=\frac{1}{2^{k-1}}a^k(\Lambda).$$

If the function Λ is odd, then, for an even $k=2r$,

$$a\bigl(\Lambda^{[2r]}\bigr)=2c\bigl(\Lambda^{[2r]}\bigr)=2c^{2r}(\Lambda)=\frac{2b^{2r}(\Lambda)}{(2i)^{2r}}=\frac{(-1)^r}{2^{2r-1}}b^{2r}(\Lambda),$$

and, for an odd $k=2r+1$,

$$b\bigl(\Lambda^{[2r+1]}\bigr)=2ic\bigl(\Lambda^{[2r+1]}\bigr)=2ic^{2r+1}(\Lambda)=\frac{2ib^{2r+1}(\Lambda)}{(2i)^{2r+1}}=\frac{(-1)^r}{2^{2r}}b^{2r+1}(\Lambda).$$

1. Let Λ satisfy the first assumption of the theorem. The analog of the Leibnitz rule for a finite difference of the product of two functions (see, e.g., [10, Chapter I, Sec. 16, pp. 19–20]),

$$\Delta_t^n(uv)(x)=\sum_{\nu=0}^{n}C_n^\nu\Delta_t^{n-\nu}u(x+\nu t)\Delta_t^\nu v(x),$$

implies that the product of three times monotone functions is three times monotone. Hence, by induction, powers of three times monotone functions are three times monotone. Consequently, if Λ is even, then, for every $k\in\mathbb{N}$, the function $\Lambda^{[k]}$ satisfies the Nagy conditions (see Lemma B and Remark 4). If Λ is odd, then the Nagy conditions are fulfilled for the functions $(-1)^r\Lambda^{[2r]}$ for $k=2r$ and $(-1)^r\Lambda^{[2r+1]}$ for $k=2r+1$. Therefore, by Lemma B and Remark 4, inequalities (3.8) are valid, whence, by Remark 3, condition W5 is fulfilled.

2. Assume that Λ satisfies the second assumption of the theorem. Changing the variable in the double integral, one can ascertain that the product of two functions of the form (1.15) has the same form [11, p. 88, Theorem 11.5]. By induction, powers of functions of the form (1.15) are of the same form. Consequently, if Λ is even, then $\Lambda^{[k]}\in\widehat{CM}_c^2(y_0)$ for all $k\in\mathbb{N}$. If Λ is odd, then $(-1)^r\Lambda^{[2r]}\in\widehat{CM}_c^2(y_0)$ and $(-1)^r\Lambda^{[2r+1]}\in\widehat{CM}_s^2(y_0)$. Thus, by Lemma C, inequalities (3.8) are valid, whence, by Remark 3, condition W5 is fulfilled.

Thus, in both cases the assumptions of Theorem 2 are satisfied. $\square$

Remark 15. The condition $\alpha_\nu \geq 0$ holds, in particular, for the operator $(-1)^{(r-\varkappa_r)/2}\delta_\theta^r$, where $\varkappa_r$ is the parity of a number r, i.e., the remainder of dividing r by 2.

4. Estimates of approximations by linear combinations of seminorms of differences

In [2], the author generalized and strengthened the upper estimate of Lemma C in the form mentioned in the title of this section. In order to strengthen Theorems 1 and 2 in a similar way, we introduce new notation and recall the previous results from [2].

Let $y_0 > 0$, $G \in \widehat{CM}^2(y_0)$, $c(G) \in C^{(2)}(\mathbb{R})$, $c_0(y) = c(G, y)$ for every y: $|y| \geq y_0$, $0 < h < \frac{2\pi}{y_0}$. We denote $c_{h0}(y) = c_0(y)$ ($|y| \geq y_0$),

$$c_{hj}(y) = \frac{1}{2i}\sum_{s\in\mathbb{Z}}\left(c_{h,j-1}(y) - c_{h,j-1}\left(\frac{2\pi s}{h}\right)\right)\frac{(-1)^{s-1}}{\pi s - hy/2} \quad (|y| \geq y_0, j \in \mathbb{N}),$$

$$c_{hj}(0) = \sum_{s\in\mathbb{Z}\setminus\{0\}}(-1)^{s-1}c_{hj}\left(\frac{2\pi s}{h}\right) \quad (j \in \mathbb{Z}_+),$$

$$K_{hj,G}(t) = \begin{cases} \frac{2\pi}{h}\sum\limits_{s\in\mathbb{Z}} c_{hj}\left(\frac{2\pi s}{h}\right)e^{i\frac{2\pi s}{h}t}, & |t| \leq \frac{h}{2}, \\ 0, & |t| > \frac{h}{2}, \end{cases} \quad (j \in \mathbb{Z}_+).$$

The functions a_{hj} and b_{hj} are defined by the general convention (1.1).

If $c_0 \in CM_c^r(y_0)$, then $c_{h1} \in CM_s^r(y_0)$, and if $c_0 \in CM_s^r(y_0)$, then $-c_{h1} \in CM_c^r(y_0)$. The functions c_{hj} and $K_{hj} = K_{hj,G}$ have the same parity as c_0 for j even, and the parity opposite to that of c_0 for j odd. For every $m \in \mathbb{Z}_+$, there exists a function $N_{hm} = N_{hm,G} \in L(\mathbb{R})$ of the same parity as c_{hm} such that $c(N_{hm}) \in C^{(2)}(\mathbb{R})$ and $c(N_{hm}, y) = c_{hm}(y)$ for every y such that $|y| \geq y_0$. The functions K_{hj} do not change their sign on $\left(0, \frac{h}{2}\right)$ and are integrable. For every $m \in \mathbb{N}$, we have the expansions

$$G = \sum_{j=0}^{m-1}\delta_h^j K_{hj} + \delta_h^m N_{hm} + M_{hm},$$

$$f = T + \varphi * M_{hm} + \delta_h^m\varphi * N_{hm} + \sum_{j=0}^{m-1}\delta_h^j\varphi * K_{hj},$$

where $M_{hm} \in \mathbf{E}_{y_0} \cap L(\mathbb{R})$, and the functions f and φ are interrelated via (1.5).

Let $\sigma \geq y_0$. We put

$$T_{\sigma hm}G = M_{hm,G} + \delta_h^m L_\sigma N_{hm,G}, \qquad \mathcal{U}_{\sigma hm,G}f = T + \varphi * T_{\sigma hm}G,$$

$$A_{hj,G} = \begin{cases} \frac{2}{\pi}\left|\sum\limits_{s=0}^{\infty}\frac{1}{2s+1}b_{hj}\left(\frac{2\pi(2s+1)}{h}\right)\right| & \text{if the parity of } j \text{ is opposite to that of } G, \\ \left|\sum\limits_{s=1}^{\infty}(-1)^{s-1}a_{hj}\left(\frac{2\pi s}{h}\right)\right| & \text{if the parity of } j \text{ is equal to that of } G, \end{cases}$$

$$B_{\sigma hm,G} = \begin{cases} \frac{2}{\pi}\left|\sum\limits_{s=0}^{\infty}\frac{1}{2s+1}b_{hm}((2s+1)\sigma)\right| & \text{if the parity of } m \text{ is opposite to that of } G, \\ \frac{2}{\pi}\left|\sum\limits_{s=0}^{\infty}\frac{(-1)^s}{2s+1}a_{hm}((2s+1)\sigma)\right| & \text{if the parity of } m \text{ is equal to that of } G. \end{cases}$$

For the operators $\mathcal{U}_{\sigma hm,G}$ a remark analogous to Remark 1 is valid.

Theorem D [2, Theorem 1]**.** *Assume that* $(\mathfrak{M}, P) \in \mathcal{B}$, $\varphi \in \mathfrak{M}$, $y_0 > 0$, $G \in \widehat{CM}^2(y_0)$, $c(G) \in C^{(2)}(\mathbb{R})$, *functions* f *and* φ *are interrelated via* (1.5), $m \in \mathbb{N}$, $0 < h < \frac{2\pi}{y_0}$, $\sigma \geq y_0$. *Then*

$$P(f - \mathcal{U}_{\sigma hm,G}f) \leq A_{h0,G}P(\varphi) + \sum_{j=1}^{m-1}A_{hj,G}P\left(\delta_h^j\varphi\right) + B_{\sigma hm,G}P\left(\delta_h^m\varphi\right).$$

If, in addition, the kernel G *is odd, then* $A_{h0,G}P(\varphi)$ *can be replaced by* $A_{h0,G}\frac{\omega_1(\varphi,h)_P}{2}$, *where* $\omega_1(\varphi, h)_P = \sup\limits_{0\leq t\leq h} P\left(\delta_t^1\varphi\right)$ *is the modulus of continuity of the first order of the function* φ *with respect to the seminorm* P.

Here and below, we only present inequalities of the types (2.12), (2.13), and (3.9), i.e., estimates of deviations of linear approximation methods. Inequalities for the best approximations are deduced from the former in the standard way, as in Theorem 1.

In the case, where the function W is even and $\mathcal{S}$ is the identity operator, the results of this section were obtained in [3].

As above, we assume that a function Φ satisfies conditions $\Phi1$ and $\Phi2$, a function W satisfies conditions W1 and W3, $\Lambda \in \widehat{CM}^2(y_0)$, $c(\Lambda) \in C^{(2)}\big(\mathbb{R} \setminus (-y_0, y_0)\big)$, and the series

$$\sum_{k=1}^{\infty} |c_k| \|\mathcal{S}^k\| \Bigg(\sum_{j=0}^{m-1} 2^j A_{hj,\Lambda^{[k]}} + 2^m B_{\sigma hm,\Lambda^{[k]}} \Bigg) \tag{4.1}$$

converges. The convergence of the series (4.1) implies, first, that condition W2 is fulfilled and, second, that the series

$$\sum_{k=1}^{\infty} c_k\, \mathcal{S}^k \big(\Lambda^{[k]} - T_{\sigma hm} \Lambda^{[k]}\big) \tag{4.2}$$

absolutely converges in $L(\mathbb{R})$. We put

$$T_{\sigma hm} G = G - \sum_{k=1}^{\infty} c_k\, \mathcal{S}^k \big(\Lambda^{[k]} - T_{\sigma hm} \Lambda^{[k]}\big).$$

This definition implies that $T_{\sigma h0} G = T_\sigma G$. As in Sec. 2, it is readily seen that $T_{\sigma hm} G \in L(\mathbb{R}) \cap \mathbf{E}_\sigma$.

If $\varphi = \Phi(\mathcal{W}) f$ and $f - \Phi(\mathcal{W}) f$ is represented by (1.3), then we put

$$\mathcal{Z}_{\sigma hm,W,\Phi} f = f * Q + \varphi * T_{\sigma hm} G.$$

In this case,

$$f - \mathcal{Z}_{\sigma hm,W,\Phi} f = \varphi + \varphi * (G - T_{\sigma hm} G).$$

A remark analogous to Remark 1 is valid for the operators $\mathcal{Z}_{\sigma hm,W,\Phi}$ as well.

Applying Theorem D, we immediately come to the following result.

Theorem 4. *Assume that* $(\mathfrak{M}, P) \in \mathcal{B}$, $f \in \mathfrak{M}$, $y_0 > 0$, $\sigma \geq y_0$, $m \in \mathbb{N}$, $0 < h < \frac{2\pi}{y_0}$, *a function* Φ *satisfies conditions* $\Phi1$ *and* $\Phi2$, *a function* W *satisfies conditions* W1 *and* W3, $\Lambda \in \widehat{CM}^2(y_0)$, $c(\Lambda) \in C^{(2)}\big(\mathbb{R} \setminus (-y_0, y_0)\big)$, *the series* (4.1) *converges,* $\varphi = \Phi(\mathcal{W}) f$. *Then*

$$P(f - \mathcal{Z}_{\sigma hm,W,\Phi} f) \leq P(\varphi) + \sum_{k=1}^{\infty} |c_k| \Bigg(\sum_{j=0}^{m-1} A_{hj,\Lambda^{[k]}} P\big(\delta_h^j \mathcal{S}^k \varphi\big) + B_{\sigma hm,\Lambda^{[k]}} P\big(\delta_h^m \mathcal{S}^k \varphi\big) \Bigg).$$

Consider the special case of Theorem 4 where $\mathcal{S} = \pm \delta_h^r$. Recall that by condition W1, the parities of r and Λ coincide.

Corollary 1. *Assume that* $(\mathfrak{M}, P) \in \mathcal{B}$, $f \in \mathfrak{M}$, $y_0 > 0$, $\sigma \geq y_0$, $r, m \in \mathbb{N}$, $0 < h < \frac{2\pi}{y_0}$, *a function* Φ *satisfies conditions* $\Phi1$ *and* $\Phi2$, *a function* W *satisfies conditions* W1 *and* W3, *where* $\mathcal{S} = \pm \delta_h^r$, $\Lambda \in \widehat{CM}^2(y_0)$, $c(\Lambda) \in C^{(2)}\big(\mathbb{R} \setminus (-y_0, y_0)\big)$,

$$\sum_{k=1}^{\infty} |c_k| \Bigg(\sum_{j=0}^{m-1} 2^{rk+j} A_{hj,\Lambda^{[k]}} + 2^{rk+m} B_{\sigma hm,\Lambda^{[k]}} \Bigg) < +\infty,$$

$\varphi = \Phi(\mathcal{W}) f$. *Then*

$$P(f - \mathcal{Z}_{\sigma hm,W,\Phi} f) \leq P(\varphi) + \sum_{k=1}^{\infty} |c_k| \Bigg(\sum_{j=0}^{m-1} A_{hj,\Lambda^{[k]}} P\big(\delta_h^{rk+j} \varphi\big) + B_{\sigma hm,\Lambda^{[k]}} P\big(\delta_h^{rk+m} \varphi\big) \Bigg).$$

In [2, Lemma 13 and Corollary 5], for kernels $\Lambda \in CM^2(y_0)$ the author proved the relations

$$B_{\sigma,\frac{\pi}{\sigma},m,\Lambda} = 2^{-m} \Bigg(\mathcal{K}_{\sigma,\Lambda} - \sum_{j=0}^{m-1} 2^j A_{\frac{\pi}{\sigma},j,\Lambda} \Bigg), \qquad \mathcal{U}_{\sigma,\frac{\pi}{\sigma},m,\Lambda} = \mathcal{X}_{\sigma,\Lambda}.$$

Consequently,

$$\mathcal{Z}_{\sigma,\frac{\pi}{\sigma},m,W,\Phi} = \mathcal{Z}_{\sigma,W,\Phi}.$$

If the parameter θ in condition W1 also is equal to $\frac{\pi}{\sigma}$ (in particular, if $\mathcal{S}$ is the identity operator), then, by Remark 13,

$$\mathcal{Z}_{\sigma,\frac{\pi}{\sigma},m,W,\Phi} = \mathcal{Y}_{\sigma,W,\Phi}.$$

Thus, for $h = \frac{\pi}{\sigma}$, Theorem 4 and Corollary 1 take the following form.

Corollary 2. *Assume that* $(\mathfrak{M}, P) \in \mathcal{B}$, $f \in \mathfrak{M}$, $y_0 > 0$, $\sigma \geq y_0$, $m \in \mathbb{N}$, *a function* Φ *satisfies conditions* $\Phi1$ *and* $\Phi2$, *a function* W *satisfies conditions* W1 *and* W3, $\Lambda \in \widehat{CM}^2(y_0)$, $c(\Lambda) \in C^{(2)}(\mathbb{R} \setminus (-y_0, y_0))$, *for* $h = \frac{\pi}{\sigma}$ *the series* (4.1) *converges,* $\varphi = \Phi(\mathcal{W})f$. *Then*

$$P(f - \mathcal{Z}_{\sigma,W,\Phi} f) \leq P(\varphi) + \sum_{k=1}^{\infty} |c_k| \left(\sum_{j=0}^{m-1} A_{\frac{\pi}{\sigma},j,\Lambda^{[k]}} P\big(\delta_{\frac{\pi}{\sigma}}^{j} \mathcal{S}^k \varphi\big) + B_{\sigma,\frac{\pi}{\sigma},m,\Lambda^{[k]}} P\big(\delta_{\frac{\pi}{\sigma}}^{m} \mathcal{S}^k \varphi\big) \right). \tag{4.3}$$

Corollary 3. *Assume that* $(\mathfrak{M}, P) \in \mathcal{B}$, $f \in \mathfrak{M}$, $y_0 > 0$, $\sigma \geq y_0$, $r, m \in \mathbb{N}$, *a function* Φ *satisfies conditions* $\Phi1$ *and* $\Phi2$, *a function* W *satisfies conditions* W1 *and* W3, *where* $\mathcal{S} = \pm\delta_{\frac{\pi}{\sigma}}^{r}$, $\Lambda \in \widehat{CM}^2(y_0)$, $c(\Lambda) \in C^{(2)}(\mathbb{R} \setminus (-y_0, y_0))$,

$$\sum_{k=1}^{\infty} |c_k| \left(\sum_{j=0}^{m-1} 2^{rk+j} A_{\frac{\pi}{\sigma},j,\Lambda^{[k]}} + 2^{rk+m} B_{\sigma,\frac{\pi}{\sigma},m,\Lambda^{[k]}} \right) < +\infty,$$

$\varphi = \Phi(\mathcal{W})f$. *Then*

$$P(f - \mathcal{Y}_{\sigma,W,\Phi} f) \leq P(\varphi) + \sum_{k=1}^{\infty} |c_k| \left(\sum_{j=0}^{m-1} A_{\frac{\pi}{\sigma},j,\Lambda^{[k]}} P\big(\delta_{\frac{\pi}{\sigma}}^{rk+j} \varphi\big) + B_{\sigma,\frac{\pi}{\sigma},m,\Lambda^{[k]}} P\big(\delta_{\frac{\pi}{\sigma}}^{rk+m} \varphi\big) \right). \tag{4.4}$$

We denote $\eta_j(\varphi, h)_P = 2^{-j} P\big(\delta_h^j \varphi\big)$. As the sequence $\{\eta_j(\varphi, h)_P\}_{j=0}^{\infty}$ decreases, the limit

$$\eta_\infty(\varphi, h)_P = \lim_{j \to \infty} \eta_j(\varphi, h)_P$$

exists. The sequence $\{\eta_j(\varphi, h)_P\}_{j=0}^{\infty}$ being decreasing, it follows that the right-hand sides of (4.3) and (4.4) decrease as functions of m. Therefore, the best estimate is obtained at the limit as $m \to \infty$.

Corollary 4. *Assume that for a certain* $m \in \mathbb{N}$ *the assumptions of Corollary* 2 *are satisfied. Then*

$$\begin{aligned} &P(f - \mathcal{Z}_{\sigma,W,\Phi} f) \\ &\quad \leq P(\varphi) + \sum_{k=1}^{\infty} |c_k| \left(\sum_{j=0}^{\infty} 2^j A_{\frac{\pi}{\sigma},j,\Lambda^{[k]}}\, \eta_j \left(\mathcal{S}^k \varphi, \frac{\pi}{\sigma} \right)_P + \left(\mathcal{K}_{\sigma,\Lambda^{[k]}} - \sum_{j=0}^{\infty} 2^j A_{\frac{\pi}{\sigma},j,\Lambda^{[k]}} \right) \eta_\infty \left(\mathcal{S}^k \varphi, \frac{\pi}{\sigma} \right)_P \right). \end{aligned} \tag{4.5}$$

Corollary 5. *Assume that for a certain* $m \in \mathbb{N}$ *the assumptions of Corollary* 3 *are satisfied. Then*

$$\begin{aligned} &P(f - \mathcal{Y}_{\sigma,W,\Phi} f) \\ &\quad \leq P(\varphi) + \sum_{k=1}^{\infty} |c_k| 2^{rk} \left(\sum_{j=0}^{\infty} 2^j A_{\frac{\pi}{\sigma},j,\Lambda^{[k]}}\, \eta_{rk+j} \left(\varphi, \frac{\pi}{\sigma} \right)_P + \left(\mathcal{K}_{\sigma,\Lambda^{[k]}} - \sum_{j=0}^{\infty} 2^j A_{\frac{\pi}{\sigma},j,\Lambda^{[k]}} \right) \eta_\infty \left(\varphi, \frac{\pi}{\sigma} \right)_P \right). \end{aligned} \tag{4.6}$$

Corollaries 4 and 5 are obtained from Corollaries 2 and 3 by passing to the limit. In order to justify this passage, we can use the following variant of the Lebesgue dominated convergence theorem: If $\{a_{km}\}$ is a double sequence of nonnegative numbers, a_{km} decreases as a function of m for every $k \in \mathbb{N}$, and if $\sum_{k=1}^{\infty} a_{km_0} < +\infty$ for a certain $m_0 \in \mathbb{N}$, then

$$\lim_{m \to \infty} \sum_{k=1}^{\infty} a_{km} = \sum_{k=1}^{\infty} \lim_{m \to \infty} a_{km}.$$

Recall that the coefficients c_k and α_ν are defined by (2.1) and (2.2).

Remark 16. Inequalities (4.3) and (4.5) strengthen (2.12). If $\theta = \frac{\pi}{\sigma}$, $c_k \geq 0$ for every $k \in \mathbb{N}$, and $\alpha_\nu \geq 0$ for every $\nu \in \mathbb{Z}$, then they also strengthen inequality (3.9) and are sharp in the cases indicated in Theorem 2. If $c_k \geq 0$ for every $k \in \mathbb{N}$ and $\mathcal{S} = (-1)^{(r-\varkappa_r)/2} \delta_{\frac{\pi}{\sigma}}^{r}$, then, by Remark 15, these assertions also apply to inequalities (4.4) and (4.6).

5. Applications

With the exception of one example, in applying Theorem 4 and its corollaries, we restrict ourselves to writing inequalities explicitly for $m=0$ (in this case, they reduce to Theorems 1–3) and for $m=1$. Recall that

$$A_{\frac{\pi}{\sigma},0,G}=2\sum_{s=1}^{\infty}(-1)^{s-1}\frac{1-(\Phi\circ w)(2s\sigma)}{(\Phi\circ w)(2s\sigma)},\quad \mathcal{K}_{\sigma,G}=\frac{4}{\pi}\sum_{s=0}^{\infty}\frac{(-1)^s}{2s+1}\frac{1-(\Phi\circ w)((2s+1)\sigma)}{(\Phi\circ w)((2s+1)\sigma)}.$$

In the assertions of this section, we do not mention the sharpness of inequalities. All of the inequalities presented and the corresponding inequalities for the best approximations are sharp in the cases listed in Theorem 2.

By Φ_q we denote the principal branch of the power function $z\mapsto(1-z)^q$. The function Φ_q is holomorphic in the unit disk $B(0,1)$. The coefficients

$$C_{k+q-1}^{q-1}=\frac{q(q+1)\dots(q+k-1)}{k!}$$

(if $k=0$, then the fraction is considered as 1) of the expansion

$$\frac{1}{(1-z)^q}=\sum_{k=0}^{\infty}C_{k+q-1}^{q-1}\,z^k,\qquad |z|<1,$$

are positive for every $q>0$. We note that if $|w(y)|<1$ for all $y:|y|\ge y_0$, then condition $\Phi2$ allows us not to assume that q is an integer.

5.1. Let $\lambda>0$ and let

$$W_\lambda(t)=\int_{-\infty}^{+\infty}e^{-\lambda y^2}e^{ity}\,dy=\frac{\sqrt{\pi}}{\sqrt{\lambda}}\,e^{-\frac{t^2}{4\lambda}}$$

be the Weierstrass kernel (the heat conduction kernel). Clearly, $W_\lambda\in\widehat{CM}_c^2(y_0)$ for every $y_0>0$. The convolution $\mathcal{W}_\lambda f=f*W_\lambda$ is the Weierstrass integral of a function f. As $y_0>0$ is arbitrary, the estimates are valid for every $\sigma>0$.

Corollary 6. *Assume that* $(\mathfrak{M},P)\in\mathcal{B}$, $f\in\mathfrak{M}$, $\lambda,\sigma,q>0$, $\varphi=(I-\mathcal{W}_\lambda)^qf$. *Then*

$$P\big(f-\mathcal{Y}_{\sigma,W_\lambda,\Phi_q}f\big)\le\left(1+2\sum_{s=1}^{\infty}(-1)^{s-1}\frac{1-\big(1-e^{-\lambda(2s\sigma)^2}\big)^q}{\big(1-e^{-\lambda(2s\sigma)^2}\big)^q}\right)P(\varphi)$$
$$+\left(\frac{2}{\pi}\sum_{s=0}^{\infty}\frac{(-1)^s}{2s+1}\frac{1-\big(1-e^{-\lambda((2s+1)\sigma)^2}\big)^q}{\big(1-e^{-\lambda((2s+1)\sigma)^2}\big)^q}-\sum_{s=1}^{\infty}(-1)^{s-1}\frac{1-\big(1-e^{-\lambda(2s\sigma)^2}\big)^q}{\big(1-e^{-\lambda(2s\sigma)^2}\big)^q}\right)P\big(\delta^1_{\frac{\pi}{\sigma}}\varphi\big),$$

$$P\big(f-\mathcal{Y}_{\sigma,W_\lambda,\Phi_q}f\big)\le\left(1+\frac{4}{\pi}\sum_{s=0}^{\infty}\frac{(-1)^s}{2s+1}\frac{1-\big(1-e^{-\lambda((2s+1)\sigma)^2}\big)^q}{\big(1-e^{-\lambda((2s+1)\sigma)^2}\big)^q}\right)P(\varphi).$$

5.2. Let $\lambda>0$ and let

$$P_\lambda(t)=\int_{-\infty}^{+\infty}e^{-\lambda|y|}e^{ity}\,dy=\frac{2\lambda}{\lambda^2+t^2}$$

be the Poisson kernel. Clearly, $P_\lambda\in\widehat{CM}_c^1(y_0)$ for every $y_0>0$. The convolution $\mathcal{P}_\lambda f=f*P_\lambda$ is the Poisson integral of a function f. As $y_0>0$ is arbitrary, the estimates are valid for every $\sigma>0$.

Corollary 7. *Assume that* $(\mathfrak{M},P)\in\mathcal{B}$, $f\in\mathfrak{M}$, $\lambda,\sigma,q>0$, $\varphi=(I-\mathcal{P}_\lambda)^qf$. *Then*

$$P\big(f-\mathcal{Y}_{\sigma,P_\lambda,\Phi_q}f\big)\le\left(1+2\sum_{s=1}^{\infty}(-1)^{s-1}\frac{1-(1-e^{-2\lambda\sigma s})^q}{(1-e^{-2\lambda\sigma s})^q}\right)P(\varphi)$$
$$+\left(\frac{2}{\pi}\sum_{s=0}^{\infty}\frac{(-1)^s}{2s+1}\frac{1-(1-e^{-(2s+1)\lambda\sigma})^q}{(1-e^{-(2s+1)\lambda\sigma})^q}-\sum_{s=1}^{\infty}(-1)^{s-1}\frac{1-(1-e^{-2\lambda\sigma s})^q}{(1-e^{-2\lambda\sigma s})^q}\right)P\big(\delta^1_{\frac{\pi}{\sigma}}\varphi\big),$$

$$P\big(f-\mathcal{Y}_{\sigma,P_\lambda,\Phi_q}f\big)\le\left(1+\frac{4}{\pi}\sum_{s=0}^{\infty}\frac{(-1)^s}{2s+1}\frac{1-(1-e^{-\lambda(2s+1)\sigma})^q}{(1-e^{-\lambda(2s+1)\sigma})^q}\right)P(\varphi).$$

5.3. Let $\lambda > 0$,

$$\Theta_\lambda(t) = \int\limits_{-\infty}^{+\infty} \frac{e^{ity}}{\operatorname{ch} \lambda y}\, dy = \frac{\pi}{\lambda \operatorname{ch} \frac{\pi t}{2\lambda}},$$

$\mathcal{T}_\lambda f = f * \Theta_\lambda$. As is known [2, formula (7.10)], $\Theta_\lambda \in \widehat{CM}_c^2(y_0)$ for every $y_0 > 0$. The kernel Θ_λ arises in describing the classes of functions analytic in the strip $\{z : |\operatorname{Im} z| < \lambda\}$ [1, Sec. 110, pp. 267–268]. As $y_0 > 0$ is arbitrary, the estimates are valid for every $\sigma > 0$.

Corollary 8. *Assume that* $(\mathfrak{M}, P) \in \mathcal{B}$, $f \in \mathfrak{M}$, $\lambda, \sigma, q > 0$, $\varphi = (I - \mathcal{T}_\lambda)^q f$. *Then*

$$P\big(f - \mathcal{Y}_{\sigma,\Theta_\lambda,\Phi_q} f\big) \le \left(1 + 2\sum_{s=1}^{\infty} (-1)^{s-1} \frac{1 - \big(1 - \frac{1}{\operatorname{ch} 2\lambda\sigma s}\big)^q}{\big(1 - \frac{1}{\operatorname{ch} 2\lambda\sigma s}\big)^q}\right) P(\varphi)$$
$$+ \left(\frac{2}{\pi}\sum_{s=0}^{\infty} \frac{(-1)^s}{2s+1} \frac{1 - \big(1 - \frac{1}{\operatorname{ch} \lambda(2s+1)\sigma}\big)^q}{\big(1 - \frac{1}{\operatorname{ch} \lambda(2s+1)\sigma}\big)^q} - \sum_{s=1}^{\infty} (-1)^{s-1} \frac{1 - \big(1 - \frac{1}{\operatorname{ch} 2\lambda\sigma s}\big)^q}{\big(1 - \frac{1}{\operatorname{ch} 2\lambda\sigma s}\big)^q}\right) P\big(\delta_{\frac{\pi}{\sigma}}^1 \varphi\big),$$

$$P\big(f - \mathcal{Y}_{\sigma,\Theta_\lambda,\Phi_q} f\big) \le \left(1 + \frac{4}{\pi}\sum_{s=0}^{\infty} \frac{(-1)^s}{2s+1} \frac{1 - \big(1 - \frac{1}{\operatorname{ch} \lambda(2s+1)\sigma}\big)^q}{\big(1 - \frac{1}{\operatorname{ch} \lambda(2s+1)\sigma}\big)^q}\right) P(\varphi).$$

For $q = 1$, Corollaries 6–8 were established in [3]. In particular, for $q = 1$ Corollaries 6 and 7 strengthen the inequalities from [12, p. 285, Corollary 2 and p. 276, Corollary 1].

5.4. Let $x_+ = x$ if $x \ge 0$, $x_+ = 0$ if $x < 0$. Let $N > 0$, $\beta \ge 2$,

$$F_{N\beta}(t) = \int_{-N}^{N} \left(1 - \frac{|y|}{N}\right)^\beta e^{ity}\, dy,$$

$\mathcal{F}_{N\beta} f = f * F_{N\beta}$. The function $c(F_{N\beta}, y) = \Big(1 - \frac{|y|}{N}\Big)_+^\beta$ is even and three times monotone on $[0, +\infty)$ in the sense of Remark 4. Consequently, for every $y_0 > 0$, we may apply Theorem 3. For $\sigma \ge N$, the estimate of the type (1.2) with the constant 1 is trivial because $F_{N\beta} \in \mathbf{E}_N$. For this reason, we state the result for $N > \sigma$ only.

Corollary 9. *Assume that* $(\mathfrak{M}, P) \in \mathcal{B}$, $f \in \mathfrak{M}$, $\sigma, q > 0$, $\beta \ge 2$, $N > \sigma$, $\varphi = (I - \mathcal{F}_{N\beta})^q f$. *Then*

$$P\big(f - \mathcal{Y}_{\sigma,F_{N\beta},\Phi_q} f\big) \le \left(1 + \frac{4}{\pi}\sum_{s=0}^{\infty} \frac{(-1)^s}{2s+1} \frac{1 - \Big(1 - \big(1 - \frac{(2s+1)\sigma}{N}\big)_+^\beta\Big)^q}{\Big(1 - \big(1 - \frac{(2s+1)\sigma}{N}\big)_+^\beta\Big)^q}\right) P(\varphi).$$

5.5. In this section, in writing sums we use the logical notation: $[A] = 1$ if an assertion A is true; $[A] = 0$ if A is false. The parity of a number r is denoted by $\varkappa_r$;

$$\mathcal{K}_r = \frac{4}{\pi}\sum_{s=0}^{\infty} \frac{(-1)^{s(r+1)}}{(2s+1)^{r+1}}$$

are the Favard constants; $I_r \in L(\mathbb{R})$ is a nonperiodic analog of the Bernoulli kernel, i.e., a kernel such that $c(I_r) \in C^{(2)}(\mathbb{R})$ and $c(I_r, y) = (iy)^{-r}$ for $|y| \ge y_0 > 0$. Clearly, $(-1)^{(r-\varkappa_r)/2} I_r \in \widehat{CM}^1(y_0)$. If $\sigma \ge y_0$, then $\mathcal{K}_{\sigma,I_r} = \mathcal{K}_r \sigma^{-r}$. This is another form of the classical Akhiezer–Krejn–Favard inequality [1, Sec. 101, pp. 237–241],

$$P\big(f - \mathcal{X}_{\sigma,I_r} f\big) \le \frac{\mathcal{K}_r}{\sigma^r} P\big(f^{(r)}\big),$$

which can be obtained by combining Lemmas A and B or Lemmas A and C. As $\mathcal{K}_{\sigma,I_r}$ does not depend on y_0, this parameter is not indicated in the notation of the kernel I_r.

Assume that $\theta > 0$, $r \in \mathbb{N}$, $\mathcal{S}_\theta^r$ is the Steklov operator of order r with step θ, i.e.,

$$\mathcal{S}_\theta^r f = \delta_\theta^r f^{(-r)},$$

where $f^{(-r)}$ denotes an arbitrary rth primitive of f. In other terms, it is the convolution operator with the kernel $\Psi_{\theta r}$, whose Fourier transform is $y \mapsto \Big(\frac{2}{\theta y} \sin \frac{\theta y}{2}\Big)^r$.

In [6], operators of the form

$$\mathcal{W}f(x) = \sum_{j\in\mathbb{N},\, i\in\mathbb{Z}} A_{ij}\mathcal{S}^r_{\lambda_j\theta} f(x+\beta_i\theta)$$

were studied; here, θ, A_{ij}, λ_j, β_i are number parameters, which satisfy some conditions, and $\sum\limits_{i\in\mathbb{Z},\, j\in\mathbb{N}} |A_{ij}| < +\infty$. Now we consider those operators whose kernels satisfy condition W1. One of the kernels, I_r, acts as Λ; the explicit form of a function V plays no role; $y_0 > 0$ is arbitrary.

Assume that $r\in\mathbb{N}$, $\theta > 0$, $\{A_{\xi\eta}\}_{\xi\in\mathbb{N},\,\eta\in\mathbb{Z}}$ is a double sequence,

$$\sum_{\xi\in\mathbb{N},\,\eta\in\mathbb{Z}} |A_{\xi\eta}| < +\infty.$$

For r even, let

$$\mathcal{W}_{\theta r} f(x) = \sum_{\xi\in\mathbb{N},\,\eta\in\mathbb{Z}} A_{\xi\eta}\mathcal{S}^r_{\xi\theta} f(x-\eta\theta)$$

and assume that $A_{\xi\eta} = A_{\xi,-\eta}$. Here and below, a symbol of the form $\xi\theta$ in the subscript of the Steklov operator means product, not the double indexing. For r odd, let

$$\mathcal{W}_{\theta r} f(x) = \sum_{\xi\in\mathbb{N},\,\eta\in\mathbb{Z}} A_{\xi\eta}\mathcal{S}^r_{\xi\theta} f\left(x - \left(\eta + \tfrac{1-\varkappa_\xi}{2}\right)\theta\right)$$

and assume that $A_{\xi\eta} = A_{\xi,-\eta}$ for ξ odd and $A_{\xi\eta} = A_{\xi,-\eta-1}$ for ξ even.

The operator $\mathcal{W}_{\theta r}$ is a convolution operator. We denote its kernel by $W_{\theta r}$, and the Fourier transform of the kernel is denoted by $w_{\theta r}$. In both cases, we have

$$\mathcal{W}_{\theta r} f = \frac{1}{\theta^r}\mathcal{V}_{\theta r} f^{(-r)},$$

where $\mathcal{V}_{\theta r}$ is a summator operator. Let us write its explicit expression. For r even,

$$\mathcal{V}_{\theta r} f(x) = \sum_{\xi\in\mathbb{N},\,\eta\in\mathbb{Z}} \frac{A_{\xi\eta}}{\xi^r}\delta^r_{\xi\theta} f(x-\eta\theta) = \sum_{\substack{\xi\in\mathbb{N},\eta\in\mathbb{Z},\\ l\in[0:r]}} (-1)^l C^l_r \frac{A_{\xi\eta}}{\xi^r} f\left(x - \eta\theta + \left(\tfrac{r}{2}-l\right)\xi\theta\right) = \sum_{\nu\in\mathbb{Z}} (-1)^\nu B_{r\nu} f(x-\nu\theta),$$

$$B_{r\nu} = \sum_{\substack{\xi\in\mathbb{N},\eta\in\mathbb{Z},\\ l\in[0:r]}} (-1)^{l+\nu} C^l_r \frac{A_{\xi\eta}}{\xi^r}\left[\eta + \left(l-\tfrac{r}{2}\right)\xi = \nu\right].$$

For r odd,

$$\mathcal{V}_{\theta r} f(x) = \sum_{\xi\in\mathbb{N},\,\eta\in\mathbb{Z}} \frac{A_{\xi\eta}}{\xi^r}\delta^r_{\xi\theta} f\left(x - \left(\eta+\tfrac{1-\varkappa_\xi}{2}\right)\theta\right) = \sum_{\substack{\xi\in\mathbb{N},\,\eta\in\mathbb{Z},\\ l\in[0;r]}} (-1)^l C^l_r \frac{A_{\xi\eta}}{\xi^r} f\left(x - \left(\eta+\tfrac{1-\varkappa_\xi}{2}\right)\theta + \left(\tfrac{r}{2}-l\right)\xi\theta\right)$$

$$= \sum_{\nu\in\mathbb{Z}} (-1)^{\nu+1} B_{r\nu} f(x-(\nu+\tfrac{1}{2})\theta),$$

$$B_{r\nu} = \sum_{\substack{\xi\in\mathbb{N},\,\eta\in\mathbb{Z},\\ l\in[0:r]}} (-1)^{l+\nu+1} C^l_r \frac{A_{\xi\eta}}{\xi^r}\left[\left(\eta - \tfrac{\varkappa_\xi}{2}\right) + \left(l-\tfrac{r}{2}\right)\xi = \nu\right].$$

On the other hand, since

$$\delta^r_{\xi\theta} f(x) = \sum_{\mu\in[0:\xi-1]^r} \delta^r_\theta f\left(x + \theta\sum_{d=1}^r \left(\mu_d - \tfrac{\xi-1}{2}\right)\right),$$

from the operator $\mathcal{V}_{\theta r}$ the difference factor δ^r_θ can be singled out,

$$\mathcal{V}_{\theta r} = \delta^r_\theta \mathcal{U}_{\theta r},$$

where

$$\mathcal{U}_{\theta r} f(x) = \sum_{\nu\in\mathbb{Z}} (-1)^\nu D_{r\nu} f(x-\nu\theta),$$

and, for r even,

$$D_{r\nu} = (-1)^\nu \sum_{\substack{\xi\in\mathbb{N},\,\eta\in\mathbb{Z},\\ \mu\in[0:\xi-1]^r}} \frac{A_{\xi\eta}}{\xi^r}\Bigl[\eta - \sum_{d=1}^{r}\Bigl(\mu_d - \tfrac{\xi-1}{2}\Bigr) = \nu\Bigr],$$

for r odd,

$$D_{r\nu} = (-1)^\nu \sum_{\substack{\xi\in\mathbb{N},\,\eta\in\mathbb{Z},\\ \mu\in[0:\xi-1]^r}} \frac{A_{\xi\eta}}{\xi^r}\Bigl[\eta + \tfrac{1-\varkappa_\xi}{2} - \sum_{d=1}^{r}\Bigl(\mu_d - \tfrac{\xi-1}{2}\Bigr) = \nu\Bigr].$$

Recall the estimates of a seminorm of a summator operator:

$$N_P\bigl(\mathcal{V}_{\theta r}^k\bigr) \le \|\mathcal{V}_{\theta r}^k\| = \sum_{\mu\in\mathbb{Z}}\Biggl|\sum_{\substack{\nu\in\mathbb{Z}^k,\\ \nu_1+\ldots+\nu_k=\mu}} B_{r\nu_1}\ldots B_{r\nu_k}\Biggr| \le \Bigl(\sum_{\nu\in\mathbb{Z}}|B_{r\nu}|\Bigr)^k.$$

Similar inequalities with $B_{r\nu}$ changed for $D_{r\nu}$ are valid for $N_P\bigl(\mathcal{U}_{\theta r}^k\bigr)$. The middle and right-hand terms of those estimates do not depend on θ, and so in the notation of norms the symbol θ is omitted, e.g., $\bigl\|\mathcal{V}_r\bigr\|$.

Corollary 10. *Assume that* $(\mathfrak{M}, P) \in \mathcal{B}$, $f \in \mathfrak{M}$, $r \in \mathbb{N}$, $\theta, \sigma, q > 0$, $\varphi = (I - \mathcal{W}_{\theta r})^q f$.

1. *If*

$$\sum_{k=0}^{\infty} C_{k+q-1}^{q-1} 2^{rk}\|\mathcal{U}_r^k\|\frac{\mathcal{K}_{rk}}{(\sigma\theta)^{rk}} < +\infty,$$

then

$$P(f - \mathcal{Z}_{\sigma,W_{\theta r},\Phi_q} f) \le \sum_{k=0}^{\infty} C_{k+q-1}^{q-1}\|\mathcal{U}_r^k\|\frac{\mathcal{K}_{rk}}{(\sigma\theta)^{rk}}\, P(\delta_\theta^{rk}\varphi). \tag{5.1}$$

2. *If*

$$\sum_{k=0}^{\infty} C_{k+q-1}^{q-1}\|\mathcal{V}_r^k\|\frac{\mathcal{K}_{rk}}{(\sigma\theta)^{rk}} < +\infty,$$

then

$$P(f - \mathcal{Z}_{\sigma,W_{\theta r},\Phi_q} f) \le \Bigl(\sum_{k=0}^{\infty} C_{k+q-1}^{q-1}\|\mathcal{V}_r^k\|\frac{\mathcal{K}_{rk}}{(\sigma\theta)^{rk}}\Bigr)P(\varphi). \tag{5.2}$$

Remark 17. If $D_{r\nu} \ge 0$ for every $\nu \in \mathbb{Z}$, then $(-1)^{(r-\varkappa_r)/2}B_{r\nu} \ge 0$ for every $\nu \in \mathbb{Z}$, and $\|\mathcal{V}_r\| = 2^r\|\mathcal{U}_r\|$. In this case, inequality (5.1) strengthens (5.2).

Corollary 11. *Assume that* $(\mathfrak{M}, P) \in \mathcal{B}$, $f \in \mathfrak{M}$, $r \in \mathbb{N}$, $\sigma, q > 0$, $\varphi = \bigl(I - \mathcal{W}_{\frac{\pi}{\sigma},r}\bigr)^q f$.

1. *If*

$$\sum_{k=0}^{\infty} C_{k+q-1}^{q-1} 2^{rk}\|\mathcal{U}_r^k\|\frac{\mathcal{K}_{rk}}{\pi^{rk}} < +\infty,$$

then

$$P\bigl(f - \mathcal{Y}_{\sigma,W_{\frac{\pi}{\sigma},r},\Phi_q} f\bigr) \le \sum_{k=0}^{\infty} C_{k+q-1}^{q-1}\|\mathcal{U}_r^k\|\frac{\mathcal{K}_{rk}}{\pi^{rk}}\, P\bigl(\delta_{\frac{\pi}{\sigma}}^{rk}\varphi\bigr). \tag{5.3}$$

2. *If*

$$\sum_{k=0}^{\infty} C_{k+q-1}^{q-1}\|\mathcal{V}_r^k\|\frac{\mathcal{K}_{rk}}{\pi^{rk}} < +\infty, \tag{5.4}$$

then

$$P\bigl(f - \mathcal{Y}_{\sigma,W_{\frac{\pi}{\sigma},r},\Phi_q} f\bigr) \le \Bigl(\sum_{k=0}^{\infty} C_{k+q-1}^{q-1}\|\mathcal{V}_r^k\|\frac{\mathcal{K}_{rk}}{\pi^{rk}}\Bigr)P(\varphi). \tag{5.5}$$

3. *If* $(-1)^{(r-\varkappa_r)/2}B_{r\nu} \ge 0$ *for every* $\nu \in \mathbb{Z}$, *then condition* (5.4) *is fulfilled,*

$$P\bigl(f - \mathcal{Y}_{\sigma,W_{\frac{\pi}{\sigma},r},\Phi_q} f\bigr) \le \Biggl(1 + \frac{4}{\pi}\sum_{s=0}^{\infty}\frac{(-1)^s}{2s+1}\frac{1 - \bigl(1 - w_{\frac{\pi}{\sigma},r}((2s+1)\sigma)\bigr)^q}{\bigl(1 - w_{\frac{\pi}{\sigma},r}((2s+1)\sigma)\bigr)^q}\Biggr)P(\varphi), \tag{5.6}$$

the right-hand sides of (5.5) *and* (5.6) *are equal, and these inequalities are sharp in the cases indicated in Theorem* 2.

Consider a particular case of Corollary 4 as an example.

Corollary 12. *Assume that* $(\mathfrak{M}, P) \in \mathcal{B}$, $f \in \mathfrak{M}$, $r \in \mathbb{N}$, $\sigma, q > 0$, *for a certain* $m \in \mathbb{N}$,

$$\sum_{k=1}^{\infty} C_{k+q-1}^{q-1} 2^{rk} \|\mathcal{U}_r^k\| \left(\sum_{j=0}^{m-1} 2^j A_{\frac{\pi}{\sigma},j,I_r^{[k]}} + \left(\frac{\mathcal{K}_{rk}}{\sigma^{rk}} - \sum_{j=0}^{m-1} 2^j A_{\frac{\pi}{\sigma},j,I_r^{[k]}} \right) \right) < +\infty,$$

$\varphi = \big(I - \mathcal{W}_{\frac{\pi}{\sigma},r}\big)^q f$. *Then*

$$P\big(f - \mathcal{Y}_{\sigma, W_{\frac{\pi}{\sigma},r},\Phi_q} f\big) \le P(\varphi) + \sum_{k=1}^{\infty} C_{k+q-1}^{q-1} 2^{rk} \|\mathcal{U}_r^k\|$$
$$\times \left(\sum_{j=0}^{\infty} 2^j A_{\frac{\pi}{\sigma},j,I_r^{[k]}} \eta_{rk+j} \left(\varphi, \frac{\pi}{\sigma}\right)_P + \left(\frac{\mathcal{K}_{rk}}{\sigma^{rk}} - \sum_{j=0}^{\infty} 2^j A_{\frac{\pi}{\sigma},j,I_r^{[k]}} \right) \eta_\infty \left(\varphi, \frac{\pi}{\sigma}\right)_P \right). \tag{5.7}$$

If $D_{r\nu} \ge 0$ for every $\nu \in \mathbb{Z}$, then inequality (5.7) strengthens (5.3) and, a fortiori, by Remark 17, it strengthens (5.4).

Let us state the results for the Steklov operators separately.

Corollary 13. *Assume that* $(\mathfrak{M}, P) \in \mathcal{B}$, $f \in \mathfrak{M}$, $r \in \mathbb{N}$, $\sigma, q > 0$, $\theta > \frac{2}{\sigma}$, $\varphi = \big(I - \mathcal{S}_\theta^r\big)^q f$. *Then*

$$P\big(f - \mathcal{Z}_{\sigma,\Psi_{\theta r},\Phi_q} f\big) \le \sum_{k=0}^{\infty} C_{k+q-1}^{q-1} \frac{\mathcal{K}_{rk}}{(\sigma\theta)^{rk}} P\big(\delta_\theta^{rk} \varphi\big).$$

In particular,

$$P\big(f - \mathcal{Y}_{\sigma,\Psi_{\frac{\pi}{\sigma},r},\Phi_q} f\big) \le \sum_{k=0}^{\infty} C_{k+q-1}^{q-1} \frac{\mathcal{K}_{rk}}{\pi^{rk}} P\big(\delta_{\frac{\pi}{\sigma}}^{rk} \varphi\big), \tag{5.8}$$

$$P\big(f - \mathcal{Y}_{\sigma,\Psi_{\frac{\pi}{\sigma},r},\Phi_q} f\big) \le \left(\sum_{k=0}^{\infty} C_{k+q-1}^{q-1} \frac{2^{rk}\mathcal{K}_{rk}}{\pi^{rk}} \right) P(\varphi). \tag{5.9}$$

Inequalities (5.8) *and* (5.9) *are sharp in the cases indicated in Theorem* 2.

Remark 18. We present two particular cases of (5.9) for $q = 1$ and $r = 1, 2$:

$$P\big(f - \mathcal{Y}_{\sigma,\Psi_{\frac{\pi}{\sigma},1},\Phi_1} f\big) \le (\operatorname{tg} 1 + \sec 1) \cdot P\big(f - \mathcal{S}_{\frac{\pi}{\sigma}}^1 f\big), \tag{5.10}$$

$$P\big(f - \mathcal{Y}_{\sigma,\Psi_{\frac{\pi}{\sigma},2},\Phi_1} f\big) \le \sec 1 \cdot P\big(f - \mathcal{S}_{\frac{\pi}{\sigma}}^2 f\big). \tag{5.11}$$

In order to obtain (5.10) and (5.11), we use the relations

$$\sum_{k=0}^{\infty} \mathcal{K}_k z^k = \operatorname{tg} \frac{\pi z}{2} + \sec \frac{\pi z}{2}, \qquad \sum_{k=0}^{\infty} \mathcal{K}_{2k} z^{2k} = \sec \frac{\pi z}{2},$$

which hold for $|z| < 1$. Usually, these expansions are written in terms of the Bernoulli and Euler numbers [13, formulas 1.411.5, 1.411.9], in terms of which the Favard constants can be expressed [13, formulas 0.233.5, 0.233.6].

Inequality (5.10) for the best approximations of periodic functions, together with the assertion about its sharpness, is contained in [5, Remark 3]. For $q \in \mathbb{N}$, the counterparts of estimates (5.1)–(5.3) and (5.5) for semiadditive functionals and, in particular, for the best approximations were obtained in [6, Theorems 2–5 and Corollary 4]. In [7, Theorems 1 and 2], for $q = 1$ and the operator

$$\mathcal{W}_{\theta,2,m} = \frac{2}{C_{2m}^m} \sum_{\xi=1}^{m} (-1)^{\xi-1} C_{2m}^{m-\xi} \mathcal{S}_{\xi\theta}^2$$

inequalities (5.5) and (5.6) were obtained by a linear method, and their sharpness was established. For $m = 1$, the operator $\mathcal{W}_{\theta,2,m}$ coincides with $\mathcal{S}_\theta^2$, and inequality (5.11) is a particular case of these estimates.

In comparison with the previous results, in the present paper, the estimates are extended to noninteger exponents and to expansions into differences of various orders; they are obtained by linear methods, and, in some cases, their sharpness is proved.

Translated by O. L. Vinogradov.

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