Journal of Mathematical Sciences, Vol. 179, No. 3, December, 2011

A SPECTRAL PROBLEM WITH INTEGRAL CONDITIONS

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Abstract. In this paper, we consider an ordinary differential operator with integral conditions. We obtain an a priori estimate of the solutions and prove the sector structure and the discreteness of the spectrum of this operator.

Ordinary differential equations with nonlocal conditions have been studied in [2–4, 7–17]. The conditions considered are of multi-point or integral type, or, in a more general case, are stated in terms of the Stieltjes integral. An interesting problem in this connection is that about the conditions on the coefficients and the nonlocal terms that ensure the spectrum of the corresponding operator to be discrete (see [16]). The case of nonlocal conditions involving the values of the unknown function and its derivatives at the end-points of the interval $(0, 1)$ (i.e., the Stieltjes integral with a nontrivial discrete measure of the points 0 and 1) has been studied most thoroughly. This is connected with the fact that nonlocal operators can be reduced to a sum of small and compact perturbations of boundary operators (see [11]). The results of these studies include a priori estimates of solutions, proof of the discreteness of the spectrum and the completeness of the system of root functions, and also asymptotic formulas for eigenvalues and eigenfunctions [3, 4, 11–14].

As it turned out, the most difficult case is that of purely integral conditions, which is mentioned in [14] as an unsolved problem. In that case, the differential operator with integral conditions has a domain which is not dense in $L_2(0, 1)$. Originally, constructive sufficient conditions for the discreteness of the spectrum and a priori estimates of solutions of such a problem for a second-order ordinary differential operator were obtained in [2, 10]. Asymptotic formulas for eigenvalues and eigenfunctions of differential operators with integral conditions were obtained in [9] in the case of smooth weight functions in the integral conditions, under an additional assumption. The problem of completeness of the system of eigenfunctions and associated functions of a differential operator with integral conditions is examined in [7, 8].

In the present paper, we study a second-order ordinary differential operator with a spectral parameter and integral conditions containing derivatives of the unknown functions. The results obtained pertain to the Fredholm solvability of the problem, its spectrum, and a priori estimates of its solutions for sufficiently large values of the parameter. The methods of this investigation are those proposed in [2, 10, 11].

1. Setting of the Problem

Consider the equation

$$Au + \lambda^2 u = -a_0(t)u''(t) + a_1(t)u'(t) + a_2(t)u(t) + \lambda^2 u(t) = f_0(t) \quad (t \in (0, 1)) \tag{1.1}$$

with the integral condition

$$B_\rho u = \int_0^1 e_\rho(t)u'(t)\, dt = f_\rho \quad (\rho = 1, 2). \tag{1.2}$$

Here, a_i $(i = 0, 1, 2)$ are real-valued functions, $a_0 \geq k > 0$ $(t \in [0, 1])$ and $a_1, a_2 \in C[0, 1]$; $f_0 \in L_2(0, 1)$ is a complex-valued function, $f_\rho \in \mathbb{C}$ $(\rho = 1, 2)$ are constants; $\lambda \in \mathbb{C}$ is a spectral parameter; e_ρ are linearly independent real-valued functions.

Translated from Trudy Seminara imeni I. G. Petrovskogo, No. 28, Part I, pp. 147–160, 2011.

In order to formulate the conditions that should be satisfied for the functions a_0 and e_ρ, we introduce the following spaces:

- $C^\alpha[a,b]$ is the Hölder space of continuous functions $v(t)$, $t \in [a,b]$, such that $|v(t_1) - v(t_2)| \le c|t_1 - t_2|^\alpha$ $(t_1, t_2 \in [a,b])$, where the constants $c \in (0,\infty)$ and $\alpha \in (0,1]$ do not depend on t_1 and t_2;
- $C^\alpha_\beta[0,1] = \{v \in L_2(0,1)\colon v \in C^\alpha[0,\beta],\ v \in C^\alpha[1-\beta,1]\}$ $(\beta \in (0,1/2))$;
- $W^1_\infty(a,b)$ is the space of absolutely continuous functions $v(t)$, $t \in [a,b]$, such that $v' \in L_\infty(a,b)$;
- $W^1_{\infty,\beta}(0,1) = \{v \in C[0,1]\colon v \in W^1_\infty(0,\beta),\ v \in W^1_\infty(1-\beta,1)\}$ $(\beta \in (0,1/2))$.

Assume that $a_0 \in W^1_{\infty,\beta}(0,1)$ and $e_\rho \in C^\alpha_\beta[0,1]$ $(\alpha \in (1/2,1])$.

In what follows, we consider the set

$$\omega_\varepsilon = \{\gamma \in \mathbb{C}\colon |\arg\gamma| \le \varepsilon \text{ or } |\arg\gamma - \pi| \le \varepsilon\}, \quad 0 < \varepsilon < \frac{\pi}{2},$$

where

$$\omega_{\varepsilon,q} = \{\gamma \in \omega_\varepsilon\colon |\gamma| \ge q\}, \quad \Delta_e = \begin{vmatrix} e_1(0) & e_1(1) \\ e_2(0) & e_2(1) \end{vmatrix}.$$

Let $\mathcal{W}[0,1] := L_2(0,1) \times \mathbb{C} \times \mathbb{C}$. In the Sobolev space $W^2(0,1)$ and in the space $\mathcal{W}[0,1]$ we introduce the respective equivalent norms depending on the parameter λ:

$$|\!|\!| u |\!|\!|_{W^2(0,1)} = \left(\|u\|^2_{W^2(0,1)} + |\lambda|^4 \|u\|^2_{L_2(0,1)}\right)^{1/2}, \tag{1.3}$$

$$|\!|\!| f |\!|\!|_{\mathcal{W}[0,1]} = \left(\|f_0\|^2_{L_2(0,1)} + |\lambda|^3(|f_1|^2 + |f_2|^2)\right)^{1/2}, \tag{1.4}$$

where $f = (f_0, f_1, f_2)$, $|\lambda| \ge 1$.

Consider a bounded linear operator

$$L(\lambda)\colon W^2(0,1) \to \mathcal{W}[0,1]$$

of the form

$$L(\lambda)u = (Au + \lambda^2 u, B_1 u, B_2 u).$$

Let us also define an unbounded operator

$$\mathcal{A}_B\colon \mathcal{D}(\mathcal{A}_B) \subset L_2(0,1) \to L_2(0,1)$$

with the domain

$$\mathcal{D}(\mathcal{A}_B) = W^2_B(0,1) := \{u \in W^2(0,1)\colon B_\rho u = 0,\ \rho = 1,2\}$$

by the formula $\mathcal{A}_B u = Au$ $(u \in \mathcal{D}(\mathcal{A}_B))$. It is easy to see that the operator $\mathcal{A}_B$ is not densely defined in $L_2(0,1)$, which prevents the utilization of the methods based on the construction of a conjugate operator.

2. Statement of the Main Result

Theorem 2.1. *Let $\Delta_e \ne 0$. Then, for any $\varepsilon \in (0,\pi/2)$, there is $q > 1$ such that for $\lambda \in \omega_{\varepsilon,q}$ any $u \in W^2(0,1)$ satisfies the inequality*

$$|\!|\!| u |\!|\!|_{W^2(0,1)} \le C|\lambda|^{1/2}\, |\!|\!| L(\lambda)u |\!|\!|_{\mathcal{W}[0,1]}, \tag{2.1}$$

where the constant $C > 0$ does not depend on λ, u.

The proof of this theorem is given in Secs. 3, 4.

Theorem 2.1 implies the following result.

Theorem 2.2. *Let $\Delta_e \ne 0$. Then the following statements hold*:

(a) *the operators*

$$L(\lambda)\colon W^2(0,1) \to \mathcal{W}[0,1], \quad \mathcal{A}_B\colon \mathcal{D}(A_B) \subset L_2(0,1) \to L_2(0,1)$$

are of Fredholm type and $\operatorname{ind} L(\lambda) = \operatorname{ind} \mathcal{A}_B = 0$ *for all* $\lambda \in \mathbb{C}$;

(b) *the spectrum* $\sigma(\mathcal{A}_B)$ *is discrete;*

(c) *for* $\mu \notin \sigma(\mathcal{A}_B)$, *the resolvent*

$$R(\mu, \mathcal{A}_B) = (\mu I - \mathcal{A}_B)^{-1}\colon L_2(0,1) \to L_2(0,1)$$

is a compact operator;

(d) *for any* $0 < \delta < \pi$, *all points of the spectrum* $\sigma(\mathcal{A}_B)$ *(possibly, except for finitely many) belong to the angle* $|\arg \eta| < \delta$ *on the complex plane.*

Proof. I. Consider problem (1.1), (1.2) for $f = 0$ and $\lambda \in \omega_{\varepsilon,q}$. Using the inequality (2.1), we find that $u = 0$. Therefore, the eigenvalues of the operator $\mathcal{A}_B$ do not belong to the set

$$\Omega_{\delta,r} = \{\mu \in \mathbb{C}\colon |\arg \mu - \pi| \le \pi - \delta,\ |\mu| \ge r\},$$

where $\delta = \pi - 2\varepsilon$, $r = q^2$.

It is not difficult to see that if $\mu = -\lambda^2$ is not an eigenvalue of the operator $\mathcal{A}_B$, then the nonhomogeneous problem (1.1), (1.2) has a unique solution for any $f \in \mathcal{W}[0,1]$. This, together with the inequality (2.1), implies that the operator

$$L(\lambda)\colon W^2(0,1) \to \mathcal{W}[0,1]$$

admits a bounded inverse

$$L^{-1}(\lambda)\colon \mathcal{W}[0,1] \to W^2(0,1)$$

for $\lambda \in \omega_{\varepsilon,q}$. Therefore, the operator

$$\mu I - \mathcal{A}_B\colon \mathcal{D}(\mathcal{A}_B) \subset L_2(0,1) \to L_2(0,1)$$

has a bounded inverse

$$(\mu I - \mathcal{A}_B)^{-1}\colon L_2(0,1) \to W^2(0,1)$$

for $\mu \in \Omega_{\delta,r}$. Since $W^2(0,1)$ is compactly imbedded into $L_2(0,1)$, it follows from the theorem about a compact resolvent (see [5, Chap. 3, Sec. 6]) that the spectrum $\sigma(\mathcal{A}_B)$ is discrete. Note that

$$\sigma(\mathcal{A}_B) \subset \{\mu \in \mathbb{C}\colon |\arg \mu| < \delta\} \cup \{\mu \in \mathbb{C}\colon |\mu| < r\}. \tag{2.2}$$

II. Let $\lambda_0 \in \omega_{\varepsilon,q}$. We have

$$L(\lambda) = \big(I + [L(\lambda) - L(\lambda_0)]L^{-1}(\lambda_0)\big)L(\lambda_0)$$

for any $\lambda \in \mathbb{C}$, where I is the identity operator in $\mathcal{W}[0,1]$. It is easy to see that $\big(L(\lambda) - L(\lambda_0)\big)u = \big((\lambda^2 - \lambda_0^2)u, 0, 0\big)$. Since $W^2(0,1)$ is compactly imbedded into $L_2(0,1)$, the operator

$$\big(L(\lambda) - L(\lambda_0)\big)L^{-1}(\lambda_0)\colon \mathcal{W}[0,1] \to \mathcal{W}[0,1]$$

is compact. Then, from (2.2) and [6, Theorem 12.2], it follows that

$$L(\lambda)\colon W^2(0,1) \to \mathcal{W}[0,1]$$

is of Fredholm type and $\operatorname{ind} L(\lambda) = 0$.

In a similar way, it is not difficult to prove that the operator

$$\mathcal{A}_B\colon \mathcal{D}(\mathcal{A}_B) \subset L_2(0,1) \to L_2(0,1)$$

is of Fredholm type and $\operatorname{ind} \mathcal{A}_B = 0$. The theorem is proved. □

Remark 2.3. Note that in contrast to the classical a priori estimates of solutions of "local" boundary value problems with a parameter (see [1]), the estimate (2.1) contains the coefficient $|\lambda|^{1/2}$. Moreover, the norm in the right-hand side of (2.1) defined by (1.4) contains the term $|\lambda|^3(|f_1|^2 + |f_2|^2)$, and for the second boundary value problem this term has the form $|\lambda|(|f_1|^2 + |f_2|^2)$. For equation (1.1) with integral conditions containing the unknown function, the corresponding term in the right-hand side of the a priori estimate takes the form $|\lambda|^4(|f_1|^2 + |f_2|^2)$ (see [3]).

3. Existence of Solutions of the Model Problem

In order to prove Theorem 2.1, we first examine the following auxiliary problem:

$$A_0 u + \lambda^2 u = -p u''(t) + \lambda^2 u(t) = f_0(t) \quad (t \in (0,1)), \tag{3.1}$$

$$e_\rho(0) \int\limits_0^{1/2} u'(t)\,dt + e_\rho(1) \int\limits_{1/2}^{1} u'(t)\,dt = f_\rho \quad (\rho = 1,2), \tag{3.2}$$

where $p > 0$ is a constant, $f_0 \in L_2(0,1)$, $f_\rho \in \mathbb{C}$.

Lemma 3.1. *Let $\Delta_e \neq 0$. Then for any $\varepsilon \in (0, \pi/2)$ there is $q_0 > 1$ such that for each $\lambda \in \omega_{\varepsilon, q_0}$ any $u \in W^2(0,1)$ satisfies the inequality*

$$\|u\|_{W^2(0,1)} \leq K \|f\|_{\mathcal{W}[0,1]}, \tag{3.3}$$

where $f = (f_0, f_1, f_2)$ and $K > 0$ does not depend on λ, u.

Proof. I. Let us rewrite problem (3.1), (3.2) in the form

$$A_0 u + \lambda^2 u = -p u''(t) + \lambda^2 u(t) = f_0(t) \quad (t \in (0,1)), \tag{3.4}$$

$$e_\rho(0) \left[u\left(\frac{1}{2}\right) - u(0) \right] + e_\rho(1) \left[u(1) - u\left(\frac{1}{2}\right) \right] = f_\rho \quad (\rho = 1,2). \tag{3.5}$$

Consider condition (3.5), which can be expressed as the following system:

$$\begin{cases} -e_1(0)u(0) + e_1(1)u(1) = f_1 - \big(e_1(0) - e_1(1)\big) u\left(\frac{1}{2}\right), \\ -e_2(0)u(0) + e_2(1)u(1) = f_2 - \big(e_2(0) - e_2(1)\big) u\left(\frac{1}{2}\right). \end{cases} \tag{3.6}$$

From system (3.6), we can express $u(0)$ and $u(1)$ in terms of $e_\rho(0)$, $e_\rho(1)$, f_ρ $(\rho = 1,2)$ and $u(1/2)$. The determinant of this system has the form

$$\bar{\Delta} = \begin{vmatrix} -e_1(0) & e_1(1) \\ -e_2(0) & e_2(1) \end{vmatrix} = -\Delta_e \neq 0.$$

Therefore, the system of linear algebraic equations for $u(0)$ and $u(1)$ has a unique solution, which can be found by Cramer's rule

$$u(0) = u\left(\frac{1}{2}\right) + \frac{(f_2 e_1(1) - f_1 e_2(1))}{\Delta_e}, \quad u(1) = u\left(\frac{1}{2}\right) + \frac{(f_2 e_1(0) - f_1 e_2(0))}{\Delta_e}.$$

Thus, the nonlocal boundary conditions (3.5) take the form

$$\begin{cases} u(0) - u\left(\frac{1}{2}\right) = \hat{f}_1, \\ u(1) - u\left(\frac{1}{2}\right) = \hat{f}_2, \end{cases} \tag{3.7}$$

where

$$\hat{f}_1 = \frac{(f_2 e_1(1) - f_1 e_2(1))}{\Delta_e}, \quad \hat{f}_2 = \frac{(f_2 e_1(0) - f_1 e_2(0))}{\Delta_e}.$$

Obviously, we have $\hat{f}_\rho \in \mathbb{C}$ $(\rho = 1,2)$.

II. Let us introduce the operator

$$L_0(\lambda) \colon W^2(0,1) \to \mathcal{W}[0,1]$$

by

$$L_0(\lambda)u = \big((A_0+\lambda^2 I)u, B_1^0 u, B_2^0 u\big),$$

where $B_1^0 u = u(0)-u(1/2)$, $B_2^0 u = u(1)-u(1/2)$.

Problem (3.4), (3.7) can be regarded as a multi-point boundary value problem. To this problem we can apply Lemma 1.1.1 and Theorem 1.1.1 from [11]. Therefore, in the case of $\Delta_e \neq 0$, for any $\varepsilon \in (0,\pi/2)$ there is $q_0 = q_0(\varepsilon) > 1$ such that for $\lambda \in \omega_{\varepsilon,q_0}$ the operator $L_0(\lambda)$ has a bounded inverse,

$$L_0^{-1}(\lambda)\colon \mathcal{W}[0,1] \to W^2(0,1),$$

and any $u \in W^2(0,1)$ satisfies the inequality

$$\|u\|_{W^2(0,1)} \le K\,\|L_0(\lambda)u\|_{\mathcal{W}[0,1]}\,, \tag{3.8}$$

where $K>0$ does not depend on λ and u.

Since problem (3.4), (3.7) is equivalent to problem (3.1), (3.2), the inequality (3.8) implies (3.3). Thus, the a priori estimate for the model problem is proved. □

4. Proof of Theorem 2.1

I. First, we transform equation (1.1), so that the coefficient of the second derivative becomes equal to a constant in a sufficiently small neighborhood of the points 0 and 1. To this end, we introduce a new variable τ, setting

$$\tau = \frac{1}{d}\int_0^t \frac{d\xi}{b(\xi)},$$

where

$$d = \int_0^1 \frac{d\xi}{b(\xi)}, \qquad b(\xi) = \begin{cases} \sqrt{a_0(\xi)}, & \xi \in [0,\beta]\cup[1-\beta,1],\\ \frac{(1-\xi-\beta)\sqrt{a_0(\beta)}+(\xi-\beta)\sqrt{a_0(1-\beta)}}{1-2\beta}, & \xi \in (\beta,1-\beta).\end{cases}$$

By assumption, we have $a_0 \in W^1_{\infty,\beta}(0,1)$ and $a_0 \ge k > 0$. Therefore,

$$0 < k_1 \le \frac{1}{b(t)} \le \frac{1}{k^{1/2}}, \qquad \frac{1}{b} \in W^1_{\infty,\beta}(0,1),$$

Thus, the function $\tau=\tau(t)$ maps the interval $[0,1]$ onto itself and

$$\frac{d\tau}{dt} = \frac{1}{db(t)} \ge k_1 k^{1/2} > 0, \qquad \frac{d\tau}{dt} \in W^1_{\infty,\beta}(0,1).$$

The substitution $\tau=\tau(t)$ reduces problem (1.1), (1.2) to

$$-\bar a_0(\tau)\bar u''(\tau) + \bar a_1(\tau)\bar u'(\tau) + \bar a_2(\tau)\bar u(\tau) + \lambda^2 \bar u(\tau) = \bar f_0(\tau) \quad (\tau \in (0,1)), \tag{4.1}$$

$$\int_0^1 \bar e_\rho(\tau)\bar u'(\tau)\,d\tau = f_\rho \quad (\rho = 1,2), \tag{4.2}$$

where

$$\bar u(\tau) = u\big(t(\tau)\big), \quad \bar f_0(\tau) = f_0\big(t(\tau)\big),$$

$$\bar a_0(\tau) = \frac{a_0(t(\tau))}{d^2 b^2(t(\tau))} \in W^1_{\infty,\beta_0}(0,1), \quad \bar a_1(\tau) = \frac{a_1(t(\tau))}{db(t(\tau))} + \frac{a_0(t(\tau))b'(t(\tau))}{db^2(t(\tau))} \in L_\infty(0,1),$$

$$\bar a_2(\tau) = a_2\big(t(\tau)\big) \in C[0,1], \quad \bar e_\rho(\tau) = e_\rho\big(t(\tau)\big) \in C^\alpha_{\beta_0}[0,1], \quad \beta_0 = \min\left\{\frac{1}{d}\int_0^\beta \frac{d\xi}{b(\xi)}, 1-\frac{1}{d}\int_0^{1-\beta}\frac{d\xi}{b(\xi)}\right\},$$

and $\bar a_0(\tau) = \frac{1}{d^2}$ for $\tau \in [0,\beta_0]\cup[1-\beta_0,1]$.

It is easy to see that the inequality (2.1) is invariant with respect to the transformation $\tau = \tau(t)$.

We have

$$\Delta_{\bar{e}} = \begin{vmatrix} \bar{e}_1(0) & \bar{e}_1(1) \\ \bar{e}_2(0) & \bar{e}_2(1) \end{vmatrix} = \begin{vmatrix} e_1(0) & e_1(1) \\ e_2(0) & e_2(1) \end{vmatrix} = \Delta_e \neq 0.$$

Therefore, without loss of generality, it can be assumed that $a_0 \in W^1_{\infty,\beta}(0,1)$, $a_i \in L_\infty(0,1)$ $(i = 1,2)$, $e_\rho \in C^\alpha_\beta[0,1]$ $(\rho = 1,2)$, $a_0(t) \geq k > 0$, $a_0(t) = a_0(0)$ $(t \in [0,\beta] \cup [1-\beta,1])$ and $\Delta_e \neq 0$.

II. To establish the inequality (2.1), we first obtain an a priori estimate of solutions of problem (1.1), (1.2) on the closed interval $[\delta/3, 1-\delta/3]$, where $\delta = \delta(\lambda) > 0$. Consider a truncating function $\zeta \in C^\infty(\mathbb{R})$ such that $0 \leq \zeta(t) \leq 1$, $\zeta(t) = 1$ for $|t| < 1/4$, $\zeta(t) = 0$ for $|t| > 1/3$. For each $\lambda \in \omega_{\varepsilon,q_1}$, we introduce the function

$$\eta(t) = \zeta\left(\frac{t}{\delta}\right) + \zeta\left(\frac{t-1}{\delta}\right),$$

where $q_1 = \max\{q_0, \beta^{-3}\}$, $\delta = \frac{1}{|\lambda|^{1/3+2\gamma}}$ and $\frac{1-\alpha}{6\alpha+3} < \gamma < \frac{1}{12}$. Note that the condition $\alpha \in (1/2, 1]$ implies the inequality $\frac{1-\alpha}{6\alpha+3} < \frac{1}{12}$.

It is not difficult to see that

$$|\eta^{(j)}(t)| \leq k_2 \delta^{-j} \quad (t \in \mathbb{R},\ j = 1,2), \tag{4.3}$$

where $k_2 > 0$ does not depend on t and δ. Since $|\lambda| \geq q_1$, we have $\delta < q_1^{-1/3} \leq \beta$.

In view of the theorem from [1, Chap. 1, Sec. 1] about a priori estimates of solutions of "local" boundary value problems with a parameter, taking into account that the functions a_i $(i = 0,1,2)$ are bounded and using the inequality (4.3) and the Leibnitz formula, we conclude that for any $\varepsilon > 0$ there is $q_2 \geq q_1$ such that for all $\lambda \in \omega_{\varepsilon,q_2}$ any solution $u \in W^2(0,1)$ of problem (1.1), (1.2) satisfies the inequality

$$\begin{aligned} |\!|\!|(1-\eta)u|\!|\!|_{W^2(0,1)} &\leq k_3 \big\|(A+\lambda^2 I)\big((1-\eta)u\big)\big\|_{L_2(0,1)} \\ &\leq k_4\Big(\|(A+\lambda^2 I)u\|_{L_2(0,1)} + \delta^{-2}\|u\|_{L_2(0,1)} + \delta^{-1}\big(\|u'\|_{L_2(0,1)} + \|u\|_{L_2(0,1)}\big)\Big) \\ &\leq k_5\big(\|(A+\lambda^2 I)u\|_{L_2(0,1)} + |\lambda|^{2/3+4\gamma}\|u\|_{L_2(0,1)} + |\lambda|^{1/3+2\gamma}\|u\|_{W^1(0,1)}\big). \end{aligned}$$

Here and in what follows, the constants $k_j > 0$ do not depend on λ and u.

Further, using the known interpolation inequality (see [1, Chap. 1, Sec. 1])

$$|\lambda|\|u\|_{W^1(0,1)} \leq c(\|u\|_{W^2(0,1)} + |\lambda|^2\|u\|_{L_2(0,1)}), \tag{4.4}$$

we find that

$$\begin{aligned} |\!|\!|(1-\eta)u|\!|\!|_{W^2(0,1)} &\leq k_6\big(\|(A+\lambda^2 I)u\|_{L_2(0,1)} + |\lambda|^{2\gamma-2/3}\|u\|_{W^2(0,1)} + (|\lambda|^{2/3+4\gamma} + |\lambda|^{4/3+2\gamma})\|u\|_{L_2(0,1)}\big) \\ &\leq k_7\Big(\|(A+\lambda^2 I)u\|_{L_2(0,1)} + |\lambda|^{2\gamma-2/3}\big(\|u\|_{W^2(0,1)} + |\lambda|^2\|u\|_{L_2(0,1)}\big)\Big). \end{aligned}$$

It follows that

$$|\!|\!|(1-\eta)u|\!|\!|_{W^2(0,1)} \leq k_8\big(|\!|\!|f|\!|\!|_{\mathcal{W}[0,1]} + |\lambda|^{-\sigma}\,|\!|\!|u|\!|\!|_{W^2(0,1)}\big), \tag{4.5}$$

where $\sigma = 2/3 - 2\gamma > 0$.

III. Now we obtain an a priori estimate for solutions of problem (1.1), (1.2) near the end-points of the interval $(0,1)$.

We introduce operators A_0 and A_1 by $A_0 u = -a_0(0)u''(t)$ and $A_1 u = -a_0(t)u''(t)$ $(t \in (0,1))$, respectively. Set $p = a_0(0)$ in equation (3.1). Then, by Lemma 3.1 we have

$$|\!|\!|\eta u|\!|\!|_{W^2(0,1)} \leq K\left(\|(A_0+\lambda^2 I)(\eta u)\|_{L_2(0,1)} + |\lambda|^{3/2}\sum_\rho\left|e_\rho(0)\int_0^{1/2}\big(\eta(t)u(t)\big)'\,dt + e_\rho(1)\int_{1/2}^{1}\big(\eta(t)u(t)\big)'\,dt\right|\right).$$

Arguing as in II, we find that

$$\|(A_0+\lambda^2 I)(\eta u)\|_{L_2(0,1)} \le k_9\big(\|(A+\lambda^2 I)u\|_{L_2(0,1)} + |\lambda|^{-\sigma}\,\|u\|_{W^2(0,1)} + \|(A_1-A_0)(\eta u)\|_{L_2(0,1)}\big).$$

From the definition of the function $\eta(t)$ and the inequality $\delta<\beta$, it follows that $\operatorname{supp}\eta \subset [0,\beta]\cup[1-\beta,1]$. But $a_0(t)=a_0(0)$ $(t\in[0,\beta]\cup[1-\beta,1])$. Therefore, $\|(A_1-A_0)(\eta u)\|_{L_2(0,1)}=0$.

We further have

$$|\lambda|^{3/2}\left|e_\rho(0)\int_0^{1/2}\big(\eta(t)u(t)\big)'\,dt + e_\rho(1)\int_{1/2}^{1}\big(\eta(t)u(t)\big)'\,dt\right|$$

$$\le |\lambda|^{3/2}\left|e_\rho(0)\int_0^{1/2}\eta(t)u'(t)\,dt + e_\rho(1)\int_{1/2}^{1}\eta(t)u'(t)\,dt\right| + k_{10}|\lambda|^{11/6+2\gamma}\|u\|_{L_2(0,1)}$$

$$\le |\lambda|^{3/2}\left|\int_0^1 e_\rho(t)u'(t)\,dt\right| + k_{10}|\lambda|^{-1/6+2\gamma}|\lambda|^2\|u\|_{L_2(0,1)} + |\lambda|^{3/2}(|I_{\rho1}|+|I_{\rho2}|+|I_{\rho3}|),$$

where

$$I_{\rho1}=\int_0^1 e_\rho(t)(\eta(t)-1)u'(t)\,dt,$$

$$I_{\rho2}=\int_0^{1/2}\big(e_\rho(0)-e_\rho(t)\big)\eta(t)u'(t)\,dt,\quad I_{\rho3}=\int_{1/2}^{1}\big(e_\rho(1)-e_\rho(t)\big)\eta(t)u'(t)\,dt.$$

Thus, we come to the inequality

$$\|\eta u\|_{W^2(0,1)} \le k_{11}\left(\|f\|_{\mathcal{W}[0,1]} + |\lambda|^{-\sigma}\,\|u\|_{W^2(0,1)} + |\lambda|^{-1/6+2\gamma}\,\|u\|_{W^2(0,1)} + |\lambda|^{3/2}\sum_\rho(|I_{\rho1}|+|I_{\rho2}|+|I_{\rho3}|)\right).$$

Let us estimate the integrals $I_{\rho j}$ $(j=1,2,3)$. To estimate $I_{\rho1}$ we introduce a new truncating function $\zeta_1\in C^\infty(\mathbb{R})$ such that $0\le\zeta_1(t)\le1$, $\zeta_1(t)=1$ for $|t|<1/8$, $\zeta(t)=0$ for $|t|>1/4$. For any $\lambda\in\omega_{\varepsilon,q_2}$ consider the function

$$\eta_1(t)=\zeta_1\left(\frac{t}{\delta}\right)+\zeta_1\left(\frac{t-1}{\delta}\right).$$

By analogy with (4.3), it is easy to see that

$$|\eta_1^{(j)}(t)|\le k_{12}\delta^{-j}\quad (t\in\mathbb{R},\ j=1,2). \tag{4.6}$$

In view of the a priori estimate for solutions of a "local" boundary value problem with a parameter, taking into account the inequalities (4.4), (4.6) and the Leibnitz formula, by analogy with II, we find that

$$|\lambda|^{3/2}|I_{\rho1}| \le k_{13}|\lambda|^{1/2}\big\|(A+\lambda^2 I)\big((1-\eta_1)u\big)\big\|_{L_2(0,1)} \le k_{14}|\lambda|^{1/2}\big(\|f\|_{\mathcal{W}[0,1]} + |\lambda|^{-\sigma}\,\|u\|_{W^2(0,1)}\big).$$

Since $\operatorname{supp}\eta\cap[0,1/2]\subset(0,\delta/3)$ and $e_\rho\in C^\alpha[0,\delta]$, using (4.4), we obtain the inequality

$$|\lambda|^{3/2}|I_{\rho2}| \le |\lambda|^{3/2}\left(\int_0^{\delta/3}|e_\rho(0)-e_\rho(t)|^2\,dt\right)^{1/2}\|u'\|_{L_2(0,1)} \le k_{15}|\lambda|^{3/2}\left(\int_0^{\delta/3}t^{2\alpha}\,dt\right)^{1/2}\|u'\|_{L_2(0,1)}$$

$$\le k_{16}|\lambda|^{3/2}(\delta^{2\alpha+1})^{1/2}\|u'\|_{L_2(0,1)} \le k_{16}|\lambda|^{3/2-(2\gamma+1/3)(\alpha+1/2)}\|u\|_{W^1(0,1)}$$

$$\le k_{15}|\lambda|^{1/2-(2\gamma+1/3)(\alpha+1/2)}\big(\|u\|_{W^2(0,1)}+|\lambda|^2\|u\|_{L_2(0,1)}\big) \le k_{17}|\lambda|^{1/2-(2\gamma+1/3)(\alpha+1/2)}\,\|u\|_{W^2(0,1)}.$$

In a similar manner, we find that

$$|\lambda|^{3/2}|I_{\rho 3}| \le k_{18}|\lambda|^{1/2-(2\gamma+1/3)(\alpha+1/2)} \|u\|_{W^2(0,1)}.$$

As a result,

$$|\lambda|^{3/2}\sum_{\rho}(|I_{\rho 1}|+|I_{\rho 2}|+|I_{\rho 3}|) \le k_{19}\big(|\lambda|^{1/2}\|f\|_{\mathcal{W}[0,1]}+|\lambda|^{1/2-\sigma}\|u\|_{W^2(0,1)}+|\lambda|^{1/2-(2\gamma+1/3)(\alpha+1/2)}\|u\|_{W^2(0,1)}\big).$$

Note that $1/2-\sigma=-1/6+2\gamma$. Then, for $\|\eta u\|_{W^2(0,1)}$ we have the following estimate:

$$\|\eta u\|_{W^2(0,1)} \le k_{20}\big(|\lambda|^{1/2}\|f\|_{\mathcal{W}[0,1]}+|\lambda|^{-\varkappa}\|u\|_{W^2(0,1)}\big),, \tag{4.7}$$

where

$$\varkappa=\min\left\{\frac{1}{6}-2\gamma, -\frac{1}{2}+\left(2\gamma+\frac{1}{3}\right)\left(\alpha+\frac{1}{2}\right)\right\}.$$

From the condition

$$\frac{1-\alpha}{6\alpha+3}<\gamma<\frac{1}{12},$$

it follows that $\varkappa>0$.

From (4.5) and (4.7), we get

$$\|u\|_{W^2(0,1)} \le k_{21}\big(|\lambda|^{1/2}\|f\|_{\mathcal{W}[0,1]}+|\lambda|^{-\varkappa}\|u\|_{W^2(0,1)}\big). \tag{4.8}$$

Therefore, taking $q>q_2$ such that $k_{21}q^{-\varkappa}<1$, we obtain the inequality (2.1) from (4.8). The proof is complete.

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