Journal of Mathematical Sciences, Vol. 179, No. 3, December, 2011

HÖLDER CONTINUITY OF SOLUTIONS OF PARABOLIC EQUATIONS WITH VARIABLE NONLINEARITY EXPONENT

Yu. A. Alkhutov and V. V. Zhikov UDC 517.946

Abstract. In this paper, we prove the Hölder continuity of solutions of parabolic equations containing the $p(x,t)$-Laplacian. The degree p must satisfy the so-called logarithmic condition.

1. Introduction

Let $Q_T = \Omega \times (0,T)$ be a cylinder whose base is a bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 1$. In Q_T, consider the equation

$$\frac{\partial u}{\partial t} - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0 \tag{1.1}$$

with variable exponent $p = p(x,t)$ satisfying the condition

$$1 < \alpha \leq p(x,t) \leq \beta < \infty. \tag{1.2}$$

There exist fairly detailed studies of the corresponding elliptic equations with variable nonlinearity exponent, especially in the case of the variable exponent $p(x)$ having a logarithmic continuity modulus in the closure of the domain Ω. This condition, discovered by X. Fan and V. V. Zhikov, ensures the density of smooth functions in the space of solutions [1], the property of increased summability of the gradient of the solution [2], Hölder continuity [3–5]) and some other properties of solutions [6,7]. Moreover, this condition plays an important role in the theory of Sobolev spaces with variable summability exponent. Parabolic equations of the form (1.1) and systems of such equations arise in various problems of mathematical physics (see [8]). Such equations have been studied much less. Thorough analysis has been carried out for the case $p(x,t) = \text{const}$, in which Hölder continuity and other properties of solutions are established in [9] (see also [10]). For the equations with $p(x,t)$ having a logarithmic continuity modulus:

$$|p(x,t) - p(y,\tau)| \leq \frac{c_0}{\ln \frac{1}{|t-\tau|+|x-y|}} \quad \text{for any} \quad (x,t),\ (y,\tau) \in Q_T, \quad |t-\tau| + |x-y| \leq \frac{1}{2}, \tag{1.3}$$

the property of increased summability of the gradient of the solution is proved in [11]. Partial regularity of solutions and increased summability of the gradient of solutions for systems of the form (1.1) whose exponent $p(x,t)$ is Lipschitz continuous in x and Hölder continuous in t are established in [12].

In the present paper, we study the problem of Hölder continuity of solutions. A solution is understood in the local sense in Q_T without any boundary or initial conditions. To define a solution, we introduce the class $W_{\text{loc}}(Q_T)$ that consists of functions $u(x,t)$ such that

$$u \in L^2_{\text{loc}}(Q_T), \quad u(\cdot,t) \in W^{1,1}_{\text{loc}}(\Omega) \ \text{ for almost all } \ t \in [0,T], \quad |\nabla u|^p \in L^1_{\text{loc}}(Q_T),$$

where ∇ is the gradient in the spatial variable.

Convergence in $W_{\text{loc}}(Q_T)$ is defined as follows: $u_j \to u$, if

$$u_j \to u \text{ for } u_j \to u \text{ in } L^2_{\text{loc}}(Q_T), \quad |\nabla u_j - \nabla u|^p \to 0 \text{ in } L^1_{\text{loc}}(Q_T). \tag{1.4}$$

There are different types of solutions of equation (1.1), which are defined next.

Translated from Trudy Seminara imeni I. G. Petrovskogo, No. 28, Part I, pp. 8–74, 2011.

Definition 1.1. A function $u \in W_{\text{loc}}(Q_T)$ is a W-solution of equation (1.1), if $u(\cdot,t)\colon (0,T) \to L^2_{\text{loc}}(\Omega)$ is a locally continuous function on $(0,T)$ and the integral identity

$$\int\limits_{Q_T} \left(-u\frac{\partial \varphi}{\partial t} + |\nabla u|^{p-2}\nabla u \cdot \nabla \varphi \right) dx\, dt = 0 \tag{1.5}$$

holds for any test function $\varphi \in W_{\text{loc}}(Q_T)$ such that

$$\frac{\partial \varphi}{\partial t} \in L^2_{\text{loc}}(Q_T), \quad \varphi = 0 \text{ in a neighborhood of } \partial Q_T. \tag{1.6}$$

On a par with $W_{\text{loc}}(Q_T)$, we introduce the class $H_{\text{loc}}(Q_T)$ generated by smooth functions. We say that $u \in H_{\text{loc}}(Q_T)$, if $u \in W_{\text{loc}}(Q_T)$ and there is a sequence $u_j \in C^\infty(Q_T)$ such that $u_j \to u$ in $W_{\text{loc}}(Q_T)$. If no constraints other than (1.2) are imposed on the exponent $p(x,t)$, then $W_{\text{loc}}(Q_T) \neq H_{\text{loc}}(Q_T)$, in general, and it makes sense to consider another class of solutions of equation (1.1).

Definition 1.2. A function $u \in H_{\text{loc}}(Q_T)$ is an H-solution of equation (1.1), if $u(\cdot,t)\colon (0,T) \to L^2_{\text{loc}}(\Omega)$ is a locally continuous function on $(0,T)$ and the integral identity (1.5) holds for any test function $\varphi \in H_{\text{loc}}(Q_T)$ satisfying the condition (1.6).

It would be interesting to find out the conditions ensuring that a W-solution of the above equation is its H-solution, which is equivalent to $H_{\text{loc}}(Q_T)$ coinciding with $W_{\text{loc}}(Q_T)$. For this purpose, we introduce the logarithmic condition (1.3). Lemma 2.2 from Sec. 2 shows that this condition ensures that the classes $W_{\text{loc}}(Q_T)$ and $H_{\text{loc}}(Q_T)$ coincide. In this case smooth functions are dense in $W_{\text{loc}}(Q_T)$ in the sense of convergence (1.4). In particular, a W-solution is also an H-solution. This statement is a version of a previous result established by X. Fan and V. V. Zhikov (see [1]) about the density of smooth functions in the corresponding Sobolev–Orlicz space.

The logarithmic condition (1.3) also ensures some other properties of the solutions. If is known (see [11]) that under the condition (stronger than (1.2))

$$\max\left(\frac{2n}{n+2}, 1\right) < \alpha \le p(x,t) \le \beta < \infty, \tag{1.7}$$

together with the logarithmic condition (1.3), any solution of equation (1.1) is locally bounded in Q_T. If (1.7) does not hold, then a solution may be unbounded in any subdomain of Q_T (see [9]) even for a constant exponent p. Let us formulate the main result.

Theorem 1. *Under the conditions* (1.3), (1.7), *any solution of equation* (1.1) *is Hölder continuous in* Q_T.

The proof of Theorem 1 is based on a modification of the technique used in [9,10], where $p(x,t) \equiv \text{const}$ and the cases $p > 2$ and $p < 2$ are treated separately. In [9, 10], the estimates of solutions and the characteristic cylinder sizes for which the oscillation lemma is proved are nonuniform with respect to p for $p \to 2$. We have been able not only to eliminate these nonuniformities for $p \equiv \text{const}$, including the case $p = 2$ into the scheme of arguments for $p > 2$, but to adjust the proof to the case of equations with variable exponent $p(x,t)$. Here, an important role is played by the value s equal to the exact lower bound for the exponent $p(x,t)$ in cylinders of small diameter with canter at an arbitrary point $(x_0,t_0) \in Q_T$. In Secs. 5 and 6 we study the cases $s \ge 2$ and $s < 2$ with the help of different techniques. In each of these cases, we prove an oscillation lemma for the corresponding "parabolic cylinders" with "vertex" at the point (x_0,t_0). The constants in the oscillation lemma and the cylinder dimensions are uniform in s for $s \to 2$. The most difficult case is that of $s < 2$, which involves essential modifications as compared with [9]. The Hölder continuity of solutions of equation (1.1) is established in Sec. 7.

Theorem 1 admits the following generalization to the case of the exponent p depending only on x and having a logarithmic continuity modulus at a point $x_0 \in \Omega$:

$$|p(x) - p(x_0)| \le \frac{c_0}{\ln \frac{1}{|x-x_0|}} \quad \text{for} \quad x \in \Omega, \quad |x - x_0| \le \frac{1}{2}. \tag{1.8}$$

In this case, smooth functions in Q_T are not dense $W_{\mathrm{loc}}(Q_T)$ and the following can be claimed.

Theorem 2. *Under conditions* (1.7) *and* (1.8), *all W-solutions and H-solutions of equation* (1.1) *are Hölder continuous at the points $(x_0, t) \in Q_T$.*

In the case under consideration, the fact that the solutions are bounded in a neighborhood of the points $(x_0, t) \in Q_T$ was established in [13].

Theorems 1 and 2 are proved by the same method. The main difference is in the choice of test functions in the integral identity (1.5), which allows us to obtain analogues of well-known local energy inequalities for $p = \mathrm{const}$ established in Sec. 4. The choice of the test functions is described in Sec. 3 and utilizes regularization operators of different structure whose properties are given in Sec. 2. For $p = p(x)$, one can use the classical arguments based on Steklov regularizations with respect to the time variable. For $p = p(x,t)$, one has to use regularization in the spatial variable, as well. In the latter case, the logarithmic condition (1.3) plays an important role.

2. Class $W_{\mathrm{loc}}(Q_T)$

Denote by $L^{p(\cdot)}_{\mathrm{loc}}(Q_T)$ the set of all measurable vector-valued functions $f\colon Q_T \to \mathbb{R}^n$ such that $|f|^p \in L^1_{\mathrm{loc}}(Q_T)$, and define the convergence by

$$|f_j - f|^p \to 0 \quad \text{in} \quad L^1_{\mathrm{loc}}(Q_T).$$

2.1. Properties of Regularization Operators. Taking the kernel

$$\rho \ge 0, \quad \rho \in C_0^\infty(\mathbb{R}^n), \quad \rho(x) \equiv 0 \quad \text{for} \quad |x| \ge 1, \quad \int\limits_{\mathbb{R}^n} \rho(x)\, dx = 1,$$

we introduce regularizations of $f \in L^{p(\cdot)}_{\mathrm{loc}}(Q_T)$ by

$$I^h f = f_h(x,t) = h^{-1} \int\limits_t^{t+h} \int\limits_{|x-y| \le h} f(y,\tau) \rho_h(x-y)\, dy\, d\tau, \quad \rho_h(x) = h^{-n} \rho(h^{-1}x), \tag{2.1}$$

and consider these inside the cylinder Q_T, i.e., in cylinders $Q' = \Omega' \times (T_1, T_2)$, where $\Omega \Subset \Omega$, $0 < T_1 < T_2 < T$.

Lemma 2.1. *If the exponent p satisfies the condition* (1.3), *then $f_h \to f$ in $L^{p(\cdot)}_{\mathrm{loc}}(Q_T)$ as $h \to 0$, for any $f \in L^{p(\cdot)}_{\mathrm{loc}}(Q_T)$.*

Proof. First, let us show that for all $(x,t) \in Q'$ and sufficiently small

$$h \le h_0 = \frac{1}{3} \min\{\mathrm{dist}(\Omega', \partial\Omega), T - T_2\}$$

we have

$$|f_h(x,t)|^{p(x,t)} \le C_1 \biggl(1 + h^{-1} \int\limits_t^{t+h} \int\limits_{|x-y| \le h} |f(y,\tau)|^{p(y,\tau)} \rho_h(x-y)\, dy\, d\tau\biggr) \tag{2.2}$$

with a constant C_1 independent of h. Set

$$\Omega'_h = \{x \colon x \in \Omega,\ \mathrm{dist}(x, \Omega') < h\}.$$

Since $\Omega'_h \Subset \Omega$ and $f \in L^{p(\cdot)}_{\mathrm{loc}}(Q_T)$, it follows that for $h \le h_0$ we have

$$|f_h(x,t)| \le h^{-n-1} \max \rho \int_t^{t+h} \int_{\Omega'_{h_0}} |f(y,\tau)|\, dy\, d\tau \le h^{-n-1} M, \quad M = M(\max \rho, \Omega', \delta).$$

Setting $p_h(x) = \min\{p(y,\tau),\ |x-y| \le h,\ \tau \in [t, t+h]\}$, we note that the following two inequalities hold. First,

$$p_h(x,t) \le p(y,\tau) \quad \text{for} \quad |x-y| < h, \quad \tau \in [t, t+h], \tag{2.3}$$

which can be verified directly. Secondly,

$$|\lambda|^{p(x)} \le 2e^{(n+1)c_0} |\lambda|^{p_h(x,t)} \quad \text{for} \quad |\lambda| \le h^{-n-1} M, \tag{2.4}$$

since $0 \le p(x,t) - p_h(x,t) \le \omega(h) = c_0 \cdot \ln^{-1}(2h)^{-1}$ and

$$|\lambda|^{p(x,t)} = |\lambda|^{p(x,t)-p_h(x,t)} |\lambda|^{p_h(x,t)} \le (|\lambda|^{\omega(h)} + 1) |\lambda|^{p_h(x,t)}$$
$$\le \big((h^{-n-1}M)^{\omega(h)} + 1\big) |\lambda|^{p_h(x,t)} \le 2e^{(n+1)c_0} |\lambda|^{p_h(x,t)} \quad \text{for} \quad h \le h_1$$

due to the logarithmic condition. Further, using the integral Jensen inequality, the Young inequality, and (2.3), we find that

$$|f_h(x,t)|^{p_h(x,t)} \le h^{-1} \int_t^{t+h} \int_{|x-y| \le h} |f(y,\tau)|^{p_h(x,t)} \rho_h(x-y)\, dy\, d\tau$$
$$\le h^{-1} \int_t^{t+h} \int_{|x-y| \le h} \big(|f(y,\tau)|^{p(y,\tau)} + 1\big) \rho_h(x-y)\, dy\, d\tau.$$

Hence, using (2.4), we obtain

$$|f_h(x,t)|^{p(x,t)} \le 2e^{(n+1)c_0} h^{-1} \int_t^{t+h} \int_{|x-y| \le h} \big(|f(y,\tau)|^{p(y,\tau)} + 1\big) \rho_h(x-y)\, dy\, d\tau,$$

which implies (2.2). The right-hand side of (2.2) is a regularization of $|f|^p + 1$, and therefore, it converges in $L^1_{\mathrm{loc}}(Q_T)$ to $|f|^p + 1$. it follows that the left-hand side of (2.2) is equi-integrable inside Q_T. Moreover, it can be assumed that $f_h(x,t)\varepsilon \to f(x,t)$ for almost all $(x,t) \in Q_T$. In this situation, the inequality

$$|f_h - f|^p \le 2^\beta (|f_h|^p + |f|^p),$$

with β being the constant from (1.2), shows that the sequence $|f_h - f|^p$ is also equi-integrable. Therefore, by the Lebesgue theorem, we have $f_h \to f$ in $L^1_{\mathrm{loc}}(Q_T)$, as required. The lemma is proved. □

Remark 2.1. If the summability exponent p does not depend on t, then instead of the regularizations (2.1) one should use the Steklov regularizations

$$f_h(x,t) = h^{-1} \int_t^{t+h} f(x,\tau)\, d\tau. \tag{2.5}$$

In this case, the statement of Lemma 2.1 holds under the sole condition (1.2) on $p(x)$.

Indeed, in this situation we immediately obtain the estimate

$$|f_h(x,t)|^{p(x)} \le h^{-1} \int_t^{t+h} |f(x,\tau)^{p(x)}\, d\tau,$$

which leads us to a relation of the form (2.2). Further reasoning is the same as above.

2.2. Density of Smooth Functions in $W_{\mathrm{loc}}(Q_T)$. Let $u \in W_{\mathrm{loc}}(Q_T)$ and suppose that the logarithmic condition (1.3) holds. We define the approximation u_h in terms of the above regularization. Then, due to the classical property of regularizations, we have

$$u_h \to u \ \text{ in } \ L^2_{\mathrm{loc}}(Q_T).$$

Moreover, in view of the approximation Lemma 2.1, we have

$$\nabla u_h \to \nabla u \ \text{ in } \ L^{p(\cdot)}_{\mathrm{loc}}(Q_T).$$

Hence, we obtain the following statement about the density of smooth functions in $W_{\mathrm{loc}}(Q_T)$.

Lemma 2.2. *Under the condition* (1.3), *the set $C^\infty(Q_T)$ is dense in $W_{\mathrm{loc}}(Q_T)$.*

This lemma implies that the logarithmic condition (1.3) ensures that the classes $W_{\mathrm{loc}}(Q_T)$ and $H_{\mathrm{loc}}(Q_T)$ coincide. In particular, the set of W-solutions of equation (1.1) coincides with the set of its H-solutions.

2.3. Locality Property. Next, we prove an auxiliary proposition that characterizes the summability exponent of functions in $W_{\mathrm{loc}}(Q_T)$ which satisfy the additional condition of being $L^2_{\mathrm{loc}}(\Omega)$-continuous on compact sets of the interval $(0, T)$. Such functions are involved in the definition of W-solutions of equation (1.1).

Proposition 2.1. *If the summability exponent $p(x, t)$ is continuous at a point $(x_0, t_0) \in Q_T$, then there is a constant $\gamma = \gamma(n) > 1$ such that $|u|^{\gamma p(x,t)} \in L^1(Q)$ for cylinders $Q \Subset Q_T$ with center at (x_0, t_0) and a sufficiently small radius depending only on n and p.*

Proof. Let $Q = B_r^{x_0} \times (t_0 - r, t_0 + r)$ and let the radius $r = r(n, p)$ of the ball $B_r^{x_0}$ with center at x_0 be so small that $Q \Subset Q_T$ and

$$\gamma p(x,t) \le \frac{s(n+2)}{n} \quad \text{for} \quad (x,t) \in Q, \quad \text{where} \quad \gamma = \frac{n+2}{n+1}, \qquad s = \inf_Q p(x,t).$$

This implies the desired summability, in view of the imbedding theorem according to which $u \in L^{sn/(n+2)}(Q)$. The proposition is proved. □

In particular, if the exponent $p(x, t)$ satisfies the logarithmic condition (1.3), then $|u|^{\gamma p(x,t)} \in L^1_{\mathrm{loc}}(Q_T)$.

Thus, if the summability exponent is continuous at some point, then the following locality property holds for W-solutions of equation (1.1): if $u \in W_{\mathrm{loc}}(Q_T)$ and the function $\eta \in C^\infty(\bar{Q})$ vanishes (identically) near the lateral surface of the cylinder Q, then $u\eta \in W_{\mathrm{loc}}(Q_T)$. A similar property holds for H-solutions.

3. Integral Identities and Inequalities with Regularization Operators

3.1. Integral Identities. In the investigation of equation (1.1) with the exponent $p(x, t)$ having a logarithmic continuity modulus, an important role is played by integral identities that involve regularizations of the form (2.1).

Lemma 3.1. *If $u \in W_{\mathrm{loc}}(Q_T)$ is a W-solution of equation* (1.1), *then for any $h < t_1 \le t_2 < T - h$ and any $\psi \in W_{\mathrm{loc}}(Q_T)$ that identically vanishes in the $2h$-neighborhood of the lateral surface of cylinder Q_T, the following relation holds*:

$$\int_{t_1}^{t_2} \int_{\Omega} \left(\frac{\partial u_h}{\partial t} \psi + (|\nabla u|^{p-2} \nabla u)_h \cdot \nabla \psi \right) dx\, dt = 0. \tag{3.1}$$

Proof. Apart from (2.1), we introduce the following regularization operator:

$$I^{-h}f = f_{-h}(x,t) = h^{-1}\int\limits_{t-h}^{t}\int\limits_{|x-y|\le h} f(y,\tau)\rho_h(x-y)\,dy\,d\tau. \tag{3.2}$$

Consider the integral identity (1.5) with

$$\varphi = I^{-h}(\psi\chi), \quad \psi\in W_{\mathrm{loc}}(Q_T), \quad \chi\in C_0^\infty(t_1,t_2),$$

where ψ has the same properties as in the statement of this lemma. Since

$$-\int\limits_{Q_T} u\frac{\partial I^{-h}(\chi\psi)}{\partial t}\,dx\,dt = \int\limits_{Q_T}\frac{\partial u_h}{\partial t}\chi\psi\,dx\,dt = \int\limits_{t_1}^{t_2}\int\limits_{\Omega}\frac{\partial u_h}{\partial t}\chi\psi\,dx\,dt,$$

it follows that

$$\int\limits_{t_1}^{t_2}\int\limits_{\Omega}\left(\frac{\partial u_h}{\partial t}\chi\psi + (|\nabla u|^{p-2}\nabla u)_h\cdot\nabla(\chi\psi)\right)dx\,dt = 0.$$

Passing here from $\chi\in C_0^\infty(t_1,t_2)$ to the characteristic function of the segment $[t_1,t_2]$, we obtain the desired relation (3.1). The lemma is proved. □

Remark 3.1. Relation (3.1) still holds, if the regularizations (2.1) are replaced by the Steklov regularizations.

This statement is proved in the same way, but instead of (3.2), one should use the regularization

$$I^{-h}f = f_{-h}(x,t) = h^{-1}\int\limits_{t-h}^{t} f(x,\tau)\,d\tau.$$

The Steklov regularizations will be used for the study of equations with the exponent $p = p(x)$ having a logarithmic continuity modulus at a fixed point $x_0\in\Omega$. For such equations, it makes sense to consider the identity (3.1) also for H-solutions. In this situation, in the conditions of Lemma 3.1 one should take $\psi\in H_{\mathrm{loc}}(Q_T)$. For an H-solution, the identity (3.1) is first verified for functions $\psi\in C^\infty(Q_T)$ that vanish near the lateral surface of the cylinder Q_T, and then, by closure, it is shown to hold for $\psi\in H_{\mathrm{loc}}(Q_T)$ with the required property.

3.2. Integral Inequalities. The proof of the oscillation lemma, which implies the Hölder continuity of solutions $u(x,t)$ of equation (1.1), is based on integral estimates for truncated functions of the form

$$(u-k)_+ = \max(u-k,0), \quad (u-k)_- = \max(-(u-k),0), \quad k\in\mathbb{R}^1. \tag{3.3}$$

One of the required inequalities is given in the following lemma.

Lemma 3.2. *If $u\in W_{\mathrm{loc}}(Q_T)$ is a W-solution of equation* (1.1) *and $w = (u-k)_\pm$ then for any $h < t_1 \le t_2 < T-h$ and any nonnegative function $\psi\in W_{\mathrm{loc}}(Q_T)$ that vanishes in the $2h$-neighborhood of the lateral surface of the cylinder Q_T, the following inequality holds*:

$$\int\limits_{t_1}^{t_2}\int\limits_{\Omega}\left(\frac{\partial w_h}{\partial t}\psi + (|\nabla w|^{p-2}\nabla w)_h\cdot\nabla\psi\right)dx\,dt \le 0. \tag{3.4}$$

Proof. Assuming the logarithmic condition (1.3) to be valid, consider the regularizations (2.1) and take the following test function in the integral identity (3.1):

$$\psi = \frac{(u_h-k)_\pm\varphi}{(u_h-k)_\pm+\varepsilon}, \quad \varphi\ge 0,$$

where φ satisfies conditions (1.6). Integrating by parts in the first term and passing to the limit, first for $h \to 0$ (this is possible due to the interpolation Lemma 2.1) and then for $\varepsilon \to 0$, we obtain

$$\int_{t_1}^{t_2}\int_{\Omega}\left(-w\frac{\partial\varphi}{\partial t}+|\nabla w|^{p-2}\nabla w\cdot\nabla\varphi\right)dx\,dt=-\liminf_{\varepsilon\searrow 0}\varepsilon\int_{t_1}^{t_2}\int_{\Omega}|\nabla w|^{p}(v+\varepsilon)^{-2}\varphi\,dx\,dt.$$

Since the limit is nonnegative and t_1, t_2 are arbitrary, it follows that

$$\int_{Q_T}\left(-w\frac{\partial\varphi}{\partial t}+|\nabla w|^{p-2}\nabla w\cdot\nabla\varphi\right)dx\,dt\le 0.$$

Now, repeating the reasoning in the proof of Lemma 3.1, we come to the desired inequality (3.4). The lemma is proved. □

A similar statement with Steklov regularizations (2.5) holds also for the equation with the exponent $p(x)$ from (1.8). In this case, the inequality (3.4), with suitably chosen test functions ψ, holds for W-and H-solutions of equation (1.1).

4. Integral Estimates of Solutions and Auxiliary Results

4.1. Notation. Below, $u(x,t)$ is a solution of equation (1.1), $K_\rho^{y_0}$ is an open n-dimensional cube with center at y_0 and edges of length 2ρ parallel to the coordinate axes, $B_R^{y_0}$ is an open n-dimensional ball of radius R with center at y_0, and $Q_{\rho,\theta}^{(y_0,\tau_0)}=K_\rho^{y_0}\times(\tau_0-\theta,\tau_0)$ is a cylinder of height θ with "vertex" at (y_0,τ_0). For $R_0\in(0,1/2)$, we fix a cylinder

$$\mathbf{Q}_{R_0}^{(x_0,t_0)}=\{(x,t)\colon |x-x_0|<R_0^{1/2},\ t_0-R_0^{1/2}<t<t_0+R_0^{1/2})\},\quad \mathbf{Q}_{R_0}^{(x_0,t_0)}\Subset Q_T, \tag{4.1}$$

and for $R\le R_0$ set

$$s=s_R=\operatorname*{ess\,inf}_{\mathbf{Q}_R^{(x_0,t_0)}}p(x,t),\quad M_0=\operatorname*{ess\,sup}_{\mathbf{Q}_{R_0}^{(x_0,t_0)}}|u|. \tag{4.2}$$

The point $x_0\in\Omega$ is chosen according to the properties of the exponent $p(x,t)$: either x_0 is the point from the condition (1.8), or x_0 is arbitrary, if condition (1.3) holds.

4.2. Integral Estimates of Solutions. All estimates proved below are valid for cylinders $Q_{\rho,\theta}^{(y_0,\tau_0)}\subset\mathbf{Q}_R^{(x_0,t_0)}$, if either condition (1.3) or (1.8) holds for the exponent p. In the latter case, these estimates hold for W-and H-solutions of equation (1.1). The proofs are based on the integral identity (3.1) with the corresponding regularizations and the utilization of the locality property from Sec. 2.3 for choosing the test function.

Next, we establish two repeatedly used statement involving truncated solutions of the form (3.3), where $\eta(x,t)$ denotes a truncating function which is Lipschitz continuous on the closure of $Q_{\rho,\theta}^{(y_0,\tau_0)}$, vanishes on the lateral surface of the cylinder $Q_{\rho,\theta}^{(y_0,\tau_0)}$, and satisfies the inequality $0\le\eta\le 1$. In what follows, $|E|$ stands for the n-dimensional (or $(n+1)$-dimensional) Lebesgue measure of a measurable set $E\subset\mathbb{R}^n$ (or $E\subset\mathbb{R}^{n+1}$), and in the next proposition

$$A_{k,\rho}^{\pm}(\tau)=\{(x,t)\colon x\in K_\rho^{y_0},\ t=\tau,\ (u(x,\tau)-k)_\pm>0\}.$$

Proposition 4.1. *The following inequality holds*:

$$\sup_{t\in(\tau_0-\theta,\tau_0)} \int\limits_{K_\rho^{y_0}\times\{t\}} (u-k)_\pm^2\eta^\beta\,dx + \int\limits_{Q_{\rho,\theta}^{(y_0,\tau_0)}} \big|\nabla\big((u-k)_\pm\eta^{\beta/s}\big)\big|^s\,dx\,dt$$

$$\leq \int\limits_{K_\rho^{y_0}\times\{\tau_0-\theta\}} (u-k)_\pm^2\eta^\beta\,dx + C(p,M_0)\int\limits_{Q_{\rho,\theta}^{(y_0,\tau_0)}} \big((u-k)_\pm^s|\nabla\eta|^p + (u-k)_\pm^2|\eta_t'|\big)\,dx\,dt + \int\limits_{\tau_0-\theta}^{\tau_0} |A_{k,\rho}^\pm(t)|\,dt, \tag{4.3}$$

where β *is the constant from condition* (1.2).

Proof. For definiteness, suppose that (1.3) holds. Let us take the test function $\psi=(u_h-k)_\pm\eta^\beta$ in the integral identity (3.1), where u_h are regularizations of the form (2.1). We obtain

$$\frac{1}{2}\int\limits_{Q_{\rho,\theta}^{(y_0,\tau_0)}} \frac{d}{dt}\big((u_h-k)_\pm^2\eta^\beta\big)\,dx\,dt + \int\limits_{Q_{\rho,\theta}^{(y_0,\tau_0)}} (|\nabla u|^{p-2}\nabla u)_h\cdot\nabla\big((u_h-k)_\pm\big)\eta^\beta\,dx\,dt$$

$$\leq \beta\int\limits_{Q_{\rho,\theta}^{(y_0,\tau_0)}} (|\nabla u|^{p-1}|)_h(u_h-k)_\pm|\nabla\eta|\eta^{\beta-1}\,dx\,dt + \frac{\beta}{2}\int\limits_{Q_{\rho,\theta}^{(y_0,\tau_0)}} (u_h-k)_\pm^2\eta^{\beta-1}\eta_t'\,dx\,dt.$$

Using well-known properties of regularizations and interpolation Lemma 2.1 and passing to the limit for $h\to0$, we get

$$\int\limits_{K_\rho^{y_0}\times\{t^0\}} (u-k)_\pm^2\eta^\beta\,dx + 2\int\limits_{Q_{\rho,\theta}^{(y_0,\tau_0)}} |\nabla(u-k)_\pm|^p\eta^\beta\,dx\,dt$$

$$\leq \int\limits_{K_\rho^{y_0}\times\{\tau_0-\theta\}} (u-k)_\pm^2\eta^\beta\,dx + \beta\int\limits_{Q_{\rho,\theta}^{(y_0,\tau_0)}} (u-k)_\pm^2\eta^{\beta-1}\eta_t'\,dx\,dt + 2\beta\int\limits_{Q_{\rho,\theta}^{(y_0,\tau_0)}} |\nabla(u-k)_\pm|^{p-1}(u-k)_\pm|\nabla\eta|\eta^{\beta-1}\,dx\,dt.$$

Hence, using the Young inequality

$$|\nabla(u-k)_\pm|^{p-1}(u-k)_\pm|\nabla\eta|\eta^{\beta-1} \leq \varepsilon|\nabla(u-k)_\pm|^p\eta^\beta + C(\varepsilon,p)(u-k)_\pm^p|\nabla\eta|^p\eta^{\beta-p},$$

and choosing a suitable ε, we find that

$$\int\limits_{K_\rho^{y_0}\times\{\tau_0\}} (u-k)_\pm^2\eta^\beta\,dx + \int\limits_{Q_{\rho,\theta}^{(y_0,\tau_0)}} |\nabla(u-k)_\pm|^p\eta^\beta\,dx\,dt$$

$$\leq \int\limits_{K_\rho^{y_0}\times\{t^0-\theta\}} (u-k)_\pm^2\eta^\beta\,dx + C(p)\int\limits_{Q_{\rho,\theta}^{(y_0,\tau_0)}} (u-k)_\pm^p|\nabla\eta|^p\eta^{\beta-p}\,dx\,dt + \beta\int\limits_{Q_{\rho,\theta}^{(y_0,\tau_0)}} (u-k)_\pm^2\eta^{\beta-1}\eta_t'\,dx\,dt.$$

Similar considerations can be carried out for cylinders $K_\rho^{y_0}\times(\tau_0-\theta,t)$, $\tau_0-\theta<t<\tau_0$, so that

$$\sup_{t\in(\tau_0-\theta,\tau_0)} \int\limits_{K_\rho^{y_0}\times\{t\}} (u-k)_\pm^2\eta^\beta\,dx + \int\limits_{Q_{\rho,\theta}^{(y_0,\tau_0)}} |\nabla(u-k)_\pm|^p\eta^\beta\,dx\,dt$$

$$\leq \int\limits_{K_\rho^{y_0}\times\{t^0-\theta\}} (u-k)_\pm^2\eta^\beta\,dx + C(p)\int\limits_{Q_{\rho,\theta}^{(y_0,\tau_0)}} (u-k)_\pm^p|\nabla\eta|^p\eta^{\beta-p}\,dx\,dt + \beta\int\limits_{Q_{\rho,\theta}^{(y_0,\tau_0)}} (u-k)_\pm^2\eta^{\beta-1}\eta_t'\,dx\,dt.$$

Now (4.3) follows from the inequalities

$$\left|\nabla\big((u-k)_\pm\big)\right|^s \le \left|\nabla\big((u-k)_\pm\big)\right|^p + 1, \quad (u-k)_\pm^p \le C(p, M_0)(u-k)_\pm^s$$

and the properties of the truncating function η that ensure the inequalities $\eta^{\beta-p} \le 1$ and $\eta^{\beta-1} \le 1$.

If the exponent $p(x)$ satisfies condition (1.8), then one should use the Steklov regularizations (2.5), which, in view of Remarks 2.1 and 3.1, leads us to the desired result. The proposition is proved. □

Let

$$\mu_k^\pm = \operatorname*{ess\,sup}_{Q_{\rho,\theta}^{(x^0,t^0)}} (u-k)_\pm,$$

$$g(u) = g^\pm(u) = \max\left\{\ln\left(\frac{\mu_k^\pm}{\mu_k^\pm - (u-k)_\pm + c}\right);\, 0\right\}, \quad 0 < c < \mu_k^\pm.$$

Proposition 4.2. *The following inequality holds*:

$$\sup_{t\in(\tau_0-\theta,\tau_0)} \int\limits_{K_\rho^{y_0}\times\{t\}} g^2(u)\eta^\beta\,dx \le \int\limits_{K_\rho^{y_0}\times\{t^0-\theta\}} g^2(u)\eta^\beta\,dx + C(p) \int\limits_{Q_{\rho,\theta}^{(y_0,\tau_0)}} g(u)|g'(u)|^{2-p}|\nabla\eta|^p\,dx\,dt, \tag{4.4}$$

where $\eta = \eta(x)$.

Proof. Assuming that condition (1.3) holds, we take the test function

$$\psi = 2g(u_h)g'(u_h)\eta^\beta(x)$$

in the integral identity (3.1), where u_h are regularizations of the form (2.1). Taking into account that

$$g'' = 2(1+g)(g')^2$$

and $g(u_h) = 0$ on the set where $(u_h - k)_\pm = 0$, performing simple calculations, and passing to the limit in the integral identity as $h \to 0$ (this is possible due to the interpolation Lemma 2.1), we come to the relation

$$\int\limits_{K_\rho^{y_0}\times\{\tau_0\}} \big(g(u)\big)^2\eta^\beta\,dx + 2\int\limits_{Q_{\rho,\theta}^{(y_0,\tau_0)}} \big(1+g(u)\big)\big(g'(u)\big)^2|\nabla(u-k)_\pm|^p\eta^\beta\,dx\,dt$$

$$\le \int\limits_{K_\rho^{y_0}\times\{\tau_0-\theta\}} g^2(u)\eta^\beta\,dx + 2\beta\int\limits_{Q_{\rho,\theta}^{(y_0,\tau_0)}} |\nabla(u-k)_\pm|^{p-1}g(u)|g'(u)|\eta^{\beta-1}|\nabla\eta|\,dx\,dt.$$

Next, we use the Young inequality

$$|\nabla(u-k)_\pm|^{p-1}g(u)|g'(u)|\eta^{\beta-1}|\nabla\eta| \le \varepsilon g(u)\big(g'(u)\big)^2|\nabla(u-k)_\pm|^p\eta^\beta + C(\varepsilon,p)g(u)|g'(u)|^{2-p}|\nabla\eta|^p\eta^{\beta-p}$$

and, choosing a suitable ε, find that

$$\int\limits_{K_\rho^{x^0}\times\{t^0\}} g^2(u)\eta^\beta\,dx \le \int\limits_{K_\rho^{x^0}\times\{t^0-\theta\}} g^2(u)\eta^\beta\,dx + C(p)\int\limits_{Q_{\rho,\theta}^{(x^0,t^0)}} g(u)|g'(u)|^{2-p}|\nabla\eta|^p\eta^{\beta-p}\,dx\,dt.$$

Hence, we obtain the inequality (4.4), if we take into account the properties of the truncating function and the possibility of establish a similar estimate in cylinders of the form $K_\rho^{y_0}\times(\tau_0-\theta, t)$, $\tau_0-\theta < t < \tau_0$. Under the condition (1.8), the proof is the same with regularizations (2.5). The proposition is proved. □

4.3. Auxiliary Results. By $\|\varphi\|_{q,E}$, $q \ge 1$, we denote the L^q-norm of a function φ on a measurable set E and use the space $V^q(Q^{(y_0,\tau_0)}_{\rho,\theta})$ that consists of functions with finite norm

$$\|\varphi\|_{V^q(Q^{(y_0,\tau_0)}_{\rho,\theta})} = \operatorname*{ess\,sup}_{\tau_0-\theta<t<\tau_0} \|\varphi(\cdot,t)\|_{q,K^{y_0}_\rho} + \|\,|\nabla\varphi|\,\|_{q,Q^{(y_0,\tau_0)}_{\rho,\theta}}, \quad q \ge 1.$$

For functions $\varphi \in V^q(Q^{(y_0,\tau_0)}_{\rho,\theta})$ with zero trace on the lateral surface of the cylinder $Q^{(y_0,\tau_0)}_{\rho,\theta}$, we use the imbedding theorem (see, for instance, [9]):

$$\|\varphi\|^q_{q,Q^{(y_0,\tau_0)}_{\rho,\theta}} \le C(n,q)|\{(x,t)\colon \varphi(x,t)>0\}|^{q/(n+q)}\|\varphi\|^q_{V^q(Q^{(y_0,\tau_0)}_{\rho,\theta})}. \tag{4.5}$$

We also use the following version of the Poincaré inequality (see [9]).

Lemma 4.1. *Let D be a bounded convex set in $\mathbb{R}^n$, $\eta \in C(\bar D)$, $0 \le \eta \le 1$, and let the sets $\{x\colon x \in D,\ \eta(x) > l\}$ be convex for all $l \in (0,1)$. If $w \in W^{1,q}(D)$ for $q \ge 1$ and the set*

$$E = \{x\colon x\in D,\ w(x)=0\} \cap \{x\colon x \in D,\ \eta(x)=0\}$$

has positive measure, then

$$\Bigl(\int\limits_D |w|^q\eta\,dx\Bigr)^{1/q} \le C(n,q)(\operatorname{diam} D)^n|E|^{-(n-1)/n}\Bigl(\int\limits_D |\nabla w|^q \eta\,dx\Bigr)^{1/q}, \tag{4.6}$$

where $\operatorname{diam} D$ *is the diameter of the domain D.*

We also need an integral inequality due to De Giorgi [14]. For a function $w(x)$, by $\mathcal{A}_{k,\rho}$ we denote the set of all $x \in B^{x^0}_\rho$ such that $w(x) > k$.

Lemma 4.2. *If $w \in W^{1,1}(B^{x^0}_\rho)$ and $l > k$, then*

$$(l-k)|\mathcal{A}_{l,\rho}| \le C(n)\frac{\rho^{n+1}}{|B^{x^0}_\rho \setminus \mathcal{A}_{k,\rho}|}\int\limits_{\mathcal{A}_{k,\rho}\setminus\mathcal{A}_{l,\rho}} |\nabla w|\,dx. \tag{4.7}$$

We utilize a result from [15] about numerical sequences satisfying some recurrent inequalities.

Proposition 4.3. *Let $\{X_n\}$, $n = 0,1,\dots,$ be a nonnegative numerical sequence that satisfies the recurrent relation*

$$X_{n+1} \le Cb^nX_n^{1+\gamma}, \quad n = 0,1,\dots,$$

with constant $C > 1$, $b > 1$, $\gamma > 0$. If

$$X_0 \le C^{-1/\gamma}b^{-1/\gamma^2},$$

then $X_n \to 0$ as $n \to \infty$.

5. Case $s \ge 2$

Below, we use the notation from (4.1), (4.2) and assume that $s \ge 2$. The point $x_0 \in \Omega$ is chosen as in Sec. 4.1. The proof of all statements follows the same lines for the equation with the exponent p satisfying condition (1.3) or (1.8), and both of these will be referred to as the logarithmic condition. Under the condition (1.8), all the results are valid for W- and H-solutions of equation (1.1).

Let (for the notation, see Sec. 4.1)

$$Q_{R,2R^2} = Q^{(x_0,t_0)}_{R,2R^2} = K^{x_0}_R \times (t_0-2R^2,t_0), \quad Q_{R,2R^2} \subset \mathbf{Q}^{(x_0,t_0)}_R,$$
$$M = \operatorname*{ess\,sup}_{Q_{R,2R^2}} u, \quad m = \operatorname*{ess\,inf}_{Q_{R,2R^2}} u, \quad \omega = \operatorname*{ess\,osc}_{Q_{R,2R^2}} u = M - m.$$

The oscillation lemma will be proved for cylinders of the form

$$Q_{R,aR^s} = Q^{(x_0,t_0)}_{R,aR^s} = K^{x_0}_R \times (t_0 - aR^s, t_0), \quad a = 2\left(\frac{\omega}{2^\lambda}\right)^{2-s} \tag{5.1}$$

with a constant $\lambda > 1$ that depends only on n, p, M_0 and will be chosen later. For brevity, we do not indicate the "vertex" in the notation of cylinders, if it coincides with the point (x_0, t_0), and in the notation of cubes we do not indicate their center, if it coincides with x_0. We also assume that $(x_0, t_0) = (0, 0)$.

Below, it is assumed that

$$\omega > 2^{\lambda} R. \tag{5.2}$$

If this inequality is violated, then the desired estimate for the oscillation of the solution in the cylinder $Q_{R,2R^2}$ does not hold (or holds). The inequality (5.2) implies the inclusion $Q_{R,aR^s} \subset Q_{R,2R^2}$ and the relation $\operatorname*{ess\,osc}_{Q_{R,aR^s}} u \le \omega$, which serves as the initial point of the iteration process.

The proof of the oscillation lemma consists in the examination of two complementary cases involving the cylinders

$$Q^{(0,\bar{t})}_{R,bR^s} = K_R \times (\bar{t} - bR^s, \bar{t}), \quad b = \Big(\frac{\omega}{2}\Big)^{2-s}, \tag{5.3}$$

that belong to Q_{R,aR^s}, provided that

$$(b-a)R^s \le \bar{t} \le 0. \tag{5.4}$$

In the following alternatives, it is important that the constant $\nu_0 \in (0,1)$ does not depend on n, p, and M_0.

Alternative 1. There is a cylinder $Q^{(0,\bar{t})}_{R,bR^s}$ defined in (5.3), (5.4) such that

$$\Big|(x,t)\colon (x,t) \in Q^{(0,\bar{t})}_{R,bR^s},\ u(x,t) < m + \frac{\omega}{2}\Big| \le \nu_0 |Q^{(0,\bar{t})}_{R,bR^s}|. \tag{5.5}$$

Alternative 2. For any cylinder $Q^{(0,\bar{t})}_{R,bR^s}$ defined in (5.3), (5.4), the following relation holds:

$$\Big|(x,t)\colon (x,t) \in Q^{(0,\bar{t})}_{R,bR^s},\ u(x,t) > M - \frac{\omega}{2}\Big| < (1-\nu_0)\big|Q^{(0,\bar{t})}_{R,bR^s}\big|. \tag{5.6}$$

5.1. Analysis of the First Alternative. In the proof of the following statements, $\chi(E)$ stands for the characteristic function of a measurable set $E \subset \mathbb{R}^{n+1}$. The next lemma defines a constant ν_0, which is crucial for all subsequent considerations.

Lemma 5.1. *There is a constant $\nu_0 \in (0,1)$ depending only on n, p, M_0 and such that the condition* (5.5) *for some cylinder $Q^{(0,\bar{t})}_{R,bR^s}$ implies that*

$$u(x,t) > m + \frac{\omega}{4} \quad \text{a.e. in} \quad Q^{(0,\bar{t})}_{R/2,b(R/2)^s}. \tag{5.7}$$

Proof. To simplify notation, assume that $\bar{t} = 0$. Set

$$R_i = \frac{R}{2} + \frac{R}{2^{i+1}}, \quad k_i = m + \frac{\omega}{4} + \frac{\omega}{2^{i+2}}, \quad i = 0, 1, \dots .$$

Let us apply to the function $(u - k_i)_-$ in Q_{R_i,bR_i^s} the estimate (4.3) with the truncating function η_i equal to unity in Q_{R_{i+1},bR_{i+1}^s}, vanishing on the parabolic boundary of Q_{R_i,bR_i^s}, and satisfying the inequalities $|\nabla\eta| \le 2^{i+1}R^{-1}$ and $|\eta_t| \le 2^{s(i+1)}b^{-1}R^{-s}$. Since the logarithmic condition ensures that $R^{s-p} \le C(p)$ in Q_{R_i,bR_i^s}, it follows from (4.3) that

$$\begin{aligned} &\sup_{t\in(-bR_i^s,0)} \int\limits_{K_{R_i}\times\{t\}} (u-k_i)_-^2 \eta_i^{\beta}\,dx + \int\limits_{Q_{R_i,bR_i^s}} \big|\nabla\big((u-k_i)_- \eta_i^{\beta/s}\big)\big|^s\,dx\,dt \\ &\le C(p,M_0) 2^{i\beta} R_i^{-s} \Bigg(\int\limits_{Q_{R_i,bR_i^s}} (u-k_i)_-^s\,dx\,dt + b^{-1} \int\limits_{Q_{R_i,bR_i^s}} (u-k_i)_-^2\,dx\,dt \Bigg) + \int\limits_{-bR_i^s}^{0} |A^-_{k_i,R_i}(t)|\,dt. \end{aligned} \tag{5.8}$$

Note that in the cylinder Q_{R_i,bR_i^s} we have $(u-k)_- \le \omega/2$ and

$$(u-k_i)_-^2 = (u-k_i)_-^{2-s}(u-k_i)_-^s \ge \left(\frac{\omega}{2}\right)^{2-s}(u-k_i)_-^s.$$

Therefore, using (5.8), we come to the inequality

$$\left(\frac{\omega}{2}\right)^{2-s} \sup_{t\in(-bR_i^s,0)} \int\limits_{K_{R_i}\times\{t\}} (u-k_i)_-^s\eta_i^\beta\,dx + \int\limits_{Q_{R_i,bR_i^s}} \left|\nabla\big((u-k_i)_-\eta_i^{\beta/s}\big)\right|^s dx\,dt$$

$$\le C(p,M_0)2^{i\beta}R^{-s}\left(\left(\frac{\omega}{2}\right)^s + \frac{1}{b}\left(\frac{\omega}{2}\right)^2\right)\int\limits_{Q_{R_i,bR_i^s}} \chi\big((u-k_i)_->0\big)\,dx\,dt + \int\limits_{-bR_i^s}^{0} |A_{k_i,R_i}^-(t)|\,dt,$$

which, being divided by b (see (5.3)), becomes

$$\sup_{t\in(-bR_i^s,0)} \int\limits_{K_{R_i}\times\{t\}} (u-k_i)_-^s\eta_i^\beta\,dx + b^{-1}\int\limits_{Q_{R_i,bR_i^s}} \left|\nabla\big((u-k_i)_-\eta_i^{\beta/s}\big)\right|^s dx\,dt$$

$$\le C(p,M_0)2^{i\beta}R^{-s}\left(\frac{\omega}{2}\right)^s b^{-1}\int\limits_{Q_{R_i,bR_i^s}} \chi\big((u-k_i)_->0\big)\,dx\,dt + b^{-1}\int\limits_{-bR_i^s}^{0} |A_{k_i,R_i}^-(t)|\,dt. \quad (5.9)$$

Next, we change the variables by letting $\tau = t/b$, which transforms the cylinders Q_{R_i,bR_i^s} into $Q_i = Q_{R_i,R_i^s}$, and set $\tilde u(\cdot,\tau) = u(\cdot,t)$, $\tilde\eta(\cdot,\tau) = \eta(\cdot,t)$,

$$A_i(\tau) = \{x\colon x\in K_{R_i},\ \tilde u(x,\tau) < k_i\}, \quad A_i = \int\limits_{-R_i^s}^{0} |A_i(\tau)|\,d\tau.$$

As a result, from (5.9) we get

$$\|(\tilde u-k_i)_-\tilde\eta_i^{\beta/s}\|_{V^s(Q_i)}^s \le C(p,M_0)2^{i\beta}R^{-s}\left(\frac{\omega}{2}\right)^s A_i + A_i. \quad (5.10)$$

Since the function $(\tilde u-k_i)_-\tilde\eta_i^{\beta/s}$ vanishes on the lateral surface of the cylinder Q_i, the imbedding theorem (4.5) yields

$$\|(\tilde u-k_i)_-\|_{s,Q_{i+1}}^s \le \|(\tilde u-k_i)_-\tilde\eta_i^{\beta/s}\|_{s,Q_i}^s \le C(n,p)A_i^{s/(n+s)}\|(\tilde u-k_i)_-\tilde\eta_i^{\beta/s}\|_{V^s(Q_i)}^s. \quad (5.11)$$

Estimating the left-hand side of (5.11) with the help of the inequality

$$\|(\tilde u-k_i)_-\|_{s,Q_{i+1}}^s \ge |k_i-k_{i+1}|^s A_{i+1} \ge 2^{-s(i+2)}\left(\frac{\omega}{2}\right)^s A_{i+1} \ge 2^{-\beta(i+2)}\left(\frac{\omega}{2}\right)^s A_{i+1},$$

and using (5.10), we find that

$$A_{i+1} \le C(n,p,M_0)4^{i\beta}\left(R^{-s}A_i^{1+s/(n+s)} + \left(\frac{\omega}{2}\right)^{-s}A_i^{1+s/(n+s)}\right).$$

Or, since $\omega \ge R$ (see (5.2)), we have

$$A_{i+1} \le C(n,p,M_0)4^{i\beta}R^{-s}A_i^{1+s/(n+s)}.$$

Dividing both sides of this inequality by $|Q_{i+1}|$ and setting

$$X_i = \frac{A_i}{|Q_i|},$$

we come to the recurrent relation

$$X_{i+1} \le C(n,p,M_0) 4^{i\beta} X_i^{1+s/(n+s)} \le C(n,p,M_0) 4^{i\beta} X_i^{1+2/(n+2)},$$

which takes into account the assumption $s \ge 2$. Let

$$X_0 \le C^{-(n+2)/2} 4^{-\beta(n+2)^2/4}.$$

Then, $X_i \to 0$ as $i \to \infty$ by Proposition 4.3. Thus, setting

$$X_0 \le \nu_0 \equiv C^{-(n+2)/2} 4^{-\beta(n+2)^2/4},$$

we obtain the desired inequality (5.7). The lemma is proved. □

Let us fix the constant ν_0 found above and extend the estimate (5.7) to the cylinder $Q_{R/8, b(R/8)^s}$.

The next step of the proof is based on estimates of the solution in cylinders of the form

$$Q_{R/2,\theta} = K_{R/2} \times (-\theta, 0), \quad -\theta = \bar{t} - b\left(\frac{R}{2}\right)^s, \tag{5.12}$$

where the height θ satisfies the inequality

$$\theta \le aR^s = \left(\frac{\omega}{2^\lambda}\right)^{2-s} R^s. \tag{5.13}$$

Lemma 5.2. *Suppose that* (5.5) *holds with the constant* ν_0 *from Lemma* 5.1*. Then, for any* $\nu_* \in (0,1)$*, there is a positive constant* q_* *depending only on* n*,* p*,* M_0*,* λ*,* ν_* *and such that for all* $q \ge q_*$ *we have*

$$|\{x\colon x \in K_{R/4},\ u(x,t) < m + 2^{-q}\omega\}| \le \nu_* |K_{R/4}| \quad \text{for all} \quad t \in (-\theta, 0). \tag{5.14}$$

Proof. Set

$$k = m + \frac{\omega}{4}, \quad c = \frac{\omega}{2^{q+2}}, \quad q > 1,$$

and consider the function

$$g\big(u(x,t)\big) = \max\left\{\ln\left(\frac{\mu_k^-}{\mu_k^- - (u(x,t)-k)_- + c}\right); 0\right\},$$

where

$$\mu_k^- = \operatorname*{ess\,sup}_{Q_{\theta,R/2}} (u-k)_- \le \frac{\omega}{4}. \tag{5.15}$$

Let us use the estimate (4.4) in the cylinder $Q_{R/2,\theta}$ for the functions $g\big(u(x,t)\big)$ and $\eta(x)$, where $\eta = 0$ on $\partial K_{R/2}$, $\eta = 1$ in $K_{R/4}$, and $|\nabla\eta| \le 8R^{-1}$. First, we note that by the estimate (5.7) from Lemma 5.1, for almost all $\theta' \in [0,\theta]$ the relation $g\big(u(x,-\theta')\big) = 0$ holds almost everywhere in $K_{R/2}$. Therefore, from (4.4) and the logarithmic condition, it follows that

$$\int\limits_{K_{R/4}\times\{t\}} g^2(u)\,dx \le C(p) R^{-s} \int\limits_{Q_{R/2,\theta'}} g|g'|^{2-p}\,dx\,dt \quad \text{for all} \quad t \in (-\theta', 0).$$

Passing here to the limit as $\theta' \to \theta$, we find that

$$\int\limits_{K_{R/4}\times\{t\}} g^2(u)\,dx \le C(p) R^{-s} \int\limits_{Q_{R/2,\theta}} g|g'|^{2-p}\,dx\,dt \quad \text{for all} \quad t \in (-\theta, 0). \tag{5.16}$$

Let us estimate the right-hand side of (5.16) from above. By (5.15), we have

$$g(u) \le \ln \frac{\omega/4}{2^{-q-2}\omega} = q \ln 2$$

and

$$|g'_u|^{2-p} = |\mu_k^- - (u-k)_- + 2^{-q-2}\omega|^{p-2} \le \left(\frac{\omega}{2}\right)^{p-2} \le C(p,M_0)\left(\frac{\omega}{2}\right)^{s-2}.$$

Now, taking into account (5.13), we obtain

$$C(p)R^{-s}\int\limits_{Q_{R/2,\theta}} g|g'|^{2-p}\,dx\,dt \le C(p,M_0)q\theta R^{-s}\Big(\frac{\omega}{2}\Big)^{s-2}|K_{R/2}| \le C(n,p,M_0)q2^{\lambda(\beta-2)}|K_{R/4}|. \tag{5.17}$$

When estimating the integral in the left-hand side of (5.16) from below, let us narrow the integration domain to the set

$$\{x\colon x\in K_{R/4},\ u(x,t)<m+2^{-q-2}\omega\},\quad t\in(-\theta,0).$$

Since $(u-k)_- > \omega/4 - 2^{-q-2}\omega$ at the points of this set, if follows that (see (5.15))

$$\frac{\mu_k^-}{\mu_k^- - (u-k)_- + 2^{-q-2}\omega} > \frac{\mu_k^-}{\mu_k^- - \omega/4 + 2^{-q-1}\omega} \ge \frac{\omega/4}{2^{-q-1}\omega} = 2^{q-1}$$

and

$$g^2(u) \ge (q-1)^2\ln^2 2. \tag{5.18}$$

Utilizing the estimates (5.17) and (5.18) in (5.16), we get

$$|\{x\colon x\in K_{R/4},\ u(x,t)<m+2^{-q-2}\omega\}| \le C(n,p,M_0)2^{\lambda(\beta-2)}q(q-1)^{-2}|K_{R/4}| \ \text{ for all }\ t\in(-\theta,0). \tag{5.19}$$

To complete the proof, we first choose $q > 1 + 2^{\lambda(\beta-2)+1}\nu_*^{-1}C$ in (5.19) and then take $q_* = q+2$. The lemma is proved. □

Let us turn to the proof of the main result that follows from the first alternative.

Lemma 5.3. *If* (5.5) *holds with the constant* ν_0 *from Lemma* 5.1*, then there is a positive constant* $l \ge \lambda$ *depending only on* n*,* p*,* M_0*,* λ *and such that the inequality*

$$\omega > 2^l R, \tag{5.20}$$

implies that

$$u(x,t) > m + 2^{-l-1}\omega \quad \text{a.e. in}\ \ Q_{R/8,\theta} \tag{5.21}$$

Proof. Let $Q_{R,bR^s}^{(0,\bar t)}$ be a cylinder for which (5.5) holds. Set

$$R_i = \frac{R}{8} + \frac{R}{2^{i+3}},\quad k_i = m + \frac{\omega}{2^{l+1}} + \frac{\omega}{2^{l+1+i}},\quad l\ge\lambda,\quad i=0,1,\dots,$$

and consider the cylinders $Q_{R_i,\theta}$ with the constant θ from (5.12). Let us chose a truncating function $\eta_i(x)$ in K_{R_i} such that $0\le\eta_i\le1$, $\eta_i=1$ in $K_{R_{i+1}}$, $|\nabla\eta_i|\le 2^{i+4}R^{-1}$, and use the estimate (4.3) for the functions $(u-k_i)_-$ and η_i. First, we note that (5.7) ensures that for almost all $\theta'\in[0,\theta]$ the function $u(x,-\theta')$ vanishes almost everywhere in K_{R_i}. Now, arguing as in the proof of (5.16) and using (4.3) and the logarithmic condition, we find that

$$\sup_{t\in(-\theta,0)}\int\limits_{K_{R_i}\times\{t\}} (u-k_i)_-^2\eta_i^\beta\,dx + \int\limits_{Q_{R_i,\theta}} \big|\nabla\big((u-k_i)_-\eta_i^{\beta/s}\big)\big|^s\,dx\,dt$$
$$\le C(p,M_0)2^{i\beta}R_i^{-s}\int\limits_{Q_{R_i,\theta}}(u-k_i)_-^s\,dx\,dt + \int\limits_{-\theta}^{0}|A_{k_i,R_i}^-(t)|\,dt. \tag{5.22}$$

Let us estimate the integrands in (5.22). Since $(u-k_i)_- \le 2^{-l}\omega$ in K_{R_i}, it follows that

$$\int\limits_{Q_{R_i,\theta}}(u-k_i)_-^s\,dx\,dt \le \Big(\frac{\omega}{2^l}\Big)^s\int\limits_{Q_{R_i,\theta}}\chi\big((u-k_i)_->0\big)\,dx\,dt,$$

and by (5.13) and the assumption, we have $l\ge\lambda$,

$$(u-k_i)_-^2 \ge \Big(\frac{\omega}{2^l}\Big)^{2-s}(u-k_i)_-^s \ge 2^{(l-\lambda)(s-2)}\theta R^{-s}(u-k_i)_-^s \ge \theta R^{-s}(u-k_i)_-^s.$$

Using the last two estimates and dividing both sides of (5.22) by θR^{-s}, we obtain

$$\sup_{t\in(-\theta,0)} \int_{K_{R_i}\times\{t\}} (u-k_i)_-^s \eta_i^\beta \, dx + R^s\theta^{-1} \int_{Q_{R_i,\theta}} \big|\nabla\big((u-k_i)_-\eta_i^{\beta/s}\big)\big|^s \, dx\, dt$$

$$\leq C(p,M_0)2^{i\beta}\theta^{-1}\left(\frac{\omega}{2^l}\right)^s \int_{Q_{R_i,\theta}} \chi\big((u-k_i)_- > 0\big)\, dx\, dt + R^s\theta^{-1}\int_{-\theta}^{0} |A^-_{k_i,R_i}(t)|\, dt. \quad (5.23)$$

Let us pass to the new variable $\tau = (R/2)^s\theta^{-1}t$, transforming the cylinders $Q_{R_i,\theta}$ into $Q_i = Q_{R_i,(R/2)^s}$. Setting $\tilde{u}(\cdot,\tau) = u(\cdot,t)$,

$$A_i = \int_{-(R/2)^s}^{0} \int_{K_{R_i}} \chi\big((\tilde{u}-k_i)_- > 0\big)\, dx\, d\tau,$$

from (5.23) we get

$$\|(\tilde{u}-k_i)_-\eta_i^{\beta/s}\|^s_{V^s(Q_i)} \leq C(p,M_0)2^{i\beta}R^{-s}\left(\frac{\omega}{2^l}\right)^s A_i + 2^s A_i.$$

Or, since

$$2^{-s(i+2)}\left(\frac{\omega}{2^l}\right)^s A_{i+1} \leq |k_i - k_{i+1}|^s A_{i+1} \leq \|(\tilde{u}-k_i)_-\|^s_{s,Q_{i+1}} \leq \|(\tilde{u}-k_i)_-\eta_i^{\beta/s}\|^s_{s,Q_i}$$

and, by the imbedding Theorem (4.5),

$$\big\|(\tilde{u}-k_i)_-\eta_i^{\beta/s}\big\|^s_{s,Q_i} \leq C(n,s)A_i^{s/(n+s)}\big\|(\tilde{u}-k_i)_-\eta_i^{\beta/s}\big\|^s_{V^s(Q_i)},$$

it follows that

$$A_{i+1} \leq C(n,p,M_0)4^{i\beta}R^{-s}A_i^{1+s/(n+s)}\left(1+\left(\frac{\omega}{2^l}\right)^{-s}\right).$$

Now, setting

$$X_i = \frac{A_i}{|Q_i|},$$

and using (5.20) and the inequality $s \geq 2$, we come to the recurrent relation

$$X_{i+1} \leq C(n,p,M_0)4^{i\beta}X_i^{1+s/(n+s)} \leq C(n,p,M_0)4^{i\beta}X_i^{1+2/(n+2)}.$$

If

$$X_0 \leq C^{-(n+2)/2}4^{-\beta(n+2)^2/4} \equiv \nu_* \in (0,1), \quad (5.24)$$

then (see Proposition 4.3) $X_i \to 0$ as $i \to \infty$ and, thus, the inequality (5.21) is proved. So far, the constant $l \geq \lambda$ has been arbitrary. Now let us chose this constant to ensure the inequality (5.24), which in the variables (x,t) has the form

$$|\{(x,t)\colon (x,t) \in Q_{R/4,\theta},\ u(x,t) < m + 2^{-l}\omega\}| \leq \nu_*|Q_{R/4,\theta}|.$$

This shows that (5.24) is always satisfied under the assumption (5.20), provided that $l = \max\{q_*, \lambda\}$, where q_* is the constant from Lemma 5.2. The lemma is proved. □

Let us formulate the main statement of this section that follows from Lemma 5.3.

Corollary 5.1. *Under the assumption* (5.5) *with the constant* $\nu_0 \in (0,1)$ *from Lemma* 5.1*, there exist constants* $\sigma_1 \in (0,1)$ *and* $l \geq \lambda$ *depending only on* n, p, M_0, λ *and such that either* $\omega \leq 2^l R$ *or*

$$\operatorname*{ess\,osc}_{Q_{R/8,b(R/8)^s}} u \leq \sigma_1\omega. \quad (5.25)$$

Proof. By Lemma 5.3, together with (5.20), we have

$$\operatorname*{ess\,inf}_{Q_{R/8,\theta}} u \geq m + \frac{\omega}{2^{l+1}}.$$

Hence, we find that

$$\operatorname*{ess\,osc}_{Q_{R/8,\theta}} u \leq (1 - 2^{-l-1})\omega.$$

Since (see (5.12)) $d(R/8)^s \leq \theta = -\bar{t} + d(R/2)^s$, $\bar{t} \leq 0$, it follows that

$$Q_{R/8,b(R/8)^s} \subset Q_{R/8,\theta}$$

and (5.25) holds with the constant $\sigma_1 = 1 - 2^{-l-1}$, which completes the proof. □

5.2. Analysis of the Second Alternative. Assume that the second alternative formulated in Sec. (5.6) holds with a constant ν_0 from Lemma 5.1. Our aim is to establish an estimate of the form (5.25).

Lemma 5.4. *If* (5.6) *holds in the cylinder* $Q^{(0,\bar{t})}_{R,bR^s}$, *then there is a constant* $t^* \in [\bar{t} - bR^s, \bar{t} - \nu_0 bR^s/2]$ *such that*

$$|\{x \colon x \in K_R,\ u(x,t^*) > M - \omega/2\}| \leq \frac{1-\nu_0}{1-\nu_0/2}|K_R|. \tag{5.26}$$

Proof. Assume the contrary. If $t \in [\bar{t} - bR^s, \bar{t} - \nu_0 bR^s/2]$, then

$$\left|\left\{(x,t) \colon (x,t) \in Q^{(0,\bar{t})}_{R,bR^s},\ u(t,x) > M - \frac{\omega}{2}\right\}\right| \geq \int\limits_{\bar{t}-bR^s}^{-\nu_0 bR^s/2} |\{x \colon x \in K_R,\ u(x,t)\}|\,dt > (1-\nu_0)\left|Q^{(0,\bar{t})}_{R,bR^s}\right|,$$

and we come to a contradiction with (5.6). The lemma is proved. □

In the proof of the next statement, t^* has the same meaning as in the previous lemma.

Lemma 5.5. *If* (5.6) *holds, then there is a constant* $q_0 > 1$ *depending only on* n, p, M_0 *and such that*

$$|\{x \colon x \in K_R,\ u(x,t) > M - 2^{-q_0}\omega\}| \leq \left(1 - \left(\frac{\nu_0}{2}\right)^2\right)|K_R| \ \textit{ for all } \ t \in \left[\bar{t} - \frac{\nu_0 bR^s}{2}, \bar{t}\right]. \tag{5.27}$$

Proof. Setting

$$k = M - \frac{\omega}{2}, \quad c = \frac{\omega}{2^{q+1}}, \quad q > 1,$$

consider the following function in the cylinder $K_R \times (t^*, \bar{t})$:

$$g\big(u(x,t)\big) = \max\left\{\ln\left(\frac{\mu_k^+}{\mu_k^+ - (u(x,t)-k)_+ + c}\right); 0\right\},$$

where

$$\mu_k^+ = \operatorname*{ess\,sup}_{K_R\times(t^*,\bar{t})} (u-k)_+ \leq \frac{\omega}{2}. \tag{5.28}$$

Let us apply (4.4) in $K_R \times (t^*, \bar{t})$ to $g\big(u(x,t)\big)$ and the truncating function $\eta(x)$ in K_R that vanishes on ∂K_R, is equal to unity on $K_{(1-\delta)R}$ for $\delta \in (0,1)$, and satisfies the inequality $|\nabla\eta| \leq (\delta R)^{-1}$. We get

$$\sup_{t\in(t^*,\bar{t})} \int\limits_{K_R\times\{t\}} g^2(u)\eta^\beta\,dx \leq \int\limits_{K_R\times\{t^*\}} g^2(u)\eta^\beta\,dx + C(p)\int\limits_{t^*}^{\bar{t}}\int\limits_{K_R} g|g'|^{2-p}|\nabla\eta|^p\,dx\,dt. \tag{5.29}$$

Let us estimate the right-hand side of (5.29) from above. By (5.28), we have

$$\frac{\mu_k^+}{\mu_k^+ - (u-k)_+ + 2^{-q-1}\omega} \leq \frac{\omega/2}{2^{-q-1}\omega} = 2^q$$

and

$$g(u) \le q \ln 2, \tag{5.30}$$

$$|g'(u)|^{2-p} = (\mu_k^+ - (u-k)_+ + 2^{-q-1}\omega)^{p-2} \le C(p, M_0)\left(\frac{\omega}{2}\right)^{s-2}. \tag{5.31}$$

Since $v = 0$ on the set $\{x\colon x \in K_R,\ u(x,\cdot) < k\}$, relations (5.30) and (5.26) allow us to conclude that

$$\int\limits_{K_R \times \{t^*\}} g^2(u)\eta^\beta \, dx \le q^2 \ln^2 2 \cdot \frac{1-\nu_0}{1-\nu_0/2}|K_R|.$$

Let us estimate the second integral in the right-hand side of (5.29) with the help of the logarithmic condition, (5.30), (5.31), and the inequality $\bar{t} - t^* \le bR^s$:

$$C(p)\int\limits_{t^*}^{\bar{t}}\int\limits_{K_R} g|g'|^{2-p}|\nabla\eta|^p \, dx\, dt \le C(p, M_0) q \ln 2 \cdot \left(\frac{\omega}{2}\right)^{s-2}(\delta R)^{-s}(\bar{t} - t^*)|K_R| \le C(p, M_0) q\delta^{-\beta}|K_R|.$$

Using (5.29) and the last two relations, we find that

$$\sup_{t \in (t^*, \bar{t})} \int\limits_{K_R \times \{t\}} g^2(u)\eta^\beta \, dx \le q^2 \ln^2 2 \cdot \frac{1-\nu_0}{1-\nu_0/2}|K_R| + C(p, M_0) q\delta^{-\beta}|K_R|. \tag{5.32}$$

In order to estimate the left-hand side of (5.32) from below, we narrow the integration region to the set

$$E(t) = \{x\colon x \in K_{(1-\delta)R},\ u(x,t) > M - 2^{-q-1}\omega\}.$$

On this set, $(u-k)_+ > \omega/2 - 2^{-q-1}\omega$, which, together with (5.28), brings us to the estimate

$$\frac{\mu_k^+}{\mu_k^+ - (u-k)_+ + 2^{-q-1}\omega} > \frac{\mu_k^+}{\mu_k^+ - \omega/2 + 2^{-q}} > \frac{\omega/2}{2^{-q}\omega} = 2^{q-1}.$$

Hence, using the relation $\eta = 1$ on $E(t)$, we get

$$\sup_{t \in (t^*, \bar{t})} \int\limits_{K_R \times \{t\}} g^2(u)\eta^\beta \, dx \ge (q-1)^2 \ln^2 2 \cdot |E(t)|.$$

Taking into account this inequality in (5.32), we obtain

$$|E(t)| \le \left(\frac{q}{q-1}\right)^2 \left(\frac{1-\nu_0}{1-\nu_0/2}\right)|K_R| + C(p, M_0)\frac{q\delta^{-\beta}}{(q-1)^2}|K_R| \quad \text{for all} \quad t \in (\bar{t}, t^*).$$

On the other hand,

$$|\{x\colon x \in K_R,\ u(x,t) > M - 2^{-q-1}\omega\}| \le |E(t)| + |K_R \setminus K_{(1-\delta)R}| \le |E(t)| + n\delta|K_R|,$$

which, combined with the previous estimate for $|E(t)|$, allows us to conclude that for all $t \in (\bar{t}, t^*)$ the following inequality holds:

$$|\{x\colon x \in K_R,\ u(x,t) > M - 2^{-q-1}\omega\}| \le \left(\left(\frac{q}{q-1}\right)^2 \left(\frac{1-\nu_0}{1-\nu_0/2}\right)|K_R| + C(p, M_0)\frac{q\delta^{-\beta}}{(q-1)^2}|K_R| + n\delta\right)|K_R|.$$

Now we first choose the constant δ so small that $n\delta \le 3\nu_0^2/8$, and then take the constant q so large that

$$\left(\frac{q}{q-1}\right)^2 \le \left(1 - \frac{\nu_0}{2}\right)(1+\nu_0), \quad C(p, M_0)\frac{q\delta^{-\beta}}{(q-1)^2} \le \frac{3\nu_0^2}{8},$$

and set $q_0 = q + 1$. The lemma is proved. □

Since the inequality (5.6) holds for all cylinders of the form (5.3) satisfying condition (5.4), the statement of Lemma 5.5 is valid for all values

$$-(a-b)R^s - \frac{\nu_0 b R^s}{2} \le t \le 0.$$

Since (see (5.1) and (5.3)) $a/b \ge 2-\nu_0$, it follows that

$$-(a-b)R^s - \frac{\nu_0 b R^s}{2} \le -\frac{aR^s}{2},$$

and we arrive at the following result.

Corollary 5.2. *Under the assumptions of Lemma 5.5, the inequality* (5.27) *holds for all* $t \in [-aR^s/2, 0]$.

Now let us chose the constant λ in (5.1).

Lemma 5.6. *Under the assumption* (5.6), *the constant* $\lambda > 1$ *can be chosen such that it depends only on* n, p, M_0 *and*

$$u(x,t) \le M - 2^{-\lambda-1}\omega \quad a.e.\ in \quad Q_{R/2,(a/2)(R/2)^s}. \tag{5.33}$$

Proof. For $i = 0, 1, \dots,$ set

$$R_i = 2^{-1}R + 2^{-i-1}R, \quad k_i = M - \frac{\omega}{2^{\lambda+1}} - \frac{\omega}{2^{\lambda+i+1}}, \quad Q_i = Q_{R_i,(a/2)(R_i/2)^s},$$

and consider a truncating function η in the cylinder Q_i such that $\eta = 0$ on the parabolic boundary of Q_i, $\eta = 1$ in Q_{i+1}, and

$$|\nabla \eta_i| \le 2^{i+1}R^{-1}, \quad 0 \le (\eta_i)'_t \le 2^{(i+1)s}(a/2)^{-1}R^{-s}.$$

From the estimate (4.3) in Q_i applied to the functions $(u-k_i)_+$ and η_i, combined with the logarithmic condition, we find that

$$\begin{aligned} &\sup_{t\in(-aR^s/2,0)} \int_{K_{R_i}\times\{t\}} (u-k_i)_+^2 \eta_i^\beta\, dx + \int_{Q_i} \left|\nabla\big((u-k_i)_+\eta_i^{\beta/s}\big)\right|^s dx\, dt \\ &\le C(p,M_0)2^{i\beta}R^{-s}\bigg(\int_{Q_i}(u-k_i)_+^s\, dx\, dt + 2a^{-1}\int_{Q_i}(u-k_i)_+^2\, dx\, dt\bigg) + \int_{-aR_i^s/2}^{0} |A^+_{k_i,R_i}(t)|\, dt. \end{aligned} \tag{5.34}$$

Since $(u-k_i)_- \le 2^{-\lambda}\omega$ and

$$(u-k_i)_+^2 = (u-k_i)_+^{2-s}(u-k_i)_+^s \ge \left(\frac{\omega}{2^\lambda}\right)^{2-s}(u-k_i)_+^s,$$

from (5.34) we get

$$\begin{aligned} &\left(\frac{\omega}{2^\lambda}\right)^{2-s} \sup_{t\in(-aR^s/2,0)} \int_{K_{R_i}\times\{t\}} (u-k_i)_+^s \eta_i^\beta\, dx + \int_{Q_i} \left|\nabla\big((u-k_i)_+\eta_i^{\beta/s}\big)\right|^s dx\, dt \\ &\le C(p,M_0)2^{i\beta}R^{-s}\left(\left(\frac{\omega}{2^\lambda}\right)^s + 2a^{-1}\left(\frac{\omega}{2^\lambda}\right)^2\right)\int_{Q_i}\chi\big((u-k_i)_+ > 0\big)\, dx\, dt + \int_{-aR_i^s/2}^{0} |A^+_{k_i,R_i}(t)|\, dt. \end{aligned} \tag{5.35}$$

Taking into account the explicit form of the constant a from (5.1) and dividing the inequality (5.35) by $a/2$, we obtain

$$\sup_{t\in(-aR^s/2,0)} \int\limits_{K_{R_i}\times\{t\}} (u-k_i)_+^s \eta_i^\beta \, dx + 2a^{-1} \int\limits_{Q_i} \left|\nabla\big((u-k_i)_+\eta_i^{\beta/s}\big)\right|^s dx\, dt$$

$$\le C(p,M_0) 2^{i\beta} R^{-s} \left(\frac{\omega}{2^\lambda}\right)^s a^{-1} \int\limits_{Q_i} \chi\big((u-k_i)_+ > 0\big)\, dx\, dt + 2a^{-1} \int\limits_{-aR_i^s/2}^{0} |A^+_{k_i,R_i}(t)|\, dt. \quad (5.36)$$

Passing to the variable $\tau = 2ta^{-1}$ and setting $\tilde u(\cdot,\tau) = u(\cdot,t)$, $\tilde\eta_i(\cdot,\tau) = \eta_i(\cdot,t)$,

$$A_i = \int\limits_{-R_i^s}^{0} \int\limits_{K_{R_i}} \chi\big((\tilde u - k_i)_+ > 0\big)\, dx\, d\tau,$$

we transform (5.36) to

$$\|(\tilde u - k_i)_+ \tilde\eta_i^{\beta/s}\|^s_{V^s(Q_i)} \le C(p,M_0) 2^{i\beta} R^{-s} \left(\frac{\omega}{2^\lambda}\right)^s A_i + A_i. \quad (5.37)$$

Let us estimate the left-hand side of (5.36) from below. Since

$$2^{-(i+2)s}\left(\frac{\omega}{2^\lambda}\right)^s A_{i+1} \le |k_i - k_{i+1}|^s A_{i+1} \le \|(\tilde u - k_i)_+\|^s_{s,Q_{i+1}} \le \|(\tilde u - k_i)_+ \tilde\eta_i^{\beta/s}\|^s_{s,Q_i}$$

and the imbedding theorem (4.5) ensures that

$$\left\|(\tilde u - k_i)_+ \tilde\eta_i^{\beta/s}\right\|^s_{s,Q_i} \le C(p,n) A_i^{s/(n+s)} \left\|(\tilde u - k_i)_+ \tilde\eta_i^{\beta/s}\right\|^s_{V^s(Q_i)},$$

the inequality (5.37) becomes

$$A_{i+1} \le C(n,p,M_0) 4^{i\beta} \left(R^{-s} A_i^{1+s/(n+s)} + \left(\frac{\omega}{2^\lambda}\right)^{-s} A_i^{1+s/(n+s)} \right).$$

Or, in view of (5.2),

$$A_{i+1} \le C(n,p,M_0) 4^{i\beta} R^{-s} A_i^{1+s/(n+s)}.$$

Now, setting

$$X_i = \frac{A_i}{|Q_i|}$$

and using the assumption $s \ge 2$, we come to the recurrent relation

$$X_{i+1} \le C(n,p,M_0) 4^{i\beta} X_i^{1+s/(n+s)} \le C(n,p,M_0) 4^{i\beta} X_i^{1+2/(n+2)}.$$

Hence, under the condition

$$X_0 \le C^{-(n+2)/2} 4^{-\beta(n+2)^2/4} \equiv \nu_* \in (0,1), \quad (5.38)$$

together with 4.3, we have

$$X_i \to 0 \quad \text{for} \quad i \to \infty. \quad (5.39)$$

Thus, under the condition (5.38), using (5.39), we come to the desired relation (5.33).

Let us establish (5.38). Set

$$E_q(t) = \{x \colon x \in K_R,\ u(x,t) > M - 2^{-q}\omega\},$$

$$E_q = \{(x,t) \colon (x,t) \in Q_{R,aR^s/2},\ u(x,t) > M - 2^{-q}\omega\}$$

and rewrite (5.38) in the form

$$|E_\lambda| \le \nu_* |Q_{R,aR^s/2}|. \quad (5.40)$$

Let $k = M - 2^{-q}\omega$, $q_0 \le q \le \lambda$, where q_0 is the constant from Lemma 5.5. Consider a truncating function η in Q_{2R,aR^s} such that $\eta = 1$ in $Q_{R,aR^s/2}$, $\eta = 0$ on the parabolic boundary of Q_{2R,aR^s}, $|\nabla\eta| \le R^{-1}$ and

$0 \le \eta_t' \le 2a^{-1}R^{-s}$. Applying the estimate (4.3) to $(u-k)_+$ and η and using the logarithmic condition, we find that

$$\int\limits_{Q_{R,aR^s/2}} |\nabla(u-k)_+|^s\,dx\,dt \le C(p,M_0)R^{-s}\Bigg(\int\limits_{Q_{2R,aR^s}} (u-k)_+^s\,dx\,dt + a^{-1}\int\limits_{Q_{2R,aR^s}} (u-k)_+^2\,dx\,dt\Bigg)$$
$$+\int\limits_{Q_{2R,aR^s}} \chi\big((u-k)_+>0\big)\,dx\,dt. \quad (5.41)$$

Let us estimate the terms in the right-hand side of (5.41) from above. Since $(u-k)_+ \le 2^{-q}\omega$ in Q_{2R,aR^s}, we have

$$\int\limits_{Q_{2R,aR^s}} (u-k)_+^s\,dx\,dt \le C(n)\left(\frac{\omega}{2^q}\right)^s |Q_{R,aR^s/2}|,$$

and due to the definition of the constant a (see (5.1)) and the choice of $q \le \lambda$ we obtain

$$a^{-1}\int\limits_{Q_{2R,aR^s}} (u-k)_+^2\,dx\,dt \le C(n)\left(\frac{\omega}{2^\lambda}\right)^{s-2}\left(\frac{\omega}{2^q}\right)^2 |Q_{R,aR^s/2}| \le C(n)\left(\frac{\omega}{2^q}\right)^s |Q_{R,aR^s/2}|.$$

Moreover (see (5.2)),

$$\int\limits_{Q_{2R,aR^s}} \chi\big((u-k)_+>0\big)\,dx\,dt \le C(n)|Q_{R,aR^s/2}| \le C(n)R^{-s}\left(\frac{\omega}{2^q}\right)^s |Q_{R,aR^s/2}|,$$

and in by (5.41), we have

$$\int\limits_{E_q} |\nabla u|^s\,dx\,dt \le C(n,p,M_0)R^{-s}\left(\frac{\omega}{2^q}\right)^s |Q_{R,aR^s/2}|. \quad (5.42)$$

To obtain an upper estimate for the measure of E_λ on the basis of (5.42), we utilize Lemma 4.2 applied to $u(x,t)$ for fixed values $t \in (-aR^s/2, 0)$ and the levels

$$k = M - 2^{-q}\omega, \quad l = M - 2^{-q-1}\omega, \quad l-k = 2^{-q-1}\omega.$$

Corollary 5.2 ensures the inequality

$$|\{x\colon x \in K_R,\ u(x,t) < k\}| \equiv |K_R| - |E_q(t)| \ge \left(\frac{\nu_0}{2}\right)^2 |K_R|,$$

and therefore, using (4.7), we get

$$\frac{\omega}{2^{q+1}}|E_{q+1}(t)| \le C(n)R\nu_0^{-2}\int\limits_{E_q(t)\setminus E_{q+1}(t)} |\nabla u|\,dx.$$

Integrating this relation in $t \in (-aR^s/2, 0)$, we find that

$$\frac{\omega}{2^{q+1}}|E_{q+1}| \le C(n)R\nu_0^{-2}\int\limits_{E_q\setminus E_{q+1}} |\nabla u|\,dx\,dt.$$

Now, using the Hölder inequality, from (5.42) we come to the estimate

$$\frac{\omega}{2^{q+1}}|E_{q+1}| \le C(n,p,M_0)\nu_0^{-2}\left(\frac{\omega}{2^q}\right)|Q_{R,aR^s/2}|^{1/s}|E_q \setminus E_{q+1}|^{(s-1)/s}.$$

Thus,

$$|E_{q+1}|^{s/(s-1)} \le C(n,p,M_0)\nu_0^{-2s/(s-1)}|Q_{R,aR^s/2}|^{1/(s-1)}|E_q \setminus E_{q+1}|.$$

Note that for q taking the values $q_0, q_0+1, \dots, \lambda-1$, we have $|E_{q+1}| \geq |E_\lambda|$. Summing the inequalities obtained over the said q and using the fact that $E_{q_0} \subset Q_{R,aR^s/2}$, we get

$$(\lambda - q_0)|E_\lambda|^{s/(s-1)} \leq C(n,p,M_0)\nu_0^{-2s/(s-1)}|Q_{R,aR^s/2}|^{1/(s-1)}|E_{q_0} \setminus E_\lambda| \\ \leq C(n,p,M_0)\nu_0^{-2s/(s-1)}|Q_{R,aR^s/2}|^{s/(s-1)}$$

and

$$|E_\lambda| \leq \frac{C(n,p,M_0)}{(\lambda-q_0)^{(s-1)/s}}\nu_0^{-2}|Q_{R,aR^s/2}|.$$

It remains to note that $s \geq 2$ and the desired inequality (5.40) is valid, if $\lambda \geq q_0+1$ is chosen from the relation

$$\frac{C(n,p,M_0)}{(\lambda-q_0)^{1/2}}\nu_0^{-2} \leq \nu_*.$$

The lemma is proved. □

The main result of this section is the following statement.

Corollary 5.3. *Under the assumption* (5.6) *with the constant* $\nu_0 \in (0,1)$ *from Lemma* 5.1*, there exist constants* $\sigma_2 \in (0,1)$ *and* $\lambda > 1$ *depending only on* n*,* p*,* M_0 *and such that either* $\omega \leq 2^\lambda R$ *or*

$$\operatorname*{ess\,osc}_{Q_{R/2,(a/2)(R/2)^s}} u \leq \sigma_2\omega. \tag{5.43}$$

The proof of this statement is similar to that of Corollary 5.1, and $\sigma_2 = 1 - 2^{-\lambda-1}$.

5.3. Oscillation Lemma. From Corollaries 5.1 and 5.3, we obtain the following result.

Proposition 5.1. *There exist constants* $\sigma \in [1/8, 1)$ *and* $l \geq \lambda > 1$ *depending only on* n*,* p*,* M_0 *and such that either* $\omega \leq 2^l R$ *or*

$$\operatorname*{ess\,osc}_{Q_{R/8,b(R/8)^s}} u \leq \sigma\omega. \tag{5.44}$$

Proof. Since $b(R/8)^s \leq a(R/2)^s$, we have

$$Q_{R/8,b(R/8)^s} \subset Q_{R/2,(a/2)(R/2)^s},$$

and, using the estimates (5.25), (5.43), we come to the inequality (5.44) with the constant $\sigma = \max\{1/8, \sigma_1, \sigma_2\}$. The proposition is proved. □

In the next statement, the constants l, λ, σ have the same meaning as above, and for the sake of uniformity of notation, we take $R_0 = R$, $a_0 = a$, $\omega_0 = \omega$, $s_0 = s$.

Lemma 5.7. *There is a constant* $\mathcal{C} > 8$ *depending only on* n*,* p*,* M_0 *and such that if*

$$R_i = \mathcal{C}^{-i}R, \quad s_i = \operatorname*{ess\,inf}_{\mathbf{Q}_{R_i}^{(x_0,t_0)}} p(x,t), \quad \omega_i = \sigma^i\omega, \tag{5.45}$$

$$a_i = 2\left(\frac{\omega_i}{2^\lambda}\right)^{2-s_i}, \quad Q_i = Q_{R_i,a_iR_i^{s_i}}, \qquad i = 0,1,\dots, \tag{5.46}$$

and $\omega_0 > 2^l R_0$*,* $s_0 \geq 2$*, then*

$$Q_{i+1} \subset Q_i \subset \mathbf{Q}_{R_i}^{(x_0,t_0)}$$

and

$$\operatorname*{ess\,osc}_{Q_i} u \leq \omega_i. \tag{5.47}$$

Proof. Recall that as the starting point in the derivation of the estimate (5.44) we have taken the relation $\operatorname*{ess\,osc}_{Q_0} u \le \omega_0$. Let us use the relation

$$b\left(\frac{R_0}{8}\right)^{s_0} = \left(\frac{\omega}{\omega_1}\right)^{2-s_0}\left(\frac{2^\lambda}{2}\right)^{2-s_0}\left(\frac{\omega_1}{2^\lambda}\right)^{2-s_0}\frac{R_0^{s_0}}{8^{s_0}} = \sigma^{s_0-2}2^{(\lambda-1)(2-s_0)-3s_0-1}\left(\frac{\omega_1}{2^\lambda}\right)^{s_1-s_0}2\left(\frac{\omega_1}{2^\lambda}\right)^{2-s_1}R_0^{s_0} \tag{5.48}$$

and chose $\mathcal{C} = \sigma^{-1}2^{(\lambda-1)+7/2}$. Since $\mathcal{C} > 8$ and $\sigma \in [1/8, 1)$, it follows that $\omega_1 > 2^l R_1$. Moreover, $s_1 \ge s_0 \ge 2$, $l \ge \lambda$, and from (5.48) we get

$$b\left(\frac{R_0}{8}\right)^{s_0} \ge \sigma^{s_0-2}2^{(\lambda-1)(2-s_0)-3s_0-1}\mathcal{C}^{s_0}a_1R_1^{s_1} > a_1R_1^{s_1}.$$

Thus, $Q_1 \subset Q_{R/8,b(R/8)^s}$, and by the estimate (5.44) from Proposition 5.1 we have

$$\operatorname*{ess\,osc}_{Q_1} u \le \operatorname*{ess\,osc}_{Q_{R/8,b(R/8)^s}} u \le \sigma\omega_0 = \omega_1.$$

Starting from the relation just obtained, together with the inequality $\omega_1 > 2^l R_1$, and repeating the above arguments, we come to the desired estimate (5.47). The lemma is proved. ☐

6. Case $s < 2$

Here we also use the notation from Sec. 4.1 and assume that $s < 2$. The choice of the point $x_0 \in \Omega$ is as above and the proof of all statements is again carried out in the same manner for equations with the exponent p that satisfies either (1.3) or (1.8). In the case under consideration, the structure of the parabolic cylinders is somewhat different and involves a positive integer $\mathcal{N} = \mathcal{N}(n, p, M_0)$ to be chosen later. Consider the cylinder

$$Q_{\mathcal{N}R^{s/2},R^s} = Q^{(x_0,t_0)}_{\mathcal{N}R^{s/2},R^s} = K^{x_0}_{\mathcal{N}R^{s/2}} \times (t_0 - R^s, t_0), \quad Q_{\mathcal{N}R^{s/2},R^s} \subset \mathbf{Q}^{(x_0,t_0)}_R,$$

and set

$$m = \operatorname*{ess\,inf}_{Q_{\mathcal{N}R^{s/2},R^s}} u, \quad M = \operatorname*{ess\,sup}_{Q_{\mathcal{N}R^{s/2},R^s}} u, \quad \omega = \operatorname*{ess\,osc}_{Q_{\mathcal{N}R^{s/2},R^s}} u = M - m.$$

The oscillation lemma is proved for cylinders of the form

$$Q_{cR,R^s} = Q^{(x_0,t_0)}_{cR,R^s} = K^{x_0}_{cR} \times (t_0 - R^s, t_0'), \quad c = \mathcal{N}\left(\frac{\omega}{2^\lambda}\right)^{(s-2)/s}, \tag{6.1}$$

where $\lambda(n, p, M_0) > 1$ is a constant to be determined later. As in Sec. 5, we do not indicate the "vertex" of cylinders, if it coincides with the point (x_0, t_0), and do not indicate the center of cubes, if it coincides with x_0. Moreover, it is assumed that $(x_0, t_0) = (0, 0)$.

In what follows, we assume that

$$\omega > 2^\lambda R^{s/2}. \tag{6.2}$$

If this inequality is violated, then the required estimate for the oscillation of the solution holds in the cylinder $Q_{\mathcal{N}R^{s/2},R^s}$. From (6.2), it follows that $Q_{cR,R^s} \subset Q_{\mathcal{N}R^{s/2},R^s}$.

The proof of the oscillation lemma is based on the examination of two alternatives for the cylinders

$$Q^{(\bar{x},0)}_{dR,R^s} = K^{\bar{x}}_{dR} \times (-R^s, 0), \quad d = \left(\frac{\omega}{2}\right)^{(s-2)/s}, \tag{6.3}$$

which belong to Q_{cR,R^s}, provided that the following condition is satisfied:

$$\bar{x} \in \bar{K}_{\mathcal{R}}, \quad \mathcal{R} = \left(2^{(\lambda-1)(2-s)/s}\mathcal{N} - 1\right)dR. \tag{6.4}$$

Below, we consider two complementary cases involving the constant $\nu_0 \in (0, 1)$, which depends only on n, p, M_0 and is crucial for the proof of the oscillation lemma.

Alternative 1. There is a cylinder $Q^{(\bar{x},0)}_{dR,R^s}$ defined in (6.3), (6.4), for which

$$\Big|\Big\{(x,t)\colon (x,t)\in Q^{(\bar{x},0)}_{dR,R^s},\ u(x,t)<m+\frac{\omega}{2}\Big\}\Big|\le \nu_0\big|Q^{(\bar{x},0)}_{dR,R^s}\big|. \tag{6.5}$$

Alternative 2. For any cylinder $Q^{(\bar{x},0)}_{dR,R^s}$ defined in (6.3), (6.4), the following inequality holds:

$$\Big|\Big\{(x,t)\colon (x,t)\in Q^{(\bar{x},0)}_{dR,R^s},\ u(x,t)>M-\frac{\omega}{2}\Big\}\Big|<(1-\nu_0)\big|Q^{(\bar{x},0)}_{dR,R^s}\big|. \tag{6.6}$$

First, we prove an auxiliary statement which is used in the analysis of both alternatives. This statement involves cylinders of the form

$$\mathcal{Q}^{(\bar{x},\bar{t})}_R=K^{\bar{x}}_{d_1R}\times(\bar{t}-2^{\mu_0(s-2)}h_0R^s,\bar{t}),\quad \mathcal{Q}^{(\bar{x},\bar{t})}_R\subset Q_{cR,R^s},$$
$$\text{where}\ \ \mu_0\ge 0,\ \ \mu_1\ge 1,\ \ h_0\in(0,1],\ \ h_1\ge 1,\ \ d_1=h_1\Big(\frac{\omega}{2^{\mu_1}}\Big)^{(s-2)/s}. \tag{6.7}$$

For $(\bar{x},\bar{t})=(0,0)$, $\mu_0=0$, $\mu_1=\lambda$, $h_0=1$, $h_1=\mathcal{N}$, we have a cylinder of the form (6.1). For $\bar{t}=0$, $\mu_0=0$, $\mu_1=1$, $h_0=1$, $h_1=1$, and a suitable chosen $\bar{x}$, we obtain cylinders defined in (6.3), (6.4).

Below, $\mu=\mu_0+\mu_1$ and, in addition to (6.2), it is assumed that

$$\omega>2^{\lambda_0}R^{s/2},\quad \lambda_0=\max\{\lambda,\mu\}. \tag{6.8}$$

Here and in what follows, $\chi(E)$ stands for the characteristic function of a measurable set $E\subset\mathbb{R}^{n+1}$.

In the next statement, it is important that the constant ν_* does not depend on μ_0, μ_1, λ.

Lemma 6.1. *Under the assumption* (6.8), *there is a constant* $\nu_*\in(0,1)$ *depending only on* n, p, M_0, h_0, h_1 *and such that the condition*

$$|\{(x,t)\colon (x,t)\in\mathcal{Q}^{(\bar{x},\bar{t})}_R,\ u(x,t)<m+2^{-\mu}\omega\}|\le\nu_*\big|\mathcal{Q}^{(\bar{x},\bar{t})}_R\big|$$

implies that

$$u(x,t)\ge m+\frac{\omega}{2^{\mu+1}}\quad a.e.\ in\ \ \mathcal{Q}^{(\bar{x},\bar{t})}_{R/2}, \tag{6.9}$$

and the condition

$$|\{(x,t)\colon (x,t)\in\mathcal{Q}^{(\bar{x},\bar{t})}_R,\ u(x,t)>M-2^{-\mu}\omega\}|\le\nu_*\big|\mathcal{Q}^{(\bar{x},\bar{t})}_R\big|, \tag{6.10}$$

implies that

$$u(x,t)\le M-\frac{\omega}{2^{\mu+1}}\quad a.e.\ in\ \ \mathcal{Q}^{(\bar{x},\bar{t})}_{R/2}. \tag{6.11}$$

Proof. We restrict ourselves to the case $(\bar{x},\bar{t})=(0,0)$ and prove (6.9). Consider the numerical sequence

$$R_i=\frac{R}{2}+\frac{R}{2^{i+1}},\quad k_i=m+\frac{\omega}{2^{\mu+1}}+\frac{\omega}{2^{\mu+i+1}},\qquad i=0,1,\dots,$$

and the cylinders

$$\mathcal{Q}_i=K_i\times\big(-2^{\mu_0(s-2)}h_0R_i^s,0\big),\quad\text{where}\ \ K_i=K_{d_1R_i}.$$

Let η_i be a truncating function such that $\eta_i=1$ in $\mathcal{Q}_{i+1}$, $\eta_i=0$ on the parabolic boundary of $\mathcal{Q}_i$, and

$$|\nabla\eta_i|\le 2^{i+2}(Rh_1)^{-1}\Big(\frac{\omega}{2^{\mu_1}}\Big)^{(2-s)/s},\quad 0\le(\eta_i)'_t\le 2^{\mu_0(2-s)}2^{s(i+2)}h_0^{-1}R^{-s}.$$

By the estimate (4.3), applied to the functions $(u-k_i)_-$, η_i, and the logarithmic condition, we have

$$\sup_{t\in(-2^{\mu_0(s-2)}R_i^s,0)} \int\limits_{K_i\times\{t\}} (u-k_i)_-^2 \eta_i^\beta \, dx + \int\limits_{Q_i} \big|\nabla\big((u-k_i)_-\eta_i^{\beta/s}\big)\big|^s \, dx\, dt$$
$$\le C(p,M_0) 2^{i\beta} R^{-s}\left(h_1^{-s}\left(\frac{\omega}{2^{\mu_1}}\right)^{2-s} \int\limits_{Q_i}(u-k_i)_-^s \, dx\, dt + 2^{\mu_0(2-s)} h_0^{-1} \int\limits_{Q_i}(u-k_i)_-^2 \, dx\, dt\right)$$
$$+ \int\limits_{Q_i} \chi\big((u-k_i)_- > 0\big)\, dx\, dt.$$

Since

$$(u-k_i)_- \le \sup_{Q_i}(u-k_i)_- \le \frac{\omega}{2^{\mu+1}} + \frac{\omega}{2^{\mu+i+1}} \le \frac{\omega}{2^\mu}$$

and

$$\left(\frac{\omega}{2^{\mu_1}}\right)^{2-s}\left(\frac{\omega}{2^{\mu}}\right)^{s} = \left(\frac{\omega}{2^{\mu}}\right)^{2} 2^{\mu_0(2-s)}, \quad h_0\in(0,1], \quad h_1 \ge 1,$$

we find that

$$\sup_{t\in(-2^{\mu_0(s-2)}R_i^s,0)} \int\limits_{K_i\times\{t\}} (u-k_i)_-^2 \eta_i^\beta \, dx + \int\limits_{Q_i} \big|\nabla\big((u-k_i)_-\eta_i^{\beta/s}\big)\big|^s \, dx\, dt$$
$$\le C 2^{i\beta} R^{-s} 2^{\mu_0(2-s)} h_0^{-1}\left(\frac{\omega}{2^{\mu}}\right)^{2} \int\limits_{Q_i} \chi\big((u-k_i)_- > 0\big)\, dx\, dt + \int\limits_{Q_i} \chi\big((u-k_i)_- > 0\big)\, dx\, dt, \quad (6.12)$$

where $C = C(p, M_0)$. To estimate the integrals in the left-hand side of (6.12), set

$$\bar{k}_i = \frac{k_i + k_{i+1}}{2} \in (k_{i+1}, k_i).$$

Then

$$\int\limits_{K_i\times\{t\}} (u-k_i)_-^2 \eta_i^\beta \, dx \ge \int\limits_{K_i\times\{t\}} (k_i-\bar{k}_i)^{2-s}(u-\bar{k}_i)_-^s \eta_i^\beta \, dx$$
$$= \left(\frac{\omega}{2^{\mu}}\right)^{2-s} 2^{(i+3)(s-2)} \int\limits_{K_i\times\{t\}} (u-\bar{k}_i)_-^s \eta_i^\beta \, dx \quad \text{for all} \quad t\in(-2^{\mu_0(s-2)}R_i^s,0)$$

and since $k_i > \bar{k}_i$, we have

$$\int\limits_{Q_i} \big|\nabla\big((u-k_i)_-\eta_i^{\beta/s}\big)\big|^s \, dx\, dt \ge \int\limits_{Q_i} \big|\nabla\big((u-\bar{k}_i)_-\eta_i^{\beta/s}\big)\big|^s \, dx\, dt.$$

Taking into account the last two inequalities in (6.12) and dividing both sides of the relation just obtained by

$$\left(\frac{\omega}{2^{\mu}}\right)^{2-s} 2^{(i+3)(s-2)},$$

we find that

$$\sup_{t\in(-2^{\mu_0(s-2)}R_i^s,0)}\int\limits_{K_i\times\{t\}}(u-\bar k_i)_-^s\eta_i^\beta\,dx+\left(\frac{\omega}{2^\mu}\right)^{s-2}2^{(i+3)(2-s)}\int\limits_{\mathcal{Q}_i}\big|\nabla\big((u-\bar k_i)_-\eta_i^{\beta/s}\big)\big|^s\,dx\,dt$$
$$\le C(p,M_0)2^{(2-s+\beta)i}R^{-s}2^{\mu_0(2-s)}h_0^{-1}\left(\frac{\omega}{2^\mu}\right)^s\int\limits_{\mathcal{Q}_i}\chi\big((u-k_i)_->0\big)\,dx\,dt$$
$$+\left(\frac{\omega}{2^\mu}\right)^{s-2}2^{(i+3)(2-s)}\int\limits_{\mathcal{Q}_i}\chi\big((u-k_i)_->0\big)\,dx\,dt,$$

and since by assumption (6.8) we have $\left(\frac{\omega}{2^\mu}\right)^{-2}\le R^{-s}$, it follows that

$$\sup_{t\in(-2^{\mu_0(s-2)}R_i^s,0)}\int\limits_{K_i\times\{t\}}(u-\bar k_i)_-^s\eta_i^\beta\,dx+\left(\frac{\omega}{2^\mu}\right)^{s-2}2^{(i+3)(2-s)}\int\limits_{\mathcal{Q}_i}\big|\nabla\big((u-\bar k_i)_-\eta_i^{\beta/s}\big)\big|^s\,dx\,dt$$
$$\le C(p,M_0)2^{(2-s+\beta)i}R^{-s}2^{\mu_0(2-s)}h_0^{-1}\left(\frac{\omega}{2^\mu}\right)^s\int\limits_{\mathcal{Q}_i}\chi\big((u-k_i)_->0\big)\,dx\,dt. \quad (6.13)$$

Let us pass to the variables

$$y=d_1^{-1}x=h_1^{-1}\left(\frac{\omega}{2^{\mu_1}}\right)^{(2-s)/s}x,\quad \tau=2^{\mu_0(2-s)}h_0^{-1}t,$$

with the cylinders $\mathcal{Q}_i$ transformed into $Q_i=K_{R_i}\times(-R_i^s,0)$, and set

$$\tilde u(y,\tau)=u(d_1y,2^{\mu_0(s-2)}h\tau),\quad \tilde\eta_i(y,\tau)=\eta_i(d_1y,2^{\mu_0(s-2)}h_0\tau),$$
$$A_i(\tau)=\{y\colon y\in K_{R_i},\ \tilde u(y,\tau)<k_i\},\quad A_i=\int\limits_{-R_i^s}^0|A_i(\tau)|\,d\tau.$$

Now (6.13) turns into

$$\sup_{\tau\in(-R_i^s,0)}\int\limits_{K_i\times\{\tau\}}(\tilde u-\bar k_i)_-^s\tilde\eta_i^\beta\,dy+2^{(i+3)(2-s)}2^{\mu_0(s-2)}h_0h_1^{-s}\left(\frac{2^\mu}{2^{\mu_1}}\right)^{2-s}\int\limits_{Q_i}\big|\nabla\big((\tilde u-\bar k_i)_-\tilde\eta_i^{\beta/s}\big)\big|^s\,dy\,d\tau$$
$$\le C(p,M_0)2^{(2-s+\beta)i}R^{-s}\left(\frac{\omega}{2^\mu}\right)^sA_i.$$

Since $\mu=\mu_0+\mu_1$, $2^{(i+3)(2-s)}>1$ and $h_1^s\le h_1^\beta$, it follows that

$$\|(\tilde u-\bar k_i)_-\tilde\eta_i^{\beta/s}\|^s_{V^s(Q_i)}\le C(p,M_0)2^{(2-s+\beta)i}h_0^{-1}h^\beta R^{-s}\left(\frac{\omega}{2^\mu}\right)^sA_i. \quad (6.14)$$

Let us estimate the left-hand side of this inequality from below. Using the inequality $k_{i+1}<\bar k_i$ and the explicit form of the constants k_i, $\bar k_i$, we get

$$2^{-(i+3)s}\left(\frac{\omega}{2^\mu}\right)^sA_{i+1}=|\bar k_i-k_{i+1}|^sA_{i+1}=\int\limits_{Q_{i+1}}(\bar k_i-k_{i+1})^s\chi\big((\tilde u<k_{i+1})\big)\,dy\,d\tau$$
$$\le\int\limits_{Q_{i+1}}(\tilde u-\bar k_i)_-^s\tilde\eta_i^\beta\,dy\,d\tau\le\|(\tilde u-\bar k_i)_-^s\tilde\eta_i^{\beta/s}\|^s_{s,Q_i},$$

and by the imbedding theorem (4.5),

$$\|(\tilde u-\bar k_i)_-^s\tilde\eta_i^{\beta/s}\|^s_{s,Q_i}\le C(n,p)A_i^{s/(n+s)}\|(\tilde u-\bar k_i)_-^s\tilde\eta_i^{\beta/s}\|^s_{V^s(Q_i)}.$$

Now from (6.14) we find that

$$A_{i+1} \le C(n,p,M_0)4^{i\beta}h_0^{-1}h_1^{\beta}R^{-s}A_i^{1+s/(n+s)}.$$

Setting

$$X_i = \frac{A_i}{|Q_i|},$$

we come to the recurrent relation

$$X_{i+1} \le C(n,p,M_0)h_1^{\beta}h_0^{-1}4^{i\beta}X_i^{1+s/(n+s)}.$$

it remains to note that if

$$X_0 \le (Ch_1^{\beta}h_0^{-1})^{-(n+s)/s}4^{-\beta(n+s)^2/s^2},$$

then $X_i \to 0$ as $i \to \infty$, according to Proposition 4.3. Setting

$$X_0 \le \nu_* \equiv (Ch_1^{\beta}h_0^{-1})^{-(n+\alpha)/\alpha}4^{-\beta(n+\alpha)^2/\alpha^2} < (Ch_1^{\beta}h_0^{-1})^{-(n+s)/s}4^{-\beta(n+s)^2/s^2},$$

where α is the constant from (1.2), we come to (6.9), since

$$X_0 = \frac{|\{(y,z)\colon (y,z)\in Q_0,\ \tilde{u}(y,\tau) < k_0\}|}{|Q_0|} = \frac{|\{(x,t)\colon (x,t)\in \mathcal{Q}_R,\ u(x,t) < m+\omega/2^{\mu}\}|}{|\mathcal{Q}_R|} \le \nu_*$$

and

$$|\{(x,t)\colon (x,t)\in \mathcal{Q}_{R/2},\ u(x,t) < m + 2^{-\mu-1}\omega\}| = 0.$$

In a similar way, we establish relation (6.11). The lemma is proved. □

In what follows, the constant ν_* from Lemma 6.1 for cylinders of the form (6.3) is denoted by ν_0. As mentioned above, for such cylinders we have $\mu_0 = 0$, $\mu_1 = 1$, $h_0 = h_1 = 1$ and the assumption (6.8) coincides with (6.2). In this case, Lemma 6.1 can be refined as follows.

Corollary 6.1. *There is a constant ν_0 depending only on n, p, M_0 and such that the alternative* (6.5) *ensures the inequality*

$$u(x,t) \ge m + \frac{\omega}{4} \quad \text{a.e. in} \quad Q^{(\bar{x},0)}_{dR/2,(R/2)^s}. \tag{6.15}$$

6.1. Analysis of the First Alternative. Assume that alternative (6.5) holds with the constant ν_0 from Corollary 6.1. Let us show that (6.15) implies a similar estimate in $Q_{cR/4,(R/8)^s}$. To this end, it suffices to extend the inequality (6.5) to the cylinder $Q_{cR/2,(R/4)^s}$ and use relation (6.9) from Lemma 6.1 in that cylinder.

Consider an auxiliary function

$$v(x,t) = 2(u(x,t)-m)\omega^{-1}, \tag{6.16}$$

which is nonnegative in $Q_{\mathcal{N}R^{s/2},R^s}$. Let us change the variables by

$$x \mapsto 4(cR)^{-1}x, \quad t \mapsto \left(\frac{8}{R}\right)^s t. \tag{6.17}$$

Thus, the cylinder $Q_{cR,(R/2)^s}$ is mapped onto $\tilde{Q} = K_4 \times (-4^s, 0)$, and the cylinder $Q^{(\bar{x},0)}_{dR/2,(R/2)^s}$ from (6.15) is mapped onto $K_e^{x^*} \times (-4^s, 0) \subset \tilde{Q}$, where x^* is the image of $\bar{x}$ and

$$e = 2dc^{-1} = 2^{2/s+\lambda(s-2)/s}\mathcal{N}^{-1}, \quad e_1(p,\lambda)\mathcal{N}^{-1} \le e \le e_2(p,\lambda)\mathcal{N}^{-1}. \tag{6.18}$$

The above estimate for e follows from condition (1.2), together with the constraint $s < 2$. In the new variables, for which we keep the previous notation, the transformed functions w and p depend on the parameter R. For the exponent p we do not specify this dependence, using the previous notation, and the transformed function v will be denoted by v_R. After the transformation (6.17), the function v_R satisfies the following equation in the image of the cylinder Q_T:

$$\frac{\partial v}{\partial t} - \operatorname{div}(a(x,t)|\nabla v_R|^{p-2}\nabla v_R) = 0, \tag{6.19}$$

where

$$a(x,t) = 2^{2p(x,t)-3s}\mathcal{N}^{-p(x,t)}\left(\frac{\omega}{2^\lambda}\right)^{p(x,t)(2-s)/s}\left(\frac{\omega}{2}\right)^{p(x,t)-2}R^{s-p(x,t)}.$$

From (1.2), (6.2), the inequality $\omega \le 2M_0$, and the logarithmic condition, it follows that

$$0 < a_1(p,\lambda,M_0,\mathcal{N}) \le a(x,t) \le a_2(p,\lambda,M_0,\mathcal{N}) < \infty, \quad (x,t) \in \tilde{Q}. \tag{6.20}$$

The inequality (6.15) established above takes the form

$$v_R(x,t) \ge \frac{1}{2} \ \text{ for almost all } \ (x,t) \in K_e^{x^*} \times (-4^s, 0). \tag{6.21}$$

Our next aim is to prove the following statement which implies an estimate for the oscillation of the solution.

Lemma 6.2. *For any $\nu \in (0,1)$, there exist constants $\gamma \in (0,1)$ and $\mathcal{R}_0 > 0$ depending only on n, p, λ, M_0, $\mathcal{N}$, ν and such that for $R \le \mathcal{R}_0$, the following inequality holds*:

$$|\{x \colon x \in K_2, \ v_R(x,t) \le \gamma\}| \le \nu|K_2| \ \text{ for all } \ t \in [-2^s, 0]. \tag{6.22}$$

For equation (1.1) with a constant growth exponent, $p(x,t) = \text{const}$, a similar lemma is proved in [9]. The estimates of solutions obtained in [9] are nonuniform in p for $p \to 2$. Utilizing the scheme proposed in [9], we modify the proof so that the constants in all estimates of solutions for $s \to 2$ are uniformly bounded in s.

We require some integral estimates of solutions in involving functions of the form

$$\Phi_k(v_R) = \int\limits_0^{(k-v_R)_+} \frac{d\tau}{((1+\delta)k - \tau + R)^{s-1}}, \tag{6.23}$$

$$\Psi_k(v_R) = \ln\left(\frac{(1+\delta)k + R}{(1+\delta)k - (k-v_R)_+ + R}\right). \tag{6.24}$$

Here $k \in (0, 1/8)$, $\delta \in (0, 1/8)$,

$$R \le k\delta^{\alpha/(\alpha-1)}, \tag{6.25}$$

and α is the constant from condition (1.2). As a prerequisite, we mention two estimates connected with $\Phi_k(v_R)$. A simple verification taking into account (6.25) shows that

$$\frac{1}{((1+\delta)k - \tau + R)^{s-1}} > \frac{(1+\delta)^{-1}}{((1+\delta)k - \tau)^{s-1}} \ \text{ for all } \ \tau \in [0,k]. \tag{6.26}$$

Moreover, direct calculation of the integral on the right-hand side of (6.23), together with the estimate $1 - (1-z)^{2-s} < (2-s)\ln(1-z)^{-1}$ for $z \in (0,1)$, yields

$$\Phi_k(v_R) \le \ln\frac{(1+\delta)k + R}{\delta k + R} \le \ln\frac{1+\delta}{\delta}. \tag{6.27}$$

For definiteness, suppose that condition (1.3) holds. Then, taking into account the structure of the function v_R and using the inequality (3.4) from Lemma 3.2, we can claim that the truncated function $w = (k - v_R)_+$, for all $-4^{-s} \le \tau_1 < \tau \le 0$ and sufficiently small $h > 0$, satisfies the integral inequality

$$\int\limits_{\tau_1}^{\tau}\int\limits_{K_4}\left(\frac{\partial w_h}{\partial t}\psi + (a|\nabla w|^{p-2}\nabla w)_h \cdot \nabla\psi \, dx\, dt\right) \le 0 \tag{6.28}$$

with regularizations of the form (2.1) and nonnegative test functions $\psi \in W(\tilde{Q})$ vanishing on the lateral surface of the cylinder $\tilde{Q}$.

We take the test function

$$\psi = \frac{\eta^\beta}{((1+\delta)k - w_h + R)^{s-1}}$$

in (6.28). Here, β is the constant from (1.2), η is a truncating function in cylinder $\tilde{Q}$; η depends on the size e of the cube $K_e^{x^*}$ in (6.18), has the form $\eta(x,t) = \eta_1(x)\eta_2(t)$, and possesses the following properties: $\eta = 1$ in $K_{4-e/2} \times (-2^s, 0)$, $\eta = 0$ on the parabolic boundary of $\tilde{Q}$, $|\nabla\eta_1| \le 2/e$, $0 \le \eta_2' \le 1$, the sets $\{x\colon \eta(x) > l\}$ are convex for any $l \in (0,1)$.

Passing to the limit for $h \to 0$ (this is possible due to the interpolation Lemma 2.1) and using simple estimates involving the Young inequality, we find that

$$\int_{K_4} \Phi_k(v_R)\eta^\beta\,dx\Big|_{\tau_1}^{\tau} + \frac{1}{2}\int_{\tau_1}^{\tau}\int_{K_4} a|\nabla\Psi_k(v_R)|^p\big((1+\delta)k - (k-v_R)_+ + R\big)^{p-s}\eta^\beta\,dx\,dt$$
$$\le C(p)\int_{\tau_1}^{\tau}\int_{K_4} a\big((1+\delta)k - (k-v_R)_+ + R\big)^{p-s}|\nabla\eta|^p\,dx\,dt + \beta\int_{\tau_1}^{\tau}\int_{K_4}\Phi_k(v_R)\eta_t'\,dx\,dt.$$

By the logarithmic condition (1.3), we have

$$C_1(p) \le \big((1+\delta)k - (k-v_R)_+ + R\big)^{p-s} \le C_2(p).$$

Using this inequality, the structure of the truncating function η, and relations (6.18), (6.20), (6.27), we come to the inequality

$$\int_{K_4} \Phi_k(v_R)\eta^\beta\,dx\Big|_{\tau_1}^{\tau} + C_3(p,\lambda,M_0,\mathcal{N})\int_{\tau_1}^{\tau}\int_{K_4}|\nabla\Psi_k(v_R)|^p\eta^\beta\,dx\,dt \le C_4(n,p,\lambda,M_0,\mathcal{N})(\tau-\tau_1)\ln\frac{1+\delta}{\delta}.$$

Since $|\nabla\Psi_k(v_R)|^s \le |\nabla\Psi_k(v_R)|^p + 1$, it follows that

$$\int_{K_4} \Phi_k(v_R)\eta^\beta\,dx\Big|_{\tau_1}^{\tau} + C_3(p,\lambda,M_0,\mathcal{N})\int_{\tau_1}^{\tau}\int_{K_4}|\nabla\Psi_k(v_R)|^s\eta^\beta\,dx\,dt \le C_4(n,p,\lambda,M_0,\mathcal{N})(\tau-\tau_1)\ln\frac{1+\delta}{\delta}. \quad (6.29)$$

Under the logarithmic condition (1.8), the integral inequality (6.28) holds for W- and H-solutions with Steklov regularizations (2.5) and suitable nonnegative test functions ψ. Further, reasoning exactly as above, we again come to (6.29).

Remark 6.1. Since the functions $\Phi_k(z)$ and $\Psi_k(z)$ are continuously differentiable for $z \in [0,k]$ and $\big(k - v_R(\cdot,t)\big)_+\colon [-4^s,0] \to L^2(K_4)$ is continuous on $[-4^s,0]$, it is not difficult to show that the integrals

$$\int_{K_4\times\{\tau\}} \Phi_k(v_R)\eta^\beta\,dx, \qquad \int_{K_4\times\{\tau\}} \Psi_k^s(v_R)\eta^\beta\,dx$$

are continuous in $\tau \in [-4^s,0]$.

The function $\Psi_k(v_R)$ vanishes in the cylinder $K_e^{x^*} \times (-4^s,0)$ (see (6.21)), and from the Poincaré inequality (4.6) applied to the integral in the right-hand side of (6.29), we deduce that

$$\int_{K_4} \Phi_k(v_R)\eta^\beta\,dx\Big|_{\tau_1}^{\tau} + C_3(p,\lambda,M_0,\mathcal{N})\int_{\tau_1}^{\tau}\int_{K_4}\Psi_k^s(v_R)\eta^\beta\,dx\,dt \le C_4(n,p,\lambda,M_0,\mathcal{N})(\tau-\tau_1)\ln\frac{1+\delta}{\delta}. \quad (6.30)$$

We will need a somewhat different form of the inequality (6.30). Fixing a $\tau \in (-4^s,0]$, we define the upper left derivative

$$\frac{d^-}{d\tau}\int_{K_4\times\tau}\Phi_k(v_R)\eta^\beta\,dx = \limsup_{\tau_1\to\tau}\frac{1}{\tau-\tau_1}\Bigg(\int_{K_4\times\{\tau\}}\Phi_k(v_R)\eta^\beta\,dx - \int_{K_4\times\{\tau_1\}}\Phi_k(v_R)\eta^\beta\,dx\Bigg).$$

Then, according to (6.30) and Remark 6.1, for all $\tau \in (-4^s, 0]$ we have

$$\frac{d^-}{d\tau} \int\limits_{K_4 \times \{\tau\}} \Phi_k(v_R)\eta^\beta \, dx + C_3(p, \lambda, M_0, \mathcal{N}) \int\limits_{K_4 \times \tau} \Psi_k^s(v_R)\eta^\beta \, dx \le C_4(n, p, \lambda, M_0, \mathcal{N}) \ln \frac{1+\delta}{\delta}. \tag{6.31}$$

Prior to formulating a statement which is crucial for the proof of Lemma 6.2, we introduce the quantities

$$Y_i = \sup_{t \in [-4^s, 0]} \int\limits_{K_4 \cap \{x \colon v_R(x,t) < \delta^i\}} \eta^\beta(x, t) \, dx, \quad i = 1, 2, \dots . \tag{6.32}$$

Lemma 6.3. *Under the assumption* (6.21), *for any* $\nu \in (0, 1)$ *there is a constant* $\delta \in (0, 1/8)$ *depending only on* n, p, λ, M_0, $\mathcal{N}$, ν *and such that for all integer* $i = 1, 2, \dots, i_*$ *such that*

$$i_* \le \ln R^{-1} \ln^{-1} \delta^{-1} - \frac{\alpha}{\alpha - 1}, \tag{6.33}$$

we have either

$$Y_i \le \nu, \tag{6.34}$$

or

$$Y_{i+1} \le \max\{\nu, (1 - \delta)Y_i\}. \tag{6.35}$$

Proof. Take $k = \delta^i$, $\delta \in (0, 1/8)$, and fix $\varepsilon \in (0, 1)$. Note that condition (6.33) implies the inequality (6.25) for $k = \delta^{i_*}$. Definition (6.32) implies that there is $t_0 \in (-4^s, 0)$ such that

$$\int\limits_{K_4 \cap \{x \colon v_R(x, t_0) < \delta^{i+1}\}} \eta^\beta(x, t_0) \, dx \ge Y_{i+1} - \varepsilon. \tag{6.36}$$

Let $\mathcal{E}^+ \subset (-4^{-s}, 0]$ be a subset on which the derivative in (6.31) is nonnegative. Fixing i and t_0, consider two possible cases: $t_0 \in \mathcal{E}^+$ and $t_0 \notin \mathcal{E}^+$. In each case we can assume that $Y_i > \nu$; otherwise, the desired statement is obvious. Take $\varepsilon \in (0, \nu/2)$ and assume first that $t_0 \in \mathcal{E}^+$. Then, by (6.31), we have

$$\int\limits_{K_4} \Psi_{\delta^i}^s\big(v_R(x, t_0)\big)\eta^\beta(x, t_0) \, dx \le C(n, p, \lambda, M_0, \mathcal{N}) \ln \frac{1+\delta}{\delta}. \tag{6.37}$$

Narrowing the integration domain in the left-hand side of (6.37) to the set $\{x \colon v_R(x, t_0) < \delta^{i+1}\} \cap K_4$ and using, on that set, the estimate (see (6.24))

$$\Psi_{\delta^i}^s\big(v_R(x, t_0)\big) \ge \ln \frac{(1+\delta)\delta^i + R}{2\delta^{i+1} + R} \ge \ln \frac{1+\delta}{3\delta},$$

which follows from (6.33), we find that

$$\int\limits_{K_4 \cap \{x \colon v_R(x, t_0) < \delta^{i+1}\}} \eta^\beta(x, t_0) \, dx \le C(n, p, \lambda, M_0, \mathcal{N}) \left(\ln \frac{1+\delta}{3\delta}\right)^{1-s}.$$

Now, from (6.36) and the constraint on ε, it follows that

$$Y_{i+1} \le \frac{\nu}{2} + C(n, p, \lambda, M_0, \mathcal{N}) \left(\ln \frac{1+\delta}{3\delta}\right)^{1-s}.$$

Now, taking δ from the inequality

$$C(n, p, \lambda, M_0, \mathcal{N}) \left(\ln \frac{1+\delta}{3\delta}\right)^{1-s} \le \frac{\nu}{2},$$

we obtain the desired relation (6.35).

Let $t_0 \notin \mathcal{E}^+$. Denote by t_* the precise upper bound of the set $\mathcal{E}^+_{t_0} = \{t\colon t \in \mathcal{E}^+,\ t < t_0\}$ and assume first that $t_0 = t_*$. Then, for any $t' \in \mathcal{E}^+_{t_0}$, as above we come to the estimate

$$\int\limits_{K_4} \Psi^s_{\delta^i}\big(v_R(x,t')\big)\eta^\beta(x,t')\,dx \le C(n,p,\lambda,M_0,\mathcal{N}) \ln\frac{1+\delta}{\delta}.$$

Using Remark 6.1 and passing to the limit as $t' \to t_0$, we go back to the case (6.37) already examined.

If $t_0 > t_*$, then

$$\int\limits_{K_4} \Phi_{\delta^i}\big(v_R(x,t_0)\big)\eta^\beta(x,t_0)\,dx \le \int\limits_{K_4} \Phi_{\delta^i}\big(v_R(x,t'')\big)\eta^\beta(x,t'')\,dx \quad \text{for all} \quad t'' \in (t_*, t_0)$$

and (see (6.31))

$$\int\limits_{K_4} \Psi^s_{\delta^i}\big(v_R(x,t')\big)\eta^\beta(x,t')\,dx \le C_0(n,p,\lambda,\delta) \quad \text{for all} \quad t' \in \mathcal{E}^+_{t_0},$$

where

$$C_0 = C(n,p,\lambda,M_0,\mathcal{N}) \ln\frac{1+\delta}{\delta}. \tag{6.38}$$

Passing to the limit for $t'' \to t_*$ and $t' \to t_*$ in the above relations, we obtain the inequalities

$$\int\limits_{K_4} \Phi_{\delta^i}\big(v_R(x,t_0)\big)\eta^\beta(x,t_0)\,dx \le \int\limits_{K_4} \Phi_{\delta^i}\big(v_R(x,t_*)\big)\eta^\beta(x,t_*)\,dx \tag{6.39}$$

and

$$\int\limits_{K_4} \Psi^s_{\delta^i}\big(v_R(x,t_*)\big)\eta^\beta(x,t_*)\,dx \le C_0. \tag{6.40}$$

By (6.40)

$$\int\limits_{K_4\cap\{x\colon (\delta^i - v_R(x,t_*))_+ > \tau\delta^i\}} \Psi^s_{\delta^i}\big(v_R(x,t_*)\big)\eta^\beta(x,t_*)\,dx \le C_0 \quad \text{for all} \quad \tau \in [0,1].$$

The function $\Psi_{\delta^i}\big(v_R(x,t_*)\big)$ on the set $K_4 \cap \big\{x\colon \big(\delta^i - v_R(x,t_*)\big)_+ > \tau\delta^i\big\}$ satisfies the inequality

$$\Psi^s_k\big(v_R(x,t_*)\big) \ge \left(\ln\frac{(1+\delta)\delta^i + R}{(1+\delta-\tau)\delta^i + R}\right)^s \ge \left(\ln\frac{1+2\delta}{1+2\delta-\tau}\right)^s,$$

which follows from (6.33), and

$$\int\limits_{K_4\cap\{x\colon (\delta^i - v_R(x,t_*))_+ > \tau\delta^i\}} \eta^\beta(x,t_*)\,dx \le C_0 \cdot \left(\ln\frac{1+2\delta}{1+2\delta-\tau}\right)^{-s} \quad \text{for all} \quad \tau \in [0,1].$$

Hence, using the definition of Y_i in (6.32), we conclude that

$$\int\limits_{K_4\cap\{x\colon (\delta^i - v_R(x,t_*))_+ > \tau\delta^i\}} \eta^\beta(x,t_*)\,dx \le \begin{cases} Y_i & \text{for} \quad \tau \in [0,\tau_*), \\ C_0 \cdot \big(\ln\frac{1+2\delta}{1+2\delta-\tau}\big)^{-s} & \text{for} \quad \tau \in [\tau_*,1), \end{cases} \tag{6.41}$$

where τ_* is the root of the equation

$$Y_i = C_0 \cdot \left(\ln\frac{1+2\delta}{1+2\delta-\tau_*}\right)^{-s}$$

and has the form

$$\tau_* = \frac{e^{(C_0/Y_i)^{1/s}} - 1}{e^{(C_0/Y_i)^{1/s}}}(1+2\delta). \tag{6.42}$$

Since $Y_i > \nu$, it follows that

$$\tau_* < \frac{e^{(C_0/\nu)^{1/s}} - 1}{e^{(C_0/\nu)^{1/s}}}(1+2\delta) \equiv \sigma_0(1+2\delta). \tag{6.43}$$

The explicit form (6.38) of the constant C_0 and condition (1.2), which ensures the inequality $s \ge \alpha > 1$, allow us to assume that

$$e^{(2C_0/\nu)^{1/s}} \le (2\delta)^{-1/2} \quad \text{for} \quad \delta \le \delta_0(n, p, \lambda, M_0, \mathcal{N}, \nu), \tag{6.44}$$

and thus,

$$\begin{aligned}\tau^* < \frac{e^{(2C_0/\nu)^{1/s}} - 1}{e^{(2C_0/\nu)^{1/s}}}(1+2\delta) \equiv \sigma_1(1+2\delta)&\\ \le (1-(2\delta)^{1/2})(1+2\delta) < 1 - \delta^{1/2} \quad \text{for} \quad \delta \le \delta_1(n, p, \lambda, M_0, \mathcal{N}, \nu).&\end{aligned} \tag{6.45}$$

Let us estimate the integral on the right-hand side of (6.39). Set

$$E(x) = \big\{\tau\colon \tau < \big(\delta^i - v_R(x, t_*)\big)_+\big\}, \quad F(\tau) = \big\{x\colon \big(\delta^i - v_R(x, t_*)\big)_+ > \tau\big\}.$$

We have (see (6.23))

$$\int\limits_{K_4} \Phi_{\delta^i}\big(v_R(x, t_*)\big)\eta^{p_2}(x, t_*)\, dx \le \int\limits_{K_4} \eta^\beta(x, t_*)\bigg(\int\limits_0^{\delta^i} \frac{\chi(E(x))}{((1+\delta)\delta^i - \tau)^{s-1}}\, d\tau\bigg)\, dx,$$

where $\chi\big(E(x)\big)$ is the characteristic function of $E(x)$. Applying the Fubini theorem and changing the variables, we find that

$$\begin{aligned}&\int\limits_{K_4} \Phi_{\delta^i}\big(v_R(x, t_*)\big)\eta^\beta(x, t_*)\, dx\\ &\le \int\limits_0^{\delta^i} \frac{1}{((1+\delta)\delta^i - \tau)^{s-1}}\bigg(\int\limits_{K_4\cap F(\tau)} \eta^\beta(x, t_*)\, dx\bigg)\, d\tau = \int\limits_0^1 \frac{\delta^{(2-s)i}}{(1+\delta-\tau)^{s-1}}\bigg(\int\limits_{K_4\cap F(\tau\delta^i)} \eta^\beta(x, t_*)\, dx\bigg)\, d\tau.\end{aligned}$$

Thus, by (6.39), we get

$$\int\limits_{K_4} \Phi_{\delta^i}(v_R(x, t_0))\eta^\beta(x, t_0)\, dx \le \int\limits_0^1 \frac{\delta^{(2-s)i}}{(1+\delta-\tau)^{s-1}}\bigg(\int\limits_{K_4\cap F(\tau\delta^i)} \eta^\beta(x, t_*)\, dx\bigg)\, d\tau$$

and (see (6.42))

$$\begin{aligned}&\int\limits_{K_4} \Phi_{\delta^i}(v_R(x, t_0))\eta^\beta(x, t_0)\, dx\\ &\le \int\limits_0^{\tau_*} \frac{\delta^{(2-s)i}}{(1+\delta-\tau)^{s-1}}\bigg(\int\limits_{K_4\cap F(\tau\delta^i)} \eta^\beta(x, t_*)\, dx\bigg)\, d\tau + \int\limits_{\tau_*}^1 \frac{\delta^{(2-s)i}}{(1+\delta-\tau)^{s-1}}\bigg(\int\limits_{K_4\cap F(\tau\delta^i)} \eta^\beta(x, t_*)\, dx\bigg)\, d\tau.\end{aligned}$$

Now from (6.41) we obtain

$$\int\limits_{K_4} \Phi_{\delta^i}\big(v_R(x, t_0)\big)\eta^\beta(x, t_0)\, dx \le \int\limits_0^{\tau_*} \frac{\delta^{(2-s)i}}{(1+\delta-\tau)^{s-1}} Y_i\, d\tau + \int\limits_{\tau_*}^1 \frac{\delta^{(2-s)i} g}{(1+\delta-\tau)^{s-1}}\bigg(\ln\frac{1+\delta}{1+\delta-\tau}\bigg)^{-s}\, d\tau.$$

A simple transformation in the right-hand side brings us to the inequality

$$\int_{K_4} \Phi_{\delta^i}\big(v_R(x,t_0)\big)\eta^\beta(x,t_0)\,dx \le \delta^{(2-s)i} Y_i G(Y_i,\delta), \tag{6.46}$$

where

$$G(Y_i,\delta) = \int_0^1 \frac{d\tau}{(1+\delta-\tau)^{s-1}} - \int_{\tau_*}^1 \left(1 - \frac{C_0}{Y_i}\left(\ln\frac{1+2\delta}{1+2\delta-\tau}\right)^{-s}\right)\frac{d\tau}{(1+\delta-\tau)^{s-1}}.$$

Using (6.43), let us increase the right-hand side in the last expression by replacing τ_* with $\sigma_0(1+2\delta)$. Now, recalling that $Y_i > \nu$, we find that

$$G(Y_i,\delta) < \int_0^1 \frac{d\tau}{(1+\delta-\tau)^{s-1}} - \int_{\sigma_0(1+2\delta)}^1 \left(1 - \frac{C_0}{\nu}\left(\ln\frac{1+2\delta}{1+2\delta-\tau}\right)^{-s}\right)\frac{d\tau}{(1+\delta-\tau)^{s-1}},$$

and by (6.46) we get

$$\int_{K_4} \Phi_{\delta^i}\big(v_R(x,t_0)\big)\eta^\beta(x,t_0)\,dx \le Y_i\big(1-f(\delta)\big)\int_0^{1-\delta} \frac{\delta^{(2-s)i}}{(1+\delta-\tau)^{s-1}}\,d\tau, \tag{6.47}$$

where

$$\begin{aligned} f(\delta) \int_0^{1-\delta} \frac{d\tau}{(1+\delta-\tau)^{s-1}} \\ = \int_{\sigma_0(1+2\delta)}^1 \left(1 - \frac{C_0}{\nu}\left(\ln\frac{1+2\delta}{1+2\delta-\tau}\right)^{-s}\right)\frac{d\tau}{(1+\delta-\tau)^{s-1}} - \int_{1-\delta}^1 \frac{d\tau}{(1+\delta-\tau)^{s-1}}. \end{aligned} \tag{6.48}$$

To estimate the left-hand side of (6.47) from below, we use the constraint (6.33) (see also (6.25)) and its consequence (6.26). Taking into account (6.32) and (6.36), we get

$$\begin{aligned} \int_{K_4} \Phi_{\delta^i}\big(v_R(x,t_0)\big)\eta^\beta(x,t_0)\,dx > (1+\delta)^{-1} \int_{K_4\cap\{x\colon v_R(x,t)<\delta^i\}} \eta^\beta(x,t_0)\,dx \int_0^{1-\delta} \frac{\delta^{(2-s)i}}{(1+\delta-\tau)^{s-1}}\,d\tau \\ \ge (1+\delta)^{-1}(Y_{i+1}-\varepsilon)\int_0^{1-\delta} \frac{\delta^{(2-s)i}}{(1+\delta-\tau)^{s-1}}\,d\tau, \end{aligned}$$

which, together with (6.47), implies the inequality

$$Y_{i+1} < \big(1-f(\delta)\big)(1+\delta)^{-1}Y_i + \varepsilon. \tag{6.49}$$

Let us estimate $f(\delta)$ from below. Note that (6.43) ensures that

$$\frac{C_0}{\nu} = \left(\ln\frac{1}{1-\sigma_0}\right)^s.$$

Therefore, the integrand of the first integral in the right-hand side of (6.48) is nonnegative. Let us narrow the integration region of this integral to the segment $(\sigma_1(1+2\delta),1)$ with $\sigma_1 > \sigma_0$ from (6.45). By the definition of σ_1, we have

$$\frac{C_0}{\nu} = \frac{1}{2}\left(\ln\frac{1}{1-\sigma_1}\right)^s,$$

and therefore,

$$\frac{C_0}{\nu}\left(\ln\frac{1+2\delta}{1+2\delta-\tau}\right)^{-s} \le \frac{1}{2} \quad \text{for} \quad \tau \in (\sigma_1(1+2\delta), 1),$$

and from (6.48) we find that

$$f(\delta)\int\limits_0^{1-\delta}\frac{d\tau}{(1+\delta-\tau)^{s-1}} > \frac{1}{2}\int\limits_{\sigma_1(1+2\delta)}^{1}\frac{d\tau}{(1+\delta-\tau)^{s-1}} - \int\limits_{1-\delta}^{1}\frac{d\tau}{(1+\delta-\tau)^{s-1}}. \tag{6.50}$$

Hence, calculating all integrals in (6.50) and taking into account the inequality for $\sigma_1(1+2\delta)$ in (6.45), we get

$$f(\delta) > \frac{(((1+\delta)\delta)^{(2-s)/2} - \delta^{2-s}) - 2((2\delta)^{2-s} - \delta^{2-s})}{2((1+\delta)^{2-s} - (2\delta)^{2-s})}. \tag{6.51}$$

Note that

$$\big((1+\delta)\delta\big)^{(2-s)/2} - \delta^{2-s} \ge 4\big((2\delta)^{2-s} - \delta^{2-s}\big) \quad \text{for} \quad \delta \le \delta_2, \tag{6.52}$$

where the constant δ_2 does not depend on s. Indeed, since $2^{2-s} < 1 + (2-s)4\ln 2$ for $s \in [1,2]$, we have

$$(2\delta)^{2-s} - \delta^{2-s} < (2-s)4\delta^{2-s}\ln 2,$$

and (6.52) follows from the inequality

$$\delta \le (1+\delta)\big(1 + (2-s)16\ln 2\big)^{2/(s-2)} \tag{6.53}$$

which holds for

$$\delta \le \frac{\gamma}{1-\gamma} \equiv \delta_2, \quad \gamma = \inf_{s\in(1,2)}\big(1 + (2-s)16\ln 2\big)^{2/(s-2)}. \tag{6.54}$$

Using (6.52), we rewrite (6.51) as

$$f(\delta) > \frac{((1+\delta)\delta)^{(2-s)/2} - \delta^{2-s}}{4((1+\delta)^{2-s} - (2\delta)^{2-s})} \tag{6.55}$$

and show that there is a constant δ_3 independent of s and such that

$$f(\delta) > \delta^2 \quad \text{for} \quad \delta \le \delta(n, p, \lambda, M_0, \mathcal{N}, \nu). \tag{6.56}$$

Indeed, the inequality

$$(1+\delta^{-1})^{2-s} < 2^{2-s} + (2-s)(1+\delta^{-1})\ln(1+\delta^{-1}),$$

which holds for $s \in [1,2]$, ensures that the denominator in (6.55) can be estimated as follows:

$$4\big((1+\delta)^{2-s} - (2\delta)^{2-s}\big) < (2-s)4\delta^{2-s}(1+\delta^{-1})\ln(1+\delta^{-1}),$$

and from (6.53), which holds under the condition (6.54), we obtain

$$\big((1+\delta)\delta\big)^{(2-s)/2} - \delta^{2-s} \ge \delta^{2-s}(2-s)16\ln 2.$$

Thus, choosing in (6.44), (6.45) and (6.54) the smallest constant, $\delta_4 = \min\{\delta_0, \delta_1, \delta_2, \delta_3, 1/8\}$, for $\delta \le \delta_4$ we come to the inequality

$$f(\delta) > \frac{16\ln 2}{(1+\delta^{-1})\ln(1+\delta^{-1})},$$

which implies (6.56). Now, by (6.49) and (6.56), we have $Y_{i+1} < (1-\delta)Y_i + \varepsilon$. Since $\varepsilon \in (0, \nu/2)$ is arbitrary, it follows that $Y_{i+1} \le (1-\delta)Y_i$, which leads us to (6.35). The lemma is proved. □

Let us turn to the *proof* of Lemma 6.2. Iterating relations (6.34), (6.35) under the condition (6.33) and using the inequality $Y_1 \le |K_4| = 2^n|K_2|$, we obtain

$$Y_i \le \max\{\nu, (1-\delta)^{i-1}Y_1\} \le \max\{\nu, 2^n(1-\delta)^{i-1}|K_2|\}, \quad i = 2, \dots, i_*. \tag{6.57}$$

Let the constant R in (6.33) be such that $(1-\delta)^{i_*-1} \le 2^{-n}\nu$ for $R \le \mathcal{R}_0(n, p, \nu)$. In (6.57), we choose the smallest i_0 for which $(1-\delta)^{i_0-1} \le 2^{-n}\nu$ and set $\delta^{i_0} = 2\gamma$. Since (see (6.32))

$$Y_{i_0} \ge |\{x\colon x \in K_2,\ v_R(x,t) < 2\gamma\}| \text{ for all } t \in [-2^s, 0],$$

we come to the relation

$$|\{x\colon x \in K_2,\ v_R(x,t) < 2\gamma\}| \le \nu|K_2| \text{ for all } t \in [-2^s, 0],$$

which leads us to (6.22). The lemma is proved.

Let us go back to the function $u(x,t)$ in (6.22) and the original coordinates (see (6.16) and (6.17)). As a result we get

$$\left|\left\{x\colon (x \in K_{cR/2},\ u(x,t) \le m + \frac{\gamma\omega}{2}\right\}\right| \le \nu|K_{cR/2}| \text{ for all } t \in \left[-\left(\frac{R}{4}\right)^s, 0\right].$$

In particular,

$$\left|\left\{x\colon (x \in K_{cR/2},\ u(x,t) < m + \frac{\gamma\omega}{2^\lambda}\right\}\right| \le \nu|K_{cR/2}| \text{ for all } t \in \left[-\left(\frac{R}{4}\right)^s, 0\right]. \tag{6.58}$$

Consider the cylinder $\mathcal{Q}_R^{(0,\bar t)}$ of the form (6.7) for which $\mu_0 = \log_2\gamma^{-1}$, $\mu_1 = \lambda$, $h_0 = 4^{-s}$, and $h_1 = \mathcal{N}/2$. By (6.58), we have

$$\left|\left\{(x,t)\colon u(x,t) < m + \frac{\omega}{2^\mu},\ (x,t) \in \mathcal{Q}_R^{(0,\bar t)}\right\}\right| \le \nu|\mathcal{Q}_R^{(0,\bar t)}|, \quad \mu = \mu_0 + \mu_1, \tag{6.59}$$

provided that $\mathcal{Q}_R^{(0,\bar t)} \subset Q_{cR/2,(R/4)^s}$, i.e., for

$$\bar t \in \left[2^{\mu_0(s-2)}\left(\frac{R}{4}\right)^s - \left(\frac{R}{4}\right)^s, 0\right]. \tag{6.60}$$

Take $\nu = \nu_*$ in (6.59), where ν_* is the constant from Lemma 6.1 corresponding to the cylinder $\mathcal{Q}_R^{(0,\bar t)}$ and depending only on n, p, M_0, $\mathcal{N}$. Then, using (6.9) from Lemma 6.1 and the assumptions (6.8), we obtain

$$u(x,t) \ge m + \frac{\omega}{2^{\mu+1}} \text{ a.e. in } \mathcal{Q}_{R/2}^{(0,\bar t)}.$$

Since $\bar t$ from the interval (6.60) is arbitrary, the last inequality holds also in the cylinder $Q_{cR/4,(R/8)^s}$, so that

$$u(x,t) \ge m + \frac{\omega}{2^{\mu+1}} \text{ a.e. in } Q_{cR/4,(R/8)^s}. \tag{6.61}$$

Setting $l = \mu + 1$, let us state the main result of the section which follows from (6.61).

Lemma 6.4. *There exist constants* $\nu_0(n, p, M_0) \in (0,1)$, $l(n, p, \lambda, M_0, \mathcal{N}) > 0$, $\mathcal{R}_0(n, p, \lambda, M_0, \mathcal{N}) > 0$ *such that condition* (6.5) *for some cylinder* $Q_{dR,R^s}^{(\bar x,0)}$, *together with the assumption* (6.8), *implies the estimate*

$$u(x,t) \ge m + \frac{\omega}{2^l} \textit{ for almost all } (x,t) \in Q_{cR/4,(R/8)^s} \tag{6.62}$$

for $R \le \mathcal{R}_0$.

The estimate (6.62) will be used in a somewhat different form, as stated next.

Corollary 6.2. *Under the assumptions of Lemma* 6.4, *the following estimate holds*:

$$\operatorname*{ess\,osc}_{Q_{cR/4,(R/8)^s}} u \le \sigma_1\omega, \tag{6.63}$$

where $\sigma_1 = 1 - 2^{-l}$.

6.2. Analysis of the Second Alternative. In what follows, it is always assumed that condition (6.6) holds with the constant $\nu_0(n,p,M_0)$ defined in the previous section. Our aim is to extend the inequality (6.6) to cylinders of the form (see (6.1)) Q_{cR,hR^s}, $h = \nu_0/2$ and then utilize the result (6.11) of Lemma 6.1 in that cylinder. This aim is realized in two steps. First, the inequality

$$\left|\left\{(x,t)\colon (x,t) \in Q^{(\bar{x},0)}_{d_*R,R^s},\ u(x,t) > M - \frac{\omega}{2}\right\}\right| < (1-\nu_0^2)|Q^{(\bar{x},0)}_{d_*R,R^s}| \tag{6.64}$$

is established for cylinders $Q^{(\bar{x},0)}_{d_*R,R^s} \subset Q_{cR,R^s}$ that contain in their interior the cylinders $Q^{(\bar{x},0)}_{dR,R^s}$ involved in the statement of the second alternative (6.6). Here

$$Q^{(\bar{x},0)}_{d_*R,R^s} = K^{\bar{x}}_{d_*R} \times (-R^s, 0), \quad d_* = \mathcal{L}\left(\frac{\omega}{2^{q+1}}\right)^{(s-2)/s}, \quad 1 \le \mathcal{L} \le \mathcal{N}, \quad 1 \le q < \lambda - 1, \tag{6.65}$$

$$\bar{x} \in \bar{K}_{\mathcal{R}_1}, \quad \mathcal{R}_1 = \left(\mathcal{N}\left(\frac{\omega}{2^\lambda}\right)^{(s-2)/s} - \mathcal{L}\left(\frac{\omega}{2^{q+1}}\right)^{(s-2)/s}\right) R, \tag{6.66}$$

and $\mathcal{N}$, λ, d are the constants from (6.1) and (6.3). The positive integer $\mathcal{L}$ and the constant q are defined in the proof of (6.64). These depend only on n, p, M_0, s and are uniformly bounded for $s \to 2$. Next, after $\mathcal{N}(n,p,M_0)$ and $\lambda(n,p,M_0)$ have been chosen in a suitable way (with the help of the Bezikovich covering theorem), relation (6.64) is extended to the cylinder Q_{cR,hR^s}.

First, assuming that in cylinders of the form (6.65) the inequality (6.64) holds a priori, we establish some preliminary results.

Lemma 6.5. *Under the assumption* (6.64), *there is a constant* $t^* \in (-R^s, -\nu_0 R^s/2)$ *such that*

$$\left|\left\{x\colon x \in K^{\bar{x}}_{d_*R},\ u(x,t^*) > M - \frac{\omega}{2^q}\right\}\right| \le \left(\frac{1-\nu_0^2}{1-\nu_0^2/2)}\right)|K^{\bar{x}}_{d_*R}| \quad \textit{for all} \ \ q \ge 2. \tag{6.67}$$

Proof. Assume the contrary: (6.67) does not hold for all $t^* \in (-R^s, -\nu_0^2 R^s/2)$ and some $q \ge 2$. Then,

$$\begin{aligned}&\left|\left\{(x,t)\colon (x,t) \in Q^{(\bar{x},0)}_{d_*R,R^s},\ u(x,t) > M - \frac{\omega}{2}\right\}\right| \\ &\qquad \ge \int_{-R^s}^{-\nu_0^2 R^s/2} \left|\left\{x\colon x \in K^{\bar{x}}_{d_*R},\ u(x,t^*) > M - \frac{\omega}{2^q}\right\}\right| dt \ge (1-\nu_0^2)|Q^{(\bar{x},0)}_{d_*R,R^s}|,\end{aligned}$$

which contradicts (6.64). The lemma is proved. □

The next statement has much in common with Lemma 5.5.

Lemma 6.6. *Under the assumption* (6.64), *there is a positive constant* $q_1 > 1$ *depending only on* n, p, ν_0, M_0 *and such that for* $q \ge q_1$, *the following inequality holds:*

$$\left|\left\{x\colon x \in K^{\bar{x}}_{d_*R},\ u(x,t) > M - \frac{\omega}{2^{q+1}}\right\}\right| < \left(1 - \frac{\nu_0^4}{2}\right)|K^{\bar{x}}_{d_*R}| \quad \textit{for all} \ \ t \in (t^*, 0), \tag{6.68}$$

where t^* *is the constant from Lemma* 6.5.

Proof. Without loss of generality, we can assume that $\bar{x} = 0$. Set $k = M - \omega/2$ and consider the function

$$g\big(u(x,t)\big) = \max\left\{\ln\left(\frac{\mu_k^+}{\mu_k^+ - (u(x,t)-k)_+ + c}\right); 0\right\}, \quad c = \frac{\omega}{2^{q+1}},$$

where $q > 0$ is the constant from (6.65) and

$$\mu_k^+ = \operatorname*{ess\,sup}_{K_{d_*R}\times(t^*,0)} (u-k)_+.$$

Let $\delta \in (0,1)$, $\gamma = (1-\delta)$, and let $\eta(x)$ be a truncating function in K_{d_*R} such that $\eta = 0$ on ∂K_{d_*R}, $\eta = 1$ in $K_{\gamma d_* R}$, $|\nabla\eta| \le (\delta d_* R)^{-1}$. Using the inequality (4.4) and the logarithmic condition, we find that

$$\int\limits_{K_{\gamma d_* R}\times\{t\}} g^2\,dx \le \int\limits_{K_{d_* R}\times\{t^*\}} g^2\,dx + C(p)\delta^{-\beta}R^{-s}\int\limits_{t^*}^{0}\int\limits_{K_{d_*R}} d_*^{-p} g|g'|^{2-p}\,dx\,dt, \tag{6.69}$$

for all $t \in (t^*, 0)$. Estimating g and g' as in (5.30) and (5.31), we obtain

$$g(u) \le q\ln 2, \quad |g'(u)|^{2-p} \le \left(\frac{\omega}{2^{q+1}}\right)^{p-2}.$$

The first integral in the right-hand side of (6.69) is equal to zero on the set of all $x \in K_{d_*R}$ such that $u(x,t^*) < M - \omega/2$, and, by Lemma 6.5, we have

$$\int\limits_{K_{d_*R}\times\{t^*\}} g^2\,dx \le q^2\ln^2 2\left(\frac{1-\nu_0^2}{1-\nu_0^2/2}\right)|K_{d_*R}|.$$

For the second integral in the right-hand side of (6.69), we have

$$\int\limits_{t^*}^{0}\int\limits_{K_{d_*R}} d_*^{-p} g|g'|^{2-p}\,dx\,dt \le q\ln 2\int\limits_{t^*}^{0}\int\limits_{K_{d_*R}} \mathcal{L}^{-p}\left(\frac{\omega}{2^{q+1}}\right)^{2(p-s)/s}dx\,dt \le C(p,M_0)qR^s|K_{d_*R}|.$$

Taking into account the last two estimates, from (6.69), we come to the inequality

$$\int\limits_{K_{\gamma d_*R}\times\{t\}} g^2\,dx \le \left(q^2\ln^2 2\left(\frac{1-\nu_0^2}{1-\nu_0^2/2}\right) + C(p,M_0)q\delta^{-\beta}\right)|K_{d_*R}|. \tag{6.70}$$

Narrowing the integration region in the left-hand side of (6.70) to the set

$$E(t) = \left\{x\colon x \in K_{\gamma d_*R},\ u(x,t) > M - \frac{\omega}{2^{q+1}}\right\}$$

and using the obvious estimate

$$g^2 \ge \ln^2\left(\frac{\omega/2}{2^{-q}\omega}\right) = (q-1)^2\ln^2 2$$

on that set, we obtain

$$\int\limits_{K_{\gamma d_*R}\times\{t\}} g^2\,dx \ge (q-1)^2\ln^2 2|E(t)|.$$

Using the last relation and dividing both parts of (6.70) by $(q-1)^2\ln^2 2$, we obtain

$$|E(t)| \le \left(\left(\frac{q}{q-1}\right)^2\left(\frac{1-\nu_0^2}{1-\nu_0^2/2}\right) + C(p,M_0)\frac{q\delta^{-\beta}}{(q-1)^2}\right)|K_{d_*R}| \quad \text{for all} \quad t \in (t^*,0).$$

On the other hand,

$$\left|\left\{x\colon x \in K_{d_*R},\ u(x,t) > M - \frac{\omega}{2^{q+1}}\right\}\right| \le |E(t)| + |K_{d_*R}\setminus K_{\gamma d_*R}| \le |E(t)| + n\delta|K_{d_*R}|,$$

and we finally get

$$\begin{aligned}&\left|\left\{x\colon x \in K_{d_*R},\ u(x,t) > M - \frac{\omega}{2^{q+1}}\right\}\right| \\ &\qquad \le \left(\left(\frac{q}{q-1}\right)^2\left(\frac{1-\nu_0^2}{1-\nu_0^2/2}\right) + C(p,M_0)\frac{q\delta^{-\beta}}{(q-1)^2} + n\delta\right)|K_{d_*R}| \quad \text{for all} \quad t \in (t^*,0).\end{aligned}$$

Choosing here δ from the inequality $\delta n \le 3\nu_0^4/8$, and then the constant q_1 satisfying the relations

$$\left(\frac{q_1}{q_1-1}\right)^2 \le \left(1-\frac{\nu_0^2}{2}\right)(1+\nu_0^2), \quad C(p,M_0)\frac{q_1\delta^{-\beta}}{(q_1-1)^2} \le \frac{3\nu_0^2}{8},$$

we obtain the desired estimate (6.68) for $q \ge q_1$. The lemma is proved. □

Next, we are going to choose the positive integer $\mathcal{L}$ and the constant q in (6.65) so that they depend only on n, p, M_0, and the constant s from (4.2), and also meet the following requirements. On the one hand, each of the closed cylinders $\bar{Q}^{(\bar{x},0)}_{d_*R,R^s}$ should be represented as a finite union of closed cylinders $\bar{Q}^{(\bar{x}_i,0)}_{dR,R^s} \subset Q_{cR,R^s}$ involved in the assumption (6.6). This representation should ensure the inequality

$$\sum_i \left|\bar{Q}^{(\bar{x}_i,0)}_{dR,R^s}\right| < (1+\nu_0)\left|\bar{Q}^{(\bar{x},0)}_{d_*R,R^s}\right|, \tag{6.71}$$

which, due to the alternative (6.6), implies the desired property (6.64). On the other hand, we need that the constants $\mathcal{N}$ and q be uniformly bounded as $s \to 2$.

First, assume that $q = q_1(n,p,M_0)$, where $q_1 > 1$ is the constant from Lemma 6.6. Consider the case in which the inequality

$$\rho = 2^{q_1(2-s)/s} \le 1+\varepsilon \tag{6.72}$$

holds with a constant $\varepsilon \in (0,1)$ depending only on n, ν_0 and defined shortly, together with $\mathcal{L} = \mathcal{L}(n,p,M_0)$. Let us utilize the relation $d_* = \mathcal{L}2^{q_1(2-s)/s}d = \mathcal{L}\rho d$, which follows from (6.65), (6.72), and recall that the edge of the cube $K^{\bar{x}}_{d_*R}$ on the base of the cylinder $Q^{(\bar{x},0)}_{d_*R,R^s}$ has length $2d_*R$.

Denote by $[z]$ the integer part of z and set $\mathcal{L} = [\varepsilon^{-1}]$. Since $(1-\varepsilon)/\varepsilon < \mathcal{L} \le \varepsilon^{-1}$, it follows, in view of (6.72), that

$$\mathcal{L}+1-\varepsilon < \mathcal{L}\rho \le \mathcal{L}+1.$$

Therefore, the segment of length $2d_*R$ can be covered by $\mathcal{L}+1$ segments of length $2dR$. This covering can be arranged in such a way that the first $\mathcal{L}$ segments, counted from one end of the covered segment, abut one another, and the last segment of the covering abuts its other end. Moreover, the measure of the intersection of the last two covering segments does not exceed $2dR\varepsilon$. Now, fix a vertex of the cube $K^{\bar{x}_i}_{d_*R}$. Along each edge issuing from that vertex, we arrange segments of length $2dR$ in the above manner and construct hyperplanes orthogonal to the edge through the end-points of the covering segments. As a result, the cube $\bar{K}^{\bar{x}}_{d_*R}$ is covered by cubes $\bar{K}^{\bar{x}_i}_{dR}$ with centers $\bar{x}_i$ satisfying condition (6.4). The number of the covering cubes is $(\mathcal{L}+1)^n$, the multiplicity of the covering does not exceed 2^n, and the measure of the common points of all neighboring cubes of the covering does not exceed $n4^nd^nR^n\varepsilon$. It follows that there is a covering of the cylinder $\bar{Q}^{(\bar{x},0)}_{d_*R,R^s}$ by cylinders $\bar{Q}^{(\bar{x}_i,0)}_{dR,R^s}$ such that

$$\sum_i \left|\bar{Q}^{(\bar{x}_i,0)}_{dR,R^s}\right| \le (1+n2^n\varepsilon)\left|\bar{Q}^{(\bar{x},0)}_{d_*R,R^s}\right|.$$

Taking here $\varepsilon = n^{-1}2^{-n}\nu_0$, we come to the desired inequality (6.71). In the case under consideration, $\mathcal{L} = \mathcal{L}_1 = [2^n n\nu_0^{-1}]$, $q = q_1$. Since $\nu_0 = \nu_0(n,p,M_0)$, it follows that the constants $\mathcal{L}_1$ and q_1 chosen above depend only on n, p, M_0. Note that due to (6.72) the constant $s < 2$ can take values infinitely close to 2.

If the assumption (6.72) does not hold for the chosen ε, i.e., if (see (1.2))

$$1 < \alpha \le s < 2\left(1 + q_1^{-1}\log_2(1+2^{-n}n^{-1}\nu_0)\right)^{-1} = s_0,$$

then we take $\mathcal{L} = \mathcal{L}_2 = 1$, and choose $q = q_2 \ge q_1$ so large that $2^{q(2-s)/s}$ takes the least possible value on the set of positive integers. In this case, the cylinder $\bar{Q}^{(\bar{x},0)}_{d_*R,R^s}$ can be covered by finitely many cylinders of the form $\bar{Q}^{(\bar{x}_i,0)}_{dR,R^s}$ with mutually disjoint interiors, and we again come to (6.71). It is not difficult to see that $\mathcal{L}_2 = \mathcal{L}_2(n, p, M_0)$, $q_2 = q_2(n, p, M_0, s)$, and the constant q_2 is uniformly bounded for $s \in (1, s_0)$. Denoting

$$\tilde{q}_2 = \sup_{s \in (\alpha, s_0)} q_2, \tag{6.73}$$

we have $\tilde{q}_2 = \tilde{q}_2(n, p, M_0)$, $\tilde{q}_2 \geq q_1$. As a result, we obtain the following statement.

Lemma 6.7. *There exist a positive integer $\mathcal{L}$ and a constant $q > 1$ depending only on n, p, M_0, s and such that for cylinders $\bar{Q}^{(\bar{x},0)}_{d_*R,R^s}$ condition* (6.64) *holds and the statements of Lemmas* 6.5 *and* 6.6 *are true.*

In what follows, an important role is played by the constants (see (6.73))

$$\mathcal{L}_* = \max\{\mathcal{L}_1, \mathcal{L}_2\} = \mathcal{L}_1, \quad q_* = \max\{q_1, \tilde{q}_2\} = \tilde{q}_2 \tag{6.74}$$

depending only on n, p, M_0.

The next statement is a preliminary result for extending the inequality (6.68) to the cylinder Q_{cR,hR^s} with $h = \nu_0/2$.

Lemma 6.8. *For any $\nu \in (0, 1)$, there is a constant $l > q_* + 1$ depending only on n, p, M_0, ν and such that for all cylinders $Q^{(\bar{x},0)}_{d_*R,hR^s} \subset Q_{cR,hR^s}$ the following inequality holds*:

$$\left|\left\{(x,t)\colon (x,t) \in Q^{(\bar{x},0)}_{d_*R,hR^s},\ u(x,t) > M - \frac{\omega}{2^l}\right\}\right| < \nu |Q^{(\bar{x},0)}_{d_*R,hR^s}|. \tag{6.75}$$

Proof. Assume that $\bar{x} = 0$ and set $j_1 = 1 + q_*$,

$$k = M - \frac{\omega}{2^j}, \quad j = j_1, j_1 + 1, \ldots, l - 1,$$

where $l \leq \lambda$ is a constant to be defined. Let $\mathcal{Q}_R = Q_{d_*R,hR^s}$. Consider a truncating function η in $\mathcal{Q}_{2R}$ such that $\eta = 1$ in $\mathcal{Q}_R$, $\eta = 0$ on the parabolic boundary of $\mathcal{Q}_{2R}$, $|\nabla \eta| \leq d_*^{-1} R^{-1}$ and $0 \leq \eta'_t \leq 2\nu_0^{-1} R^{-s}$. From the estimate (4.3), the logarithmic condition, and the inequality $(u - k)_+ \leq 2^{-j}\omega$, which holds almost everywhere in $\mathcal{Q}_{2R}$, it follows that

$$\int\limits_{\mathcal{Q}_R} |\nabla (u - k)_+|^s \, dx\, dt \leq R^{-s} C(p, M_0) \left(d_*^{-s} \left(\frac{\omega}{2^j}\right)^s + 2\nu_0^{-1} \left(\frac{\omega}{2^j}\right)^2 \right) |\mathcal{Q}_{2R}| + |\mathcal{Q}_{2R}|.$$

Since $\left(\frac{\omega}{2^j}\right)^{2-s} \leq \mathcal{L}^s d_*^{-s}$ and by (6.2) $\mathcal{L}^s (d_*R)^{-s} \left(\frac{\omega}{2^j}\right)^s \geq 1$, we have

$$\begin{aligned}\int\limits_{\mathcal{Q}_R} |\nabla (u - k)_+|^s \, dx\, dt &\leq (C(p, M_0)(1 + 2\nu_0^{-1}\mathcal{L}^s) + \mathcal{L}^s)(d_*R)^{-s} \left(\frac{\omega}{2^j}\right)^s |\mathcal{Q}_{2R}| \\ &\leq C(n, p, M_0)(d_*R)^{-s} \left(\frac{\omega}{2^j}\right)^s |\mathcal{Q}_R| \end{aligned} \tag{6.76}$$

for all $j = j_1, j_1 + 1, \ldots, l - 1$.

The estimate (6.76) will be needed a bit later, but now we use the inequality (4.7) from Lemma 4.2 in the cube K_{d_*R} for the function $u(x, \cdot)$ for fixed $t \in (-hR^s, 0)$ and the levels

$$l = M - \frac{\omega}{2^{j+1}}, \quad k = M - \frac{\omega}{2^j}, \quad j \geq j_1.$$

First, we note that according to the estimate (6.68) from Lemma 6.6, we have

$$\left|\left\{x\colon x \in K_{d_*R},\ u(x,t) < M - \frac{\omega}{2^j}\right\}\right| \geq \frac{1}{2}\nu_0^4 |K_{d_*R}| \ \text{ for all } \ t \in (-hR^s, 0).$$

Therefore, setting

$$A_j(t) = \left\{x\colon x \in K_{d_*R},\ u(x,t) > M - \frac{\omega}{2^j}\right\}, \quad A_j = \int\limits_{-hR^s}^{0} |A_j(t)|\, dt,$$

from (4.7), we obtain

$$\left(\frac{\omega}{2^j}\right)|A_{j+1}(t)| \le C(n)\nu_0^{-4} d_* R \int\limits_{A_j(t)\setminus A_{j+1}(t)} |\nabla u|\,dx \quad \text{for all} \quad t \in (-hR^s, 0). \tag{6.77}$$

Let us integrate both sides of (6.77) over the interval $(-hR^s, 0)$. Then, using the Hölder inequality and raising the resulting relation to the power s from (6.76), we obtain

$$\left(\frac{\omega}{2^j}\right)^s |A_{j+1}|^s \le C(n,\nu_0)(d_*R)^s |A_j\setminus A_{j+1}|^{s-1} \int\limits_{-hR^s}^{0} \int\limits_{A_j(t)} |\nabla u|^s\,dx\,dt \le C(n,p,M_0)\left(\frac{\omega}{2^j}\right)^s |\mathcal{Q}_R||A_j\setminus A_{j+1}|^{s-1}.$$

Therefore,

$$|A_{j+1}|^{s/(s-1)} \le C(n,p,M_0)|\mathcal{Q}_R|^{1/(s-1)}|A_j \setminus A_{j+1}| \quad \text{for all} \quad j = j_1, j_1+1, \dots, l-1.$$

Summing these inequalities over j, we find that

$$(l - j_1)|A_l|^{s/(s-1)} \le C(n,p,M_0)|\mathcal{Q}_R|^{1/(s-1)} \sum_{j=j_1}^{l-1} |A_j \setminus A_{j+1}| \le C(n,p,M_0)|\mathcal{Q}_R|^{s/(s-1)}.$$

Since $s \ge \alpha > 1$ (see (1.2)), it follows that $\nu^{-\alpha/(\alpha-1)} \ge \nu^{-s/(s-1)}$. Now, choosing the constant l from the inequality

$$l - j_1 \ge C(n,p,M_0)\nu^{-\alpha/(\alpha-1)},$$

we come to (6.75). The lemma is proved. □

Let us show that (see (6.74))

$$\mathcal{N} = \mathcal{L}_* = \mathcal{L}_1, \quad \lambda = l,$$

where l is the constant from Lemma 6.8 depending only on $\nu \in (0,1)$. In order to choose ν, let us show that

$$\left|\left\{(x,t)\colon (x,t) \in Q_{cR,hR^s},\ u(x,t) > M - \frac{\omega}{2^l}\right\}\right| < C(n)\nu|Q_{cR,hR^s}|. \tag{6.78}$$

To this end, it suffices to use (6.75) for constructing a finite covering of cube $\bar{K}_{cR}$ by cubes $\bar{K}^{\bar{x}_i}_{d_*R}$ with centers satisfying condition (6.66), and ensure that the multiplicity of the covering depends only on the space dimension n.

For this purpose, we first consider only the cubes $\bar{K}^{\bar{x}_i}_{d_*R}$ with centers on the boundary $\partial K_{\mathcal{R}_1}$ of the cube $K_{\mathcal{R}_1}$ from (6.66). By the Bezikovich theorem, there is a finite covering of $\partial K_{\mathcal{R}_1}$ by such cubes ($\partial K_{\mathcal{R}_1}$ is a compact set), and the multiplicity of that covering depends only on the space dimension n. Let us supplement this covering by 2^n cubes $\bar{K}^{\bar{x}_i}_{d_*R}$ with centers at the vertices of the cube $K_{\mathcal{R}_1}$. As a result, for $d_* \ge \mathcal{R}_1$, we obtain the desired covering of $\bar{K}_{cR}$, and for $d_* < \mathcal{R}_1$, we obtain a covering of the cubic layer $\bar{K}_{cR} \setminus K_{\mathcal{R}_1 - d_*}$. In the latter case, again using the Bezikovich theorem, we construct the required covering of $\bar{K}_{\mathcal{R}_1-d_*}$ by cubes $\bar{K}^{\bar{x}_i}_{d_*R}$ with centers on $\bar{K}_{\mathcal{R}_1-d_*}$. Now from the estimate (6.75) of Lemma 6.8 applied to each cube of the constructed covering we obtain (6.78).

Recalling that the constant h in (6.78) is equal to $\nu_0/2$, let us use Lemma 6.1 for the cylinder $\mathcal{Q}_R$ of the form (6.7), which coincides with Q_{cR,hR^s} for $\mu_0 = 0$, $\mu_1 = l = \lambda$, $h_0 = \nu_0/2$ and $h_1 = \mathcal{N}$. In (6.78), take $\nu = C^{-1}(n)\nu_*$, where ν_* is the constant from the inequality (6.10) of Lemma 6.1. The constant ν_* does not depend on λ, but depends only on n, p, M_0, h_0, $\mathcal{N}$, and since $h_0 = h_0(n,p,M_0)$ and $\mathcal{N} = \mathcal{N}(n,p,M_0)$, it follows that $\nu_* = \nu_*(n,p,M_0)$. Therefore, the constant ν to be chosen in (6.78) and the constant $l = \lambda$ depending on ν also depend only on n, p, M_0.

By (6.11) from Lemma 6.1, we obtain

$$u(x,t) \le M - \frac{\omega}{2^{\lambda+1}} \quad \text{a.e. in} \quad Q_{cR/2,h(R/2)^s}, \tag{6.79}$$

which leads us to the following statement.

Lemma 6.9. *If condition* (6.6) *holds with the constant* $\nu_0(n,p,M_0)$ *determined above, then there exist a constant* $\lambda > 1$ *and a positive integer* $\mathcal{N}$ *from* (6.1) *depending only on* n, p, M_0 *and such that the estimate* (6.79) *holds with* $h = \nu_0/2$.

It would be more convenient to reformulate Lemma 6.9 as follows.

Corollary 6.3. *Under the assumptions of Lemma* 6.9, *the following estimate holds*:

$$\operatorname*{ess\ osc}_{Q_{cR/2,h(R/2)^s}} \le \sigma_2\omega, \tag{6.80}$$

where $\sigma_2 = 1 - 2^{-\lambda-1}$.

Now, having chosen the constants λ and $\mathcal{N}$ from Corollaries 6.2 and 6.3, we obtain the following statement.

Proposition 6.1. *There exist constants* $\sigma \in [1/8, 1)$, $\lambda_0 \ge \lambda > 1$, $\mathcal{R}_0 > 0$ *and a positive integer* $\mathcal{N}$ *depending only on* n, p, M_0 *and such that for* $R \le \mathcal{R}_0$ *we have either* $\omega \le 2^{\lambda_0} R^{s/2}$ *or*

$$\operatorname*{ess\ osc}_{Q_{cR/A,(R/A)^s}} u \le \sigma\omega, \tag{6.81}$$

where $A = 8h^{-1/\alpha}$ *and* α *is the constant from* (1.2).

Proof. Due to our choice of the constant A, we have

$$Q_{cR/A,(R/A)^s} \subset Q_{cR/2,h(R/2)^s}, \quad Q_{cR/A,(R/A)^s} \subset Q_{cR/4,l(R/8)^s}.$$

Now, using (6.63), (6.80) and setting $\sigma = \max\{1/8, \sigma_1, \sigma_2\}$, we come to (6.81). The proposition is proved. □

In the next lemma, which is similar to Lemma 5.7, the constants λ_0, λ, σ, $\mathcal{N}$, A have the same meaning as above, and for convenience it is assumed that $R_0 = R$, $c_0 = c$, $\omega_0 = \omega$, $s_{-1} = s_0 = s$.

Lemma 6.10. *There exist constants* $\mathcal{C} > 8A$ *and* $\mathcal{R}_0$ *depending only on* n, p, M_0 *and such that if*

$$R_i = \mathcal{C}^{-i}R, \quad s_i = \operatorname*{ess\ inf}_{\mathbf{Q}_{R_i}^{(x_0,t_0)}} p(x,t), \quad \omega_i = \sigma^i\omega,$$

$$c_i = \mathcal{N}\left(\frac{\omega_i}{2^\lambda}\right)^{(s_i-2)/s_i}, \quad Q_i = Q_{c_iR_i,R_i^{s_i}}, \quad i = 0, 1, \dots, \tag{6.82}$$

and $R_0 \le \mathcal{R}_0$, $\omega_0 \ge 2^{\lambda_0} R_0^{s_0/2}$, $s_{i-1} < 2$, *then* $Q_{i+1} \subset Q_i \subset \mathbf{Q}_{R_i}^{(x_0,t_0)}$ *and*

$$\operatorname*{ess\ osc}_{Q_i} u \le \omega_i. \tag{6.83}$$

Proof. Recall that in the proof of the estimate (6.81) our starting point was the inequality $\operatorname*{ess\ osc}_{Q_0} u \le \omega_0$. In order to find the constant χ, we use the relations

$$\frac{c_0R_0}{A} = \mathcal{N}\left(\frac{\omega}{\omega_1}\right)^{(s_0-2)/s_0}\left(\frac{\omega_1}{2^\lambda}\right)^{(s_0-2)/s_0}\frac{R_0}{A} = \sigma^{(2-s_0)/s_0}\left(\frac{\omega_1}{2^\lambda}\right)^{2/s_1-2/s_0}\mathcal{N}\left(\frac{\omega_1}{2^\lambda}\right)^{(s_1-2)/s_1}\frac{\chi R_1}{A}$$

and

$$\left(\frac{R_0}{A}\right)^{s_0} = A^{-s_0}\mathcal{C}^{s_1}R_0^{s_0-s_1}R_1^{s_1}.$$

Since $\sigma \in [1/8, 1)$, $s_1 \ge s_0 > 1$, $A > 8$ and, a priori, $\chi > 8A > 64$, it follows that

$$\omega_1 = \sigma\omega_0 > \frac{\mathcal{C}^{1/2}}{8}2^{\lambda_0}R^{s_1/2} > 2^{\lambda_0}R_1^{s_1/2}. \tag{6.84}$$

Moreover, we have $(2 - s_0)/s_0 < 1$, $\omega_1 \le 2M_0$, and thus come to the inequalities

$$\frac{c_0 R_0}{A} > \sigma (2M_0)^{2/s_1 - 2/s_0} A^{-1} \mathcal{C} c_1 R_1 \ge \frac{1}{8} (2M_0)^{2/s_1 - 2/s_0} A^{-1} \mathcal{C} c_1 R_1,$$
$$\left(\frac{R_0}{A}\right)^{s_0} \ge A^{-s_0} \mathcal{C}^{s_0} R_1^{s_1}.$$

Without loss of generality, we assume that $M_0 \ge 1$ and denote by w the oscillation of the exponent $p(x,t)$ in the cylinder $\mathbf{Q}_{R_0}^{(x_0,t_0)}$. Using the continuity of $p(x,t)$ at (x_0,t_0), let us chose the constant $\mathcal{R}_0$ so small that $(2M_0)^{-w/\alpha^2}$ 4/5 for $R_0 \le \mathcal{R}_0$. Then, $(2M_0)^{2/s_1 - 2/s_0}$ 4/5 and, setting $\chi = 10A$ and using the above inequalities, we get

$$\frac{c_0 R_0}{A} > c_1 R_1, \quad \left(\frac{R_0}{A}\right)^{s_0} > R_1^{s_1}, \quad Q_1 \subset Q_{cR/A,(R/A)^s}.$$

Therefore, according to the estimate (6.81) from Proposition 6.1, we have

$$\operatorname*{ess\,osc}_{Q_1} u \le \operatorname*{ess\,osc}_{Q_{cR/A,(R/A)^s}} u \le \sigma \omega_0 = \omega_1. \tag{6.85}$$

On the basis of (6.84) and (6.85), repeating the above arguments, we come to the desired relation (6.83). The lemma is proved. □

7. Hölder Continuity of Solutions

The Hölder continuity of solutions follows from the oscillation Lemmas 5.7 and 6.10, whose notation is used here. The proof of these lemmas shows that their results remain valid with increased constants $\sigma \in (0,1)$ and $\mathcal{C} > 1$. In what follows, it is assumed that each of these constants coincides with the largest of the two possible values in Lemmas 5.7 and 6.10.

Proof of Theorems 1 and 2. For $s = s_0$ in (4.2) consider two cases: (a) $s_0 \ge 2$; (b) $s_0 < 2$. The above oscillation lemmas have been established for cylinders $Q_i \subset \mathbf{Q}_{R_i}^{(x_0,t_0)}$ of the form (5.46), (6.82) with the "vertex" at (x_0,t_0). Fixing $\rho \in (0,R]$, we find the smallest i for which $R_{i+1} < \rho \le R_i$. Here as above, $R_i = \mathcal{C}^{-i} R$, $i = 0,1,\dots$, and $R_0 = R$.

In case (a), the exact lower bound s_j from (5.45) in cylinders $Q_j \subset \mathbf{Q}_{R_j}^{(x_0,t_0)}$ of the form (5.46) also satisfies the inequality $s_j \ge 2$ for all $j = 1,2,\dots$. By Lemma 5.7, the inequality $\omega_0 > 2^l R_0$ allows us to obtain the estimate 5.47, from which, setting $\gamma = -\ln \frac{1}{\sigma} \cdot \ln^{-1} \mathcal{C}$, we find that (recall that $\sigma > \mathcal{C}^{-1}$)

$$\operatorname*{ess\,osc}_{Q_{\rho,a_i\rho^{s_i}}} u \le \operatorname*{ess\,osc}_{Q_{R_i,a_iR_i^{s_i}}} u \le \sigma^{-1} \omega_0 \left(\frac{\rho}{R_0}\right)^{\gamma}, \quad \gamma \in (0,1).$$

Let us use here the inequality $\omega_i \le 2M_0$.

From the explicit form of the constant a_i (see (5.46)), it follows that $Q_{\rho,\rho^{\beta+1}} \subset Q_{\rho,a_i\rho^{s_i}}$ for $R_0 \le \mathcal{R}_0(p, M_0)$, where β is the constant from condition (1.2). Therefore, since $\sigma = \sigma(n,p,M_0)$, we have

$$\operatorname*{ess\,osc}_{Q_{\rho,\rho^{\beta+1}}} u \le C(n,p,M_0) \left(\frac{\rho}{R_0}\right)^{\gamma} \quad \text{for all} \quad \rho \in (0,R_0]. \tag{7.1}$$

If in case (a) it turns out that $\omega_0 \le 2^l R_0$, then we should check the inequalities $\omega_j \le 2^l R_j$ for $j = 1,2,\dots$. For $\omega_j \le 2^l R_j$, $j = 1,2,\dots,i-1$, we obtain the estimate

$$\operatorname*{ess\,osc}_{Q_{\rho,\rho^{\beta+1}}} u \le C(n,p,M_0)\rho,$$

which implies (7.1). If $\omega_j \le 2^l R_j$ for $j = 1, 2, \dots, k$, where $k < i - 1$, and $\omega_{k+1} > 2^l R_{k+1}$, then, using the estimate (5.47) from Lemma 5.7, we come to the relation

$$\operatorname*{ess\,osc}_{Q_{R_j, a_i R_j^{s_j}}} u \le \sigma^{j-k}\omega_{k+1} \le \sigma^{j-k-1} 2^l R_{k+1}, \quad j = k+2, k+3, \dots .$$

Since $\sigma > \chi^{-1}$, we again have (7.1).

Consider case (b), in which $s_0 < 2$, and for simplicity assume that $\omega_0\, 2^{\lambda_0} R_0^{s_0/2}$. Now we use the estimate (6.83) from Lemma 6.10 in cylinders Q_j of the form (6.82). Under the assumption that $s_i < 2$, as in the proof of (7.1), for $R_0 \le \mathcal{R}_0(n, p, M_0)$ we have $Q_{\rho^\alpha, \rho^\beta} \subset Q_{c_i R_i, R_i^{s_i}}$, where α is the constant from (1.2) and

$$\operatorname*{ess\,osc}_{Q_{\rho^\alpha, \rho^\beta}} u \le C(n, p, M_0) \left(\frac{\rho}{R_0}\right)^\gamma \quad \text{for all} \quad \rho \in (0, R_0]. \tag{7.2}$$

Now let $s_j < 2$ for $j = 1, 2, \dots, k$, where $k < i$, and $s_{k+1} \ge 2$. Then

$$\omega_k \le \operatorname*{ess\,osc}_{Q_{c_k R_k, R_k^{s_k}}} u \le \sigma^k \omega_0.$$

Let us take the smallest integer $m_0 > 0$ starting from which the inclusion $Q_{R_j, 2R_j^2} \subset Q_{c_k R_k, R_k^{s_k}}$ holds for $j = m_0 + k$.

Clearly, m_0 depends only on n, p, M_0, and due to the preceding relation, we have

$$\operatorname*{ess\,osc}_{Q_{R_j, 2R_j^2}} \le \omega_k \le \sigma^k \omega_0, \quad j = m_0 + k. \tag{7.3}$$

It is not difficult to see that if $R_j \le \rho$, then for $R_0 \le \mathcal{R}_0(n, p, M_0)$ we come to the estimate (7.2). If $R_j > \rho$, then we start from (7.3) and use the estimate (5.47) from Lemma 5.7. Repeating the above arguments for the case $s_0 \ge 2$, we obtain (7.1). Now from (7.1), (7.2) it follows that in the cylinder $\mathcal{Q}_\rho = K_{\rho^\alpha}^{x_0} \times (t_0 - \rho^{\beta+2}, t_0)$ for $R_0 \le \mathcal{R}_0(n, p, M_0)$, we have the estimate

$$\operatorname*{ess\,osc}_{\mathcal{Q}_\rho} u \le C(n, p, M_0) \left(\frac{\rho}{R_0}\right)^\gamma \quad \text{for all} \quad \rho \in (0, R_0]. \tag{7.4}$$

If condition (1.3) holds, then the point $(x_0, t_0) \in Q_T$ is arbitrary and (7.4)) implies the Hölder continuity of the solution in the entire cylinder Q_T. If condition (1.8) holds, then (7.4) implies the Hölder continuity of W- and H-solutions at points $(x_0, t) \in Q_T$. The theorems are proved. □

This work has been supported by the Russian Foundation for Basic Research, Projects 09-0100446-a and 08-01-9007-a.

REFERENCES

1. V. V. Zhikov, "On Lavrentiev's Phenomenon," *Russ. J. Math. Phys.*, **3**, No. 2, 249–269 (1994).
2. V. V. Zhikov, "On some variational problems," *Russ. J. Math. Phys.*, **5**, No. 1, 105–116 (1996).
3. X. Fan, *A Class of De Giorgi Type and Holder Continuity of Minimizers of Variational with $m(x)$-Growth Condition*, Lanzhou Univ. (1995).
4. Yu. A. Alkhutov, "Harnaack's inequality and Hölder continuity of solutions of nonlinear elliptic equations with nonstandard condition of growth," *Differ. Uravn.*, **33**, No. 12, 1651–1660 (1997).
5. O. V. Krasheninnikova, "On point-wise continuity of solutions of elliptic equations with nonstandard condition of growth," *Tr. Mat. Inst. Steklova*, **236**, 204–211 (2002).
6. E. Acerbi and G. Mingione, "Regularity results for a class of functionals with non-standard growth," *Arch. Ration. Mech. Anal.*, **156**, 121–140 (2001).
7. Yu. A. Alkhutov and O. V. Krasheninnikova, "Continuity at boundary points of solutions of quasi-linear elliptic equations with nonstandard condition of growth," *Izv. Ross. Akad. Nauk, Ser. Mat.*, **68**, No. 6, 3–60 (2004).

8. M. Rùžhička, *Electrorheological Fluids: Modeling and Mathematical Theory*, Lect. Notes Math., Vol. 1748, Springer, Berlin (2000).
9. E. DiBenedetto, *Degenerate Parabolic Equations*, Springer, New York (1993).
10. E. DiBenedetto, M. J. Urbano, and V. Vespri, *Current Issues on Singular and Degenerate Evolution Equations*, Preprint No. 03 24, Pré-Publicações do Departamento de Matemática Universidade de Coimbra (2003).
11. N. S. Antontsev and V. V. Zhikov, "Higher integrability for parabolic equations of $p(x,t)$-Laplacian type," *Adv. Differen. Equ.*, **10**, No. 9, 1053–1080 (2005).
12. E. Acerbi, G. Mingione, and G. A. Seregin, "Regularity results for parabolic systems related to a class of non-Newtonian fluids," *Ann. Inst. Henri Poincaré, Anal. Non Linéare*, **21**, No. 1, 25–60 (2004).
13. Yu. A. Alkhutov, S. N. Antontsev, and V. V. Zhikov, "Parabolic equations with variable order of nonlinearity," *Zbirnik Prats Inst. Mat. NAN Ukr.*, **6**, No. 1, 23–50 (2009).
14. E. De Giorgi, "Sulla differenziabilitá e l'analicitá delle estremali degli integrali multipli regolari," *Mem. Acc. Sci. Torino, Cl. Sc. Fis. Mat. Nat.*, **3**, No. 3, 25–43 (1957).
15. A. S. Agranovich and M. I. Vishik, "Elliptic problems with a parameter and parabolic problems of general type," *Usp. Mat. Nauk*, **19**, No. 3, 53–161 (1964).

Yu. A. Alkhutov
Vladimir State University for the Humanities, Vladimir, Russia
E-mail: `alkhutov@vladggu.ru`

V. V. Zhikov
Vladimir State University for the Humanities, Vladimir, Russia
E-mail: `zhikov@vladggu.ru`