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CALCULATIONS IN EXCEPTIONAL GROUPS OVER RINGS

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In the present paper, we discuss a major project whose goal is to develop theoretical background and working algorithms for calculations in exceptional Chevalley groups over commutative rings. We recall some basic facts concerning calculations in groups over fields, and indicate complications arising in the ring case. Elementary calculations as such are no longer conclusive. We describe the basics of calculations with elements of exceptional groups in their minimal representations, which allow one to reduce calculations in the group itself to calculations in subgroups of smaller rank. For all practical purposes, such calculations are much more efficient than localization methods. Bibliography: 147 titles.

The present paper is based on our talks at the ACA-2008, SNSC-2008, and PCA-2009. In these talks, we described a major project whose goal is to systematically develop both theoretical background and working algorithms for calculations in exceptional groups over rings [17, 18, 50–53, 68–72, 91, 92, 112, 113, 127–145].

This project was started some 10 years ago, in cooperation with Mikhail Gavrilovich, Roozbeh Hazrat, Victoria Kazakevich, Sergei Nikolenko, Viktor Petrov, Igor Pevzner, Nikita Semenov, and Anastasia Stavrova; see also [78, 79, 81–87, 95, 102, 103]. It is a successor of the previous project carried forward by the second and the third authors jointly with Eugene Plotkin since the late 1980s [143, 125, 142, 112, 126].

Due to space limitations, in the present paper we can only scratch the surface and give a very rough outline of our methods. In part, this is indemnified by an extensive bibliography, where one can find references addressing both the theoretical background and technical details of calculations.

§1. The groups

Groups of Lie type are subdivided into two large classes.

• **Classical groups**, such as linear, orthogonal, symplectic, and unitary.

• **Exceptional groups.** In the first approximation, when we are looking at the split forms (= Chevalley groups), there are 5 types of exceptional groups: G_2, F_4, E_6, E_7, and E_8. Actually, groups of type E_6 and E_7 come in two denominations, as adjoint and simply connected ones.

It is easy to calculate in classical groups using their small degree matrix representations. In the present paper, we mostly focus on calculations with elements of the three large exceptional groups of types E_6, E_7, E_8 represented as matrices of size 27×27 or 78×78, 56×56 or 133×133, and 248×248, respectively. The message is that with a right approach such calculations can be efficiently carried out over an arbitrary commutative ring. The group of type F_4 is also most naturally considered in the 27-dimensional representation, as the twisted group of type E_6.

However, simultaneously we pursue also alternative approaches, based on reduction to classical subgroups by elementary calculations and/or on localization and dimension reduction, which eventually lead to groups over semilocal rings, where everything can again be handled by elementary calculations.

On the other hand, the group of type G_2 is not very interesting in this context. First, it is so closely related to the groups of types B_3 and D_4 that it should be considered a classical group. Second, for this group, direct matrix calculations in the 7-dimensional or 8-dimensional representation can be performed by (a very clever) hand, see [41, 15, 101]. Third, many other results we are interested in simply do not hold for the group of type G_2, since it is too small.

More precisely, let Φ be a reduced irreducible root system and R be a commutative ring with 1. We study the following three closely related groups associated to the pair (Φ, R):

• the (simply connected) Chevalley group $G(\Phi, R)$;

• the (simply connected) elementary Chevalley group $E(\Phi, R)$;

• the Steinberg group $\mathrm{St}(\Phi, R)$.

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For definitions of these groups and further references, see [1–11, 20, 26–33, 57, 62, 73–76, 88–92, 100, 104, 106–109, 117, 118, 125, 126, 129, 130, 142, 145].

Recall that $G(\Phi, R)$ is the group of points of an affine group scheme $G(\Phi, -)$, the so-called Chevalley–Demazure group scheme. In other words, $G(\Phi, R)$ is defined by algebraic equations.

On the other hand, the elementary subgroup $E(\Phi, R) \leq G(\Phi, R)$ is generated by elementary generators $x_\alpha(\xi)$, where $\alpha \in \Phi$ and $\xi \in R$. These generators are subject to the following Steinberg relations.

(R1) Additivity:

$$x_\alpha(\xi)x_\alpha(\eta) = x_\alpha(\xi + \eta);$$

(R2) Chevalley commutator formula:

$$[x_\alpha(\xi), x_\beta(\eta)] = \prod x_{i\alpha+j\beta}(N_{\alpha\beta ij}\xi^i\eta^j),$$

where $[x, y] = xyx^{-1}y^{-1}$ denotes the commutator of x and y, the product is taken over all roots of the form $i\alpha + j\beta \in \Phi$, $i, j \in \mathbb{N}$, in a fixed order, and the structure constants $N_{\alpha\beta ij}$ do not depend on ξ and η, see [20, 26, 107, 141].

In fact, the elementary generators $x_\alpha(\xi)$ may be subject to *further* relations, not implied by (R1) and (R2). For $\mathrm{rk}(\Phi) \geq 2$, the Steinberg group $\mathrm{St}(\Phi, R)$ is an abstract group admitting a presentation in terms of the generators $y_\alpha(\xi)$, $\alpha \in \Phi$, $\xi \in R$, subject to the *defining* relations (R1) and (R2). Since we are only interested in exceptional cases, we do not discuss the extra relations necessary in the case $\mathrm{rk}(\Phi) = 1$.

The interrelations among these groups may be described as follows. By definition, $E(\Phi, R)$ is a subgroup of $G(\Phi, R)$:

$$1 \longrightarrow E(\Phi, R) \longrightarrow G(\Phi, R) \longrightarrow K_1(\Phi, R) \longrightarrow 1,$$

where

$$K_1(\Phi, R) = G(\Phi, R)/E(\Phi, R)$$

is the K_1-functor.

In turn, by the von Dyck theorem, $E(\Phi, R)$ is a quotient of $\mathrm{St}(\Phi, R)$:

$$1 \longrightarrow K_2(\Phi, R) \longrightarrow \mathrm{St}(\Phi, R) \longrightarrow E(\Phi, R) \longrightarrow 1,$$

the kernel of the natural projection being the K_2-functor.

§2. On the field side

Calculations in groups of Lie type over a field, especially over a finite field, form a vast area in computational group theory. There are several specialized computer algebra systems (CAS), such as `GAP`, `Lie`, `Chevie`, `Magma`, `MeatAxe`, and various packages developed within general-purpose CAS, such as `Mathematica` or `Maple`; see references in [21, 35–39, 43, 44, 48, 54, 55, 63, 93, 94, 127, 138, 140].

These calculations rely on several fundamental results, which we now recall very briefly. In fact, most of these results readily generalize to semilocal rings and other rings of dimension 0. First of all, the following theorem (see [1, 11, 73, 106]) implies that in this case calculations in Chevalley groups reduce to calculations with the elementary generators $x_\alpha(\xi)$, $\alpha \in \Phi$, $\xi \in R$.

Theorem 1. *Let R be a semilocal ring. Then $\mathrm{K}_1(\Phi, R) = 1$. In other words, $G(\Phi, R) = E(\Phi, R)$.*

Better still, over fields and semilocal rings there are canonical forms, such as the Bruhat decomposition and the Gauß decomposition, which give sharp upper bounds for the length of such expressions. Namely, let $N = |\Phi^+|$ be the number of positive roots. The following results provide upper bounds for the length of elementary expressions, with the leading terms $2N$ or $3N$, respectively.

Fix a split maximal torus $T(\Phi, R)$ of $G(\Phi, R)$ and a fundamental root system Π in Φ. As usual, $B(\Phi, R)$ denotes the standard Borel subgroup of $G(\Phi, R)$ corresponding to such a choice, $U(\Phi, R)$ denotes its unipotent radical, $U^-(\Phi, R)$ denotes the unipotent radical of the standard opposite Borel subgroup $B^-(\Phi, R)$. Further, we fix a[ny] lifting of the Weyl group $W(\Phi)$ to the normalizer $N(\Phi, R)$ of $T(\Phi, R)$.

Theorem 2. *Let K be a field. Then*

$$G(\Phi, K) = B(\Phi, K)W(\Phi)U(\Phi, K).$$

Theorem 3. *Let R be a semilocal ring. Then*

$$G(\Phi, R) = B(\Phi, R)U^-(\Phi, R)U(\Phi, R).$$

In other words, over fields and semilocal rings there are VERY VERY VERY SHORT expressions of arbitrary elements in terms of elementary generators.

Similar facts hold at the level of K_2. As usual, for any $\alpha \in \Phi$ and $\varepsilon \in R^*$, we set

$$w_\alpha(\varepsilon) = x_\alpha(\varepsilon)x_{-\alpha}(-\varepsilon^{-1})x_\alpha(\varepsilon), \quad h_\alpha(\varepsilon) = w_\alpha(\varepsilon)w_\alpha(1)^{-1}.$$

A calculation in $\mathrm{SL}(2, R)$ shows that these semisimple root elements $h_\alpha(\varepsilon)$ are subject to the following relation.

(R3) Multiplicativity: $h_\alpha(\varepsilon)h_\alpha(\eta) = h_\alpha(\varepsilon\eta)$.

The first Steinberg theorem [107–110] asserts that over a field these (together with the definition of $w_\alpha(\varepsilon)$ and $h_\alpha(\varepsilon)$) are the only relations that should be added to (R1) and (R2) to obtain a presentation of $G(\Phi, R)$. This result can be phrased slightly differently. Namely, define elements $h_\alpha(\varepsilon)$ of the Steinberg group $\mathrm{St}(\Phi, R)$ by the same formulas as above, but in terms of $y_\alpha(\varepsilon)$ instead of $x_\alpha(\varepsilon)$. The Steinberg symbols

$$\{\varepsilon, \eta\} = h_\alpha(\varepsilon\eta)h_a(\varepsilon)^{-1}h_\varepsilon(\eta)^{-1}$$

measure the failure of these new $h_\alpha(\varepsilon)$ to be commutative.

Theorem 4. *For any field K, one has that $K_2(\Phi, K)$ is central and is generated by the Steinberg symbols $\{\varepsilon, \eta\}$, where $\varepsilon, \eta \in K^*$.*

The second Steinberg theorem [107, 108] asserts that over a *finite* field the Steinberg symbols are trivial.

Theorem 5. *For a finite field $K = \mathbb{F}_q$ one has $K_2(\Phi, \mathbb{F}_q) = 1$.*

Essentially, this amounts to saying that for Chevalley groups of rank ≥ 2 over fields, all relations among the elementary generators follow from VERY VERY VERY SHORT ones. Similar, but slightly fancier results in terms of Dennis–Stein or similar symbols hold also over semilocal rings.

These facts allow us to completely reduce calculations in Chevalley groups over fields and semilocal rings to what we call **elementary calculations** – calculations with elementary generators using only Steinberg-type relations [20, 26, 104, 108, 142].

In turn, canonical forms readily reduce such calculations to calculations in the unipotent groups $U(\Phi, R)$ and in the Weyl group $W(\Phi)$; this is essentially the viewpoint adopted by Cohen, de Graaf, Haller, Murray, Rónyai, Taylor in [38, 39, 48, 93].

§3. ON THE RING SIDE

However, many applications outside of group theory, say in algebraic geometry, the theory of algebraic groups, algebraic K-theory, number theory, topology, and physics, require calculations over rings, especially over such classes of rings as

- semilocal rings,
- Hasse domains R_S,
- polynomial rings $K[x_1, \dots, x_n]$ and $\mathbb{Z}[x_1, \dots, x_n]$,
- rings of geometric origin $\mathcal{O}_X$, etc.

As we have seen in the preceding section, semilocal rings are not very much different from fields. However, over rings of dimension ≥ 1 the above beautiful picture breaks spectacularly. Let us mention some of the most striking phenomena.

Pitfall 1. *The group $\mathrm{K}_1(\Phi, R)$ is usually nontrivial.*

In fact, it may be nontrivial already for principal ideal domains. Let us mention two such examples, [46, 56, 64].

- Let $R = \mathbb{Z}[x]$, and let $S \subseteq R$ be the multiplicative system generated by the cyclotomic polynomials Φ_n, $n \in \mathbb{N}$. Then $S^{-1}R$ is a principal ideal ring with $\mathrm{SK}_1(S^{-1}R) \neq 1$.
- Let K be a field of algebraic functions in one variable over a perfect field k. Then $R = K \otimes_k k(x_1, \dots, x_m)$ is a principal ideal ring. If, moreover, $m \geq 2$, and the genus of K is distinct from 0, then $\mathrm{SK}_1(R) \neq 1$.

In other words, there are more elements in a Chevalley group than mere the products of the elementary generators. However, over rings the situation is *terribly* much more tragic than that. Even at the level of elementary groups there are no analogs of the Bruhat and Gauß decompositions.

Pitfall 2. *Even when the group* $\mathrm{K}_1(\Phi, R)$ *is trivial, there are no canonical forms expressing an arbitrary element in terms of the elementary generators.*

In fact, no such forms can possibly exist, since even over principal ideal domains there is no bound for the length of such an expression [59]. Even when there is such a bound, it may be unreasonably large, [12, 24, 25, 77, 98, 119–121].

Pitfall 3. *The group* $\mathrm{K}_2(\Phi, R)$ *may be far too large.*

Moreover, even when $\mathrm{K}_2(\Phi, R)$ is not too large, complete sets of its generators are not known, except in some very special cases. In other words, there is little hope to abstractly control relations among the elementary generators.

In fact, there is another very serious complication. To explain it, we have to recall the notion of relative subgroups. For type F_4, one should define relative subgroups in terms of two ideals rather than one, but in this exposition we limit ourselves to the classical case of one relative parameter.

Namely, let $I \trianglelefteq R$ be an ideal of R. Then I defines the *reduction homomorphism*

$$\rho_I : G(\Phi, R) \to G(\Phi, R/I).$$

The kernel of ρ_I is denoted by $G(\Phi, R, I)$ and is called the *principal congruence subgroup* of level I. The group

$$E(\Phi, R, I) = \langle x_\alpha(\xi),\ \alpha \in \Phi,\ \xi \in I \rangle^{E(\Phi, R)}$$

is called the *relative elementary subgroup* of level I. The quotient

$$\mathrm{K}_1(\Phi, R, I) = G(\Phi, R, I)/E(\Phi, R, I)$$

is called the relative K_1-functor.

There are many important examples where the absolute $K_1(n, R)$, $n \geq 3$, itself is trivial. Let us cite two such paramount examples.

• $R = \mathcal{O}_{K,S}$ is a Hasse domain, i.e., the ring of S-integers of an algebraic number field K. This is part of the solution of the congruence subgroup problem by Bass, Milnor, and Serre [19].

• $R = K[x_1, \dots, x_m]$ is a polynomial ring over a field. This is the positive solution of the K_1-analog of Serre's problem due to Suslin, see [114, 115, 61, 80, 3, 4].

However, here comes a mighty final blow.

Pitfall 4. *Even when the group* $\mathrm{K}_1(\Phi, R)$ *itself is trivial, the corresponding relative groups* $\mathrm{K}_1(\Phi, R, I)$ *for an ideal* $I \trianglelefteq R$ *are usually not.*

At this point one gets impression that apart from zero-dimensional rings and some very special examples, there is no hope to achieve anything useful with the help of elementary calculations.

§4. Theoretical background

In fact, there are some very powerful structure results that make situation *slightly* less desperate than it seems.

First of all, we have the following crucial theorem, originally established by Suslin, Kopeiko, and Taddei [114, 115, 61, 117, 118], which essentially reduces calculations in $G(\Phi, R)$ to calculations in $E(\Phi, R)$.

Theorem 6. *Let* $\mathrm{rk}(\Phi) \geq 2$. *Then the elementary subgroup* $\mathrm{E}(\Phi, R, I)$ *is normal in* $G(\Phi, R)$.

Several different proofs of this result are discussed in [16–18, 49, 51, 52, 112, 116, 125]. The *very* final step towards the proof of the definitive result with *two* relative parameters was – modulo the previous work by Abe and Stein [1, 5, 6, 8, 11, 104] – accomplished in [50]. In [53] one can find further generalizations with several relative parameters and many further references.

Thus $\mathrm{K}_1(\Phi, R)$ and $\mathrm{K}_1(n, R, I)$ are indeed groups, and a conjugate $g x_\alpha(\xi) g^{-1}$ of an elementary generator can be expressed as a product of elementary generators.

In fact, for finite-dimensional rings, reduction from $G(\Phi, R)$ to $E(\Phi, R)$ is further enhanced by the following powerful result due to Bak, Hazrat, and the third author, [16, 17, 49, 51, 52]. If you are not sure what the Bass–Serre dimension $\delta(R)$ of a ring is, you are welcome to substitute it by the Jacobson dimension $\text{j-dim}(R) = \dim(\mathrm{Max}(R))$.

Theorem 7. *Let* $\mathrm{rk}(\Phi) \geq 2$*, and let the Bass–Serre dimension* $\delta(R)$ *be finite. Then the group* $\mathrm{K}_1(\Phi, R, I)$ *is nilpotent.*

In fact, in these papers we establish much more specific results, which *in particular* imply that $E(\Phi, R)$ is fully characteristic in $G(\Phi, R)$.

Another very powerful and useful result is the description of $E(\Phi, R)$-normalized subgroups of $G(\Phi, R)$, first established by Abe, Suzuki, and Vaserstein [1, 5, 6, 8, 11, 124]. Not to go into technical details related to relativization with several parameters, we only state it for the case of simply-laced systems.

Theorem 8. *Let* Φ *be a simply-laced system such that* $\mathrm{rk}(\Phi) \geq 2$*. Then for any subgroup* $H \leq G(\Phi, R)$ *normalized by* $E(\Phi, R)$ *there exists a unique ideal* $I \trianglelefteq R$ *such that*

$$E(\Phi, R, I) \leq H \leq C(\Phi, R, I).$$

Of course, for classical groups similar results are known since the 1970s, while the original proofs by Abe, Suzuki, and Vaserstein used localization. Similar results for multiply-laced systems should be stated in terms of admissible pairs, or ideals of the corresponding Chevalley algebra. We refer to [41, 112, 125, 133, 134, 139, 141, 144], where one can find new-generation proofs of these results for arbitrary commutative rings and many further references.

Further, it turns out that most elementary calculations do not depend on the size of $\mathrm{K}_2(\Phi, R)$ should it be central. The following outstanding result was established by van der Kallen, Tulenbaev, Bak, and Tang.

Theorem 9. *Let* $\Phi = \mathrm{A}_l$*,* $l \geq 3$ *or* $\Phi = \mathrm{C}_l, D_l$*,* $l \geq 4$*, Then the group* $\mathrm{K}_2(\Phi, R)$ *is central in* $\mathrm{St}(\Phi, R)$*.*

However, as of today, a complete proof of this fact is published in [58, 33] only for the case $\Phi = \mathrm{A}_l$. The proof for other classical cases was announced some 10 years ago, but due to its enormous size and complexity is still not published. See comments on the status of this proof in [52].

In [130, 132] the third author launched a subproject that [hopefully!] should result in a proof of the centrality for the cases $\Phi = \mathrm{E}_6, \mathrm{E}_7$. We are certain that a proof of the centrality for E_8 can also be obtained along these lines, though at present there is little to no hope to treat F_4 with the existing methods.

§5. Weyl modules

As opposed to the field case, most of our calculations are carried through using *representations* of groups rather than their presentations. Observe that such a viewpoint is sometimes taken by the authors working over fields, for instance, by Howlett, Rylands, Taylor, Ryba, and Testerman in [55, 93, 94] and [122].

Let $P(\Phi)_{++}$ be the cone of integral dominant weights. Fix a weight $\omega \in P(\Phi)_{++}$ and consider the Weyl module $V = V(\omega)$ of the group $G(\Phi, R)$. It is obtained by reduction of the (integral form of the) irreducible module with highest weight ω in characteristic 0. The resulting module needs not remain irreducible over a field of small positive characteristic, but for small modules it usually is. Usually, only characteristics 2 and 3 may constitute noticeable trouble.

In fact, for applications in structure theory one needs only to have a thorough understanding of one module for each group. We mostly work in the following modules.

- The 27-dimensional microweight module $V(\varpi_1)$ or $V(\varpi_6)$, for the simply connected group of type E_6.
- The 78-dimensional adjoint module $V(\varpi_2)$, for the adjoint group of type E_6.
- The 56-dimensional microweight module $V(\varpi_7)$, for the simply connected group of type E_7.
- The 133-dimensional adjoint module $V(\varpi_1)$, for the adjoint group of type E_7.
- The 248-dimensional adjoint module $V(\varpi_8)$, for the group of type E_8.
- The 27-dimensional module that is the restriction of $V(\varpi_1)$ from E_6 to F_4, for the group of type F_4.

This last module is *not* irreducible, generically it decomposes into the direct sum of the 26-dimensional short-root module and the trivial 1-dimensional module. However, the larger symmetry of defining equations and the possibility to use results for E_6 more than compensate for the increase in dimension.

Let $V = V(\omega)$ be such a module, and let $\Lambda(\omega)$ be the set of its weights *with multiplicities*. Recall that for microweight representations, all weights are extremal, in other words, $\Lambda(\omega) = W\omega$. Thus in this case all weights have multiplicity 1. Similarly, for adjoint representations, the *nonzero* weights are the roots of Φ and have multiplicity 1. On the other hand, in this case the zero weight has multiplicity $l = \mathrm{rk}(\Phi)$. For actual calculations, we fix an appropriate admissible base v^λ, $\lambda \in \Lambda(\omega)$, in V. Practically, it is very advantageous to make the following choices.

- For microweight representations, a **crystal base**.
- For adjoint representations, a **positive Chevalley base**.

Recall that the action of root elements in a crystal base is described by the following simple formula:

$$x_\alpha(\xi)v^\lambda = v^\lambda + c_{\lambda,\alpha}v^{\lambda+\alpha},$$

where $v^{\lambda+\alpha} = 0$ if $\lambda + \alpha$ is not a weight. The structure constants $c_{\lambda,\alpha}$ of the action enjoy the following nice properties:

- $c_{\lambda,\alpha} = \pm 1$ for all $\lambda \in \Lambda(\omega)$ and $\alpha \in \Phi$ provided that $\lambda + \alpha \in \Lambda(\omega)$;
- $c_{\lambda,\alpha} = 1$ if $\alpha \in \pm\Pi$ is a fundamental or a negative fundamental root.

The action of root elements in a Chevalley base is expressed by similar but slightly more complicated formulas, see [20, 26, 125, 128, 142] for details.

Calculations in terms of the action of G on a rational module are referred to as **stable calculations**. The name is due to the fact that over rings techniques of such calculations were first developed by Matsumoto [73] and Stein [106] in the context of stability of lower K-functors.

We have started to implement practical algorithms based on our methods in `Mathematica` and `Maple`, and to tabulate explicit results concerning the explicit action of the groups on their minimal modules, including action constants, root elements, multilinear invariants, equations, and the like, [127, 128, 131, 138, 140, 72].

§6. Equations

In the natural coordinates, the membership of an individual matrix $g \in GL(V)$ to the Chevalley group $G(\Phi, R)$ is described by equations of degrees 3 and 4. These equations can be explicitly retrieved from the multilinear invariants of the G-action on V. In characteristic 0 such invariants are classically known, but making them characteristic-free oftentimes requires considerable work, cf. [13, 14, 40, 47, 67]. A thorough discussion of these invariants and many further references can be found in [70–72, 85–87, 91, 96, 125, 126, 131, 136, 137, 142].

As an illustration, let us reproduce from [136] explicit equations defining the normalizer of $G(\mathrm{E}_6, R)$ in the 27-dimensional representation $V(\varpi_1)$.

Theorem 10. *A matrix $g \in \mathrm{GL}(27, R)$ lies in $N(G(\mathrm{E}_6, R))$ if and only if it satisfies the following types of equations.*

- *Equations on pairs of adjacent columns. For all $\lambda, \mu, \nu \in \Lambda$ such that $d(\mu, \nu) \le 1$,*

$$f_\lambda(g_{*\mu}, g_{*\nu}) = 0.$$

- *Equations on two pairs of nonadjacent columns. For all $\lambda, \mu, \nu, \rho, \sigma, \tau \in \Lambda$ such that $d(\mu, \nu) = d(\sigma, \tau) = 2$,*

$$(-1)^{h(\mu\circ\nu,\mu,\nu)} g'_{\mu\circ\nu,\lambda} f_\rho(g_{*\sigma}, g_{*\tau}) = (-1)^{h(\sigma\circ\tau,\sigma,\tau)} g'_{\sigma\circ\tau,\rho} f_\lambda(g_{*\mu}, g_{*\nu}).$$

The occurring polynomials f_λ are in fact first partial derivatives of the invariant cubic form F on $V(\varpi_1)$. Below we list them according to the natural (= height-lexicographic) numbering of weights:

$f_1(x) = x_{13}x_{27} - x_{16}x_{26} + x_{18}x_{25} - x_{20}x_{24} + x_{22}x_{23}$,

$f_2(x) = -x_{11}x_{27} + x_{14}x_{26} - x_{17}x_{25} + x_{19}x_{24} - x_{21}x_{23}$,

$f_3(x) = x_9x_{27} - x_{12}x_{26} + x_{15}x_{25} - x_{19}x_{22} + x_{20}x_{21}$,

$f_4(x) = -x_7x_{27} + x_{10}x_{26} - x_{15}x_{24} + x_{17}x_{22} - x_{18}x_{21}$,

$f_5(x) = x_6x_{27} - x_8x_{26} + x_{15}x_{23} - x_{17}x_{20} + x_{18}x_{19}$,

$f_6(x) = x_5x_{27} - x_{10}x_{25} + x_{12}x_{24} - x_{14}x_{22} + x_{16}x_{21}$,

$f_7(x) = -x_4x_{27} + x_8x_{25} - x_{12}x_{23} + x_{14}x_{20} - x_{16}x_{19}$,

$f_8(x) = -x_5x_{26} + x_7x_{25} - x_9x_{24} + x_{11}x_{22} - x_{13}x_{21}$,

$f_9(x) = x_3x_{27} - x_8x_{24} + x_{10}x_{23} - x_{14}x_{18} + x_{16}x_{17}$,

$$f_{10}(x) = x_4x_{26} - x_6x_{25} + x_9x_{23} - x_{11}x_{20} + x_{13}x_{19},$$
$$f_{11}(x) = -x_2x_{27} + x_8x_{22} - x_{10}x_{20} + x_{12}x_{18} - x_{15}x_{16},$$
$$f_{12}(x) = -x_3x_{26} + x_6x_{24} - x_7x_{23} + x_{11}x_{18} - x_{13}x_{17},$$
$$f_{13}(x) = x_1x_{27} - x_8x_{21} + x_{10}x_{19} - x_{12}x_{17} + x_{14}x_{15},$$
$$f_{14}(x) = x_2x_{26} - x_6x_{22} + x_7x_{20} - x_9x_{18} + x_{13}x_{15},$$
$$f_{15}(x) = x_3x_{25} - x_4x_{24} + x_5x_{23} - x_{11}x_{16} + x_{13}x_{14},$$
$$f_{16}(x) = -x_1x_{26} + x_6x_{21} - x_7x_{19} + x_9x_{17} - x_{11}x_{15},$$
$$f_{17}(x) = -x_2x_{25} + x_4x_{22} - x_5x_{20} + x_9x_{16} - x_{12}x_{13},$$
$$f_{18}(x) = x_1x_{25} - x_4x_{21} + x_5x_{19} - x_9x_{14} + x_{11}x_{12},$$
$$f_{19}(x) = x_2x_{24} - x_3x_{22} + x_5x_{18} - x_7x_{16} + x_{10}x_{13},$$
$$f_{20}(x) = -x_1x_{24} + x_3x_{21} - x_5x_{17} + x_7x_{14} - x_{10}x_{11},$$
$$f_{21}(x) = -x_2x_{23} + x_3x_{20} - x_4x_{18} + x_6x_{16} - x_8x_{13},$$
$$f_{22}(x) = x_1x_{23} - x_3x_{19} + x_4x_{17} - x_6x_{14} + x_8x_{11},$$
$$f_{23}(x) = x_1x_{22} - x_2x_{21} + x_5x_{15} - x_7x_{12} + x_9x_{10},$$
$$f_{24}(x) = -x_1x_{20} + x_2x_{19} - x_4x_{15} + x_6x_{12} - x_8x_9,$$
$$f_{25}(x) = x_1x_{18} - x_2x_{17} + x_3x_{15} - x_6x_{10} + x_7x_8,$$
$$f_{26}(x) = -x_1x_{16} + x_2x_{14} - x_3x_{12} + x_4x_{10} - x_5x_8,$$
$$f_{27}(x) = x_1x_{13} - x_2x_{11} + x_3x_9 - x_4x_7 + x_5x_6.$$

Projectively, these quadratic forms are in fact equations defining the Cayley plane (= projective octave plane). Explicit equations defining $G(\mathrm{F}_4, R)$ easily follow, see [69].

We have obtained similar geometric and combinatorial descriptions of such equations for all other relevant cases. This, as well as their connection with Groebner bases and standard bases, is thoroughly discussed in [34, 72]. Explicit equations defining groups of type E_7 and E_8 are derived in our papers [71, 72, 131, 137].

Classically, for (E_7, ϖ_7) the characteristic 2 presented considerable difficulties. Lately, these difficulties were completely surmounted by Lurie and the first author in [67, 70].

§7. Decomposition of unipotents, and beyond

Most types of calculations in $G(\Phi, R)$ can be reduced to calculations in groups $G(\Phi, \mathbb{Z}[x_1, \dots, x_m])$ over integer polynomials. Even so, straightforward calculations with 27×27, 56×56, or 248×248 matrices using equations of degree ≥ 3 almost immediately run into formidable difficulties:

- large number of variables, like 61504 variables for E_8,
- large number of equations, ≈200000 equations for E_8,
- 24360 monomials in the quartic form for E_7,
- growth of coefficients in the intermediate results.

What do you do with these things? Here we describe a crucial idea that allows one to *dramatically* reduce the complexity of calculations actually occurring in the majority of practical situations.

Many usual calculations in Chevalley groups can be reduced to calculations with elements of the form $gx_\alpha(\xi)g^{-1}$ or $[g, x_\alpha(\xi)]$, where $g \in G(\Phi, R)$, $\alpha \in \Phi$, $\xi \in R$. As we know, over fields g itself is a product of elementary generators. Let us list various ideas that have been used to treat such elements in general.

• Historically, the first idea that would work in reasonable generality was proposed by Bass, and was based on a decomposition of g. However, such decompositions only exist under stability conditions, see references in [106, 89–92, 52, 125].

• Suslin's initial approach "**factorization and patching**" [114, 115, 61] consisted in decomposing $gx_\alpha(\xi)g^{-1}$ itself. It works *marvelously* for classical groups. However, it requires the whole matrix $gx_\alpha(\xi)g^{-1}$, and, despite considerable effort, no one succeeded in pushing this approach to exceptional groups.

• Localization methods consist in decomposing ξ. The two most popular such methods are Quillen and Suslin's **localization and patching** [114] and Bak's **localization-completion** [16].

• Finally, in **decomposition of unipotents**, proposed by the second and the third authors [112] and developed jointly with Plotkin [143, 125, 126, 142], one decomposes $x_\alpha(\xi)$.

The following result expresses the gist of decomposition of unipotents in the simplest cases (E_6, ϖ_1) and (E_7, ϖ_7), see [143, 125, 126].

Theorem 11. *Let $\Phi = \mathrm{E}_6, \mathrm{E}_7$, and, further, let $\omega = \varpi_1(\mathrm{E}_6)$ or $\omega = \varpi_7(\mathrm{E}_7)$, respectively. Then for any fixed element $g \in \pi(G)$, the elementary group $E(\Phi, R)$ is generated by the root type elements $z \in E(\Phi, R)$ such that $zg_{*\mu} = g_{*\mu}$ for some column of the matrix g.*

More precisely, there exists a formula polynomial in ξ and the entries of g and g^{-1} that expresses $x_\alpha(\xi)$ as a product of 27 or 56 root type elements z_λ, $\lambda \in \Lambda(\omega)$, respectively, such that the conjugates gzg^{-1} of these elements by g fall into parabolic subgroups of type P_1 or P_7.

To an outsider, this result might seem to be a whimsical formal exercise. In reality, it is an *extremely* powerful structure result, which, within half a page, implies most of the classical structure theorems. Moreover, it gives a *working* method of calculations in exceptional groups.

Its true significance consists in the fact that A LARGE CLASS OF CALCULATIONS IN THE GROUP $G = G(\Phi, R)$ ITSELF ARE REDUCED TO THE FOLLOWING THREE FAMILIAR TYPES:

- elementary calculations,
- stable calculations,
- calculations in $G(\Delta, R)$, for proper subsystems $\Delta \subseteq \Phi$.

Originally, the decomposition of unipotents as described in [125, 142] depended on extensive computer calculations. Later we succeeded in getting rid of most of them [126], replacing them by visual aids such as weight diagrams [91].

Specifically, the proof of Theorem 11 in [126] is an A_5-proof for E_6 and an A_7-proof for E_7; see [134] for a thorough discussion of these concepts. In other words, this proof gives a *working algorithm* for the following reductions.

- Use calculations in $G(\mathrm{A}_5, R) \cong \mathrm{SL}(6, R)$ to reduce calculations in $G(\mathrm{E}_6, R)$ to those in $G(\mathrm{D}_5, R) \cong \mathrm{Spin}(10, R)$.
- Use calculations in $G(\mathrm{A}_7, R) \cong \mathrm{SL}(8, R)$ to reduce calculations in $G(\mathrm{E}_7, R)$ to those in $G(\mathrm{E}_6, R)$.

What is really amazing here is that such reductions work for *arbitrary* commutative rings, without any finiteness or stability conditions whatsoever.

Another striking aspect of this result is that it only depends on the *quadratic* equations on an individual column or row of a matrix $g \in G$ and does not invoke fancier cubic or quartic equations mentioned in the previous section!

For adjoint representations, a similar result is *much* more demanding, both in terms of prerequisites and in terms of actual computational complexity; see [142], many details of actual calculations are still not fully published.

- Use calculations in $G(\mathrm{D}_5, R) \cong \mathrm{Spin}(10, R)$ to reduce calculations in $G(\mathrm{E}_6, R)$ to those in $G(\mathrm{A}_5, R) \cong \mathrm{SL}(6, R)$.
- Use calculations in $G(\mathrm{D}_6, R) \cong \mathrm{Spin}(12, R)$ to reduce calculations in $G(\mathrm{E}_7, R)$ to those in $G(\mathrm{D}_6, R)$.
- Use calculations in $G(\mathrm{D}_8, R) \cong \mathrm{Spin}(16, R)$ to reduce calculations in $G(\mathrm{E}_8, R)$ to those in $G(\mathrm{E}_7, R)$.

For F_4 there is a similar interplay between subsystems of types B_3 and C_3. However, in this case we could only prove generation (and even that with terribly much work, and only under the additional assumption $2 \in R^*$), and could never get an actual polynomial formula, see [143, 112] and [141] for some indications concerning this case.

Unfortunately, the above form of the decomposition of unipotents relied on the presence of *huge* classical subgroups. Recently we discovered two amazing twists to this approach, one together with Gavrilovich and Nikolenko, another one together with Kazakevich, see [133, 134, 141, 130, 132] and [135].

- These approaches refer only to the presence of such small classical subgroups as A_2, A_3, or D_3.
- They invoke only *linear* equations on the Lie algebra of G.
- They give a much larger flexibility in terms of how to actually decompose unipotents, by varying not only a subsystem, but also its parabolic subgroup.

Thus, from our current viewpoint, all the above proofs are (A_l, P_1)-proofs or (D_l, P_1)-proofs, respectively. Already an (A_3, P_2)-proof, as described in [135], gives enormous additional freedom.

It is impossible to discuss any further details here, but after these improvements CALCULATIONS WITH ELEMENTS OF E_6 AND F_4 REPRESENTED AS 27×27 MATRICES OR WITH ELEMENTS OF E_7 REPRESENTED AS 56×56 MATRICES CAN BE PERFORMED BY HAND, with some determination [28, 33]. We still have some trouble with elements of E_8 represented as 248×248 matrices, though.

§8. Stability, and localization methods

Simultaneously we developed, *with variable success*, competing approaches. Their essence consists in reducing dimension rather than rank.

For small-dimensional rings, one can use stability conditions [106, 88–90, 92] to obtain decompositions such as the Bass–Kolster decomposition of $G(\Phi, R)$ or the Dennis–Vaserstein decomposition of $E(\Phi, R)$.

Thereby one usually can completely reduce calculations in the group $G(\Phi, R)$ itself to calculations in its parabolic subgroups. However, this reduction involves shortening of long products of factors from opposite unipotent radicals. Such an exercise, performed by Loos and Stavrova [66, 103], is rather nontrivial even at the conceptual level, not to mention implementation.

There is another approach, localization, which works for arbitrary commutative rings and in many cases gives shorter, and more uniform, proofs than the geometric methods described in the previous section.

Namely, localization consists in reduction of calculations in a Chevalley group $G(\Phi, R)$ over a commutative ring R to similar calculations in $G(\Phi, R_M)$ over the localizations R_M of the ring R at all prime ideals $M \in \mathrm{Max}(R)$. As we know from §3, over local rings one can solve most of the occurring problems by elementary calculations.

Standard tricks reduce most problems to *principal* localizations R_s constructed by inverting *one* non-nilpotent element $s \in R$. Let $F_s : R \longrightarrow R_s$ be the corresponding reduction homomorphism.

The papers [51, 17, 113, 53] develop conjugation calculus and commutator calculus (also known as the yoga of conjugation and the yoga of commutators, respectively). Several new simplifications and strengthenings were proposed in [53]. Let us state a typical result, which embodies the gist of these methods, see [51].

Theorem 12. *Let* $\mathrm{rk}(\Phi) \geq 2$. *Then for any finite number of elements* $g_1, \dots, g_n \in G(\Phi, R)$ *and any* $k \geq 2$ *there exists* $m \geq 0$ *such that*

$$\Big[g_i, F_s(G(\Phi, R, s^m R))\Big] \leq E(\Phi, F_s(s^k R)).$$

Again, though *seemingly* of a technical nature, this is an *extremely* powerful result, which generalizes many classical results. For example, Theorem 6 is a very special case of Theorem 12, when $n = 1$ and $s = 1$.

For example, using this result we obtained formulas expressing commutators as products of elementary generators [113]. However, for all practical purposes, LOCALIZATION IS DRASTICALLY LESS EFFICIENT THAN GEOMETRIC METHODS. Namely, for classical groups, using decomposition of unipotents, one gets expressions of polynomial length [97]. For exceptional groups, all such expressions we succeeded in obtaining so far via localization methods are of hyperexponential length [113], and unfeasible for any practical purpose.

Let us cite one typical problem that accounts for this staggering difference. Denote by $E^m(\Phi, R)$ the subset (in general, not a subgroup) of $E(\Phi, R)$ consisting of products of not more than m elementary root unipotents.

Problem. *Let*

$$x \in E^1\left(\Phi, \frac{t^l}{s^k} R\right), \quad y \in E^1\left(\Phi, \frac{s^n}{t^m} R\right),$$

the exponents l and n being at our disposal. Compute the exact bound for the number L of factors necessary to rewrite the commutator $[x, y]$ without denominators.

Let us mention two *estimates*, one of them based on elementary calculations alone, another one using explicit matrix calculations in small representations.

• The use of elementary calculations [113] based on the Chevalley commutator formula gives the ridiculous bounds

$$L \leq 585, \qquad L \leq 61882, \qquad L \leq 797647204,$$

depending on whether the system is simply-laced, doubly-laced, or triply-laced.

• The use of representations gives $L \leq 42$, in all cases.

We believe that this example alone amply illustrates the advantages of representation-theoretic/geometric techniques.

§9. Conclusion

The above can be summarized as follows.

• Calculations in exceptional groups over rings, represented in matrices of degrees 27, 56, and 248, are possible.

• Such calculations are inevitable, since purely ring-theoretic methods give unrealistic bounds.

• We have already applied these methods to the proof of the main structure theorems and the study of various classes of subgroups.

• We are positive that these methods may have many further applications in the study of automorphisms, generators and relations, etc., etc.

We are amateur programmers, for us the system of a few thousand equations of degree 3 in 729 variables (arising in the *junior* case of E_6) is a *huge* system of equations. Thus we would be MOST INTERESTED in starting a long-term cooperation with accomplished programmers able to convert our methods and algorithms into efficient implementations.

The following outstanding problem is not solved in general, not even *over fields*.

Problem. *Develop working methods of calculations in the groups of points of all* (*isotropic*) **reductive groups**, *not just the split ones* (= *Chevalley groups*).

We would like to mention two recent contributions. First, the thesis by Haller [48], which addresses computations in forms of algebraic groups *over fields*. Second, with a completely different slope, the paper by Petrov and Stavrova [81], which [under some mild restrictions on ranks] proves the normality of the elementary subgroup in isotropic reductive groups over arbitrary commutative rings. With this end, they develop extensive fragments of a general theory. Their work clearly indicates that with some effort and determination one could obtain in this setting all general results on which our approach relies.

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