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FOREWORD

The theory of *dessins d'enfants*, initiated by Alexander Grothendieck during the Montpellier period (1970–1984) of his mathematical activity, provides a fascinating possibility of *visualizing* arithmetic objects. Moreover, it turns out that all the arithmetical information is encoded in simple combinatorial images; technically speaking, the corresponding *categories* are equivalent.

The first (it would be better to say the zeroth) instance of an equivalence of this kind is known to everybody from early childhood: counting establishes the equivalence between 0-dimensional topological objects (discrete finite spaces) and natural numbers. To the best of my knowledge, dimension 1 is skipped by the modern math: we are unaware of any arithmetical objects, corresponding to *abstract finite graphs*. However, as soon as we change the abstract graphs for the *ribbon graphs* or, equivalently, fix an embedding of a graph into an oriented surface, we find that such an object represents a complicated arithmetic structure. It is called a *Belyi pair* and includes the number field, the algebraic curve over it, and the special rational function — the *Belyi function* — on this curve. The exact definitions and formulations can be found in several papers of this volume.

Unfortunately, it turned out to be impossible to unify the terminology and notation in the papers of this volume. Basically, a Belyi pair consists of a complex algebraic curve and a nonconstant rational function β on it with no more than three critical values; usually these values are normalized, and sometimes the additional restrictions are imposed on the orders of ramifications. Any such curve is the complexification of a curve over $\bar{\mathbb{Q}}$. The embedded graph, i.e., the *dessin d'enfant* itself, is the β-preimage of the real segment joining two of the three critical values of β. Sometimes the graph is assumed to be bi-colored; in the case of spherical trees the Belyi functions are called the *generalized Chebyshev* (or sometimes *Shabat*) polynomials, etc.

The dessins d'enfants theory is well out of infancy now: the appearance of Grothendieck's *Esquisse d'un programme* dates back to 1984. However, due to certain nonmathematical aspects of Grothendieck's life and to the very special style of the *Esquisse* (it might seem to belong rather to poetry or philosophy than to mathematics, almost without exact definitions and theorems), years were needed for the mathematical community to recognize its mathematical depth. Perhaps the first "official" and systematic discussion of Grothendieck's ideas from *Esquisse* took place at I. Gelfand's seminar at Moscow State University in the late 1980s. The first publication where some of the constructions of the *Esquisse* were represented in the traditional mathematical form was the paper of 1990 by G. Shabat and V. Voevodsky,[1] devoted to Grothendieck's 60th anniversary. The first monograph by S. Lando and A. Zvonkin,[2] containing a systematic exposition of the dessins d'enfants theory, appeared only in 2004.

However, the activity in the dessins d'enfants domain has been gradually growing in several countries starting from 1990s; the proceedings of two international conferences[3] give some idea about the state of the theory before the turn of the century. Reports of the group GTEM (*Galois Theory and explicit methods in Arithmetic*), coordinated by L. Schneps (see `www.math.jussieu.fr/~leila/gtem/FinalReport.pdf`), as well as many others, demonstrate the current state.

The seminar *Graphs on surfaces and curves over number fields* that I am conducting at Moscow State University dates back to 1990. The present volume consists of papers by its participants (with obvious exclusion of the short paper by B. Birch). Its material represents the main research directions of the seminar: our main interests are related to the very beginnings of the theory (according to Grothendieck, to *une identité profonde entre la combinatoire des cartes finies d'une part, et la géométrie algébriques définies sur des corps des nombres, de l'autre*), to the explicit calculations connecting combinatorial-topological and arithmetic-geometrical objects. The relations with other branches of mathematics, especially with the theory of moduli spaces of curves and with Teichmüller theory, are also considered.

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The papers by N. M. Adrianov "On plane trees with a prescribed number of valency set realizations" (pp. 5–10) and I. V. Artamkin "Combinatorics of trivalent ribbon graphs with two faces" (pp. 81–86) formally belong to 2-dimensional combinatorial topology but the true sense is found in the context of their relations with the dessins theory.

Adrianov's result is motivated by the study of the absolute Galois group action on the isotopy classes of plane trees (they constitute the simplest class of dessins d'enfants, on which, however, the absolute Galois group acts faithfully). This action has an obvious invariant: the set of valencies of the vertices. Therefore, if a set of valencies admits only one realization by a plane tree, the Galois orbit of such a tree consists of the single element and the tree is defined over $\mathbb{Q}$. The number of such trees is infinite: e.g., it contains chains, corresponding to classical Chebyshev polynomials. However, it turns out that chains and some other simple trees constitute special types. With the exception of trees belonging to these types there is only a finite number of valency sets with the unique realization. Adrianov presents a complete proof of the generalization of this result to the case of an arbitrary number of realizations, which was announced by him and myself long ago.

Artamkin's paper gives a beautiful, nontrivial, and self-contained description of the 2-faced dessins with trivalent graphs. One of the possible applications of this description is related to the geometry of moduli spaces $\mathcal{M}_{g,2}$ whose cells of maximal dimensions in the well-known cellular decomposition of Strebel–Kontsevich–Penner–Witten–... correspond exactly to the dessins considered by Artamkin. It should be noted that the flips (that play the crucial role in his description) correspond to moving to the "neighboring" cell. The paper also contains the elementary explanation of the particular case of Kontsevich identities with rational functions in which the monomials in the denominators cancel by somewhat mysterious reasons.

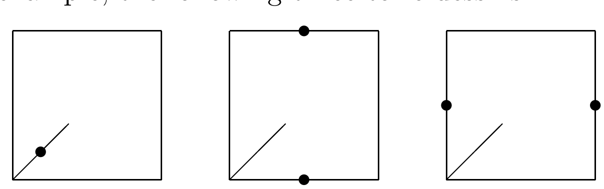
B. Birch's paper "Shabat trees of diameter 4: Appendix to a paper of Zvonkin" (pp. 94–96) is somehow isolated in the volume and belongs to polynomial algebra. It explains (as Birch writes, "using only 19-century algebra") one of the first phenomenon in the calculation of Belyi function, discovered by myself and A. Zvonkin at the beginning of the 1990's: the discriminants of fields of definitions of diameter 4 plane trees split into the product of linear forms in valencies (of very special type). Though the paper is a 12-year-old preprint, its constructions deserve attention in the context of more recent calculations.

The central part of the volume is occupied by three papers, devoted to the actual calculations of Belyi pairs.

Two of them belong to the special genre of "catalogs" — the complete lists of Belyi pairs corresponding to the dessins (sometimes of certain type) of bounded complexity. This genre was started by Shabat in the 1991 IHES preprint and by Bétréma, Péré, and Zvonkin in the 1992 Bordeaux preprint; both were never published. Shabat calculated the Belyi pairs corresponding to all the dessins with ≤ 3 edges while Bétréma, Péré, and Zvonkin covered the plane trees with ≤ 8 edges.

As is seen from the titles, the two catalogs in the present volume — "Catalog of dessins d'enfants with no more than 4 edges" by N. M. Adrianov, N. Ya. Amburg, V. A. Dremov, Yu. Yu. Kochetkov, E. M. Kreines, Yu. A. Levitskaya, V. F. Nasretdinova, and G. B. Shabat (pp. 22–80) and "Plane trees with nine edges. Catalog" by Yu. Yu Kochetkov (pp. 114–140) — go one step beyond the two mentioned preprints. These papers contain only answers: some calculations of Belyi pairs are terribly long; however, it is relatively easy to check the answers.

The complexity of some of the answers can be shortened by better choices of normalization; however, generally it is not so. As the catalog shows, very simple four-edged dessins correspond to extremely complicated Belyi pairs. One of the indications of it is by j-invariants of elliptic curves that do not depend on any choices. For example, the following three toric dessins



(the opposite sides of the squares are identified) constitute a Galois orbit. The j-invariants of the corresponding curves are the roots of the polynomial

$$2^{15}5^{14}7^{10}j^3 - 31562956092228535000000000j^2 \\ + 748295885321347996073297265625j - 5640551353206681359387213998281 28.$$

The third paper of this cycle, “The computation of Belyi pairs of 6-edged dessins d’enfants of genus 3 with automorphism groups of order 12 and 3” by B. S. Bychkov, V. A. Dremov, and E. M. Epifanov (pp. 97–105), is just the first step into genus 3 calculations. Three Belyi pairs corresponding to the smallest possible number 6 of edges and with symmetries of order ≥ 3 are calculated. In this paper the process of calculations is explained.

The remaining six papers are devoted to various questions related to dessins theory.

The paper “On the generalized Chebyshev polynomials corresponding to plane trees of diameter 4” by N. M. Adrianov (pp. 11–21) is an overview of various properties of these polynomials. The author discusses their discriminants, fields of definitions, and Galois orbits, giving a detailed and self-contained exposition of one famous “hidden” Galois invariant. In some cases, the fields of definition are given by hypergeometric polynomials. Some interesting particular classes of diameter 4 plane trees are described.

The paper “Strebel differentials on families of hyperelliptic curves” by I. V. Artamkin, Yu. A. Levitskaya, and G. B. Shabat (pp. 87–93) is devoted to a certain analog of dessins: the considered graphs (separatrices of Strebel differentials) cut the surfaces not into discs, but into cylinders. The existence of such differentials on any Riemann surface was known long ago; the authors, however, present a simple explicit construction on the real 1-parametric family of hyperelliptic curves of arbitrary even genus.

The paper “The Chekhov–Fock parametrization of Teichmüller spaces and dessins d’enfants” by G. B. Shabat and V. I. Zolotarskaia (pp. 155–161) contains mathematical reformulations of some results that have been published in the physical papers. These reformulations are based on Grothendieck’s cartographic techniques, which is one of the central tools in the dessins theory. The authors reformulate the trivalent version of the dessins theory in terms of uniformization. The dessins correspond to the subgroups of $\mathrm{PSL}_2(\mathbb{Z})$, while physicists have defined their deformations in $\mathrm{PSL}_2(\mathbb{R})$. The main result of the paper is that the principle “every dessin sits in its cell” (which makes sense for any of the known decompositions of moduli spaces into pieces labelled by dessins) is confirmed once again.

The paper “Geometry of plane trees” by Yu. Yu. Kochetkov (pp. 106–113) is devoted to the so-called “true shape of plane trees.” One of the consequences of the dessins theory says that among all the isotopically equivalent plane trees with a fixed combinatorial structure there is a canonical representative defined uniquely up to similarity — it is the preimage of the segment joining the critical values of the corresponding generalized Chebyshev polynomial. The differential geometry of the true shape of the trees is almost unexplored. Kochetkov’s paper contains some results on the true shapes of the trees with a small number of edges and known generalized Chebyshev polynomials. These results constitute the basis for some general conjectures and questions, mostly concerning the types of concavity of the edges of the true shapes of plane trees.

The paper “Foliations generated by differentials of Abelian type” by Yu. Yu. Kochetkov (pp. 141–147) is related to the dessins theory in some complicated way. In fact, the considered foliations are the projections to the complex plane of the horizontal foliations on the elliptic curves defined by quadratic differentials of the Strebel–Kontsevich type. The exposition is self-contained and reveals the conditions under which the considered foliations have closed leaves.

The paper “On trees covering chains or stars” by F. B. Pakovich (pp. 148–154) contains a simple combinatorial criterion allowing one to decide whether a given plane tree covers the simplest ones. Being quite elementary, this result, combined with some results from the author’s theses, has applications to subtle questions of arithmetic geometry such as the field of definition of torsion points on the Jacobians of hyperelliptic curves. Using the results of Mazur and Merel, Pakovich also gets the lower bounds for the degree of fields of definitions of some simple classes of trees and lists all of them defined over $\mathbb{Q}$.

The papers in the volume are ordered according to the Russian names of the authors.

At the present stage of research we rather *describe* things than *explain* them. However, the general idea underlying our work can already be formulated: the integrity of the mathematical world (according to Grothendieck, *une réalité mystérieuse au-delà des mots*) can be observed and studied. Hopefully, this idea will be shaped and realized in our further publications.

Notes

[1]G. B. Shabat and V. A. Voevodsky, "Drawing curves over number fields," in: *The Grothendieck Festschrift*, Vol. III, Progress Math., Vol. 88, Birkhäuser (1990), pp. 199–227.

[2]S. K. Lando and A. K. Zvonkin, *Graphs on Surfaces and Their Applications*, Encyclopedia Math. Sci., Vol. 141, Springer, Berlin (2004).

[3]L. Schneps and P. Lochak, eds., *Geometric Galois Actions*, Vol. 1, *Around Grothendieck's Esquisse d'un Programme*, Vol. 2, *The Inverse Galois Problem, Moduli Spaces and Mapping Class Groups*, London Math. Soc. Lect. Note Ser., Vols. 242, 243, Cambridge Univ. Press, Cambridge (1997); L. Schneps, ed., *The Grothendieck Theory of Dessins d'Enfants*, London Math. Soc. Lect. Note Ser., Vol. 200, Cambridge Univ. Press (1994).

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