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ORIGINAL PAPER

A Continuity Result for the Adjusted Normal Cone Operator

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Abstract

The concept of adjusted sublevel set for a quasiconvex function was introduced by Aussel and Hadjisavvas who proved the local existence of a norm-to-weak* upper semicontinuous base-valued submap of the normal operator associated with the adjusted sublevel set. When the space is finite-dimensional, a globally defined upper semicontinuous base-valued submap is obtained by taking the intersection of the unit sphere, which is compact, with the normal operator, which is closed. Unfortunately, this technique does not work in the infinite-dimensional case. We propose a partition of unity technique to overcome this problem in Banach spaces. An application is given to a quasiconvex quasioptimization problem through the use of a new existence result for generalized quasivariational inequalities which is based on the Schauder fixed point theorem.

Keywords Cone upper semicontinuity · Normal operator · Quasiconvexity · Quasivariational inequality · Quasioptimization

Mathematics Subject Classification 47H04 · 49J40 · 49J52 · 49J53 · 90C26

1 Introduction

A quasiconvex function is characterized by the convexity of its sublevel sets. This is why normal cones to the sublevel sets or the strict sublevel sets of a quasiconvex function were initially studied in a finite-dimensional setting [13] by Borde and Crouzeix. To study the continuity properties of the normal cone operator, the authors first observed that the notion of upper semicontinuity is inappropriate for cone-valued

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maps, and therefore, they introduced the concept of cone upper semicontinuity. Subsequently, characterizations of various classes of quasiconvex functions in terms of the generalized quasimonotonicity of the normal cone operators were obtained in [7]. Unfortunately, these normal operators do not satisfy quasimonotonicity and upper semicontinuity at the same time, even if the quasiconvex function is lower semicontinuous. Indeed, these operators fit well with the family of quasiconvex functions without flat parts, i.e., quasiconvex functions such that each local minimum is a global minimum.

Aussel and Hadjisavvas [8] proposed the concept of adjusted sublevel set to treat all kinds of quasiconvex functions defined on a Banach space. The authors proved that the normal cone operator to the adjusted sublevel sets of a quasiconvex function is both quasimonotone and cone upper semicontinuous. In particular, they showed that the normal cone operator admits a locally defined base-valued submap being norm-to-weak* upper semicontinuous. In [6], the authors obtained a globally defined upper semicontinuous base-valued submap when the space is Euclidean taking the convex hull of the normalized normal operator which is the intersection of the unit sphere, which is compact, with the normal operator, which is closed. Since the unit sphere is not compact in the infinite-dimensional case, this approach is unsuccessful in a Banach space.

The first aim of this paper is to overcome this problem by using a partition of unity technique. Theorem 2.5 states the existence of a norm-to-weak* upper semicontinuous map whose values are nonempty weak* compact convex sets not containing the origin and generating the normal cone to the adjusted sublevel set. Subsequently, we establish an existence result (Theorem 3.3) for a generalized quasivariational inequality which improves the famous Tan's result [22]. Finally, combining both results, we present an application to quasioptimization problems.

We conclude by presenting some preliminary notions and results. For a subset C of a topological vector space Z, $\operatorname{cl} C$ and $\operatorname{int} C$ are used for denoting the closure and the interior of C, respectively. The set $K \subseteq Z$ is a cone if for each $z \in K$ and scalar $t > 0$, the product $tz \in K$. (Note that some authors define cone with the scalar t ranging over all nonnegative scalars.) Clearly, the empty set is a cone. Let K be a cone. A convex subset A of K is called a base if $K = \{tz : t \geq 0,\ z \in A\}$ and $0 \notin \operatorname{cl} A$. Clearly, the empty set is a base of the empty cone. Vice versa, if K admits a nonempty base then K is a convex cone such that $\{0\} \subsetneq K$. In particular, if the base is compact then K is closed.

The domain and the graph of a set-valued map $\Phi : Y \rightrightarrows Z$ between topological spaces are denoted by $\operatorname{dom} \Phi$ and $\operatorname{gph} \Phi$, respectively. The map Φ is lower semicontinuous at $y \in Y$ if for every open set Ω such that $\Phi(y) \cap \Omega$, there exists a neighborhood U_y of y such that $\Phi(y') \cap \Omega$, for all $y' \in U_y$. The characterization of lower semicontinuity in terms of nets is the following: If $\{y_\alpha\}$ converges to y and $z \in \Phi(y)$, then there exist a subnet $\{y_{\alpha_\beta}\}$ of $\{y_\alpha\}$ and a net $\{z_\beta\}$ converging to z such that $(y_{\alpha_\beta}, z_\beta) \in \operatorname{gph} \Phi$ for each β. If the topologies of Y and Z are first countable, it is possible to replace nets with sequences. The map Φ is upper semicontinuous at $y \in Y$ if for every open set Ω such that $\Phi(y) \subseteq \Omega$, there exists a neighborhood U_y of y such that $\Phi(y') \subseteq \Omega$, for all $y' \in U_y$. It is closed at y if for each net $\{(y_\alpha, z_\alpha)\} \subseteq \operatorname{gph} \Phi$

which converges to (y, z), we have that $(y, z) \in \operatorname{gph} \Phi$. Moreover, we say that Φ is compact if there exists a compact set containing all the values of Φ.

In this paper, we will consider a real Banach space X with norm $\|\cdot\|$, its topological dual space X^* with norm $\|\cdot\|_*$, and the duality pairing $\langle\cdot,\cdot\rangle$ between X^* and X. The closed unit balls in X and X^* will be denoted by B and B^*, respectively. Furthermore, in order to avoid misunderstanding, the topology that X^* is endowed with will be specified, at each time.

2 Continuity of the Adjusted Normal Cone Operator

The polar cone of a set C in X is the set

$$C^\circ = \{x^* \in X^* : \langle x^*, z\rangle \leq 0,\ \forall z \in C\}.$$

In this section, we start studying the closedness properties of the polar map $C^\circ : Y \rightrightarrows X^*$ defined $C^\circ(x) = C(x)^\circ$ where $Y \subseteq X$ and $C : Y \rightrightarrows X$ is a given set-valued map. Clearly, C° has nonempty convex weak* closed values. The first result extends to the infinite-dimensional case Corollary 1 in [9].

Theorem 2.1 *If C is lower semicontinuous at $x \in Y$ then the polar map C° is norm-to-norm closed at x.*

Proof Let $\{(x_n, x_n^*)\}$ be a sequence which converges to (x, x^*) and $x_n^* \in C^\circ(x_n)$ for all n. Since C is lower semicontinuous, for each fixed $z \in C(x)$, there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and elements $z_k \in C(x_{n_k})$ for each k such that $\{z_k\}$ converges to z. The conclusion follows from the fact that the duality pairing is jointly norm continuous and $\langle x_{n_k}^*, z_k\rangle \leq 0$, for all k. □

Unfortunately, this result does not hold if the norm topology in the dual space is replaced by the weak* topology as the example shows.

Example 2.1 Fix $x^* \in X^*$ and $x \in X$ such that

$$\|x^*\|_* = 1/2 \quad \text{and} \quad \langle x^*, x\rangle = 1/4.$$

Let $\mathcal{A}$ be the collection of all nonempty finite subsets of the bidual X^{**} containing x^{**} where $\langle x^{**}, z^*\rangle = \langle z^*, x\rangle$, for all $z^* \in X^*$. The set $\mathcal{A}$ is directed by the set inclusion, $\alpha \geq \beta$ whenever $\alpha \supseteq \beta$. For each $\alpha = \{x_1^{**}, \dots, x_n^{**}\}$, there exists some $z_\alpha^* \in \bigcap_{i=1}^n \ker x_i^{**}$ such that $\|z_\alpha^*\|_* = n$. Hence, since $x^{**} \in \alpha$, the net $\{z_\alpha^*\}$ satisfies the following properties:

$$\begin{aligned} &z_\alpha^* \overset{*}{\rightharpoonup} 0 \\ &\|z_\alpha^*\|_* = n \\ &\langle z_\alpha^*, x\rangle = 0. \end{aligned}$$

By definition, for each α there exists $y_\alpha \in X$ such that

$$\|y_\alpha\| \leq 1 \quad \text{and} \quad \langle z^*_\alpha, y_\alpha \rangle = n/2.$$

Put $z_\alpha = 2y_\alpha/n$, then

$$\|z_\alpha\| \to 0 \quad \text{and} \quad \langle z^*_\alpha, z_\alpha \rangle = 1.$$

Now, let us consider the lower semicontinuous map $C : X \rightrightarrows X$ defined by

$$C(z) = [-z, (x-z)/2] = \{u = -tz + (1-t)(x-z)/2 : t \in [0, 1]\}$$

and the net $\{(x_\alpha, x^*_\alpha)\} \subseteq X \times X^*$ such that

$$x_\alpha = z_\alpha - x \quad \text{and} \quad x^*_\alpha = x^* + z^*_\alpha.$$

Clearly, the net converges to $(-x, x^*)$. We show that $(x_\alpha, x^*_\alpha) \in \operatorname{gph} C^\circ$ for each α. Since $C(x_\alpha) = [x - z_\alpha, x - z_\alpha/2] = (x - z_\alpha) + [0, z_\alpha/2]$, for each $t \in [0, 1/2]$ we have

$$\begin{aligned}\langle x^*_\alpha, x - z_\alpha + tz_\alpha \rangle &= (t-1)\langle x^*_\alpha, z_\alpha \rangle + \langle x^*_\alpha, x \rangle \\ &= (t-1)\Big(\langle x^*, z_\alpha \rangle + \langle z^*_\alpha, z_\alpha \rangle\Big) + \langle x^*, x \rangle.\end{aligned}$$

Since $\langle x^*, z_\alpha \rangle \geq -1/2$, $\langle z^*_\alpha, z_\alpha \rangle = 1$ and $\langle x^*, x \rangle = 1/4$, we have $x^*_\alpha \in C^\circ(x_\alpha)$. Nevertheless, the limit $(-x, x^*) \notin \operatorname{gph} C^\circ$ since $C(-x) = \{x\}$ and $\langle x^*, x \rangle = 1/4 > 0$.

The lack of closedness of C° is due to the norm unboundedness of the convergent net $\{z^*_\alpha\}$. However, the duality pairing restricted to $B^* \times X$ is jointly continuous, where X has its norm topology and B^* has its weak* topology [2, Corollary 6.40]. This allows us to obtain the following which will be useful in the sequel.

Theorem 2.2 *Let $H \subseteq B^*$ be a weak* closed set. If C is lower semicontinuous at $x \in Y$, then the map $C^\circ \cap H$ is norm-to-weak* closed at x.*

Proof Let $\{(x_\alpha, x^*_\alpha)\}$ be a net which converges to (x, x^*) with respect to the norm$\times$weak* topology and $x^*_\alpha \in C^\circ(x_\alpha) \cap H$ for all α. The limit $x^* \in H$ since H is weak* closed and now the proof follows the same line of reasoning of the proof of Theorem 2.1. □

Now, let us consider the particular case of the pointwise polar to the shifted sublevel set of a quasiconvex function. More precisely, let $f : X \to \mathbb{R} \cup \{+\infty\}$ be an extended-valued function. Define for any $\lambda \in \mathbb{R} \cup \{+\infty\}$ the sublevel and the strict sublevel set of f at level λ by $S_\lambda = \{x \in X : f(x) \leq \lambda\}$ and $S^<_\lambda = \{x \in X : f(x) < \lambda\}$, respectively. Clearly $S_\infty = X$ and $S^<_\infty = \operatorname{dom} f$. The function f is quasiconvex if S_λ is convex for all $\lambda \in \mathbb{R}$. Now, we recall the notion of adjusted level set introduced in [8].

Definition 2.1 Let $f : X \to \mathbb{R} \cup \{+\infty\}$ and $x \in X$. The adjusted sublevel set of f at x is

$$S_f^a(x) = \begin{cases} S_{f(x)} & \text{if } x \in \arg\min f \\ S_{f(x)} \cap B(S_{f(x)}^<, \rho_x) & \text{if } x \notin \arg\min f, \end{cases}$$

where $\rho_x = \inf\{\|y - x\| : y \in S_{f(x)}^<\}$ and

$$B(S_{f(x)}^<, \rho_x) = \{z \in X : \|y - z\| \le \rho_x, \ \forall y \in S_{f(x)}^<\}.$$

Note that $S_{f(x)}^< \subseteq S_f^a(x) \subseteq S_{f(x)}$ for all $x \in X$. Moreover, the convexity of the adjusted sublevel sets characterizes the quasiconvexity of the function.

Theorem 2.3 (Proposition 2.4 in [8]) *The extended-valued function f is quasiconvex if and only if $S_f^a(x)$ is convex, for every $x \in X$.*

In general, even if f is continuous on a finite-dimensional space, the map $S_f^a : X \rightrightarrows X$ need not to be upper semicontinuous (see Example 3.1 in [1]). However, mirroring the proof of Theorem 3.1 in [1], the lower semicontinuity of S_f^a can be shown in the infinite-dimensional case.

Theorem 2.4 *Let f be quasiconvex. If $S_{f(x)}$ is closed for all $x \in X$, then the map S_f^a is lower semicontinuous.*

To any quasiconvex function f, we associate the set-valued map $N^a : X \rightrightarrows X^*$ defined by

$$N^a(x) = (S_f^a(x) - x)^\circ = \{x^* \in X^* : \langle x^*, y - x\rangle \le 0, \ \forall y \in S_f^a(x)\}.$$

The map N^a has nonempty convex weak* closed values, and if $\operatorname{int} S_{f(x)}^< \neq \emptyset$, then $N^a(x)$ is different from $\{0\}$. Indeed, $S_{f(x)}^< \subseteq S_f^a(x)$ implies $\operatorname{int} S_f^a(x) \neq \emptyset$ and, since $x \notin \operatorname{int} S_f^a(x)$, by the separation theorem, there exists $x^* \in X^* \setminus \{0\}$ such that $\langle x^*, y\rangle \le \langle x^*, x\rangle$ for each $y \in S_f^a(x)$.

When we are dealing with a cone-valued map, the concept of upper semicontinuity is not appropriate to picture the behavior of the map and it is convenient to slightly alter the definition. A cone-valued map $\Phi : Y \rightrightarrows Z$ between the topological space Y and the topological vector space Z is called

- cone upper semicontinuous [13] at $y \in Y$ if for every open cone K such that $\Phi(x) \subseteq K \cup \{0\}$, there exists a neighborhood U_y of y such that $\Phi(y') \subseteq K \cup \{0\}$, for all $y' \in U_y$;
- base upper semicontinuous [8] at $y \in Y$ if there exist a neighborhood U_y of y and a set-valued map $A : U_y \rightrightarrows Z$ such that $A(y')$ is a base of $\Phi(y')$ for each $y' \in U_y$ and A is upper semicontinuous at y.

Some remarks are needed. If Φ is base upper semicontinuous at $y \in Y$, then there exists a neighborhood U_y of y such that $\Phi(y') \neq \{0\}$ for each $y' \in U_y$. Instead, if Φ is cone upper semicontinuous at $y \notin \operatorname{dom} \Phi$ then there exists a neighborhood U_y of y

such that $\Phi(y') \subseteq \{0\}$ for each $y' \in U_y$. Therefore, if Φ is cone upper semicontinuous and $\Phi(y)$ admits a base for each $y \in Y$ then dom Φ is closed. Moreover, the base upper semicontinuity of Φ at y implies the cone upper semicontinuity at the same point.

The reverse implication has been shown in [8] when Y is a Banach space X, Z is its dual space X^* endowed with the weak* topology, $\Phi(y)$ admits a base, and $\Phi(y') \neq \{0\}$ for all y' in a suitable neighborhood of y. Moreover, the norm-to-weak* cone upper semicontinuity of Φ implies its norm-to-weak* closedness if the map admits a compact base at every point [12, Proposition 2.3].

Focusing on the normal operator, in [8, Proposition 3.5] the authors showed that N^a is norm-to-weak* base upper semicontinuous. The main result of this section consists in passing from the local existence to the global existence.

Theorem 2.5 *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be quasiconvex and lower semicontinuous. Assume that*

$$\text{int } S^<_{f(x)} \neq \emptyset, \quad \forall x \notin \arg\min f. \tag{1}$$

Then

(i) *N^a is norm-to-weak* closed at any $x \notin \arg\min f$;*
(ii) *there exists a norm-to-weak* upper semicontinuous set-valued map $T : X \rightrightarrows B^*$ such that $T(x)$ is a weak* compact base of $N^a(x)$, for all $x \notin \arg\min f$.*

Proof Let $z \notin \arg\min f$ be fixed. The first step of the proof consists in finding a suitable bounded base of N^a in a neighborhood of z.

We distinguish two cases. If there exists $w \in \text{int } S^<_{f(z)} \setminus \arg\min f$, set $\lambda = f(w)$. Otherwise, the interior of $\arg\min f$ is nonempty and fix λ with $\min f < \lambda < f(z)$. In both cases, we have $\lambda < f(z)$ and there exists $w_z \in \text{int } S^<_\lambda$.

Now, we proceed similarly as in [8, Lemma 3.6]. Since f is lower semicontinuous, there exists $\varepsilon_z > 0$ such that

$$w_z + 2\varepsilon_z B \subseteq S^<_\lambda \subseteq S^<_{f(x)}, \quad \forall x \in z + \varepsilon_z B.$$

Notice that $z + \varepsilon_z B \subseteq X \setminus \arg\min f$. Thus, for every $x \in z + \varepsilon_z B$ and for every

$$x^* \in N^<(x) = \{x^* \in X^* : \langle x^*, y - x\rangle \leq 0,\ \forall y \in S^<_{f(x)}\},$$

we obtain the following:

$$\langle x^*, w_z + 2\varepsilon_z u - x\rangle \leq 0, \quad \forall u \in B.$$

It follows that

$$\begin{aligned} 2\varepsilon_z \|x^*\|_* = 2\varepsilon_z \sup_{u\in B}\langle x^*, u\rangle &\leq \langle x^*, x - w_z\rangle \\ &= \langle x^*, z - w_z\rangle + \langle x^*, x - z\rangle \end{aligned}$$

$$\leq \langle x^*, z - w_z \rangle + \varepsilon_z \|x^*\|_*.$$

Thus,

$$\langle x^*, z - w_z \rangle \geq \varepsilon_z \|x^*\|_*, \quad \forall x \in z + \varepsilon_z B, \ x^* \in N^<(x).$$

Set $H_z = \{x^* \in X^* : \langle x^*, z - w_z \rangle = \varepsilon_z\}$. Obviously, for every $x \in z + \varepsilon_z B$ we have $N^<(x) \cap H_z \subseteq B^*$. Since $N^a(x) \subseteq N^<(x)$ and $N^a(x) \setminus \{0\} \neq \emptyset$, the nonempty set $N^a(x) \cap H_z \subseteq B^*$ is a weak* compact base for the cone $N^a(x)$. From now on, let $A_z : z + \varepsilon_z B \rightrightarrows X^*$ be the set-valued map defined by $A_z(x) = N^a(x) \cap H_z$, for all $x \in z + \varepsilon_z B$.

We prove (i) that is the norm-to-weak* closedness of N^a at z. Let $\{(z_\alpha, z_\alpha^*)\}$ be a net norm×weak* convergent to (z, z^*) such that $z_\alpha^* \in N^a(z_\alpha)$, and without loss of generality, assume that $z_\alpha \in z + \varepsilon_z B$ for each α. Hence, there exist two nets $\{t_\alpha\} \subseteq \mathbb{R}$ and $\{y_\alpha^*\}$ such that $t_\alpha \geq 0$, $y_\alpha^* \in A_z(z_\alpha)$ and $z_\alpha^* = t_\alpha y_\alpha^*$, for each α. Therefore, $t_\alpha = \langle z_\alpha^*, z - w_z \rangle / \varepsilon_z$ converges to $t = \langle z^*, z - w_z \rangle / \varepsilon_z$. Moreover, since $\{y_\alpha^*\}$ is bounded, by passing to a subnet if necessary, we may assume that y_α^* converges to $y^* \in X^*$. Thanks to Theorems 2.2 and 2.4, we have $y^* \in A_z(z)$ which implies $z^* = t y^* \in N^a(z)$.

We prove (ii) and the last step of the proof consists in finding a selection A as a convex combination of the local maps A_z through a partition of unity technique (see [2, Sect. 2.19] for more details). The family $\{z + \varepsilon_z \operatorname{int} B : z \in X \setminus \arg\min f\}$ is an open cover of the space $X \setminus \arg\min f$. Since this space is paracompact, there exists an open refinement cover $\mathcal{U} = \{U_i : i \in I\}$, i.e., every $U_i \in \mathcal{U}$ is a subset of some ball $z + \varepsilon_z \operatorname{int} B$, which is locally finite, i.e., each point x has a neighborhood that meets at most finitely many U_i. Now, for each i take a z such that $U_i \subseteq z + \varepsilon_z \operatorname{int} B$ and denote by A_i the map A_z corresponding to the ball $z + \varepsilon_z B$. Moreover, there is a partition of unity $\{\lambda_i : i \in I\}$ subordinate to $\mathcal{U}$ such that each $\lambda_i : X \setminus \arg\min f \to [0, 1]$ is continuous, the finite sum $\sum_{i \in I} \lambda_i(x) = 1$ for any x and $\lambda_i(x) = 0$ for each $x \notin U_i$. For every $x \notin \arg\min f$, let $I(x) = \{i \in I : \lambda_i(x) > 0\}$, which is nonempty and finite, and define the map $A : X \setminus \arg\min f \rightrightarrows X^*$ as follows

$$A(x) = \sum_{i \in I(x)} \lambda_i(x) A_i(x).$$

Clearly, $A(x)$ is a weak* compact base of $N^a(x)$, for all x. Moreover, since all the values of A are contained in the weak* compact ball B^*, the norm-to-weak* upper semicontinuity of A is equivalent to prove that gph A is norm × weak* closed. Assume that the net $\{x_\alpha\}$ converges to x. Since all the λ_i are continuous, it is not restrictive to assume that $I(x) \subseteq I(x_\alpha)$ for all α and we get

$$A(x_\alpha) = \sum_{i \in I(x)} \lambda_i(x_\alpha) A_i(x_\alpha) + \sum_{i \in I(x_\alpha) \setminus I(x)} \lambda_i(x_\alpha) A_i(x_\alpha).$$

Moreover, from the continuity of the functions λ_i, we deduce

$$\lim_\alpha \sum_{i \in I(x_\alpha)\setminus I(x)} \lambda_i(x_\alpha) = 1 - \lim_\alpha \sum_{i \in I(x)} \lambda_i(x_\alpha) = 0. \tag{2}$$

Now, let $\{x_\alpha^*\}$ be a net which weakly* converges to x^* and such that $x_\alpha^* \in A(x_\alpha)$, for any α. Then, there exist $x_{i,\alpha}^* \in A_i(x_\alpha)$ for every $i \in I(x_\alpha)$ such that

$$x_\alpha^* = \sum_{i \in I(x)} \lambda_i(x_\alpha) x_{i,\alpha}^* + \sum_{i \in I(x_\alpha)\setminus I(x)} \lambda_i(x_\alpha) x_{i,\alpha}^*. \tag{3}$$

The second addend of (3) weakly* converges to zero since, thanks to (2), it converges to zero in norm

$$\left\| \sum_{i \in I(x_\alpha)\setminus I(x)} \lambda_i(x_\alpha) x_{i,\alpha}^* \right\|_* \leq \sum_{i \in I(x_\alpha)\setminus I(x)} \lambda_i(x_\alpha) \|x_{i,\alpha}^*\|_* \leq \sum_{i \in I(x_\alpha)\setminus I(x)} \lambda_i(x_\alpha).$$

On the other hand, without loss of generality, we may assume that $\{x_{i,\alpha}^*\}$ weakly* converges to some x_i^*, for every $i \in I(x)$. Since Theorem 2.2 guarantees that A_i is norm$\times$weak* closed at x, we obtain $x_i^* \in A_i(x)$ and $x^* \in A(x)$ follows from (3) taking the weak* limit. Finally, let $T : X \rightrightarrows X^*$ be defined as

$$T(x) = \begin{cases} B^* & \text{if } x \in \arg\min f \\ A(x) & \text{if } x \notin \arg\min f. \end{cases}$$

Since $\arg\min f$ is closed and $A(x) \subseteq B^*$, then T is norm-to-weak* upper semicontinuous and the proof is completed. □

Remark 2.1 For proving Proposition 3.5 in [8], the authors required that $\operatorname{int} S_\lambda \neq \emptyset$, for all $\lambda > \inf f$. Our assumption (1) is clearly weaker. Consider, for example, the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = |x|$ if $x \neq 0$ and $f(0) = -1$. Then (1) holds but $\operatorname{int} S_\lambda = \emptyset$ for any level λ such that $-1 = \inf f < \lambda \leq 0$.

Statement (i) has been proved in [6] where $x \in \operatorname{dom} f \setminus \arg\min f$. The assumption in [6, Proposition 4.3], that is, for each $x \notin \arg\min f$ there exists $\lambda < f(x)$ such that $\operatorname{int} S_\lambda \neq \emptyset$, coincides with ours. Recently, the same result has been proved avoiding assumption (1) but in the finite-dimensional case [1].

Taking advantage of the compactness of the unit sphere S in $\mathbb{R}^n$ and the closedness of N^a, Aussel and Cotrina deduced [6, Proposition 4.4] the upper semicontinuity of the normalized submap $N^a \cap S : \mathbb{R}^n \setminus \arg\min f \rightrightarrows B$. Unfortunately, their technique does not work in the infinite-dimensional case, since the sphere is not weak* compact. A workaround for this issue in a Banach space is to consider firstly the intersection with a hyperplane in a neighborhood of each point and then to use a partition of unity technique as done in the proof of statement (ii) of Theorem 2.5.

3 An Existence Result for Generalized Quasivariational Inequalities

One of the main advantages of the normal operator approach is that it provides a sufficient optimality condition for a quasiconvex optimization problem expressed by a variational inequality. Given a nonempty subset C of X and a map $T : C \rightrightarrows X^*$, the generalized variational inequality $GVI(T, C)$ consists in finding

$$x \in C \text{ such that } \exists x^* \in T(x) \text{ with } \langle x^*, y - x\rangle \geq 0, \quad \forall y \in C.$$

This problem has its origins with Stampacchia and Fichera and it provides a broad unifying setting for the study of optimization, complementarity problems, and, more in general, equilibrium problems. In a recent paper [5], the authors established a sufficient optimality condition based on the following continuity-type hypothesis.

Definition 3.1 A quasiconvex function $f : X \to \mathbb{R}$ is said to be sub-boundarily constant on $C \subseteq X$ if, for every $x, y \in C$, one has that

$$y \in S^<_{f(x)} \implies]y, x[\subseteq \operatorname{int} S^a_f(x).$$

As observed in [5], if f is radially continuous or $X = \mathbb{R}$, then f is sub-boundarily constant. Anyway, the family of the sub-boundarily constant functions is quite large.

Example 3.1 The function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = \begin{cases} x^2 + y^2 & \text{if } (x, y) \in B \\ x^2 + y^2 + 1 & \text{otherwise} \end{cases}$$

is quasiconvex, lower semicontinuous and sub-boundarily constant. Clearly, it is not radially continuous.

Nevertheless, not all the quasiconvex functions are sub-boundarily constant.

Example 3.2 The function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = \begin{cases} 2x/(x + 1) & \text{if } x, y \geq 0 \\ 3 & \text{otherwise} \end{cases}$$

is quasiconvex, but it is not sub-boundarily constant on $C = \{(x, 0) : x \geq 0\}$. Indeed, take $(0, 0)$ and $(1, 0)$ in C. Then $f(0, 0) = 0 < 1 = f(1, 0)$ and each element $(t, 0)$ with $t \in (0, 1)$ does not belong to $\operatorname{int} S^a_f(1, 0) = (0, 1) \times (0, +\infty)$.

The following result was proved when $X = \mathbb{R}^n$ [5, Proposition 2.9], but the same proof works in the infinite-dimensional case and it is omitted.

Theorem 3.1 *Let C be a nonempty subset of X and $f : X \to \mathbb{R}$ be a quasiconvex and sub-boundarily constant function on C. Then, any solution of $GVI(N^a_f \setminus \{0\}, C)$ is a global minimizer of f over C.*

This variational approach may be useful also for studying the existence of solutions for optimization problems where the feasible region of each agent depends by the choices of the other agents. For this reason, we introduce the generalized quasi-variational inequality problem, that is, a generalized variational inequality where the constraint set is subject to modifications depending on the considered point. In particular, let C be a nonempty subset of X, $T : C \rightrightarrows X^*$ and $K : C \rightrightarrows C$ be two set-valued maps, the generalized quasivariational inequality $GQVI(T, K)$ consists in finding

$$x \in K(x) \text{ such that } \exists x^* \in T(x) \text{ with } \langle x^*, y - x\rangle \geq 0, \quad \forall y \in K(x),$$

that is, $x \in \operatorname{fix} K$ and it solves the generalized variational inequality $GVI(T, K(x))$. One of the most classic existence results for $GQVI(T, K)$ in the infinite-dimensional setting is due to Tan and it was stated for locally convex topological vector spaces.

Theorem 3.2 (Theorem 1 in [22]) *Let C be compact and convex and K be closed and lower semicontinuous with nonempty convex values. Assume that T is norm-to-norm upper semicontinuous with nonempty norm compact convex values, then $GQVI(T, K)$ has a solution.*

The existence of solutions for $GQVI(T, K)$ can be obtained with weaker assumptions on T if the space X is normed. To this purpose, we need to recall the notion of inside point of a convex set that appeared in 1956 in the famous Michael's paper [20]. The convex set $S \subseteq C$ is a face of C if $x_1, x_2 \in C$, $t \in (0, 1)$ and $tx_1 + (1-t)x_2 \in S$ imply $x_1, x_2 \in S$. Let $\mathcal{F}_C$ be the (possibly empty) collection of all proper closed faces of $\operatorname{cl} C$.

Definition 3.2 Let C be a convex subset of X. A point $x \in C$ is said to be an inside point of C if it is not in any proper closed face of $\operatorname{cl} C$. Denote by

$$I(C) = C \setminus \bigcup_{S \in \mathcal{F}_C} S$$

the set of the inside points of C and by $\mathcal{D}(X)$ the following family of convex sets

$$\mathcal{D}(X) = \{C \subseteq X : C \text{ is convex and } I(\operatorname{cl} C) \subseteq C\}.$$

The family $\mathcal{D}(X)$ contains all the convex sets which are either closed or with nonempty relative interior. In particular, when X is finite-dimensional $\mathcal{D}(X)$ coincides with the family of all convex sets. For further details and a comparison with other notions of relative interior, the interested reader can refer to [14, 15] and the references therein. Now, we are in a position to state and prove our existence result.

Theorem 3.3 *Let C be convex and K be a compact and lower semicontinuous set-valued map with nonempty values in $\mathcal{D}(X)$, and assume that* fix K *is closed. If T is norm-to-weak* upper semicontinuous with nonempty weak* compact convex values, then $GQVI(T, K)$ has a solution.*

Proof Notice that K admits a continuous selection thanks to [15, Theorem 3.2]. Hence, the Schauder fixed point theorem as formulated in [18, Proposition 6.3.2] guarantees $\operatorname{fix} K \neq \emptyset$.

Let us consider the set-valued map $F : \operatorname{fix} K \rightrightarrows X$ defined as

$$F(x) = \bigcap_{x^* \in T(x)} \{y \in X : \langle x^*, y - x\rangle < 0\} = \left\{ y \in X : \max_{x^* \in T(x)} \langle x^*, y - x\rangle < 0 \right\}.$$

Clearly, F has convex values. To prove that F has open graph in $\operatorname{fix} K \times X$, it is sufficient to show that the function $m : \operatorname{fix} K \times X \to \mathbb{R}$ defined as

$$m(x, y) = \max_{x^* \in T(x)} \langle x^*, y - x\rangle$$

is upper semicontinuous. First, $\operatorname{fix} K$ is compact since closed subset of the compact set which contains $K(C)$. From Aliprantis and Border [2, Lemma 17.8], the subset $T(\operatorname{fix} K)$ is weak* compact; hence, it is norm bounded. Thanks to Aliprantis and Border [2, Corollary 6.40], the duality pairing $\langle \cdot, \cdot \rangle$ restricted to $T(\operatorname{fix} K) \times X$ is jointly continuous, where X has its norm topology and X^* has its weak* topology; hence, Aliprantis and Border [2, Lemma 17.30] guarantee the upper semicontinuity of m.

By contradiction, assume that $F(x) \cap K(x) \neq \emptyset$ for all $x \in \operatorname{fix} K$. Fix $(x_0, y_0) \in \operatorname{gph} K$ and define the map $K_0 : C \rightrightarrows C$ as

$$K_0(x) = \begin{cases} K(x) & \text{if } x \neq x_0 \\ \{y_0\} & \text{if } x = x_0. \end{cases}$$

K_0 is compact and lower semicontinuous, and $K_0(x) \in \mathcal{D}(X)$ for every $x \in C$. From Castellani and Giuli [15, Theorem 3.2], the map K_0 admits a continuous selection. From Aubin and Cellina [3, Proposition 1.10.4], we deduce that $F \cap K$ is locally selectionable, that is, for all $z \in \operatorname{fix} K$ and $y \in F(z) \cap K(z)$, there exists a neighborhood U_z of z such that the restriction of $F \cap K$ to U_z admits a continuous selection f_z passing through (z, y). Since $\operatorname{fix} K$ is paracompact, there exists a locally finite open covering $\mathcal{U} = \{U_i : i \in I\}$ where every $U_i \in \mathcal{U}$ is a subset of some U_z: let us denote by f_i the map f_z corresponding to U_z. Moreover, there is a partition of unity $\{\lambda_i : i \in I\}$ subordinate to $\mathcal{U}$ such that each $\lambda_i : \operatorname{fix} K \to [0, 1]$ is continuous, the finite sum $\sum_{i \in I} \lambda_i(x) = 1$ for any x and $\lambda_i(x) = 0$ for each $x \notin U_i$. For every $x \in \operatorname{fix} K$, let $I(x) = \{i \in I : \lambda_i(x) > 0\}$, which is nonempty and finite, and define the map $f : \operatorname{fix} K \to C$ as follows

$$f(x) = \sum_{i \in I(x)} \lambda_i(x) f_i(x).$$

Clearly, f is continuous. Furthermore, f is a selection of $F \cap K$. Indeed, for $x \in \operatorname{fix} K$ we have $x \in U_i$, for all $i \in I(x)$, so $f_i(x) \in F(x) \cap K(x)$ which implies $f(x) \in$

$F(x) \cap K(x)$ since $F(x) \cap K(x)$ is convex. Therefore, the set-valued map $\Upsilon : C \rightrightarrows C$ defined as

$$\Upsilon(x) = \begin{cases} K(x) & \text{if } x \notin \text{fix } K \\ \{f(x)\} & \text{if } x \in \text{fix } K \end{cases}$$

is lower semicontinuous [15, Lemma 2.3] with values in the class $\mathcal{D}(X)$. Hence, Castellani and Giuli [15, Theorem 3.2] guarantee that f can be extended to a continuous selection φ for Υ. The Schauder fixed point theorem guarantees that φ has a fixed point, that is, there exists $x \in C$ such that $x = \varphi(x) \in \Upsilon(x)$. Clearly $x \in \text{fix } K$ and this implies $x = f(x) \in F(x)$ which is absurd. Therefore, there exists $x \in \text{fix } K$ such that $F(x) \cap K(x) = \emptyset$, that is,

$$\inf_{y \in K(x)} \max_{x^* \in T(x)} \langle x^*, y - x \rangle \geq 0.$$

Invoking the Sion's minimax theorem [21], we deduce that

$$\max_{x^* \in T(x)} \inf_{y \in K(x)} \langle x^*, y - x \rangle \geq 0,$$

which means that x solves the generalized quasivariational inequality. □

Remark 3.1 Let us compare our result with Theorem 3.2 due to Tan. The first difference is about the setting: Tan's result works in a locally convex topological vector space; instead, Theorem 3.3 is stated in a Banach space. Nevertheless, the other assumptions of Theorem 3.3 are rather weaker than the ones in Theorem 3.2. Maybe, the most significant improvement consists in requiring the norm-to-weak* upper semicontinuity of T instead of the stronger norm-to-norm upper semicontinuity. Moreover, the values of T are assumed weakly* compact instead of norm compact. Also, the assumptions on K are weaker. In Theorem 3.2, the map K is closed, which implies the closedness of $K(x)$, for all x. Conversely, in Theorem 3.3 we require only the closedness of fix K, which is necessary for the closedness of K, and $K(x)$ may not be closed but belonging to the class $\mathcal{D}(X)$ only. Lastly, we do not assume the compactness of C, not even its closedness, but only the fact that $K(C)$ is contained in a compact set.

The following example highlights the improvements of Theorem 3.3.

Example 3.3 Let $C = B$ be the closed unit ball in an infinite-dimensional Banach space X and fix $x_0 \in X$, $x_0^* \in X^*$ such that $\|x_0\| = 1$, $\|x_0^*\|_* = 2$ and $\langle x_0^*, x_0 \rangle < -3/2$. Define $K : C \rightrightarrows C$ by

$$K(x) = \begin{cases}]-x_0, \|x\|x_0] & \text{if } x \neq -x_0 \\ [-x_0, x_0] & \text{if } x = -x_0 \end{cases}$$

and $T : C \rightrightarrows X^*$ by

$$T(x) = \{x^* \in X^* : \|x^* - x_0^*\|_* \leq \|x\|\}.$$

Then all the assumptions of Theorem 3.3 are satisfied. In particular, the compact set fix $K = [-x_0, x_0]$ contains all the values of the lower semicontinuous map K, and T is norm-to-weak* upper semicontinuous with weak* compact values. The solution set of $GQVI(T, K)$ is $[0, x_0]$. Nevertheless, Theorem 3.2 cannot be applied since C is not compact, K is not closed and T has no norm compact values. Moreover, if the Banach space X is not reflexive, the map T is not even norm-to-norm upper semicontinuous. Indeed, X^* is not reflexive and James [19, Theorem 2] affirms that there is a linear continuous functional $x^{**} \in X^{**}$ of norm 1 which does not attain its norm. Hence, fixed $x \in C$ with $\|x\| = 1/2$, the open set

$$V = x_0^* + \{x^* \in X^* : \langle x^{**}, x^* \rangle < 1/2\}$$

contains $T(x_0)$, but the distance of $T(x_0)$ to the boundary of V is zero. So $T(x') \nsubseteq V$ whenever $\|x'\| > 1/2$.

Generalized quasivariational inequality problems over product sets are of great interest in game theory. This particular format is when

$$X = \prod_{i \in I} X_i \quad C = \prod_{i \in I} C_i \quad K = \prod_{i \in I} K_i,$$

where I is a finite index set, and for each $i \in I$, X_i is a normed space with X_i^* its topological dual, $C_i \subseteq X_i$ is a nonempty set, and $K_i : C \rightrightarrows C_i$ is a set-valued map. Denote by x_i the i-component of an element $x \in X$, and by $\langle \cdot, \cdot \rangle_i$ the duality pairing of (X_i^*, X_i). Here, the product map $K : C \rightrightarrows X$ is defined as $K(x) = \prod_{i \in I} K_i(x)$.

The problem consists in finding a fixed point $x \in K(x)$ such that for each $i \in I$ there exists $x_i^* \in T_i(x)$ with

$$\sum_{i \in I} \langle x_i^*, y_i - x_i \rangle_i \geq 0, \quad \forall y \in K(x),$$

where $T_i : X \rightrightarrows X_i^*$. If we denote by $T : X \rightrightarrows \prod_{i \in I} X_i^*$ the product map $T = \prod_{i \in I} T_i$, then the designation of this problem as $GQVI(T, K)$ is a certain abuse of notation due to the fact that the range space of T is the product of the dual spaces instead of the dual of the product $X^* = (\prod_{i \in I} X_i)^*$. However, we stress the fact that these two vector spaces are isomorphic taking the bijection

$$x^* \mapsto \sum_{i \in I} x_i^*$$

and that this map is a homeomorphism when considering the product of the weak* topologies on $\prod_{i \in I} X_i^*$ and the weak* topology on X^*.

The study of the existence of solutions to $GQVI(T, K)$ by requiring the regularity of the component set-valued maps T_i and K_i only is not always possible if certain generalized monotonicity and continuity assumptions are needed. For a comprehensive analysis of the problem, the interested reader can refer to Aussel et al. [4] and the references therein. Working without monotonicity conditions, we obtain an existence result for product-type generalized quasivariational inequalities as a natural consequence of Theorem 3.3.

Corollary 3.1 *Let X_i be Banach spaces, $C_i \subseteq X_i$ be convex, and K_i be compact and lower semicontinuous set-valued maps with nonempty values in $\mathcal{D}(X_i)$. Assume that* fix *K is closed. If T_i are norm-to-weak* upper semicontinuous with nonempty weak* compact convex values, then $GQVI(T, K)$ has a solution.*

Proof The Tychonoff's theorem guarantees the compactness of the map K, and Lemma 3.3 in [15] implies that K has nonempty values in $\mathcal{D}(X)$. Moreover, according to Berge [11, Theorems VI.2.4 and VI.2.4′], the lower semicontinuity and the norm-to-weak* upper semicontinuity are preserved by the product of set-valued maps. Hence, all the assumptions of Theorem 3.3 are verified and the conclusion follows. □

We conclude with an application of Theorems 2.5 and 3.3 to a quasioptimization problem which is not a standard optimization problem since, analogously to the quasivariational inequalities, the constraint set is subject to modifications. Given $C \subseteq X$ nonempty, $K : C \rightrightarrows C$ and $f : C \to \mathbb{R}$, a quasioptimization problem consists in finding

$$x \in K(x) \text{ such that } f(x) \leq f(y), \quad \forall y \in K(x).$$

Clearly, if $K(x) = C$ for all $x \in C$, the quasioptimization problem reduces to the classical optimization problem. The quasioptimization problem highlights the parallelism to quasivariational inequalities that express the optimality conditions. As pointed out in [17], if f stands for the gap function of the Nikaido–Isoda function associated with a convex generalized Nash equilibrium problem, then x is a solution of the quasioptimization problem with $f(x) = 0$ if and only if x solves the generalized Nash equilibrium problem.

Theorem 3.4 *Let C be convex and K be a compact and lower semicontinuous set-valued map with nonempty values in $\mathcal{D}(X)$ and* fix *K be closed. Assume that $f : X \to \mathbb{R}$ is lower semicontinuous, sub-boundarily constant on C, quasiconvex and* (1) *holds. Then the quasioptimization problem has a solution.*

Proof Let $T : X \rightrightarrows X^*$ be the norm-to-weak* upper semicontinuous set-valued map obtained in Theorem 2.5. In this way, thanks to Theorem 3.3, it follows that $GQVI(T, K)$ has a solution $x \in C$. Hence, $x \in \text{fix } K$ and it solves the generalized variational inequality $GVI(T, K(x))$. Clearly, if $x \in \arg\min f$, then $f(x) \leq f(y)$ for all $y \in K(x)$. Instead, if $x \notin \arg\min f$, since

$$T(x) \subseteq N^a(x) \setminus \{0\},$$

x is a solution to the generalized variational inequality associated with the operator $N^a \setminus \{0\}$ and the feasible set $K(x)$. The thesis follows from Theorem 3.1. □

In [6], Aussel and Cotrina proposed two existence results for a quasioptimization problem with a quasiconvex objective function: Proposition 4.2, when X is a Banach space and Proposition 4.5 when $X = \mathbb{R}^n$. Such results were extended to locally convex topological vector spaces in [16, Corollary 3.2]. Very recently, Theorem 8 in [10] has improved slightly the result in [16] since the compactness of the feasible region C is replaced by the compactness of K.

Theorem 3.4 is stated in a Banach space but with weaker assumptions on f and K. Indeed, the continuity of f has been replaced by some continuity-like properties of its sublevel sets, the values of K are not necessarily closed and the upper semicontinuity K is not required as done in [10, 16].

4 Conclusions

In this work, we prove the existence of a globally defined upper semicontinuous and base-valued submap of the normal cone operator to the adjusted sublevel sets of a quasiconvex function. The result is stated in a Banach space and it opens the door to potential applications in optimization and economics via a reformulation in terms of variational inequalities. An existence result for quasivariational inequalities is proved in this regard. In particular, our result is obtained without any assumption of monotonicity which is not usually inherited by the product of set-valued maps. This allows us to cover the case of quasivariational inequalities over product sets. Hence, although it is out of the scope of this paper, applications to generalized Nash equilibrium problems could be a natural extension of this work. In the end, an application to quasioptimization problems is considered.

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