Journal of Optimization Theory and Applications (2023) 197:383–385
https://doi.org/10.1007/s10957-023-02190-8

CORRECTION

# Correction to: Convexifactors, Generalized Convexity, and Optimality Condition

J. Dutta[1] · S. Chandra[2] · Rimpi[3] · C. S. Lalitha[4]

Published online: 24 March 2023


**Correction to: J Optim Theory Appl**
https://doi.org/10.1023/A:1014853129484

In this article the below author names are included.
Rimpi and C.S. Lalitha

## 1 Introduction

We correct an error in the proof of Theorem 3.5 in [1].

The original article can be found online at https://doi.org/10.1023/A:1014853129484.

✉ J. Dutta
jdutta@iitk.ac.in

✉ S. Chandra
sureshiitdelhi@gmail.com

Rimpi
baloda.rimpi@gmail.com

C. S. Lalitha
cslalitha@maths.du.ac.in

1 Department of Economic Sciences, Indian Institute of Technology Kanpur, Kanpur 208016, India

2 Department of Mathematics, Indian Institute of Technology Delhi, New Delhi 110016, India

3 Department of Mathematics, University of Delhi, New Delhi, Delhi 110007, India

4 Department of Mathematics, University of Delhi South Campus, Benito Juarez Road, New Delhi 110021, India

## 2 Mathematical Details

Theorem 3.5 in [1] says, if $\bar{x} \in S$ is a local minimum of (P) and $f$ is locally Lipschitz which admits an USRC $\partial^* f(\bar{x})$ at $\bar{x}$, then

$$0 \in \mathrm{cl}(\overline{\mathrm{co}}(\partial^* f(\bar{x})) + T^\circ(S, \bar{x})). \tag{1}$$

The following example justifies that the above theorem fails to hold.

***Example 2.1*** Let $f : \mathbb{R}^2 \to \mathbb{R}$ and the feasible set $S$ be defined as $f(x_1, x_2) = -x_1 + |x_2|$ and $S = \{(x_1, x_2) \in \mathbb{R}^2 : 0 \leq x_1 \leq |x_2|\}$. Clearly, $\bar{x} = (0, 0)$ is a global minimum, $T(S, \bar{x}) = S$ and $\mathrm{co}(T(S, \bar{x})) = \mathrm{co}(S) = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0\}$. Now, for any $v = (v_1, v_2) \in \mathbb{R}^2$, $f_d^+(\bar{x}, v) = -v_1 + |v_2|$. Also it can be seen that $\partial^* f(\bar{x}) = \{(-1, 1), (-1, -1)\}$ is an upper semi-regular convexificator of $f$ at $\bar{x}$. As $T(S, \bar{x}) = S$, it follows that $f_d^+(\bar{x}, v) \geq 0$, for all $v \in T(S, \bar{x})$. Thus $\sup_{\zeta \in \partial^* f(\bar{x})} \langle \zeta, v \rangle \geq 0$, for all $v \in T(S, \bar{x})$. Clearly, $\sup_{\zeta \in \partial^* f(\bar{x})} \langle \zeta, \overline{v} \rangle = f_d^+(\bar{x}, \overline{v}) = -1 < 0$ for $\overline{v} = (2, 1) \in \mathrm{co}(T(S, \bar{x}))$. Moreover, $T^\circ(S, \bar{x}) = \{(x, 0) \in \mathbb{R}^2 : x \leq 0\}$ and $\overline{\mathrm{co}}(\partial^* f(\bar{x})) = \{(-1, t) \in \mathbb{R}^2 : -1 \leq t \leq 1\}$. Hence $0 \notin \mathrm{cl}(\overline{\mathrm{co}}(\partial^* f(\bar{x})) + T^\circ(S, \bar{x}))$. Thus Theorem 3.5 in [1] fails to hold for $f$ at $\bar{x}$.

We rectify the error in the above theorem by assuming the tangent cone to be convex. For this we first recall the notion of support functions from [2]. The *support function* $\sigma_A(x) : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ of a nonempty set $A \subseteq \mathbb{R}^n$ is defined as $\sigma_A(x) := \sup_{a \in A} \langle x, a \rangle$.

A correct statement of Theorem 3.5 in [1] should be as follows.

**Theorem 2.1** *If $\bar{x} \in S$ is a local minimum of* (P)*, $T(S, \bar{x})$ is a convex cone and $f$ is locally Lipschitz which admits an USRC $\partial^* f(\bar{x})$ at $\bar{x}$, then* (1) *holds. Further, if $\partial^* f(\bar{x})$ is bounded, then*

$$0 \in \mathrm{co}(\partial^* f(\bar{x})) + T^\circ(S, \bar{x}). \tag{2}$$

***Proof*** As $\bar{x}$ is a local minimum of (P), there exists $\epsilon > 0$ such that $f(\bar{x}) \leq f(x)$ for all $x \in B(\bar{x}, \epsilon) \cap S$. For $v \in T(S, \bar{x})$, there exist sequences $(t_k)_{k \in \mathbb{N}}$ and $(v_k)_{k \in \mathbb{N}}$ with $t_k \downarrow 0$ and $v_k \to v$ such that $\bar{x} + t_k v_k \in S$. Thus there exists $k_0 \in \mathbb{N}$ such that $\bar{x} + t_k v_k \in B(\bar{x}, \epsilon) \cap S$ for all $k \geq k_0$, which implies $f(\bar{x}) \leq f(\bar{x} + t_k v_k)$ for all $k \geq k_0$. As $f$ is locally Lipschitz with Lipschitz constant say, L, hence for every $v \in T(S, \bar{x})$ we have

$$\begin{aligned} f_d^+(\bar{x}, v) &= \limsup_{t \downarrow 0} \frac{f(\bar{x} + tv) - f(\bar{x})}{t} \\ &\geq \limsup_{k \to \infty} \left[ \frac{f(\bar{x} + t_k v) - f(\bar{x} + t_k v_k)}{t_k} + \frac{f(\bar{x} + t_k v_k) - f(\bar{x})}{t_k} \right] \\ &\geq \lim_{k \to \infty} [-L \|v_k - v\|] + \limsup_{k \to \infty} \left[ \frac{f(\bar{x} + t_k v_k) - f(\bar{x})}{t_k} \right] \geq 0. \end{aligned}$$

As $\partial^* f(\bar{x})$ is an USRC of $f$ at $\bar{x}$, it follows from [2, Proposition 2.2.1 (p. 211)] that

$$\sigma_{\mathrm{co}(\partial^* f(\bar{x}))}(v) = \sigma_{\partial^* f(\bar{x})}(v) = \sup_{\zeta \in \partial^* f(\bar{x})} \langle \zeta, v \rangle \geq 0, \text{ for all } v \in T(S, \bar{x}). \tag{3}$$

As $T(S, \bar{x})$ is convex, hence by applying [2, Example 2.3.1 (p. 215)] for $K = T^\circ(S, \bar{x})$ we deduce that

$$\sigma_{T^\circ(S,\bar{x})}(v) = \begin{cases} 0, & \text{if } v \in T(S, \bar{x}), \\ +\infty, & \text{otherwise.} \end{cases} \tag{4}$$

In view of [2, Theorem 3.3.3(i) (p. 226)] the support function of the set $U = \mathrm{cl}(\mathrm{co}(\partial^* f(\bar{x})) + T^\circ(S, \bar{x}))$ is

$$\sigma_U(v) = \sigma_{\mathrm{co}(\partial^* f(\bar{x}))}(v) + \sigma_{T^\circ(S,\bar{x})}(v), \text{ for all } v \in T(S, \bar{x}). \tag{5}$$

Using (3)–(5) and the fact that $\sigma_K(v)$ is infinite for $v \notin T(S, \bar{x})$, we conclude that $\sigma_U(v) \geq 0$ for all $v \in \mathbb{R}^n$. Thus, by [2, Theorem 2.2.2 (p. 211)], $0 \in U = \mathrm{cl}(\mathrm{co}(\partial^* f(\bar{x})) + T^\circ(S, \bar{x}))$.

If $\partial^* f(\bar{x})$ is a bounded set then $\mathrm{co}(\partial^* f(\bar{x}))$ is compact as $\partial^* f(\bar{x})$ is a closed set. Hence (1) reduces to (2) as $\mathrm{co}(\partial^* f(\bar{x})) + T^\circ(S, \bar{x})$ is a closed set. □

## References


1. Dutta, J., Chandra, S.: Convexifactors, generalized convexity, and optimality conditions. J. Optim. Theory Appl. **113**(1), 41–64 (2002)
2. Hiriart-Urruty, J.-B., Lemaréchal, C.: Convex Analysis and Minimization Algorithms I. Springer, Berlin (1993)