



# A New Portfolio Optimization Model Under Tracking-Error Constraint with Linear Uncertainty Distributions

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## Abstract

Enhanced index tracking problem is the issue of selecting a tracking portfolio to outperform the benchmark return with a minimum tracking error. In this paper, we address the enhanced index tracking problem based on uncertainty theory where stock returns are treated as uncertain variables instead of random variables. First, we propose a non-linear uncertain optimization model, i.e., uncertain mean-absolute downside deviation enhanced index tracking model. Then, we give the analytical solution of the proposed optimization model when stock returns take linear uncertainty distributions. Based on the solution, we find that tracking portfolio frontier is a continuous curve composed of at most  $n - 1$  different line segments. Furthermore, we give the condition that tracking portfolio return and risk increase with benchmark return and risk, respectively. Finally, we offer some experiments and show that our proposed model is effective in controlling the tracking error.

**Keywords** Portfolio selection · Uncertain programming · Enhanced index tracking model · Uncertainty theory

**Mathematics Subject Classification** C61

## 1 Introduction

In the portfolio field, there are two investment strategies based on the index. One is index tracking which is designed to replicate the performance of a given bench-

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mark. Another is enhanced index tracking (EIT) which aims to beat the benchmark return while having a minimum tracking error. Since the enhanced index funds perform well and their scale grow rapidly in recent years [5, 15], the EIT strategy is highly praised and has attracted scholars' attention. Nowadays many scholars and fund managers use optimization models to solve the EIT problem. An important work is given by Beasley et al. [3] who propose a mixed-integer nonlinear programming to track the index with fewer stocks whilst limiting total transaction cost. In addition, Canakgoz and Beasley [4] present a mixed-integer linear programming formulation for enhanced index tracking and propose a two-stage solution procedure. Then, various EIT optimization models are proposed to help investors build their portfolios [8, 9, 27].

According to the way the tracking error is measured, EIT optimization models are divided into different categories. The popular ones among them are mean-variance EIT model [28] and mean-absolute deviation EIT model [30]. Compared with the mean-variance EIT model, the mean-absolute deviation EIT model has an advantage. The absolute deviation tracking error gives a more accurate description of the investors' risk attitude than the quadratic deviation [30]. Under the mean-absolute deviation framework, scholars formulate many EIT optimization models. Guastaroba and Speranza [10] minimize the absolute deviation between the historical values of the tracking portfolio and the index and propose the mixed-integer linear programming formulations for the EIT problem. Filippi et al. [5] propose an EIT model by setting up the bi-objective programming: minimizing the absolute deviation tracking error and maximizing a linear excess return subject to some real-life constraints. Strub and Baumann [32] put forward a mixed-integer linear programming in which the sum of absolute deviation tracking error and transaction cost is minimized subject to real-life constraints.

The absolute deviation used in the above models is a symmetric tracking error measure which contains upside (positive) and downside (negative) deviations from the benchmark. In enhanced index tracking strategy, investors welcome the upside deviation and do not want downside deviation. So the symmetric tracking error measure is no longer suitable. For this reason, some scholars begin to use downside risk measures such as absolute downside deviation. The absolute downside deviation is consistent with the investors' perception of risk because most investors understand risk as the potential underachievement of a target return by an asset [25]. By using absolute downside deviation as the tracking error measure, Rudolf et al. [30] propose a linear EIT model and compare it with EIT models using other tracking error measures. Lejeune [17] provides a game theoretical formulation for the EIT problem in which the minimum excess return over all allowable probability distributions is maximized subject to the absolute downside deviation tracking error constraint. Gaustaroba et al. [8] put forward an EIT model in which they maximize the ratio of the upside and downside deviations between the tracking portfolio and the benchmark returns.

In addition to constructing various EIT models, scholars also focus on the returns and risks of the tracking portfolios because these two parameters are what investors care about most. Roll [29] shows that the tracking portfolio is risky for investors compared with standard Markowitz mean-variance portfolio and uses a constraint

on the beta of the tracking portfolio to alleviate the overly risk problem. Then, scholars begin to focus on how to reduce the risk of the tracking portfolio. Their approaches are to impose a limit on the amount of risk that investors can take, but differ on the measure of risk used. For example, papers [16], [1] and [2] add tracking portfolio's variance constraint, value-at-risk constraint and ex ante alpha constraint, respectively, to reduce the tracking portfolio's risk. Moreover, Maxwell et al. [26] propose an alternative tracking portfolio which has lower absolute risk relative to the benchmark risk and which has the highest Sharpe ratio—i.e., located on a capital market line tangential to the constant tracking error frontier, so it generates the highest excess return per unit of absolute risk. These works provide insightful findings on reducing tracking portfolio risk and improving tracking portfolio performance.

In the above studies, probability theory is used as the mathematical tool. The application premise of probability theory is that probability distributions estimated from historical data are close enough to the real frequencies. However, there exists the situation in financial market where either no historical data are available (e.g., stocks are newly listed) or the historical data cannot well reflect the future frequencies. For example, the outbreak of COVID-19 has led to a series of unexpected events so that historical data become invalid and distributions estimated from historical data cannot be close enough to the frequencies of future returns. Besides, some empirical tests show that in many cases observed data in financial market are not random variables. For example, using two-sample Kolmogorov–Smirnov test, Ye and Liu [34] show US Dollar to Chinese Yuan (USD-CNY) exchange rates cannot be treated as random variables because residuals from different parts are neither from the same population nor white noise in the sense of probability theory. Similar tests also show that Alibaba stock prices (see [23]) and GDP (see [35]) are not suitable to be treated as random variables. Therefore, there does exist the situation where probability theory is not applicable.

In order to handle this problem, Liu [19] proposes a new theory, i.e., uncertainty theory. Uncertainty theory is a new branch of mathematics and is developed based on four axioms. Like probability theory, uncertainty theory can help investors make decisions in the state of indeterminacy but in different situation. Liu [22] points out that people should use probability theory when the distribution is close enough to the frequency. Otherwise, they should use uncertainty theory. We provide an example here to show it. Consider that there are 20 stocks whose returns distribute uniformly between 2 to 5 percent which is the real frequency and a benchmark return is 5 percent. An investor uniformly allocates his/her money to the 20 stocks. Since the maximum return of each stock is 5 percent, it is easy to get that the return of the portfolio composed of these 20 stocks cannot exceed 5 percent, i.e.,

$$\Pr\{\text{“ The return of the portfolio exceeds the benchmark ”}\} = 0.$$

If the 20 stock returns estimated from historical data distribute uniformly between 2 to 14 which is not close enough to the real frequency, treating stock returns as random variables, we can get by simulation (3000 times) that

$$\Pr\{\text{"The return of the portfolio exceeds the benchmark"}\} = 1$$

which says that it is sure that the portfolio return will exceed the benchmark. Then, what if the 20 stock return distributions are treated as linear uncertainty distributions on [2%, 14%]? Based on operational law of uncertain variables, we can infer that the chance, expressed by uncertain measure, is 75%, i.e.,

$$\mathcal{M}\{\text{"The return of the portfolio exceeds the benchmark"}\} = 0.75.$$

Misusing probability theory, an event that is sure not to happen becomes sure to happen. This is dangerous because people will not be alert to and prepare for a sure to happen event and being off guard may lead to disastrous result. Though the result obtained by uncertainty theory still deviates from the real case, it is due to the great errors in the input. Uncertainty theory does not further magnify the errors in the input, and the 25% chance can still alert the investors to prepare for the likely risk event. Thus, we suggest employing uncertainty theory when the application premise of probability theory cannot be satisfied.

Nowadays uncertainty theory has been applied in solving various optimization problems [7, 21, 31]. Particularly, Huang [11] is the first to use uncertainty theory to study portfolio selection systematically. Subsequently, scholars have studied many portfolio problems based on uncertainty theory. Wang and Huang [33] discuss the effect of option on the portfolio performance and find that portfolios with options gain higher expected returns than those without options. Huang and Yang [13] study how the background risk affects investment decisions. They give properties of the portfolio efficient frontier when stock and background asset returns all take normal uncertainty distributions and tell how background risk affects portfolio selection decision. In order to reflect different attitudes toward risk in one portfolio investment, Huang and Di [12] apply mental accounts to uncertain optimization model. Besides, there are uncertain portfolio optimization models considering the bankruptcy constraint [18], the borrowing constraint [24] and the entropy constraint [6]. Yet so far, no paper studies the EIT problem based on uncertainty theory. However, there exist some situations in reality that the application premise of probability theory cannot be satisfied and uncertainty theory is more suitable. This motivates us to do the research of EIT model based on uncertainty theory in which stock returns are treated as uncertain variables instead of random variables.

The rest of the paper is organized as follows. In Sect. 2, we provide the necessary knowledge of uncertainty theory for easy understanding of our paper. In Sect. 3, we propose an uncertain mean-absolute downside deviation EIT model. In Sect. 4, we study the form of the tracking portfolio frontier and its properties when stock returns take linear uncertainty distributions. In Sect. 5, we show how the optimal tracking portfolio's return and risk change with the benchmark return distribution and the tracking error tolerance level. In Sect. 6, we report on the computational experiments. Finally, we conclude the paper in Sect. 7.

## 2 Preliminaries

Uncertainty theory is found by Liu [19] in 2007 and subsequently studied by many researchers. Nowadays uncertainty theory has become a branch of mathematics concerned with the analysis of uncertain phenomena. To indicate the chance that an uncertain event happens, a concept of the uncertain measure is defined.

**Definition 2.1** [19] Let  $\Gamma$  be a nonempty set and  $\mathcal{L}$  a  $\sigma$ -algebra over  $\Gamma$ . Each element  $\Lambda \in \mathcal{L}$  is called an event. A set function  $M\{\Lambda\}$  is called an uncertain measure if it satisfies the following axioms:

- (i) (Normality axiom)  $M\{\Gamma\} = 1$  for the universal set  $\Gamma$ .
- (ii) (Duality axiom)  $M\{\Lambda\} + M\{\Lambda^c\} = 1$  for any  $\Lambda \in \mathcal{L}$ .
- (iii) (Subadditivity axiom) For every countable sequence of events  $\{\Lambda_i\}$ , we have

$$M\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \leq \sum_{i=1}^{\infty} M\{\Lambda_i\}.$$

The triplet  $(\Gamma, \mathcal{L}, M)$  is called an uncertainty space. Furthermore, Liu [20] defined an uncertain product measure which produces the fourth axiom:

- (iv) (Product axiom) Let  $(\Gamma_k, \mathcal{L}_k, M_k)$  be uncertainty spaces for  $k = 1, 2, \dots, n$ . Then, the product uncertain measure is an uncertain measure satisfying

$$M\left\{\prod_{k=1}^{\infty} \Lambda_k\right\} = \bigwedge_{k=1}^{\infty} M_k\{\Lambda_k\},$$

where  $\Lambda_k$  are arbitrarily chosen events from  $\mathcal{L}_k$  for  $k = 1, 2, \dots, n$ , respectively.

Though probability measure meets the above normality, self-duality, and countable subadditivity axioms, it is not a special case of uncertainty theory because the product probability does not satisfy the product measure axiom of the uncertainty theory.

**Theorem 2.1** [21] *The uncertain measure is a monotone increasing set function. That is, for any events  $\Lambda_1 \subset \Lambda_2$ , we have*

$$M\{\Lambda_1\} \leq M\{\Lambda_2\}.$$

**Definition 2.2** [19] An uncertain variable is a measurable function  $\xi$  from an uncertainty space  $(\Gamma, \mathcal{L}, M)$  to the set of real numbers, i.e., for any Borel set  $B$  of real numbers, the set

$$\{\xi \in B\} = \{\gamma \in \Gamma \mid \xi(\gamma) \in B\}$$

is an event.

**Definition 2.3** [19] Let  $\xi$  and  $\eta$  be uncertain variables. We say  $\xi > \eta$  if  $\xi(\gamma) > \eta(\gamma)$  for almost all  $\gamma \in \Gamma$ .

In order to characterize uncertain variable, uncertainty distribution is defined as follows:

**Definition 2.4** [19] The uncertainty distribution  $\Phi : \Re \rightarrow [0, 1]$  of an uncertain variable  $\xi$  is defined by

$$\Phi(t) = M\{\xi \leq t\}$$

for any real number  $t$ .

An uncertainty distribution  $\Phi$  is called regular if it is a continuous and strictly increasing function concerning  $t$  at which  $0 < \Phi(t) < 1$ , and  $\lim_{t \rightarrow -\infty} \Phi(t) = 0$ ,  $\lim_{t \rightarrow +\infty} \Phi(t) = 1$ .

There are some popularly used uncertain variables, one of which is a linear uncertain variable. An uncertain variable is called a linear uncertain variable if it has the following linear uncertainty distribution:

$$\Phi(t) = \begin{cases} 0, & \text{if } t < a \\ \frac{t-a}{b-a}, & \text{if } a \leq t \leq b \\ 1, & \text{otherwise.} \end{cases}$$

The linear uncertain variable is denoted by  $\mathcal{L}(a, b)$  where  $a$  and  $b$  are real numbers with  $a < b$ .

When we have the uncertainty distributions of the uncertain variables  $\xi_1, \xi_2, \dots, \xi_n$ , the operational law of them is given by [21] as follows:

**Theorem 2.2** [21] Let  $\xi_1, \xi_2, \dots, \xi_n$  be independent uncertain variables with regular uncertainty distributions  $\Phi_1, \Phi_2, \dots, \Phi_n$ , respectively. If  $f(\xi_1, \xi_2, \dots, \xi_n)$  is strictly decreasing with respect to  $\xi_1, \xi_2, \dots, \xi_m$  and strictly increasing with respect to  $\xi_{m+1}, \xi_{m+2}, \dots, \xi_n$ , then  $\xi = f(\xi_1, \xi_2, \dots, \xi_n)$  is an uncertain variable with inverse uncertainty distribution function

$$\Psi^{-1}(\alpha) = f(\Phi_1^{-1}(1-\alpha), \dots, \Phi_m^{-1}(1-\alpha), \Phi_{m+1}^{-1}(\alpha), \dots, \Phi_n^{-1}(\alpha)), \alpha < 1. \quad (1)$$

The expected value and variance of an uncertain variable are defined as follows:

**Definition 2.5** [19] Let  $\xi$  be an uncertain variable. Then, the expected value of  $\xi$  is defined by

$$E[\xi] = \int_0^\infty M\{\xi \geq r\}dr - \int_{-\infty}^0 M\{\xi \leq r\}dr \quad (2)$$

provided that at least one of the two integrals is finite.

**Theorem 2.3** [21] Let  $\xi$  be an uncertain variable with a regular uncertainty distribution  $\Phi$ . If its expected value exists, then

$$E[\xi] = \int_0^1 \Phi^{-1}(\alpha)d\alpha. \quad (3)$$

**Definition 2.6** [19] Let  $\xi$  be an uncertain variable with finite expected value  $e$ . Then, the variance of  $\xi$  is defined by

$$V[\xi] = E[(\xi - e)^2]. \quad (4)$$

We offer relevant knowledge of uncertainty theory that will be used in this paper. For more expositions on uncertainty theory, the interested readers can consult Liu [21]. Uncertainty theory is a branch of mathematics concerned with the analysis of uncertain phenomena. The uncertain variable is used to represent quantities with uncertainty. In this paper, we treat the stock return and the benchmark return as uncertain variables. According to knowledge of uncertainty theory and portfolio theory, we can derive the tracking portfolio return and tracking error represented by uncertain variables. By calculating the center and dispersion of these uncertain variables, we show investors the tracking portfolio's return and risk.

### 3 The Uncertain Mean-Absolute Downside Deviation EIT Model with Linear Uncertainty Distributions

#### 3.1 The Notations and Assumptions

As discussed in Introduction, we study the problem in the situation where stock returns are treated as uncertain variables in this paper. Suppose there are  $n$  different stocks in the asset universe. Different stocks mean that they have different returns and risks and stocks with higher returns also have higher risks. This assumption is based on a logic. If two stocks have the same return and risk, we think they are the same stock, and if two stocks have the same return but different risks, the one with higher risk is not good enough to enter the investor's universe; similarly the one with lower return and higher risk is also not good enough to enter the investor's universe. Short-selling is not allowed, which is the requirement of the stock market in many countries. Let  $X_I$  denote the benchmark with uncertain return  $r_I$ . Let  $n \times 1$  vector  $X_P$  denote the tracking portfolio with uncertain return  $r_P$  which is independent of  $r_I$ . For example,  $r_P$  represents the return of a portfolio from Shanghai Stock Exchange and  $r_I$  represents NASDAQ-100 Index return, and they can be converted into independent variables by using the factor method in [14]. Let  $x_i$  denote the investment weight on stock  $i$  ( $i = 1, \dots, n$ ) in  $X_P$  and  $x_i$  are decision variables. Let  $\xi_i$  represent the uncertain return of stock  $i$ . So  $r_P = \sum_{i=1}^n x_i \xi_i$ . The expected values of  $r_P$  and  $r_I$  are  $\mu_P$  and  $\mu_I$ , respectively.

#### 3.2 The Uncertain Model

The uncertain mean-absolute downside deviation EIT model is designed to maximize the expected excess return over the benchmark subject to a fixed level of tracking error. The expected excess return is  $E[r_P] - E[r_I] = E[\sum_{i=1}^n x_i \xi_i] - E[r_I]$ . The tracking error is measured by absolute downside deviation between  $r_P$  and  $r_I$ . For

simplicity, we write  $(r_P - r_I)^- = \min(r_P - r_I, 0)$ . The tracking error function is  $E[|(r_P - r_I)^-|] = E[|(\sum_{i=1}^n x_i \xi_i - r_I)^-|]$ . So we formulate the uncertain mean-absolute downside deviation EIT model as follows:

$$\begin{aligned} & \max E\left[\sum_{i=1}^n x_i \xi_i\right] - E[r_I] \\ & \text{subject to: } E\left[\left|\left(\sum_{i=1}^n x_i \xi_i - r_I\right)^-\right|\right] = D \\ & \sum_{i=1}^n x_i = 1 \\ & x_i \geq 0, i = 1, 2, \dots, n, \end{aligned} \tag{5}$$

where  $D$  is the tolerance level of the tracking error. In order to get the solution, we give the deterministic form of the model below.

**Theorem 3.1** *Suppose stock returns  $\xi_i$  and the benchmark return  $r_I$  have regular uncertainty distributions  $\Phi_i, i = 1, 2, \dots, n$ , and  $\Phi_I$ , respectively. Then, model (5) is equivalent to the following model:*

$$\begin{aligned} & \max \int_0^1 \sum_{i=1}^n x_i \Phi_i^{-1}(\alpha) d\alpha - \int_0^1 \Phi_I^{-1}(\alpha) d\alpha \\ & \text{subject to: } \int_0^\beta \left( \Phi_I^{-1}(1 - \alpha) - \sum_{i=1}^n x_i \Phi_i^{-1}(\alpha) \right) d\alpha = D \\ & \sum_{i=1}^n x_i \Phi_i^{-1}(\beta) - \Phi_I^{-1}(1 - \beta) = 0 \\ & \sum_{i=1}^n x_i = 1 \\ & x_i \geq 0, i = 1, 2, \dots, n. \end{aligned} \tag{6}$$

**Proof** (i) Since  $x_i > 0$  and  $\xi_i$  have regular uncertainty distributions  $\Phi_i$ , according to Theorem 2.2, the inverse uncertainty distribution of  $\sum_{i=1}^n x_i \xi_i$  is  $\sum_{i=1}^n x_i \Phi_i^{-1}(\alpha)$ . Then, according to Theorem 2.3, we have  $E[\sum_{i=1}^n x_i \xi_i] = \int_0^1 \sum_{i=1}^n x_i \Phi_i^{-1}(\alpha) d\alpha$  and  $E[r_I] = \int_0^1 \Phi_I^{-1}(\alpha) d\alpha$ .

(ii) Let  $\eta = \sum_{i=1}^n x_i \xi_i - r_I$  and  $\Psi$  denote the uncertainty distribution of  $\eta$ . Since we suppose that  $\xi_i$  is independent of  $r_I$ , according to Theorem 2.2, we can have the inverse uncertainty distribution of  $\eta$  is  $\Psi^{-1}(\alpha) = \sum_{i=1}^n x_i \Phi_i^{-1}(\alpha) - \Phi_I^{-1}(1 - \alpha)$ . Then, from Definition 2.5, we have



$$\begin{aligned}
 E[|\eta^-|] &= \int_0^{+\infty} M\{|\eta^-| \geq t\}dt - \int_{-\infty}^0 M\{|\eta^-| \leq t\}dt \\
 &= \int_0^{+\infty} M\{\eta \leq -t\}dt \\
 &= \int_{-\infty}^0 \Psi(t)dt \\
 &= - \int_0^\beta \Psi^{-1}(\alpha)d\alpha,
 \end{aligned}
 \tag{7}$$

where  $\Psi^{-1}(\beta) = 0$ .

So we have

$$E[|(r_P - r_I)^-|] = \int_0^\beta \left( \Phi_I^{-1}(1 - \alpha) - \sum_{i=1}^n x_i \Phi_i^{-1}(\alpha) \right) d\alpha,$$

where  $\sum_{i=1}^n x_i \Phi_i^{-1}(\beta) - \Phi_I^{-1}(1 - \beta) = 0$ . □

When stock returns all take linear uncertainty distributions, we further give the deterministic of the model (5) below. In the following, a linear uncertainty distribution is denoted by  $\mathcal{L}(e - s, e + s)$  where  $e$  and  $s$  represent the distribution center and distribution spread, respectively.

**Theorem 3.2** *Suppose stock returns  $\xi_i$  and the benchmark return  $r_I$  take linear uncertainty distributions, i.e.,  $\xi_i \sim \mathcal{L}(e_i - s_i, e_i + s_i)$  and  $r_I \sim \mathcal{L}(e_I - s_I, e_I + s_I)$ , respectively. Then, model (5) can be transformed into the following form:*

$$\begin{aligned}
 &\max \sum_{i=1}^n x_i e_i - e_I \\
 \text{subjectto :} &\frac{(\sum_{i=1}^n x_i e_i - e_I - \sum_{i=1}^n x_i s_i - s_I)^2}{4(\sum_{i=1}^n x_i s_i + s_I)} = D \\
 &\sum_{i=1}^n x_i = 1 \\
 &x_i \geq 0, i = 1, 2, \dots, n.
 \end{aligned}
 \tag{8}$$

**Proof** (i) Let  $\Phi_i$  and  $\Phi_I$  denote the uncertainty distributions of  $\xi_i$  and  $r_I$ , respectively. Since  $\xi_i \sim \mathcal{L}(e_i - s_i, e_i + s_i)$ , from the definition of linear uncertain variable, it can be derived that  $\Phi_i^{-1}(\alpha) = 2s_i\alpha + e_i - s_i$ . Similarly, we can have  $\Phi_I^{-1}(\alpha) = 2s_I\alpha + e_I - s_I$ . According to the proof of Theorem 3.1, we have the objective function.

(ii) When  $\eta \sim \mathcal{L}(e - s, e + s)$ , we can get the uncertainty distribution of  $\eta$ . Then, according to equation (7), we calculate and get

$$E[|\eta^-|] = \begin{cases} \frac{(e-s)^2}{4s} & \text{when } e > 0, s > 0, e < s, \\ 0 & \text{when } e > 0, s > 0, e > s. \end{cases}$$

Remember that  $\eta = r_P - r_I$ . If  $e > 0, s > 0$  and  $e > s$ , it means that  $r_P - r_I > 0$  which is equivalent to  $r_P > r_I$ . According to Definition 2.3, the situation that  $r_P - r_I > 0$  is rare in reality. So the following discussions are in the situation that  $E[|\eta^-|] = \frac{(e-s)^2}{4s}$  where  $e > 0, s > 0$  and  $e < s$ . Since  $\xi_i$  and  $r_I$  are linear uncertain variables and  $x_i > 0$ , it can be proven that

$$\begin{aligned} \sum_{i=1}^n x_i \xi_i - r_I &\sim \mathcal{L} \left( \sum_{i=1}^n x_i e_i - e_I - \left( \sum_{i=1}^n x_i s_i + s_I \right), \right. \\ &\left. \sum_{i=1}^n x_i e_i - e_I + \left( \sum_{i=1}^n x_i s_i + s_I \right) \right). \end{aligned}$$

So we have

$$E \left[ \left| \left( \sum_{i=1}^n x_i \xi_i - r_I \right)^- \right| \right] = \frac{(\sum_{i=1}^n x_i e_i - e_I - \sum_{i=1}^n x_i s_i - s_I)^2}{4(\sum_{i=1}^n x_i s_i + s_I)}. \quad \square$$

### 3.3 The Solution

In order to solve model (8), we introduce the following constraint:

$$\theta = \frac{1}{4(\sum_{i=1}^n x_i s_i + s_I)}. \tag{9}$$

Adding Eq. (9) to model (8), model (8) is equivalent to the following form:

$$\begin{aligned} &\max \sum_{i=1}^n x_i e_i - e_I \\ &\text{subject to: } \left( \sum_{i=1}^n x_i e_i - e_I - \sum_{i=1}^n x_i s_i - s_I \right)^2 \theta = D \\ &\qquad \qquad \qquad \sum_{i=1}^n x_i = 1 \\ &\qquad \qquad \qquad 4 \left( \sum_{i=1}^n x_i s_i + s_I \right) \theta = 1 \\ &\qquad \qquad \qquad x_i \geq 0, i = 1, 2, \dots, n. \end{aligned} \tag{10}$$

To solve the model (10), we first need to obtain the KT points. Using Lagrange multipliers for the constraints in model (10), respectively, we have the Lagrangian:

$$L(x_i, \rho, \lambda, \theta) = - \left( \sum_{i=1}^n x_i e_i - e_I \right) - \rho_1 \left( \left( \sum_{i=1}^n x_i e_i - e_I - \sum_{i=1}^n x_i s_i - s_I \right)^2 \theta - D \right) - \rho_2 \left( \sum_{i=1}^n x_i - 1 \right) - \rho_3 \left( 4 \left( \sum_{i=1}^n x_i s_i + s_I \right) \theta - 1 \right) - \lambda_1 x_1 - \dots - \lambda_n x_n. \tag{11}$$

The necessary KT optimality conditions are

$$\frac{\partial L}{\partial x_i} = -e_i - 2\rho_1\theta(e_i - s_i) \left( \sum_{i=1}^n x_i e_i - e_I - \sum_{i=1}^n x_i s_i - s_I \right) - \rho_2 - 4\theta\rho_3 s_i - \lambda_i = 0, \tag{12}$$

$$\frac{\partial L}{\partial \rho_1} = \left( \sum_{i=1}^n x_i e_i - e_I - \sum_{i=1}^n x_i s_i - s_I \right)^2 \theta - D = 0, \tag{13}$$

$$\frac{\partial L}{\partial \rho_2} = \sum_{i=1}^n x_i - 1 = 0, \tag{14}$$

$$\frac{\partial L}{\partial \rho_3} = 4 \left( \sum_{i=1}^n x_i s_i + s_I \right) \theta - 1 = 0, \tag{15}$$

$$\frac{\partial L}{\partial \theta} = -\rho_1 \left( \sum_{i=1}^n x_i e_i - e_I - \sum_{i=1}^n x_i s_i - s_I \right)^2 - 4\rho_3 \left( \sum_{i=1}^n x_i s_i + s_I \right) = 0, \tag{16}$$

$$\lambda_i x_i = 0, \tag{17}$$

$$x_i \geq 0, \tag{18}$$

$$\lambda_i \geq 0, i = 1, 2, \dots, n. \tag{19}$$

Note that there are  $2n + 2$  unknown quantities  $(x_i, \lambda_i, \rho_2, \rho_3, i = 1, 2, \dots, n)$  in  $2n + 2$  simultaneous equations (Formulas (12),(14),(15),(17)-(19)). Inspired by the solving method in [13], we can get the solution of the  $2n + 2$  simultaneous equations, i.e., KT point shown by formula (20). In order to get  $\theta$ , we substitute (20) into Eq. (13). Then, we have equations (21)–(23) and  $\theta$  is decided by them. Note that  $\theta$  has two values, so we have two KT points. Next, we need to judge whether the KT points are the optimal solution of the model (10). According to second-order sufficient condition, the one that makes  $\nabla_{xx}^2 L$  positive definite is the one we need, and the corresponding KT point is the optimal solution. So we get the optimal solution of model (10), i.e.,

$$\mathbf{X} \mathbf{P}^* = [0 \dots x_j^* \dots x_k^* \dots 0]^T, \text{ where } \begin{bmatrix} x_j^* \\ x_k^* \end{bmatrix} = \begin{bmatrix} \frac{4\theta s_k - (1 - 4\theta s_I)}{(1 - 4\theta s_I) - 4\theta s_j} \\ \frac{4\theta(s_k - s_j)}{4\theta(s_k - s_j)} \end{bmatrix}, \tag{20}$$

$$16(P - e_I - s_I N)^2 \theta^2 + 8((P - e_I - s_I N)(N - 1) - 2D)\theta + (N - 1)^2 = 0, \tag{21}$$

$$P = \frac{e_j s_k - e_k s_j}{s_k - s_j}, \tag{22}$$

$$N = \frac{e_k - e_j}{s_k - s_j}. \quad (23)$$

Observing (20), the optimal tracking portfolio contains two stocks: the  $j$ -th and the  $k$ -th. Note that stock weights are affected by  $\theta$  that is decided by (21)–(23). So the parameters  $e_I$ ,  $s_I$  and  $D$  affect the proportions of stocks  $j$  and  $k$ . If any of the above parameters changes, the weights of stocks  $j$  and  $k$  will change. If these parameters change too much, the optimal tracking portfolio may choose another two stocks instead of stocks  $j$  and  $k$ .

## 4 The Return and Risk of the Tracking Portfolio

In Sect. 3, we have derived the optimal tracking portfolio's composition. In this section, we are concerned with the return and risk of the optimal tracking portfolio. As we mentioned in Introduction, the downside deviation is risk to investors and upside deviation is desired by investors. So portfolio risk is measured by the absolute downside deviation between the portfolio's return and its expected return (hereinafter referred to as ADD) in this section.

### 4.1 The Risk of the Tracking Portfolio

Consider an uncertain variable  $\xi$  with a finite expected return  $E[\xi]$ . The absolute downside deviation between the uncertain variable and its expected value is given by

$$\text{ADD}[\xi] = E\left[|(\xi - E[\xi])^-|\right],$$

where  $(\xi - E[\xi])^- = \min(\xi - E[\xi], 0)$ . As a tool of risk measure, ADD has the following property.

**Theorem 4.1** *Suppose  $\xi_1$  and  $\xi_2$  are two independent uncertain variables. Then,  $\text{ADD}[\xi_1 + \xi_2] \leq \text{ADD}[\xi_1] + \text{ADD}[\xi_2]$ .*

**Proof** We rewrite the ADD as follows:

$$\text{ADD}[\xi] = E[\max\{0, E[\xi] - \xi\}].$$

Consider,

$$\begin{aligned} \text{ADD}[\xi_1 + \xi_2] &= E[\max\{0, E[\xi_1 + \xi_2] - \xi_1 - \xi_2\}] \\ &= E[\max\{0, E[\xi_1] - \xi_1 + E[\xi_2] - \xi_2\}]. \end{aligned}$$

It is a straightforward that,  $\max\{0, a + b\} \leq \max\{0, a\} + \max\{0, b\} \quad \forall a, b \in \mathbb{R}$ . Thus,

$$\begin{aligned} \text{ADD}[\xi_1 + \xi_2] &= E[\max\{0, E[\xi_1] - \xi_1 + E[\xi_2] - \xi_2\}] \\ &\leq E[\max\{0, E[\xi_1] - \xi_1\}] + E[\max\{0, E[\xi_2] - \xi_2\}] \end{aligned}$$

$$= \text{ADD}[\xi_1] + \text{ADD}[\xi_2]. \quad \square$$

If  $\xi_1$  and  $\xi_2$  in Theorem 4.1 are regarded as stocks’ uncertain returns, it gives an implication that the ADD of a diversified portfolio cannot be greater than the sum of ADD of the individual assets. This is consistent with the real-world behavior of portfolios that diversification leads to the reduction of risk.

Remember that  $r_P$  is tracking portfolio’s uncertain return and  $\mu_P$  is its expected return. So we can get the risk of the tracking portfolio as

$$\text{ADD}[r_P] = E[|(r_P - \mu_P)^-|].$$

Then, according to Definition 2.5, the ADD of the tracking portfolio can be calculated via

$$\text{ADD}[r_P] = \int_{-\infty}^{\mu_P} \Phi(r) dr, \tag{24}$$

where  $\Phi(\cdot)$  is the uncertainty distribution of  $r_P$ . Formula (24) will facilitate the calculation of tracking portfolio’s risk.

### 4.2 The Frontier of the Tracking Portfolio

In this section, we analyze the tracking portfolio frontier in the mean-ADD space. The frontier can give investors a panoramic view of portfolio return and risk. From Sect. 3.3, we know that the optimal tracking portfolio contains stocks  $j$  and  $k$ . So the optimal tracking portfolio’ uncertain return is  $r_P^* = x_j^* \xi_j + x_k^* \xi_k$ . Since  $\xi_j \sim \mathcal{L}(e_j - s_j, e_j + s_j)$  and  $\xi_k \sim \mathcal{L}(e_k - s_k, e_k + s_k)$ , it can be proven that

$$r_P^* \sim \mathcal{L}\left(x_j^* e_j + x_k^* e_k - (x_j^* s_j + x_k^* s_k), x_j^* e_j + x_k^* e_k + (x_j^* s_j + x_k^* s_k)\right). \tag{25}$$

According to (25), we have known the uncertainty distribution of  $r_P^*$ . Then, according to Theorem 2.3, we can get the expected return of the optimal tracking portfolio

$$\mu_P^* = x_j e_j + x_k e_k. \tag{26}$$

And according to formula (24), we can get the ADD of the optimal tracking portfolio

$$\text{ADD}[r_P^*] = (x_j s_j + x_k s_k)/4. \tag{27}$$

Substitute Eq. (20) into Eqs. (26) and (27), then we have

$$\mu_P^* = P + N \left( \frac{1}{4\theta} - s_I \right), \tag{28}$$

$$\text{ADD}[r_P^*] = \frac{1}{4} \left( \frac{1}{4\theta} - s_I \right), \tag{29}$$

where  $P$ ,  $N$  and  $\theta$  are decided by Eqs. (21)–(23). Equations (28) and (29) give the optimal tracking portfolio's return and risk, respectively. According to Eqs. (28) and (29), we can easily get the tracking portfolio frontier

$$\text{ADD}[r_p^*] = \frac{\mu_p^* - P}{4N}. \quad (30)$$

Note that Eq. (30) shows the tracking portfolio frontier when the optimal tracking portfolio contains stocks  $j$  and  $k$ . If the optimal tracking portfolio chooses another stocks, what happens to the frontier? The following theorem will answer it.

**Theorem 4.2** *Consider there are  $n > 2$  candidate stocks in the asset pool. The tracking portfolio frontier of model (10) is a continuous curve composed of at most  $n - 1$  different line segments in the mean-ADD space.*

**Proof** Denote  $e_1 < \dots < e_j < \dots < e_k < \dots < e_l < \dots < e_n$ . When optimal tracking portfolio contains the  $j$ -th and  $k$ -th stocks, we have Eq. (26). Since  $x_j > 0$  and  $x_k > 0$ , according to Eq. (26), we get  $e_j \leq \mu_p^* \leq e_k$ . When  $\mu_p^* \in [e_j, e_k]$ , we know that the frontier takes the form of (30) where  $P = P_1 = \frac{e_j s_k - e_k s_j}{s_k - s_j}$  and

$$N = N_1 = \frac{e_k - e_j}{s_k - s_j}.$$

As  $\mu_p^*$  increases, e.g.,  $\mu_p^* \in [e_k, e_l]$ , model (10) may select the  $k$ -th and the  $l$ -th stocks. And we can prove that the frontier takes the form of (30) where  $P = P_2 = \frac{e_k s_l - e_l s_k}{s_l - s_k}$  and  $N = N_2 = \frac{e_l - e_k}{s_l - s_k}$  when  $\mu_p^* \in [e_k, e_l]$ .

When  $\mu_p^* = e_k$ ,  $\text{ADD}[r_p^*]$  calculated by  $\frac{\mu_p^* - P_1}{4N_1}$  and  $\frac{\mu_p^* - P_2}{4N_2}$  are the same. It means that the frontier is a continuous curve that is composed of two line segments when  $\mu_p^* \in [e_j, e_l]$ . As  $\mu_p^*$  changes from  $e_1$  to  $e_n$ , the frontier becomes a continuous curve composed of different line segments.

Next prove that the frontier is composed of at most  $n - 1$  line segments. Note that points  $e_1 \dots e_j \dots e_n$  divide the interval  $[e_1, e_n]$  into  $n - 1$  intervals. And model (10) has only one optimal solution when  $\mu_p^*$  belongs to each interval. Meanwhile, there exists the possibility that model (10) has the same optimal solution when  $\mu_p^*$  belongs to several different adjacent intervals. So the tracking portfolio frontier is composed of at most  $n - 1$  different line segments.  $\square$

As shown in Theorem 4.2, the tracking portfolio frontier is linear shape in the mean-ADD space. It is different from traditional parabolic shape like the Markowitz efficient frontier in the mean-variance space. We believe that this linear relationship is due to the use of ADD as risk measure. Since both mean and ADD are the first-order moments of an uncertain variable, it is reasonable to be a linear relationship between them.

## 5 Comparative Statics

We are interested in how the optimal tracking portfolio’s return and risk respond to the changes in benchmark and tracking error tolerance level  $D$ . All the discussions are based on the situation that model (10) still selects the same stocks  $j$  and  $k$  when changes occur in benchmark and  $D$ . If model (10) selects other stocks, e.g., stocks  $k$  and  $l$ , a similar conclusion will be obtained.

### 5.1 Changes in the Benchmark

Remember that the benchmark return  $r_I$  takes a linear uncertainty distribution  $\mathcal{L}(e_I - s_I, e_I + s_I)$  where  $e_I$  and  $s_I$  represent the distribution center and distribution spread, respectively. The analysis is to address how the optimal tracking portfolio’s return and risk change with the benchmark return’s distribution center  $e_I$ . First, we get the derivatives of Eqs. (28) and (29) with respect to  $e_I$  which are

$$\frac{\partial \mu_P^*}{\partial e_I} = \frac{\partial \mu_P^*}{\partial \theta} \frac{\partial \theta}{\partial e_I} = -\frac{N}{4\theta^2} \frac{\partial \theta}{\partial e_I}, \tag{31}$$

$$\frac{\partial \text{ADD}[r_P^*]}{\partial e_I} = \frac{\partial \text{ADD}[r_P^*]}{\partial \theta} \frac{\partial \theta}{\partial e_I} = -\frac{1}{16\theta^2} \frac{\partial \theta}{\partial e_I}. \tag{32}$$

Then, we rewrite Eq. (21) as follows:

$$F(\theta) = 16(P - e_I - s_I N)^2 \theta^2 + 8\left((P - e_I - s_I N)(N - 1) - 2D\right)\theta + (N - 1)^2 = 0. \tag{33}$$

After some rearrangements, implicit differentiation of (33) with respect to  $e_I$  yields

$$\frac{\partial \theta}{\partial e_I} = -\frac{F'_{e_I}(\theta)}{F'_\theta(\theta)} = \frac{2\theta}{P - e_I - s_I N - (N - 1)/4\theta}. \tag{34}$$

Please remember the assumption we make in Sect. 3 that stocks with higher returns also have higher risks. So we have  $N > 0$  according to Eq. (23). And from Eq. (9), we know  $\theta > 0$ . So we have the following results.

(i) Situation A:  $P - e_I - s_I N - (N - 1)/4\theta < 0$  holds. Then, we get  $\frac{\partial \theta}{\partial e_I} < 0$ .  
Therefore,  $\frac{\partial \mu_P^*}{\partial e_I} > 0$  and  $\frac{\partial \text{ADD}[r_P^*]}{\partial e_I} > 0$ .

(ii) Situation B:  $P - e_I - s_I N - (N - 1)/4\theta > 0$  holds. Then, we get  $\frac{\partial \theta}{\partial e_I} > 0$ .  
Therefore,  $\frac{\partial \mu_P^*}{\partial e_I} < 0$  and  $\frac{\partial \text{ADD}[r_P^*]}{\partial e_I} < 0$ .

We now turn to analyze how the optimal tracking portfolio’s return and risk change with benchmark return’s distribution spread  $s_I$ . The derivatives of equations (28) and

(29) with respect to  $s_I$  are

$$\frac{\partial \mu_P^*}{\partial s_I} = \frac{\partial \mu_P^*}{\partial \theta} \frac{\partial \theta}{\partial s_I} - N, \quad (35)$$

$$\frac{\partial \text{ADD}[r_P^*]}{\partial s_I} = \frac{\partial \text{ADD}[r_P^*]}{\partial \theta} \frac{\partial \theta}{\partial s_I} - \frac{1}{4}. \quad (36)$$

Implicit differentiation of (33) with respect to  $s_I$  yields

$$\frac{\partial \theta}{\partial s_I} = \frac{2\theta N}{P - e_I - s_I N - (N - 1)/4\theta}. \quad (37)$$

Substituting Eq. (37) into Eqs. (35) and (36) and rearranging, we have

$$\frac{\partial \mu_P^*}{\partial s_I} = -N \cdot \frac{P - e_I - s_I N + (N + 1)/4\theta}{P - e_I - s_I N - (N - 1)/4\theta}, \quad (38)$$

$$\frac{\partial \text{ADD}[r_P^*]}{\partial s_I} = -\frac{1}{4} \cdot \frac{P - e_I - s_I N + (N + 1)/4\theta}{P - e_I - s_I N - (N - 1)/4\theta}. \quad (39)$$

Substitute Eq. (20) into  $\sum_{i=1}^n x_i^* e_i$  and  $\sum_{i=1}^n x_i^* s_i$ , then we get  $\sum_{i=1}^n x_i^* e_i = P - s_I N + N/4\theta$  and  $\sum_{i=1}^n x_i^* s_i = 1/4\theta - s_I$ . After some rearrangements, we can have

$$P - e_I - s_I N + (N + 1)/4\theta = \sum_{i=1}^n x_i^* e_i - e_I + \left( \sum_{i=1}^n x_i^* s_i + s_I \right). \quad (40)$$

Remember that the benchmark return  $r_I$  takes a linear uncertainty distribution  $\mathcal{L}(e_I - s_I, e_I + s_I)$ . If  $e_I > s_I$ , it means that  $r_I > 0$  which is rare in reality. We only consider the case where  $e_I < s_I$ . In equation (40), it is easy to prove that  $\sum_{i=1}^n x_i^* e_i - e_I + (\sum_{i=1}^n x_i^* s_i + s_I) > 0$ . So we can get  $P - e_I - s_I N + (N + 1)/4\theta > 0$ . Note that  $N > 0$ . So the following results are obtained.

- (i) When Situation A holds, we get  $\frac{\partial \mu_P^*}{\partial s_I} > 0$  and  $\frac{\partial \text{ADD}[r_P^*]}{\partial s_I} > 0$ .
- (ii) When Situation B holds, we get  $\frac{\partial \mu_P^*}{\partial s_I} < 0$  and  $\frac{\partial \text{ADD}[r_P^*]}{\partial s_I} < 0$ .

## 5.2 Changes in the Tracking Error Tolerance Level

Now, we discuss how the optimal tracking portfolio's return and risk change with tracking error tolerance level  $D$ . We get the derivatives of equations (28) and (29) with respect to  $D$  which are

$$\frac{\partial \mu_P^*}{\partial D} = \frac{\partial \mu_P^*}{\partial \theta} \frac{\partial \theta}{\partial D} = -\frac{N}{4\theta^2} \frac{\partial \theta}{\partial D}, \quad (41)$$

$$\frac{\partial \text{ADD}[r_P^*]}{\partial D} = \frac{\partial \text{ADD}[r_P^*]}{\partial \theta} \frac{\partial \theta}{\partial D} = -\frac{1}{16\theta^2} \frac{\partial \theta}{\partial D}. \quad (42)$$



Implicit differentiation of (33) with respect to  $D$  yields

$$\frac{\partial \theta}{\partial D} = -\frac{F'_D(\theta)}{F'_\theta(\theta)} = \frac{1}{(P - e_I - s_I N + \frac{N-1}{4\theta})(P - e_I - s_I N - \frac{N-1}{4\theta})}. \tag{43}$$

Similarly, we substitute Eq. (20) into  $\sum_{i=1}^n x_i^* e_i$  and  $\sum_{i=1}^n x_i^* s_i$ . After some rearrangement, we get the following formula

$$P - e_I - s_I N + \frac{N - 1}{4\theta} = \sum_{i=1}^n x_i^* e_i - e_I - \left( \sum_{i=1}^n x_i^* s_i + s_I \right). \tag{44}$$

Remember that the stock return  $\xi_i \sim \mathcal{L}(e_i - s_i, e_i + s_i), i = 1, 2, \dots, n$ . If  $e_i > s_i$ , it means that  $\xi_i > 0$  which is rare in reality. Then, we only consider the case where  $e_i < s_i$ . In Eq. (44), since  $e_i < s_i$  and  $e_I < s_I$ , it is easy to prove that  $\sum_{i=1}^n x_i^* e_i - e_I - (\sum_{i=1}^n x_i^* s_i + s_I) < 0$ . Then, we get  $P - e_I - s_I N + \frac{N - 1}{4\theta} < 0$ . So the following results are obtained.

(i) When Situation A holds, we get  $\frac{\partial \theta}{\partial D} > 0$ . Therefore,  $\frac{\partial \mu_P^*}{\partial D} < 0$  and  $\frac{\partial ADD[r_P^*]}{\partial D} < 0$ .

(ii) When Situation B holds, we get  $\frac{\partial \theta}{\partial D} < 0$ . Therefore,  $\frac{\partial \mu_P^*}{\partial D} > 0$  and  $\frac{\partial ADD[r_P^*]}{\partial D} > 0$ .

The above analysis shows that when  $D$  changes,  $\mu_P^*$  and  $ADD[r_P^*]$  both change. But there is one thing to note. The change of  $D$  does not affect the slope of tracking portfolio frontier in mean-ADD space when tracking portfolio contains stocks  $j$  and  $k$ . Observing frontier formula (30), we see that the slope is independent of  $D$ . If tracking portfolio contains other stocks, e.g., stocks  $k$  and  $l$ , a similar conclusion will be obtained.

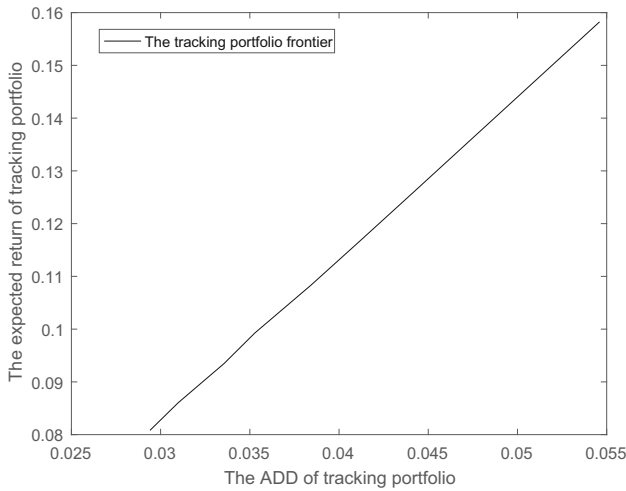
## 6 Numerical Examples

### 6.1 Computational Results of Uncertain Mean-Absolute Downside Deviation EIT Model with Linear Uncertainty Distributions

In order to clearly illustrate the modeling idea and the research results, we present some numerical examples. Suppose stock and benchmark returns take linear uncertainty distributions which are shown in Table 1. According to four prospectuses of enhanced index funds coded 008593, 015148, 007994, 001556 in China’s securities market, the annual tracking error should not exceed 7.75%. So we set the tolerance level of tracking error at 5% in our proposed model, i.e.,  $D = 0.05$ . First, we give the computational results of the optimal tracking portfolio. By solving model (8) with computer, we obtain the optimal tracking portfolio’s composition,

**Table 1** Uncertain distributions of stock and the benchmark returns

Stock <i>i</i>	<i>e</i>	<i>s</i>	Uncertainty distributions
1	0.059	0.089	$\mathcal{L}(-0.030, 0.148)$
2	0.075	0.110	$\mathcal{L}(-0.035, 0.185)$
3	0.114	0.156	$\mathcal{L}(-0.042, 0.270)$
4	0.120	0.169	$\mathcal{L}(-0.049, 0.289)$
5	0.150	0.205	$\mathcal{L}(-0.055, 0.355)$
6	0.168	0.231	$\mathcal{L}(-0.063, 0.399)$
Benchmark	0.080	0.120	$\mathcal{L}(-0.040, 0.200)$



**Fig. 1** The tracking portfolio frontier

i.e.,  $X_P^* = (0, 0, 0, 0.2031, 0, 0.7969)^T$ . The expected return of the optimal tracking portfolio is  $\mu_P^* = 0.1582$ , and the ADD is  $ADD[r_P^*] = 0.0546$ .

Then to give investors a panorama of the relationship between portfolio risk and return, we give the tracking portfolio frontier. Figure 1 shows the tracking portfolio frontier in mean-ADD space. In Fig. 1, the tracking portfolio frontier is an increasing curve. With the increase in ADD, the optimal tracking portfolio’s expected return increases.

In order to test how the relevant parameters affect the optimal tracking portfolio’s return and risk, we change the values of  $e_I, s_I$  and  $D$  and do the sensitivity analysis. There is one thing to note. When one parameter changes, the others remain unchanged. Table 2 shows the effect of  $e_I$  on the optimal tracking portfolios. As shown in Table 2,  $\mu_P^*$  and  $ADD[r_P^*]$  increase with  $e_I$ . Table 3 shows the effect of  $s_I$  on the optimal tracking portfolio. As shown in Table 3,  $\mu_P^*$  and  $ADD[r_P^*]$  increase with  $s_I$ . Table 4 shows the effect of  $D$  on the optimal tracking portfolio. As the increase of  $D$ ,  $\mu_P^*$  and  $ADD[r_P^*]$  both decrease. The above experimental results are consistent with our theoretical analysis.

**Table 2** The effect of  $e_I$  on the optimal tracking portfolio

$e_I$	<i>Obj.</i>	$\mu_P^*$	$ADD[r_P^*]$
0.075	0.0601	0.1351	0.0471
0.076	0.0636	0.1396	0.0486
0.077	0.0671	0.1441	0.0500
0.078	0.0707	0.1487	0.0515
0.079	0.0744	0.1534	0.0530
0.080	0.0782	0.1582	0.0546

Note:  $D = 0.05$  and  $s_I = 0.12$ **Table 3** The effect of  $s_I$  on the optimal tracking portfolio

$s_I$	<i>Obj.</i>	$\mu_P^*$	$ADD[r_P^*]$
0.115	0.0640	0.1440	0.0500
0.116	0.0667	0.1467	0.0509
0.117	0.0695	0.1495	0.0518
0.118	0.0724	0.1524	0.0527
0.119	0.0753	0.1553	0.0536
0.120	0.0783	0.1558	0.0546

Note:  $D = 0.05$  and  $e_I = 0.08$ **Table 4** The effect of  $D$  on the optimal tracking portfolio

$D$	<i>Obj.</i>	$\mu_P^*$	$ADD[r_P^*]$
0.050	0.0783	0.1582	0.0546
0.051	0.0662	0.1462	0.0507
0.052	0.0555	0.1355	0.0472
0.053	0.0457	0.1257	0.0441
0.054	0.0367	0.1167	0.0412
0.055	0.0283	0.1083	0.0384

Note:  $e_I = 0.08$  and  $s_I = 0.12$ 

## 6.2 Comparison between Uncertain Mean-Absolute Downside Deviation EIT Model and Other Uncertain EIT Models

As mentioned in Introduction, EIT models can be classified according to the tracking error measures. In this section, we compare our proposed model with uncertain mean-absolute deviation EIT model (hereinafter referred to as Model I) and uncertain mean-standard deviation EIT model (hereinafter referred to as Model II). In Model I, tracking error is measured by absolute deviation between tracking portfolio and benchmark returns. And in Model II, tracking error is measured by standard deviation of the difference between these two quantities. Models I and II are in the following forms:

$$\text{Model I} \left\{ \begin{array}{l} \max E\left[\sum_{i=1}^n x_i \xi_i\right] - E[r_I] \\ \text{subject to:} \\ E\left[\left|\sum_{i=1}^n x_i \xi_i - r_I\right|\right] = D_1 \\ \sum_{i=1}^n x_i = 1 \\ x_i \geq 0, i = 1, 2, \dots, n, \end{array} \right.$$

$$\text{Model II} \left\{ \begin{array}{l} \max E\left[\sum_{i=1}^n x_i \xi_i\right] - E[r_I] \\ \text{subject to:} \\ \sqrt{V\left[\sum_{i=1}^n x_i \xi_i - r_I\right]} = D_2 \\ \sum_{i=1}^n x_i = 1 \\ x_i \geq 0, i = 1, 2, \dots, n. \end{array} \right.$$

Table 5 shows the comparison results between different uncertain EIT models. In Table 5, the first column is the objective value which represents the expected excess return of the optimal tracking portfolio over the benchmark. The second column is the tracking error in our proposed model, and the third and fourth columns are the tracking errors in Models I and II, respectively. As shown in Table 5, the tracking error of our proposed model is smaller than that of Model I or II with the same expected excess return. Considering that we use absolute downside deviation to measure the tracking error, even if we divide  $D_1$  and  $D_2$  by 2, the tracking error of our model is still smaller than that of the other two models. This shows that our model is effective in controlling the tracking error.

### 6.3 The Discussion on the Cardinal Constraint

The cardinal constraint is a common constraint in EIT model (see [4]). To show the effect of the cardinal constraint, we do more experiments. We increase the number of candidate stocks from the original 6 to 100. Table 6 shows the returns of the 100 stocks. The benchmark return and tracking error tolerance level remain unchanged. We add the cardinal constraint in our uncertain EIT model and do the experiments in two cases.

**Table 5** The comparison results between different uncertain EIT models

$Obj.$	$D$	$D_1$	$D_2$
0.0783	0.0500	0.1776	0.1946
0.0662	0.0510	0.1668	0.1847
0.0555	0.0520	0.1577	0.1763
0.0457	0.0530	0.1495	0.1686
0.0367	0.0540	0.1423	0.1615
0.0283	0.0550	0.1360	0.1554
0.0206	0.0560	0.1307	0.1499
0.0135	0.0570	0.1259	0.1449
0.0069	0.0580	0.1212	0.1403
0.0008	0.0590	0.1178	0.1360

$e_I = 0.08$  and  $s_I = 0.12$

**Case 1:** The proposed uncertain EIT model is solved with the cardinal constraint that there should be an investment in at most 30 stocks with at least a 2% allocation in each stock.

**Case 2:** The proposed uncertain EIT model is solved with the cardinal constraint that there should be an investment in at most 50 stocks with at least a 1% allocation in each stock.

The obtained results in two cases are presented in Table 7. Here,  $K$  represents the cardinal constraint. For example,  $K = 30$  means that the number of stocks in the tracking portfolio is 30 for the given problem. By comparing the results, we find that when the number of stocks in the tracking portfolio increases, the risk of the tracking portfolio decreases. This is in line with the statement “diversification reduces portfolio risk.”

## 7 Conclusions

In financial market, there exists the situation where probability theory is not applicable. In this situation, it is more appropriate to treat the stock return as uncertain variable and employ the uncertainty theory. Under uncertainty theory framework, this paper has studied the EIT problem. We have proposed an uncertain mean-absolute downside deviation EIT model and given the optimal tracking portfolio when stock returns take linear uncertainty distributions. By using the ADD as the portfolio risk measure, we have given the form of the tracking portfolio frontier in mean-ADD space and found that the frontier is a continuous curve composed of different line segments. Moreover, we have analyzed the effects of benchmark return distribution and tracking error tolerance level on the optimal tracking portfolio’s return and risk. The experimental results are consistent with our theoretical analysis.

There are many things to do in the future. Under uncertainty theory framework, we will formulate a bi-objective EIT model considering reality constraints and present an algorithm to solve the proposed model.

**Table 6** Uncertain returns of 100 stocks

Stock i	<i>e</i>	<i>s</i>	Stock i	<i>e</i>	<i>s</i>	Stock i	<i>e</i>	<i>s</i>
1	0.013	0.093	2	0.014	0.094	3	0.015	0.095
4	0.016	0.096	5	0.017	0.097	6	0.018	0.098
7	0.019	0.099	8	0.020	0.1	9	0.030	0.11
10	0.035	0.115	11	0.046	0.126	12	0.047	0.127
13	0.049	0.129	14	0.050	0.13	15	0.059	0.139
16	0.061	0.141	17	0.062	0.142	18	0.063	0.143
19	0.084	0.164	20	0.09	0.17	21	0.092	0.172
22	0.094	0.174	23	0.098	0.178	24	0.107	0.187
25	0.115	0.195	26	0.118	0.198	27	0.128	0.208
28	0.138	0.218	29	0.152	0.232	30	0.157	0.237
31	0.159	0.239	32	0.160	0.24	33	0.161	0.241
34	0.165	0.245	35	0.170	0.25	36	0.183	0.263
37	0.184	0.264	38	0.185	0.265	39	0.187	0.267
40	0.197	0.277	41	0.198	0.278	42	0.2	0.28
43	0.202	0.282	44	0.206	0.286	45	0.208	0.288
46	0.212	0.292	47	0.215	0.295	48	0.217	0.297
49	0.220	0.3	50	0.233	0.313	51	0.236	0.316
52	0.239	0.319	53	0.242	0.322	54	0.244	0.324
55	0.249	0.329	56	0.252	0.332	57	0.256	0.336
58	0.260	0.34	59	0.264	0.344	60	0.265	0.345
61	0.267	0.347	62	0.27	0.35	63	0.274	0.354
64	0.276	0.356	65	0.278	0.358	66	0.281	0.361
67	0.285	0.365	68	0.290	0.37	69	0.291	0.371
70	0.296	0.376	71	0.298	0.378	72	0.300	0.38
73	0.303	0.383	74	0.306	0.386	75	0.308	0.388
76	0.322	0.402	77	0.324	0.404	78	0.325	0.405
79	0.328	0.408	80	0.346	0.426	81	0.352	0.432
82	0.354	0.434	83	0.360	0.44	84	0.367	0.447
85	0.370	0.45	86	0.373	0.453	87	0.375	0.455
88	0.378	0.458	89	0.380	0.46	90	0.381	0.461
91	0.382	0.462	92	0.383	0.463	93	0.384	0.464
94	0.386	0.466	95	0.387	0.467	96	0.389	0.469
97	0.390	0.47	98	0.391	0.471	99	0.392	0.472
100	0.399	0.479						

**Table 7** Results for uncertain EIT model with the cardinal constraint

K	Return/risk	1	2	3	4	5	6	7	8	9	10
30	0.276/0.089	0.02	0	0	0	0	0	0	0	0.02	0
50	0.269/0.087	0.01	0	0	0.01	0.01	0	0.01	0	0	0.02
		11	12	13	14	15	16	17	18	19	20
		0	0	0.02	0	0	0	0.03	0	0	0
		0	0	0	0	0	0	0	0.02	0.02	0.02
		21	22	23	24	25	26	27	28	29	30
		0.03	0	0	0	0	0	0	0	0	0
		0	0	0	0.01	0	0	0	0.02	0.02	0
		31	32	33	34	35	36	37	38	39	40
		0.02	0	0	0.02	0	0	0.03	0	0	0.04
		0.02	0	0	0	0	0.02	0	0.03	0	0.01
		41	42	43	44	45	46	47	48	49	50
		0.02	0	0	0	0	0	0	0	0	0
		0	0.01	0.03	0	0.02	0	0	0	0.03	0.03
		51	52	53	54	55	56	57	58	59	60
		0.04	0	0	0	0	0	0	0	0.04	0.05
		0	0.01	0	0.01	0.02	0.01	0	0.01	0.02	0
		61	62	63	64	65	66	67	68	69	70
		0	0	0	0.02	0.03	0	0	0.03	0.05	0.03
		0	0.01	0.02	0.02	0	0.02	0	0.01	0.03	0
		71	72	73	74	75	76	77	78	79	80
		0	0.04	0	0.05	0.04	0	0	0	0	0
		0.01	0	0	0.02	0	0.03	0.01	0.02	0	0
		81	82	83	84	85	86	87	88	89	90
		0.02	0	0	0.04	0.02	0.03	0	0	0	0
		0.03	0.03	0	0	0.03	0.02	0.02	0.02	0.02	0.03
		91	92	93	94	95	96	97	98	99	100
		0.05	0	0	0	0.02	0	0.06	0.04	0	0.05
		0	0.03	0	0.03	0	0	0	0.03	0.03	0.03

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