Journal of Optimization Theory and Applications (2022) 193:877–909
https://doi.org/10.1007/s10957-021-01995-9

Coupled Variational Inequalities: Existence, Stability and Optimal Control

Jinjie Liu[1] · Xinmin Yang[1] · Shengda Zeng[2,3,4] · Yong Zhao[5,6]

Received: 6 May 2021 / Accepted: 29 December 2021 / Published online: 3 February 2022

Abstract
In this paper, we introduce and investigate a new kind of coupled systems, called coupled variational inequalities, which consist of two elliptic mixed variational inequalities on Banach spaces. Under general assumptions, by employing Kakutani-Ky Fan fixed point theorem combined with Minty technique, we prove that the set of solutions for the coupled variational inequality (CVI, for short) under consideration is nonempty and weak compact. Then, two uniqueness theorems are delivered via using the monotonicity arguments, and a stability result for the solutions of CVI is proposed, through the perturbations of duality mappings. Furthermore, an optimal control problem governed by CVI is introduced, and a solvability result for the optimal control problem is established. Finally, to illustrate the applicability of the theoretical results, we study a coupled elliptic mixed boundary value system with nonlocal effect and multivalued boundary conditions, and a feedback control problem involving a least energy condition with respect to the control variable, respectively.

Keywords Coupled variational inequality · Existence · Uniqueness · Optimal control · Mixed boundary value system · Feedback control problem

Mathematics Subject Classification 47J20 · 49J53 · 35J87 · 35J66 · 58E35 · 46Txx

1 Introduction and Mathematical Prerequisites

In numerous complicated natural phenomenon, physical constitutive laws, chemical processes, and economic models are often leaded to inequalities rather than

Dedicated to Professor Franco Giannessi on the occasion of his 85th birthday.

Communicated by Massimo Pappalardo.

✉ Shengda Zeng
zengshengda@163.com

Extended author information available on the last page of the article

the more commonly seen equations. In this context, variational inequalities, as a powerful mathematical tool, have been widely studied. Essentially speaking, variational inequalities emerge from applied models with an underlying convex structure and have been studied extensively since 1960s. Some representative references include on mathematical theories, numerical treatment and application analysis, see e.g., Kinderlehrer-Stampacchia [18], Migórski-Sofonea-Zeng [34], Glowinski-Lions-Trémoliéres [15], Facchinei-Pang [10], Liu-Motreanu-Zeng [26–28], Duvaut-Lions [9], Hlaváček-Haslinger-Nečas-Lovíšek [17], Giannessi [14], Han-Sofonea [16], Fukushima [11,12], Migórski-Khan-Zeng [35,36].

Recently, many scholars noticed that various comprehensive physical phenomenon and engineering applications could be, eventually, modeled by the complicated systems governed by variational inequalities, for example, Nash equilibrium problems of multiple players with shared constraints and dynamic decision processes, contact mechanics problems with adhesion (or wear) effect, and convection diffusion models in porous materials. Among the results we mention: Pang-Stewart [38] in 2008 systematically introduced and studied a class of dynamical systems on finite-dimensional spaces, which is formulated as a combination of ordinary differential equations and time-dependent variational inequalities. They represent powerful mathematical tools with applications to various problems involving both dynamics and constraints arising in mechanical impact processes, electrical circuits with ideal diodes, Coulomb friction for contacting bodies, economical dynamics, dynamic traffic networks. Cojocaru-Matei [6] introduced a Lagrange multiplier system which is composed of a variational inequality of elliptic type and a linear equation, and applied this system to study a boundary value problem involving p-Laplace operator and nonsmooth boundary conditions. By using a surjectivity result for multivalued maps and a fixed point argument for a history-dependent operator, Migórski [31] proved the unique solvability of a system of coupled nonlinear first order history-dependent evolution inclusions in the framework of evolution triples of spaces, and applied these abstract results to a dynamic frictional contact problem in mechanics. For more details on this topic, the reader is referred to Migórski-Zeng [32,33], Liu-Migórski-Zeng [23], Liu-Xu-Lin [22], Liou-Yang-Yao [21], Li-Yang [19], Chen-Wang [4,5], Zeng-Migórski-Liu [44,45], Wang-Huang [42] and the cited references therein. However, it should be pointed out that these results aforementioned cannot be used to study the coupled systems which are made up of two mixed variational inequalities of elliptic type, called coupled variational inequalities. But, coupled variational inequalities could be a useful mathematical tool for investigating numerous coupled mixed boundary value problems, feedback control problems and so forth. Therefore, to fill this gap, the main goal of the present paper is to introduce a new kind of coupled variational inequalities and to deliver the theoretical results concerning existence, uniqueness, stability, optimal control and applications to coupled variational inequalities under consideration.

Before any advancement, let us first introduce the problem that will play the central role in this paper. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two reflexive Banach spaces with its dual spaces $(X^*, \|\cdot\|_{X^*})$ and $(Y^*, \|\cdot\|_{Y^*})$, respectively. In what follows, we denote by $\langle\cdot,\cdot\rangle_X$ (resp., $\langle\cdot,\cdot\rangle_Y$) the duality pairing between X^* and X (resp., the duality pairing between Y^* and Y). We formulate the following coupled system which consists of two mixed variational inequalities on Banach spaces.

Problem 1 *Find* $(x, y) \in K \times L$ *such that*

$$\langle G(y, x), v - x\rangle_X + \varphi(v) - \varphi(x) \geq \langle f, v - x\rangle_X \text{ for all } v \in K, \tag{1.1}$$

and

$$\langle F(x, y), w - y\rangle_Y + \phi(w) - \phi(y) \geq \langle g, w - y\rangle_Y \text{ for all } w \in L. \tag{1.2}$$

To highlight the general form of our problem, we list the following particular cases of Problem 1

(i) if $\phi \equiv 0$ and $L = Y$, then Problem 1 becomes to the coupled system, which consists of a mixed variational inequality and a nonlinear equation: find $(x, y) \in K \times Y$ such that

$$\begin{aligned}&\langle G(y, x), v - x\rangle_X + \varphi(v) - \varphi(x) \geq \langle f, v - x\rangle_X \text{ for all } v \in K\\ &F(x, y) = g.\end{aligned}$$

(ii) if $\phi \equiv 0$, $\varphi \equiv 0$, $L = Y$ and $K = X$, then Problem 1 becomes to the coupled nonlinear equations: find $(x, y) \in K \times Y$ such that

$$\begin{aligned}G(y, x) &= f\\ F(x, y) &= g.\end{aligned}$$

(iii) when $\phi \equiv 0$, $L = Y$ and F is independent of x, then Problem 1 becomes to the parameter control system: find $x \in K$ and $y \in U$ such that

$$\langle G(y, x), v - x\rangle_X + \varphi(v) - \varphi(x) \geq \langle f, v - x\rangle_X \text{ for all } v \in K,$$

where $U := \{y \in Y \mid F(y) = g\}$.

(iv) if G is independent of y, $F \equiv 0$, $\phi \equiv 0$ and $g = 0$, then Problem 1 reduces the elliptic variational inequality of the first kind studied in [20,40]: find $x \in K$ such that

$$\langle G(x), v - x\rangle_X + \varphi(v) - \varphi(x) \geq \langle f, v - x\rangle_X \text{ for all } v \in K.$$

(v) when G is independent of y, $F \equiv 0$, $\phi \equiv 0$, $g = 0$, and $K = X$, then Problem 1 reduces the elliptic variational inequality of the second kind investigated in [3, 20,37]: find $x \in X$ such that

$$\langle G(x), v - x\rangle_X + \varphi(v) - \varphi(x) \geq \langle f, v - x\rangle_X \text{ for all } v \in X.$$

The main contribution of this paper is threefold. First, in Sect. 2, we apply Kakutani-Ky Fan fixed point theorem and Minty approach to show the nonemptiness and compactness of solution set to Problem 1, and then use the arguments of monotonicity to establish two uniqueness results for Problem 1. Second, in Sect. 3, we propose a stability result to Problem 1 and consider an optimal control driven by CVI. Finally, in Sect. 4, we provide novel applications of our abstract results to a coupled elliptic mixed boundary value system with nonlocal effect and multivalued boundary conditions, and a feedback control problem involving a least energy condition with respect to the control variable, respectively.

We end the section by recalling a preliminary material to be used in the next sections. More details can be found in [1,7,8,13,30,43].

Throughout the text, the symbols " $\xrightarrow{w}$ " and "$\to$" stand for the weak and the strong convergence, respectively. Let $(X, \|\cdot\|_X)$ be a Banach space with its dual X^* and denote by $\langle\cdot,\cdot\rangle_X$ the duality pairing between X^* and X. Recall that a function $f\colon X \to \overline{\mathbb{R}} := \mathbb{R}\cup\{+\infty\}$ is called proper, convex, and lower semicontinuous, if it fulfills the conditions

$$\begin{aligned}
&\mathrm{D}(f) := \{u \in X \mid f(u) < +\infty\} \neq \emptyset,\\
&f(\lambda u + (1-\lambda)v) \le \lambda f(u) + (1-\lambda) f(v) \quad \text{for all } \lambda \in [0,1] \text{ and } u, v \in X,\\
&f(u) \le \liminf_{n\to\infty} f(u_n) \quad \text{for all sequences } \{u_n\} \subset X \text{ with } u_n \to u,
\end{aligned}$$

respectively.

We recall the following important result for the proper convex and l.s.c. functions, see e.g., [2, Proposition 1.10].

Proposition 2 *Let $(X, \|\cdot\|_X)$ be a Banach space. Assume that $\varphi\colon X \to \overline{\mathbb{R}}$ is convex, l.s.c. and $\varphi \not\equiv +\infty$. Then, φ is bounded below by an affine continuous function, i.e., there exist $l \in X^*$ and $c_\varphi \in \mathbb{R}$ such that*

$$\varphi(v) \ge \langle l, v\rangle_X + c_\varphi \text{ for all } v \in X.$$

Remark 3 It is not difficult to see that if $\varphi\colon X \to \overline{\mathbb{R}}$ is convex, l.s.c. and $\varphi \not\equiv +\infty$, then we are able to find constants $\alpha_\varphi, \beta_\varphi \ge 0$ such that

$$\varphi(v) \ge -\alpha_\varphi \|v\|_X - \beta_\varphi \text{ for all } v \in X.$$

Let K be a nonempty subset of X, $\varphi\colon K \to \overline{\mathbb{R}}$ be a proper convex and l.s.c. function, and $A\colon K \to X^*$. We say that A is

(i) monotone, if it holds $\langle Au - Av, u - v\rangle_X \ge 0$ for all $u, v \in K$;
(ii) strictly monotone, if it holds $\langle Au - Av, u - v\rangle_X > 0$ for all $u, v \in K$ and $u \ne v$;
(iii) strongly monotone with constant $m_A > 0$, if it holds $\langle Au - Av, u - v\rangle_X \ge m_A\|u - v\|_X^2$ for all $u, v \in K$;
(iv) pseudomonotone, if for any $u, v \in K$ we have $\langle Au, v - u\rangle_X \ge 0$, then it entails that$\langle Av, v - u\rangle_X \ge 0$;
(v) stable pseudomonotone with respect to the set $W \subset X^*$, if A and $u \mapsto Au - w$ are pseudomonotone for all $w \in W$;
(vi) φ-pseudomonotone, if for any $u, v \in K$ we have $\langle Au, v-u\rangle_X + \varphi(v) - \varphi(u) \ge 0$, then it entails that $\langle Av, v - u\rangle_X + \varphi(v) - \varphi(u) \ge 0$;
(vii) stable φ-pseudomonotone with respect to the set $W \subset X^*$, if A and $u \mapsto Au - w$ are φ-pseudomonotone for each $w \in W$.

Remark 4 The following diagram reveals the essential implications of these monotonicity and generalized monotonicity.

$$
\begin{array}{ccccc}
 & & \varphi\text{-pseudomonotonicity} & \Leftarrow & \text{stable } \varphi\text{-pseudomonotonicity} \\
 & & & & \Uparrow \\
\text{strong monotonicity} & \Rightarrow & \text{strict monotonicity} & \Rightarrow & \text{monotonicity} \\
 & & & & \Downarrow \\
 & & \text{pseudomonotonicity} & \Leftarrow & \text{stable pseudomonotonicity}
\end{array}
$$

However, it should be mentioned that the inverse direction of each implication relationship in the diagram above does not hold in general (for more details, the reader is welcome to consult the examples, [24, Examples 1 and 2], [25, Example 1], [41, Example 3.1], and [29, Examples 3.4 and 3.5]).

Let Z and Y be topological spaces and $V \subset Z$ be a nonempty set. In what follows, we denote by 2^V the collection of its subsets. Given a set-valued mapping $F\colon Z \to 2^Y$, we use the symbol $Gr\,F$ to stand for the graph of F, i.e.,

$$Gr\,F := \{(x, y) \in Z \times Y \mid y \in F(x)\} \subset Z \times Y.$$

We say that the graph of F is sequentially closed (or F is sequentially closed) in $Z \times Y$, if for any sequence $\{(x_n, y_n)\} \subset Gr\,F$ is such that

$$(x_n, y_n) \to (x, y) \text{ as } n \to \infty$$

for some $(x, y) \in Z \times Y$, then we have $(x, y) \in Gr\,F$ (i.e., $y \in F(x)$).

Finally, we recall the Kakutani-Ky Fan theorem for a reflexive Banach space, see e.g., [39, Theorem 2.6.7].

Theorem 5 *Let Y be a reflexive Banach space and $D \subseteq Y$ be a nonempty, bounded, closed and convex set. Let $\Lambda\colon D \to 2^D$ be a set-valued map with nonempty, closed and convex values such that its graph is sequentially closed in $Y_w \times Y_w$ topology. Then, Λ has a fixed point.*

2 Existence and Uniqueness

In the section, we are devoted to the study of existence and uniqueness of solution to the abstract coupled variational inequalities, Problem 1. More precisely, under mild assumptions, an existence theorem for the solutions of CVI is established by employing Kakutani-Ky Fan fixed point theorem, Theorem 5, and Minty method. Moreover, we apply the monotonicity arguments to deliver two uniqueness results to Problem 1.

Let us introduce the set-valued mappings $S\colon L \to 2^K$ and $T\colon K \to 2^L$ defined by

$$S(y) := \{x \in K \mid x \text{ is a solution of problem (1.1) corresponding to } y\} \text{ for all } y \in L,$$

and

$$T(x) := \{y \in L \mid y \text{ is a solution of problem (1.2) corresponding to } x\} \text{ for all } x \in K,$$

respectively.

In order to examine the existence of solutions to Problem 1, we now impose the following assumptions.

$\underline{H(0)}$: $K \subset X$ and $L \subset Y$ are both nonempty, closed and convex.

$\underline{H(1)}$: $f \in X^*$ and $g \in Y^*$.

$\underline{H(\varphi)}$: $\varphi : K \to \overline{\mathbb{R}}$ is a proper, convex and lower semicontinuous function.

$\underline{H(G)}$: $G : Y \times X \to X^*$ is such that

(i) for each $y \in Y$, $x \mapsto G(y, x)$ is stable φ-pseudomonotone with respect to $\{f\}$ and satisfies

$$\limsup_{\lambda \to 0} \langle G(y, \lambda v + (1-\lambda)x), v - x\rangle_X \le \langle G(y, x), v - x\rangle_X$$

for all $y \in Y$ and $x, v \in X$;

(ii) it holds

$$\limsup_{n \to \infty} \langle G(y_n, v), v - x_n\rangle_X \le \langle G(y, v), v - x\rangle_X,$$

whenever $v \in X$, $(x, y) \in X \times Y$, $\{y_n\} \subset Y$ and $\{x_n\} \subset X$ are such that

$$y_n \xrightarrow{w} y \text{in } Y \text{and} x_n \xrightarrow{w} x \text{in} X \text{as } n \to \infty;$$

(iii) there exists a function $r : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ such that

$$\langle G(y, x), x\rangle_X \ge r(\|x\|_X, \|y\|_Y)\|x\|_X \text{ for all } x \in X \text{ and } y \in Y,$$

and

- for each nonempty and bounded set $D \subset \mathbb{R}_+$, we have $r(t, s) \to +\infty$ as $t \to +\infty$ for all $s \in D$,
- for any constants $c_1, c_2 \ge 0$, it holds $r(t, c_1 t + c_2) \to +\infty$ as $t \to +\infty$.

(iv) there exists a constant $c_G > 0$ such that

$$\|G(y, x)\|_{X^*} \le c_G(1 + \|x\|_X + \|y\|_Y)$$

for all $(x, y) \in X \times Y$.

$\underline{H(\phi)}$: $\phi : L \to \overline{\mathbb{R}}$ is a proper, convex and lower semicontinuous function.

$\underline{H(F)}$: $F : X \times Y \to Y^*$ is such that

(i) for each $x \in X$, $y \mapsto F(x, y)$ is stable ϕ-pseudomonotone with respect to $\{g\}$ and satisfies

$$\limsup_{\lambda \to 0} \langle F(x, \lambda w + (1-\lambda)y), w - y\rangle_Y \le \langle F(x, y), w - y\rangle_Y$$

for all $w, y \in Y$ and $x \in X$;

(ii) it holds

$$\limsup_{n \to \infty} \langle F(x_n, w), w - y_n\rangle_Y \le \langle F(x, w), w - y\rangle_Y,$$

whenever $w \in Y$, $(x, y) \in X \times Y$, $\{y_n\} \subset Y$ and $\{x_n\} \subset X$ are such that

$$y_n \xrightarrow{w} y \text{ in } Y \text{ and } x_n \xrightarrow{w} x \text{ in } X \text{ as } n \to \infty;$$

(iii) there exists a function $l \colon \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ such that

$$\langle F(x, y), y\rangle_Y \ge l(\|y\|_Y, \|x\|_X)\|y\|_Y \text{ for all } x \in X \text{ and } y \in Y,$$

and

- for each nonempty and bounded set $D \subset \mathbb{R}_+$, we have $l(t, s) \to +\infty$ as $t \to +\infty$ for all $s \in D$,
- for any constants $c_1, c_2 \ge 0$, it holds $l(t, c_1 t + c_2) \to +\infty$ as $t \to +\infty$.

(iv) there exists a constant $c_F > 0$ such that

$$\|F(x, y)\|_{Y^*} \le c_F(1 + \|x\|_X + \|y\|_Y)$$

for all $(x, y) \in X \times Y$.

Remark 6 Particularly, if function r given in $H(G)$(iii) (resp. l given in $H(F)$(iii)) is independent of its second variable, then condition $H(G)$(iii) (resp. $H(F)$(iii)) reduces to the following uniformly coercive condtion

$H(G)$(iii)': there exists a function $r \colon \mathbb{R}_+ \to \mathbb{R}$ with $r(s) \to +\infty$ as $s \to +\infty$ such that

$$\langle G(y, x), x\rangle_X \ge r(\|x\|_X)\|x\|_X \text{ for all } x \in X \text{ and } y \in Y$$

(resp. $H(F)$(iii)': there exists a function $l \colon \mathbb{R}_+ \to \mathbb{R}$ with $l(s) \to +\infty$ as $s \to +\infty$ such that

$$\langle F(x, y), y\rangle_Y \ge l(\|y\|_Y)\|y\|_Y$$

for all $x \in X$ and $y \in Y$).

The first main result of this paper concerning the existence of solutions to Problem 1 is provided as follows.

Theorem 7 *Assume that $H(G)$, $H(F)$, $H(0)$, $H(1)$, $H(\varphi)$ and $H(\phi)$ are satisfied. Then, the solution set of Problem 1 corresponding to $(f, g) \in X^* \times Y^*$, denoted by $\Gamma(f, g)$, is nonempty and weakly compact in $X \times Y$.*

To prove this theorem, we need the following lemmas.

Lemma 8 *Suppose that $H(0)$, $H(1)$, $H(G)$ and $H(\varphi)$ are fulfilled. Then, the statements hold*

(i) *for each $y \in Y$ fixed, $x \in K$ is a solution of problem (1.1), if and only if, x solves the following Minty inequality: find $x \in K$ such that*

$$\langle G(y, v), v - x\rangle_X + \varphi(v) - \varphi(x) \geq \langle f, v - x\rangle_X \text{ for all } v \in K; \tag{2.1}$$

(ii) *for each $y \in Y$ fixed, the solution set of problem (1.1), denoted by $S(y)$, is nonempty, bounded, closed and convex;*

(iii) *the graph of the set-valued mapping $S\colon L \to 2^K$ is sequentially closed in $Y_w \times X_w$, i.e., S is sequentially closed from Y endowed with the weak topology into the subsets of X with the weak topology;*

(iv) *for each $y \in Y$ fixed, if the mapping $x \mapsto G(y, x)$ is strictly monotone, then S is a single-valued mapping and weakly continuous.*

Proof The assertions (i) and (ii) are the direct consequences of [24, Theorem 3.3] and [25, Lemma 3.3 and Theorem 3.4].

Next, we show the conclusion (iii). Let $\{(y_n, x_n)\} \subset Gr\, S$ be such that

$$y_n \xrightarrow{w} y \text{ in } Y \text{ and } x_n \xrightarrow{w} x \text{ in } X \text{ as } n \to \infty \tag{2.2}$$

for some $(x, y) \in X \times Y$. Then, for each $n \in \mathbb{N}$, we have $x_n \in S(y_n)$, i.e.,

$$\langle G(y_n, x_n), v - x_n\rangle_X + \varphi(v) - \varphi(x_n) \geq \langle f, v - x_n\rangle_X$$

for all $v \in K$. Assertion (i) indicates

$$\langle G(y_n, v), v - x_n\rangle_X + \varphi(v) - \varphi(x_n) \geq \langle f, v - x_n\rangle_X \tag{2.3}$$

for all $v \in K$. Passing to the upper limit as $n \to \infty$ to (2.3), we use hypothesis $H(G)$(ii) and weak lower semicontinuity of φ (due to the convexity and lower semicontinuity of φ) to find

$$\begin{aligned}
&\langle G(y, v), v - x\rangle_X + \varphi(v) - \varphi(x) \\
&\quad \geq \limsup_{n\to\infty} \langle G(y_n, v), v - x_n\rangle_X + \varphi(v) - \liminf_{n\to\infty} \varphi(x_n) \\
&\quad \geq \limsup_{n\to\infty} \left[\langle G(y_n, v), v - x_n\rangle_X + \varphi(v) - \varphi(x_n)\right]
\end{aligned}$$

$$\begin{aligned}
&\geq \limsup_{n\to\infty} \langle f, v - x_n \rangle_X \\
&= \langle f, v - x \rangle_X
\end{aligned}$$

for all $v \in K$. Applying assertion (i) again, we conclude that $x \in S(y)$. Therefore, $(y, x) \in Gr S$, namely, the graph of the set-valued mapping $S: L \to 2^K$ is sequentially closed in $Y_w \times X_w$.

Moreover, assume that $x \mapsto G(y, x)$ is strictly monotone. Let $x_1, x_2 \in K$ be two solutions to problem (1.1). Then, it has

$$\langle G(y, x_i), v - x_i \rangle_X + \varphi(v) - \varphi(x_i) \geq \langle f, v - x_i \rangle_X$$

for all $v \in K$ and $i = 1, 2$. Inserting $v = x_2$ and $v = x_1$ into the inequalities above for $i = 1$ and $i = 2$, respectively, we sum up the resulting inequalities to get

$$\langle G(y, x_1) - G(y, x_2), x_1 - x_2 \rangle_X \leq 0.$$

Hence, the strict monotonicity of $x \mapsto G(y, x)$ guarantees that $x_1 = x_2$. So, S is a single-valued mapping. But, by virtue of assertion (iii), we can see that S is weakly continuous. □

Likewise, for problem (1.2), we have the following lemma.

Lemma 9 *Suppose that $H(0)$, $H(1)$, $H(F)$ and $H(\phi)$ are fulfilled. Then, the statements hold*

(i) *for each $x \in X$ fixed, $y \in L$ is a solution of problem (1.2), if and only if, y solves the following Minty inequality: find $y \in L$ such that*

$$\langle F(x, w), w - y \rangle_Y + \phi(w) - \phi(y) \geq \langle g, w - y \rangle_Y \text{ for all } w \in L; \tag{2.4}$$

(ii) *for each $x \in X$ fixed, the solution set of problem (1.2), denoted by $T(x)$, is nonempty, bounded, closed and convex;*
(iii) *the graph of the set-valued mapping $T: K \to 2^L$ is sequentially closed in $X_w \times Y_w$;*
(iv) *for each $x \in X$ fixed, if the mapping $y \mapsto F(x, y)$ is strictly monotone, then T is a single-valued mapping and weakly continuous.*

Furthermore, we provide a priori estimates for the solutions of Problem 1.

Lemma 10 *Assume that $H(0)$, $H(1)$, $H(G)$, $H(F)$, $H(\varphi)$ and $H(\phi)$ are fulfilled. If the solution set of Problem 1 is nonempty, thus, $\Gamma(f, g) \neq \emptyset$, then there exists a constant $M > 0$ such that*

$$\|x\|_X \leq M \text{ and } \|y\|_Y \leq M \tag{2.5}$$

for all $(x, y) \in \Gamma(f, g)$.

Proof Assume that $\Gamma(f, g) \neq \emptyset$. Let $(x, y) \in \Gamma(f, g)$ be arbitrary and $(x_0, y_0) \in (D(\varphi) \cap K) \times (D(\phi) \cap L)$. Inserting $v = x_0$ and $w = y_0$ into (1.1) and (1.2), respectively, we have

$$\langle G(y, x), x\rangle_X \leq \langle G(y, x), x_0\rangle_X + \varphi(x_0) - \varphi(x) + \langle f, x_0 - x\rangle_X \tag{2.6}$$

and

$$\langle F(x, y), y\rangle_Y \leq \langle F(x, y), y_0\rangle_Y + \phi(y_0) - \phi(y) + \langle g, y_0 - y\rangle_Y. \tag{2.7}$$

Recall that φ and ϕ are both proper, convex and l.s.c., it follows from Proposition 2 that there are constants $\alpha_\varphi, \alpha_\phi, \beta_\varphi, \beta_\phi \geq 0$ satisfying

$$\varphi(v) \geq -\alpha_\varphi \|v\|_X - \beta_\varphi \text{ and } \phi(w) \geq -\alpha_\phi \|w\|_Y - \beta_\phi \tag{2.8}$$

for all $(v, w) \in X \times Y$ (see Remark 3). Taking account of (2.6)–(2.8), we use hypotheses $H(G)$(iii)–(iv) and $H(F)$(iii)–(iv) to obtain

$$\begin{aligned} r(\|x\|_X, \|y\|_Y)\|x\|_X &\leq \langle G(y, x), x\rangle_X \\ &\leq \langle G(y, x), x_0\rangle_X + \varphi(x_0) - \varphi(x) + \langle f, x_0 - x\rangle_X \\ &\leq \|G(y, x)\|_{X^*}\|x_0\|_X + \varphi(x_0) + \alpha_\varphi\|x\|_X + \beta_\varphi + \|f\|_{X^*}(\|x_0\|_X + \|x\|_X) \\ &\leq c_G(1 + \|x\|_X + \|y\|_Y)\|x_0\|_X + \varphi(x_0) + \alpha_\varphi\|x\|_X + \beta_\varphi + \|f\|_{X^*}(\|x_0\|_X + \|x\|_X), \end{aligned}$$

and

$$\begin{aligned} &l(\|y\|_Y, \|x\|_X)\|y\|_Y \\ &\quad \leq c_F(1 + \|x\|_X + \|y\|_Y)\|y_0\|_Y + \phi(y_0) + \alpha_\phi\|y\|_Y + \beta_\phi + \|g\|_{Y^*}(\|y_0\|_Y + \|y\|_Y). \end{aligned}$$

Hence,

$$\begin{aligned} &r(\|x\|_X, \|y\|_Y) \\ &\quad \leq \frac{c_G(1 + \|x\|_X + \|y\|_Y)\|x_0\|_X}{\|x\|_X} + \frac{\varphi(x_0) + \|f\|_{X^*}\|x_0\|_X + \beta_\varphi}{\|x\|_X} + \alpha_\varphi + \|f\|_{X^*}, \end{aligned} \tag{2.9}$$

and

$$
\begin{aligned}
& l(\|y\|_Y, \|x\|_X) \\
& \quad \le \frac{c_F(1+\|x\|_X+\|y\|_Y)\|y_0\|_Y}{\|y\|_Y} + \frac{\phi(y_0)+\|g\|_{Y^*}\|y_0\|_Y+\beta_\phi}{\|y\|_Y} + \alpha_\phi + \|g\|_{Y^*}.
\end{aligned} \tag{2.10}
$$

Arguing by contradiction, suppose that $\Gamma(f, g)$ is unbounded. Then, taking a subsequence if necessary, we are able to find a sequence $\{(x_n, y_n)\} \subset K \times L$ such that it holds

$$
\|x_n\|_X \uparrow +\infty \text{ as } n \to \infty, \tag{2.11}
$$

or

$$
\|y_n\|_Y \uparrow +\infty \text{ as } n \to \infty. \tag{2.12}
$$

Let us distinguish the following cases:

① (2.11) holds and $\{y_n\}$ is bounded in Y;
② (2.12) holds and $\{x_n\}$ is bounded in X;
③ (2.11) and (2.12) hold.

Assume that ① is valid, then we take $x = x_n$ and $y = y_n$ to (2.9) for getting

$$
\begin{aligned}
& r(\|x_n\|_X, \|y_n\|_Y) \\
& \quad \le \frac{c_G(1+\|x_n\|_X+\|y_n\|_Y)\|x_0\|_X}{\|x_n\|_X} + \frac{\varphi(x_0)+\|f\|_{X^*}\|x_0\|_X+\beta_\varphi}{\|x_n\|_X} + \alpha_\varphi + \|f\|_{X^*}.
\end{aligned}
$$

Letting $n \to \infty$ for the inequality above and using (2.11) as well as $H(G)$(iii) turn out

$$
\begin{aligned}
+\infty &= \lim_{n\to\infty} r(\|x_n\|_X, \|y_n\|_Y) \\
&\le \lim_{n\to\infty} \Bigg[\frac{c_G(1+\|x_n\|_X+\|y_n\|_Y)\|x_0\|_X}{\|x_n\|_X} \\
&\qquad + \frac{\varphi(x_0)+\|f\|_{X^*}\|x_0\|_X+\beta_\varphi}{\|x_n\|_X} + \alpha_\varphi + \|f\|_{X^*} \Bigg] \\
&= c_G\|x_0\|_X + \alpha_\varphi + \|f\|_{X^*}.
\end{aligned}
$$

This generates a contradiction. Similarly, for the case ②, we could use (2.10) to get a contradiction as well. However, suppose ③ occurs, we, further, consider the following two situations:

(a) $\frac{\|y_n\|_Y}{\|x_n\|_X} \to +\infty$ as $n \to \infty$;
(b) there exist $n_0 \in \mathbb{N}$ and $\widehat{c}_0 > 0$ such that $\frac{\|y_n\|_Y}{\|x_n\|_X} \le \widehat{c}_0$ for all $n \ge n_0$.

If item (a) is true, then we put $x = x_n$ and $y = y_n$ into (2.10) to yield

$$l(\|y_n\|_Y, \|x_n\|_X) \leq \frac{c_F(1 + \|x_n\|_X + \|y_n\|_Y)\|y_0\|_Y}{\|y_n\|_Y} + \frac{\phi(y_0) + \|g\|_{Y^*}\|y_0\|_Y + \beta_\phi}{\|y_n\|_Y} + \alpha_\phi + \|g\|_{Y^*}.$$

Passing to the limit as $n \to \infty$ for the inequality above, it gives

$$\begin{aligned} +\infty &= \lim_{n\to\infty} l(\|y_n\|_Y, \|x_n\|_X) \\ &\leq \lim_{n\to\infty} \Bigg[\frac{c_F(1 + \|x_n\|_X + \|y_n\|_Y)\|y_0\|_Y}{\|y_n\|_Y} \\ &\quad + \frac{\phi(y_0) + \|g\|_{Y^*}\|y_0\|_Y + \beta_\phi}{\|y_n\|_Y} + \alpha_\phi + \|g\|_{Y^*} \Bigg] \\ &= c_F\|y_0\|_Y + \alpha_\phi + \|g\|_{Y^*}. \end{aligned}$$

Obviously, it is impossible, whereas in terms of the situation (b), it follows from (2.9) that

$$\begin{aligned} &+\infty \leftarrow r(\|x_n\|_X, \|y_n\|_Y) \quad (\text{as } n \to \infty) \\ &\leq \frac{c_G(1 + \|x_n\|_X + \|y_n\|_Y)\|x_0\|_X}{\|x_n\|_X} + \frac{\varphi(x_0) + \|f\|_{X^*}\|x_0\|_X + \beta_\varphi}{\|x_n\|_X} + \alpha_\varphi + \|f\|_{X^*} \\ &\leq c_G(2 + \widehat{c}_0)\|x_0\|_X + \varphi(x_0) + \|f\|_{X^*}\|x_0\|_X + \beta_\varphi + \alpha_\varphi + \|f\|_{X^*} \end{aligned}$$

for $n \geq n_1$, where $n_1 \geq n_0$ is such that $\|x_{n_1}\|_X > 1$. This also triggers a contradiction.

To summary, we conclude that $\Gamma(f, g)$ is bounded in $X \times Y$. Consequently, we are able to find a constant $M > 0$ such that (2.5) is valid. □

Consider the set-valued mapping $\Lambda\colon K \times L \to 2^{K\times L}$ given by

$$\Lambda(x, y) := (S(y), T(x)) \text{ for all } (x, y) \in K \times L. \tag{2.13}$$

Invoking Lemmas 8 and 9, we can see that Λ is well-defined. The following lemma points out that there exists a bounded, closed and convex set D in $K \times L$ such that Λ maps D into itself.

Lemma 11 *Assume that $H(0)$, $H(1)$, $H(G)$, $H(F)$, $H(\varphi)$ and $H(\phi)$ are fulfilled. Then, there exists a constant $\widehat{M} > 0$ satisfying $\Lambda(\overline{B(0, \widehat{M})}) \subset \overline{B(0, \widehat{M})}$, where $\overline{B(0, \widehat{M})} \subset X \times Y$ is defined by*

$$\overline{B(0, \widehat{M})} := \left\{(x, y) \in K \times L \mid \|x\|_X \leq \widehat{M} \text{ and } \|y\|_Y \leq \widehat{M}\right\}.$$

Proof We use the proof by contradiction. Suppose that for each $n \in \mathbb{N}$, it holds $\Gamma(\overline{B(0, n)}) \not\subset \overline{B(0, n)}$. Then, for every $n \in \mathbb{N}$, we are able to find $(x_n, y_n) \in \overline{B(0, n)}$

and $(z_n, w_n) \in \Gamma(x_n, y_n)$ (i.e., $z_n \in S(y_n)$ and $w_n \in T(x_n)$) such that

$$\|z_n\|_X > n \text{ or } \|w_n\|_Y > n. \tag{2.14}$$

Therefore, passing to a relabeled subsequence if necessary, we may assume that $\|z_n\|_X > n$ for each $n \in \mathbb{N}$ (since the proof for the case that $\|w_n\|_Y > n$ for each $n \in \mathbb{N}$ is similar). Using (2.9), it finds

$$\begin{aligned} & r(\|z_n\|_X, \|y_n\|_Y) \\ & \quad \le \frac{c_G(1 + \|z_n\|_X + \|y_n\|_Y)\|x_0\|_X}{\|z_n\|_X} + \frac{\varphi(x_0) + \|f\|_{X^*}\|x_0\|_X + \beta_\varphi}{\|z_n\|_X} + \alpha_\varphi + \|f\|_{X^*}. \end{aligned}$$

Note that $\|y_n\|_Y \le n < \|z_n\|_X$, so, letting $n \to \infty$ to the inequality above, we have

$$\begin{aligned} +\infty &= \lim_{n\to\infty} r(\|z_n\|_X, \|y_n\|_Y) \\ &\le \lim_{n\to\infty} \left[\frac{c_G(1+\|z_n\|_X+\|y_n\|_Y)\|x_0\|_X}{\|z_n\|_X} + \frac{\varphi(x_0)+\|f\|_{X^*}\|x_0\|_X+\beta_\varphi}{\|z_n\|_X} + \alpha_\varphi + \|f\|_{X^*} \right] \\ &\le 2c_G\|x_0\|_X + \alpha_\varphi + \|f\|_{X^*}. \end{aligned}$$

This results in a contradiction. Consequently, there exists a constant $\widehat{M} > 0$ satisfying $\Lambda(\overline{B(0, \widehat{M})}) \subset \overline{B(0, \widehat{M})}$. $\square$

Proof of Theorem 7. Observe that if (x^*, y^*) is a fixed point of Λ (that is, $(x^*, y^*) \in \Lambda(x^*, y^*)$), then we have $x^* \in S(y^*)$ and $y^* \in T(x^*)$. By the definitions of S and T, it gives

$$\langle G(y^*, x^*), v - x^*\rangle_X + \varphi(v) - \varphi(x^*) \ge \langle f, v - x^*\rangle_X \text{ for all } v \in K,$$

and

$$\langle F(x^*, y^*), w - y^*\rangle_Y + \phi(w) - \phi(y^*) \ge \langle g, w - y^*\rangle_Y \text{ for all } w \in L.$$

Then, it is obvious that (x^*, y^*) is also a solution to Problem 1. Based on this important fact, we are going to apply Kakutani-Ky Fan fixed point theorem, Theorem 5, for examining the existence of a fixed point of Λ.

Indeed, it follows from Lemmas 8, 9 and 11 that $\Lambda\colon \overline{B(0, \widehat{M})} \to 2^{\overline{B(0,\widehat{M})}}$ has nonempty, closed and convex values and the graph of Λ is sequentially closed in $(X \times Y)_w \times (X \times Y)_w$. So, all conditions of Theorem 5 are verified. Using this theorem, we conclude that there exists $(x^*, y^*) \in K \times L$ such that $(x^*, y^*) \in \Lambda(x^*, y^*)$. Therefore, (x^*, y^*) is a solution to Problem 1, that is, $\Gamma(f, g) \ne \emptyset$.

From Lemma 10, we can see that $\Gamma(f, g)$ is bounded in $X \times Y$. Next, we shall show that $\Gamma(f, g)$ is weakly closed. Let $\{(x_n, y_n)\} \subset \Gamma(f, g)$ be such that

$$(x_n, y_n) \xrightarrow{w} (x, y) \text{ in } X \times Y \text{ as } n \to \infty \tag{2.15}$$

for some $(x, y) \in K \times L$. It is not difficult to see that for each $n \in \mathbb{N}$, it holds $(x_n, y_n) \in \Lambda(x_n, y_n)$. Keeping in mind that Λ is sequentially closed from $(X \times Y)_w$ to $(X \times Y)_w$ (see Lemmas 8 and 9), we, therefore, imply that $(x, y) \in \Lambda(x, y)$. This means that $(x, y) \in \Gamma(f, g)$. Consequently, from the boundedness of $\Gamma(f, g)$, we conclude that $\Gamma(f, g)$ is weakly compact. □

Theorem 7 has revealed the nonemptiness and weak compactness of the solution set of Problem 1. Naturally, a problem arises: whether can we prove the uniqueness to Problem 1 under necessary assumptions? The following theorems give a positive answer for the issue.

Theorem 12 *Assume that $H(G)$, $H(F)$, $H(0)$, $H(1)$, $H(\varphi)$ and $H(\phi)$ are satisfied. If, in addition, the following inequality holds,*

$$\langle G(y_1, x_1) - G(y_2, x_2), x_1 - x_2\rangle_X + \langle F(x_1, y_1) - F(x_2, y_2), y_1 - y_2\rangle_Y > 0 \tag{2.16}$$

for all $(x_1, y_1), (x_2, y_2) \in X \times Y$ with $(x_1, y_1) \neq (x_2, y_2)$, then Problem 1 has a unique solution.

Proof Theorem 7 ensures that $\Gamma(f, g) \neq \emptyset$. We now show the uniqueness of Problem 1. Let $(x_1, y_1), (x_2, y_2) \in \Gamma(f, g)$. Then, we have

$$\langle G(y_i, x_i), v - x_i\rangle_X + \varphi(v) - \varphi(x_i) \geq \langle f, v - x_i\rangle_X \text{ for all } v \in K,$$

and

$$\langle F(x_i, y_i), w - y_i\rangle_Y + \phi(w) - \phi(y_i) \geq \langle g, w - y_i\rangle_Y \text{ for all } w \in L.$$

A simple calculation gives

$$\langle G(y_1, x_1) - G(y_2, x_2), x_1 - x_2\rangle_X + \langle F(x_1, y_1) - F(x_2, y_2), y_1 - y_2\rangle_Y \leq 0.$$

This combined with the condition (2.16) implies that $x_1 = x_2$ and $y_1 = y_2$. Therefore, Problem 1 has a unique solution. □

The following theorem also provides a uniqueness result for Problem 1 by using an alternative condition to (2.16).

Theorem 13 *Assume that $H(G)$, $H(F)$, $H(0)$, $H(1)$, $H(\varphi)$ and $H(\phi)$ are satisfied. If, in addition, the following conditions hold*

- *for each $y \in Y$, the function $x \mapsto G(y, x)$ is strongly monotone with constant $m_G > 0$, and for each $x \in X$ the function $y \mapsto G(y, x)$ is Lipschitz continuous with constant $L_G > 0$,*
- *for each $x \in X$, the function $y \mapsto F(x, y)$ is strongly monotone with constant $m_F > 0$, and for each $y \in Y$ the function $x \mapsto F(x, y)$ is Lipschitz continuous with constant $L_F > 0$,*

- $\dfrac{L_G L_F}{m_G m_F} < 1$,

then Problem 1 has a unique solution.

Proof Let (x_1, y_1) and (x_2, y_2) be two solutions to Problem 1. Then, it has

$$\langle G(y_1, x_1) - G(y_2, x_2), x_1 - x_2\rangle_X \leq 0.$$

Hence, we have

$$\begin{aligned} &m_G\|x_1 - x_2\|_X^2 \leq \langle G(y_1, x_1) - G(y_1, x_2), x_1 - x_2\rangle_X \\ &\leq \langle G(y_2, x_2) - G(y_1, x_2), x_1 - x_2\rangle_X \leq L_G\|y_1 - y_2\|_Y\|x_1 - x_2\|_X. \end{aligned}$$

Analogously, it gets

$$\begin{aligned} &m_F\|y_1 - y_2\|_Y^2 \leq \langle F(x_1, y_1) - F(x_1, y_2), y_1 - y_2\rangle_Y \\ &\leq \langle F(x_2, y_2) - F(x_1, y_2), y_1 - y_2\rangle_Y \leq L_F\|y_1 - y_2\|_Y\|x_1 - x_2\|_X. \end{aligned}$$

The last two inequalities imply

$$\|x_1 - x_2\|_X \leq \frac{L_G L_F}{m_G m_F}\|x_1 - x_2\|_X.$$

But, the inequality $\frac{L_G L_F}{m_G m_F} < 1$ derives that $x_1 = x_2$ and $y_1 = y_2$. Consequently, Problem 1 has a unique solution. □

3 Stability and Optimal Control for Coupled Variational Inequalities

In the present section, we move our attention to explore the stability and optimal control for coupled variational inequalities. More precisely, we, first, introduce a family regularized problems corresponding to Problem 1 which are perturbated by duality mappings. Then, a stability result, which shows that any sequence of solutions to regularized problems has at least a subsequence to converge to some solution of the original problem, Problem 1, is obtained. Furthermore, we consider an optimal control problem driven by CVI, and prove the solvability of the optimal control problem.

Recall that X and Y are two reflexive Banach spaces, so, they can be renormed such that X and Y become strictly convex. So, without loss of generality, we may assume that X and Y are strictly convex. Let $J_X : X \to X^*$ and $J_Y : Y \to Y^*$ be the duality mappings of the spaces X and Y, respectively, namely:

$$\begin{aligned} J_X(x) &:= \{x^* \in X^* \mid \langle x^*, x\rangle_X = \|x\|_X^2 = \|x^*\|_{X^*}^2\}, \\ J_Y(y) &:= \{y^* \in Y^* \mid \langle y^*, y\rangle_Y = \|y\|_Y^2 = \|y^*\|_{Y^*}^2\}. \end{aligned}$$

Let real sequences $\{\varepsilon_n\}$ and $\{\delta_n\}$ be such that

$$\varepsilon_n > 0,\ \delta_n > 0,\ \varepsilon_n \to 0 \text{ and } \delta_n \to 0. \tag{3.1}$$

For each $n \in \mathbb{N}$, consider the following perturbated problem corresponding to Problem 1.

Problem 14 *Find $(x_n, y_n) \in K \times L$ such that*

$$\langle G(y_n, x_n) + \varepsilon_n J_X(x_n), v - x_n\rangle_X + \varphi(v) - \varphi(x_n) \geq \langle f, v - x_n\rangle_X \text{ for all } v \in K, \tag{3.2}$$

and

$$\langle F(x_n, y_n) + \delta_n J_Y(y_n), w - y_n\rangle_Y + \phi(w) - \phi(y_n) \geq \langle g, w - y_n\rangle_Y \text{ for all } w \in L. \tag{3.3}$$

We make the following assumptions.
$\underline{H(2)}$: $x \mapsto G(y, x)$ and $y \mapsto F(x, y)$ are monotone, and satisfy

$$\limsup_{\lambda\to 0}\langle G(y, \lambda v + (1-\lambda)x), v - x\rangle_X \leq \langle G(y, x), v - x\rangle_X,$$
$$\limsup_{\lambda\to 0}\langle F(x, \lambda w + (1-\lambda)y), w - y\rangle_Y \leq \langle F(x, y), w - y\rangle_Y$$

for all $w, y \in Y$ and $v, x \in X$.
$\underline{H(3)}$: $x \mapsto G(y, x)$ and $y \mapsto F(x, y)$ are strongly monotone with constants $m_G > 0$ and $m_F > 0$, respectively, and satisfy

$$\limsup_{\lambda\to 0}\langle G(y, \lambda v + (1-\lambda)x), v - x\rangle_X \leq \langle G(y, x), v - x\rangle_X,$$
$$\limsup_{\lambda\to 0}\langle F(x, \lambda w + (1-\lambda)y), w - y\rangle_Y \leq \langle F(x, y), w - y\rangle_Y$$

for all $w, y \in Y$ and $v, x \in X$.

The following theorem delivers the existence and convergence of solutions to Problem 14.

Theorem 15 *Assume that $H(G)$(ii)–(iv), $H(F)$(ii)–(iv), $H(0)$, $H(1)$, $H(\varphi)$ and $H(\phi)$ hold. Then, we have*

(i) *if, in addition, $H(2)$ holds, then for each $n \in \mathbb{N}$, Problem 14 has at least a solution $(x_n, y_n) \in K \times L$;*
(ii) *if, in addition, $H(2)$ holds, then for any sequence of solutions $\{(x_n, y_n)\}$ of Problem 14, there exists a subsequence of $\{(x_n, y_n)\}$, still denoted by the same way, such that*

$$(x_n, y_n) \xrightarrow{w} (x, y) \text{ in } X \times Y \text{ as } n \to \infty, \tag{3.4}$$

where $(x, y) \in K \times L$ *is a solution of Problem 1;*

(iii) *if, in addition,* $H(3)$ *holds, then for any sequence of solutions* $\{(x_n, y_n)\}$ *of Problem 14, there exists a subsequence of* $\{(x_n, y_n)\}$*, still denoted by the same way, such that*

$$(x_n, y_n) \to (x, y) \text{ in } X \times Y \text{ as } n \to \infty, \tag{3.5}$$

where $(x, y) \in K \times L$ *is a solution of Problem 1.*

Proof (i) Set $G_n(y, x) = G(y, x) + \varepsilon_n J_X(x)$ and $F_n(x, y) = F(x, y) + \delta_n J_Y(y)$ for all $(x, y) \in X \times Y$. We shall verify that G_n and F_n satisfy hypotheses $H(G)$ and $H(F)$, respectively. Note that J_X is demicontinuous and

$$0 \le (\|x\|_X - \|v\|_X)^2 \le \langle J_X(x) - J_X(v), x - v\rangle_X \text{ for all } x, v \in X,$$

we use hypotheses $H(2)$ to find that for each $y \in Y$, $x \mapsto G_n(y, x)$ satisfies $H(G)$(i). By using the facts, $\|J_X(x)\|_X = \|x\|_X$ and $\langle J_X(x), x\rangle_X = \|x\|_X^2$ for all $x \in X$, it is not difficult to prove that G_n enjoys the conditions $H(G)$(ii)–(iv). Analogously, F_n satisfies hypotheses $H(F)$. Therefore, we use Theorem 7 to conclude that Problem 14 admits a solution.

(ii) Let $\{(x_n, y_n)\}$ be an arbitrary sequence of solutions of Problem 14. Then, a careful computation gives

$$\begin{aligned} r(\|x_n\|_X, \|y_n\|_Y) &\le r(\|x_n\|_X, \|y_n\|_Y) + \frac{\varepsilon_n\langle J_X(x_n), x_n\rangle_X}{\|x_n\|_X} \\ &\le \frac{c_G(1 + \|x_n\|_X + \|y_n\|_Y)\|x_0\|_X}{\|x_n\|_X} \\ &\quad + \frac{(\varepsilon_n\|J_X(x_n)\|_{X^*} + \|f\|_{X^*})\|x_0\|_X + \varphi(x_0) + \beta_\varphi}{\|x_n\|_X} \\ &\quad + \alpha_\varphi + \|f\|_{X^*} \\ &= \frac{c_G(1 + \|x_n\|_X + \|y_n\|_Y)\|x_0\|_X}{\|x_n\|_X} \\ &\quad + \frac{(\varepsilon_n\|x_n\|_X + \|f\|_{X^*})\|x_0\|_X + \varphi(x_0) + \beta_\varphi}{\|x_n\|_X} \\ &\quad + \alpha_\varphi + \|f\|_{X^*}, \end{aligned}$$

and

$$\begin{aligned} &l(\|y_n\|_Y, \|x_n\|_X) \\ &\quad \le \frac{c_F(1 + \|x_n\|_X + \|y_n\|_Y)\|y_0\|_Y}{\|y_n\|_Y} + \frac{(\delta_n\|y_n\|_Y + \|g\|_{Y^*})\|y_0\|_Y + \phi(y_0) + \beta_\phi}{\|y_n\|_Y} \\ &\quad\; + \alpha_\phi + \|g\|_{Y^*}. \end{aligned}$$

It could be carried out by using the same arguments as in the proof of Lemma 10 that $\{(x_n, y_n)\}$ is bounded in $X \times Y$.

Taking to a relabeled subsequence if necessary, we may assume that

$$(x_n, y_n) \xrightarrow{w} (x, y) \text{ in } X \times Y \text{ as } n \to \infty \tag{3.6}$$

for some $(x, y) \in K \times L$. Applying the monotonicity of $x \mapsto G(y, x)$ and $y \mapsto F(x, y)$, we have

$$\langle G(y_n, v) + \varepsilon_n J_X(x_n), v - x_n\rangle_X + \varphi(v) - \varphi(x_n) \geq \langle f, v - x_n\rangle_X \text{ for all } v \in K,$$

and

$$\langle F(x_n, w) + \delta_n J_Y(y_n), w - y_n\rangle_Y + \phi(w) - \phi(y_n) \geq \langle g, w - y_n\rangle_Y \text{ for all } w \in L.$$

Passing to the upper limit as $n \to \infty$ and using hypotheses $H(G)$(ii) and $H(F)$(ii) imply

$$\langle G(y, v), v - x\rangle_X + \varphi(v) - \varphi(x) \geq \langle f, v - x\rangle_X \text{ for all } v \in K,$$

and

$$\langle F(x, w), w - y\rangle_Y + \phi(w) - \phi(y) \geq \langle g, w - y\rangle_Y \text{ for all } w \in L,$$

where we have used the boundedness of $\{(x_n, y_n)\}$ in $X \times Y$. Employing Minty technique, we conclude that $(x, y) \in K \times L$ is a solution of Problem 1, i.e., $(x, y) \in \Gamma(f, g)$.

(iii) It follows from assertion (ii) that for any sequence of solutions $\{(x_n, y_n)\}$ of Problem 14, there exists a subsequence of $\{(x_n, y_n)\}$, still denoted by the same way, such that (3.4) is valid. We assert that $\{(x_n, y_n)\}$ converges strongly to (x, y). It is not difficult to obtain that

$$\begin{aligned} m_G\|x_n - x\|_X^2 &\leq \langle G(y_n, x_n) - G(y_n, x), x_n - x\rangle_X \leq \langle G(y, x) - G(y_n, x), x_n - x\rangle_X \\ &\quad + \varepsilon_n\langle J_X(x_n), x - x_n\rangle_X. \end{aligned}$$

Passing to the upper limit as $n \to \infty$ to the inequality above and using hypothesis $H(G)$(ii), we get

$$\begin{aligned} 0 &\leq \liminf_{n\to\infty} m_G\|x_n - x\|_X^2 \leq \limsup_{n\to\infty} m_G\|x_n - x\|_X^2 \\ &\leq \limsup_{n\to\infty}\langle G(y, x) - G(y_n, x), x_n - x\rangle_X + \limsup_{n\to\infty} \varepsilon_n\|x_n\|_X\|x - x_n\|_X \\ &\leq 0. \end{aligned}$$

This means that $x_n \to x$ in X as $n \to \infty$. Analogically, it has $y_n \to y$ in Y as $n \to \infty$. □

Let Z_1, Z_2 be two Banach spaces such that the embeddings from X into Z_1 and from Y in Z_2 are both continuous. Given two target profiles $x_0 \in Z_1$ and $y_0 \in Z_2$, let U and V be subspaces of X^* and Y^*, respectively, such that the embeddings from U to X^* and V to Y^* are compact. Next, we focus our attention on the investigation of the following optimal control problem:

Problem 16 *Find $(f^*, g^*) \in U \times V$ such that*

$$I(f^*, g^*) = \inf_{(f,g)\in U\times V} I(f, g), \tag{3.7}$$

where the cost function $I : U \times V \to \mathbb{R}$ is defined by

$$I(f, g) = \inf_{(x,y)\in\Gamma(f,g)} \left(\frac{\rho}{2}\|x - x_0\|^2_{Z_1} + \frac{\theta}{2}\|y - y_0\|^2_{Z_2}\right) + h(f, g). \tag{3.8}$$

Here, $\Gamma(f, g)$ is the solution set of Problem 1 associated with $(f, g) \in X^* \times Y^*$, and $\rho > 0$, $\theta > 0$ are two regularized parameters.

For the function h, we assume that it reads the following conditions.
$\underline{H(h)}$: $h : U \times V \to \mathbb{R}$ is such that

(i) h is bounded from below;
(ii) h is coercive on $U \times V$, namely it holds

$$\lim_{(f,g)\in U\times V, \|f\|_U+\|g\|_V\to\infty} h(f, g) \to +\infty;$$

(iii) h is weakly lower semicontinuous on $U \times V$, i.e., $\liminf_{n\to\infty} h(f_n, g_n) \geq h(f, g)$, whenever $\{(f_n, g_n)\} \subset U \times V$ and $(f, g) \in U \times V$ are such that $(f_n, g_n) \xrightarrow{w} (f, g)$ in $U \times V$ as $n \to \infty$.

We examine the following existence result for Problem 16.

Theorem 17 *Assume that $H(G)$(ii)–(iv), $H(F)$(ii)–(iv), $H(0)$, $H(1)$, $H(\varphi)$ and $H(\phi)$ hold. If, in addition, $H(h)$ and $H(2)$ are fulfilled, then Problem 16 admits an optimal control pair.*

Proof For every $(f, g) \in U \times V$ fixed, the closedness of $\Gamma(f, g)$ (see Theorem 7) guarantees that there exists $(\hat{x}, \hat{y}) \in \Gamma(f, g)$ such that

$$\frac{\rho}{2}\|\hat{x} - x_0\|^2_{Z_1} + \frac{\theta}{2}\|\hat{y} - y_0\|^2_{Z_2} = \inf_{(x,y)\in\Gamma(f,g)} \left(\frac{\rho}{2}\|x - x_0\|^2_{Z_1} + \frac{\theta}{2}\|y - y_0\|^2_{Z_2}\right), \tag{3.9}$$

i.e., $\inf_{(x,y)\in\Gamma(f,g)} \left(\frac{\rho}{2}\|x - x_0\|^2_{Z_1} + \frac{\theta}{2}\|y - y_0\|^2_{Z_2}\right)$ is attainable.

It follows from the definition of I and hypothesis $H(h)$(i) that there exists a minimizing sequence $\{(f_n, g_n)\} \subset U \times V$ such that

$$\lim_{n\to\infty} I(f_n, g_n) = \inf_{(f,g)\in U\times V} I(f, g). \tag{3.10}$$

We assert that the sequence $\{(f_n, g_n)\}$ is bounded in $U \times V$. Arguing by contradiction, we suppose that

$$\|f_n\|_U + \|g_n\|_V \to +\infty \text{ as } n \to \infty.$$

The latter together with hypothesis $H(h)$(ii) deduces

$$\inf_{(f,g)\in U\times V} I(f, g) = \lim_{n\to\infty} I(f_n, g_n) \geq \lim_{n\to\infty} h(f_n, g_n) = +\infty.$$

This leads to a contradiction, so, $\{(f_n, g_n)\}$ is bounded in $U \times V$. Passing to a relabeled subsequence if necessary, we may assume that

$$(f_n, g_n) \xrightarrow{w} (f^*, g^*) \text{ in } U \times V \text{as } n \to \infty \tag{3.11}$$

for some $(f^*, g^*) \in U \times V$.

Let sequence $\{(x_n, y_n)\} \subset K \times L$ be such that (3.9) holds by taking $\hat{x} = x_n$, $\hat{y} = y_n$, and $(f, g) = (f_n, g_n)$. Next, we are going to show that $\{(x_n, y_n)\} \subset K \times L$ is uniformly bounded in $X \times Y$. A direct computation finds

$$\begin{aligned} &r(\|x_n\|_X, \|y_n\|_Y)\\ &\quad\leq \frac{c_G(1+\|x_n\|_X+\|y_n\|_Y)\|x_0\|_X}{\|x_n\|_X} + \frac{\varphi(x_0)+\|f_n\|_{X^*}\|x_0\|_X+\beta_\varphi}{\|x_n\|_X} + \alpha_\varphi + \|f_n\|_{X^*}, \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} &l(\|y_n\|_Y, \|x_n\|_X)\\ &\quad\leq \frac{c_F(1+\|x_n\|_X+\|y_n\|_Y)\|y_0\|_Y}{\|y_n\|_Y} + \frac{\phi(y_0)+\|g_n\|_{Y^*}\|y_0\|_Y+\beta_\phi}{\|y_n\|_Y} + \alpha_\phi + \|g_n\|_{Y^*}. \end{aligned} \tag{3.13}$$

Since the embeddings from U to X^* and from V to Y^* are both continuous, so, we can apply the same arguments as in the proof of Lemma 10 to obtain that $\{(x_n, y_n)\} \subset K \times L$ is uniformly bounded in $X \times Y$. Without loss of generality, we may suppose that

$$(x_n, y_n) \xrightarrow{w} (x^*, y^*) \text{ in } X \times Y \text{ and } Z_1 \times Z_2 \text{ as } n \to \infty \tag{3.14}$$

for some $(x^*, y^*) \in K \times L$. Employing Minty approach derives

$$\langle G(y_n, v), v - x_n\rangle_X + \varphi(v) - \varphi(x_n) \geq \langle f_n, v - x_n\rangle_X \text{ for all } v \in K, \tag{3.15}$$

and

$$\langle F(x_n, w), w - y_n\rangle_Y + \phi(w) - \phi(y_n) \geq \langle g_n, w - y_n\rangle_Y \text{ for all } w \in L. \tag{3.16}$$

The compactness of the embedding from (U, V) into (X^*, Y^*) and (3.11) indicate that $(f_n, g_n) \to (f^*, g^*)$ in $X^* \times Y^*$ as $n \to \infty$. Passing to the upper limit as $n \to \infty$ for inequalities (3.15)–(3.16), we have

$$\langle G(y^*, v), v - x^*\rangle_X + \varphi(v) - \varphi(x^*) \geq \langle f^*, v - x^*\rangle_X \text{ for all } v \in K,$$

and

$$\langle F(x^*, w), w - y^*\rangle_Y + \phi(w) - \phi(y^*) \geq \langle g^*, w - y^*\rangle_Y \text{ for all } w \in L,$$

where we have applied the conditions $H(F)$(ii) and $H(G)$(ii). Using Minty trick again, it leads to $(x^*, y^*) \in \Gamma(f^*, g^*)$.

But, the weak lower semicontinuity of $\|\cdot\|_{Z_1}$ and $\|\cdot\|_{Z_2}$ implies

$$\frac{\rho}{2}\|x^* - x_0\|^2_{Z_1} + \frac{\theta}{2}\|y^* - y_0\|^2_{Z_2} \leq \liminf_{n\to\infty}\left[\frac{\rho}{2}\|x_n - x_0\|^2_{Z_1} + \frac{\theta}{2}\|y_n - y_0\|^2_{Z_2}\right]. \tag{3.17}$$

Recall that h is weakly lower semicontinuous on $U \times V$, it yields

$$h(f^*, g^*) \leq \liminf_{n\to\infty} h(f_n, g_n). \tag{3.18}$$

Taking account of (3.17) and (3.18), we have

$$\begin{aligned}
&\liminf_{n\to\infty} I(f_n, g_n)\\
&\quad\geq \liminf_{n\to\infty} \inf_{(x,y)\in\Gamma(f_n,g_n)}\left(\frac{\rho}{2}\|x - x_0\|^2_{Z_1} + \frac{\theta}{2}\|y - y_0\|^2_{Z_2}\right) + \liminf_{n\to\infty} h(f_n, g_n)\\
&\quad= \liminf_{n\to\infty}\left[\frac{\rho}{2}\|x_n - x_0\|^2_X + \frac{\theta}{2}\|y_n - y_0\|^2_Y\right] + \liminf_{n\to\infty} h(f_n, g_n)\\
&\quad\geq \frac{\rho}{2}\|x^* - x_0\|^2_{Z_1} + \frac{\theta}{2}\|y^* - y_0\|^2_{Z_2} + h(f^*, g^*)\\
&\qquad\text{(due to } (x^*, y^*) \in \Gamma(f^*, g^*))\\
&\quad\geq \inf_{(x,y)\in\Gamma(f^*,g^*)}\left(\frac{\rho}{2}\|x - x_0\|^2_{Z_1} + \frac{\theta}{2}\|y - y_0\|^2_{Z_2}\right) + h(f^*, g^*)\\
&\quad= I(f^*, g^*).
\end{aligned}$$

This combined with (3.10) concludes that

$$I(f^*, g^*) \leq \inf_{(f,g)\in U\times V} I(f, g),$$

namely (f^*, g^*) is an optimal control of Problem 16. □

4 Applications

The goal of the section is to illustrate the applicability of the theoretical results established in Sections 3 and 4 to the study of two elliptic partial differential systems. The first application is a coupled elliptic mixed boundary value system with nonlocal effect and a multivalued boundary condition which is described by subgradient for a convex superpotential. But, the second application is a feedback control problem involving a least energy condition with respect to the control variable.

4.1 A Coupled Mixed Boundary Value System

Given a bounded domain Ω in $\mathbb{R}^N$ $(N \geq 2)$ such that its boundary $\Gamma = \partial\Omega$ is locally Lipschitz and

$$\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 = \Gamma_a \cup \Gamma_b \cup \Gamma_c$$

with $\Gamma_i \cap \Gamma_j = \emptyset$ for $i, j = 1, 2, 3, i \neq j$, and $\Gamma_p \cap \Gamma_q = \emptyset$ for $p, q = a, b, c, p \neq q$, and $\text{meas}(\Gamma_1) > 0$ and $\text{meas}(\Gamma_a) > 0$. In what follows, we denote by ν the outward unit normal to the boundary Γ. The classical form of the coupled mixed boundary value system is given as follows.

Problem 18 *Find functions $x : \Omega \to \mathbb{R}$ and $y : \Omega \to \mathbb{R}$ such that*

$$-m_1(\|y\|_{L^2(\Omega)})\Delta x(z) + s(z, x(z)) = f_0(z) \qquad in\ \ \Omega, \tag{4.1}$$

$$x(z) = 0 \qquad \text{on}\ \ \Gamma_1, \tag{4.2}$$

$$\frac{\partial x(z)}{\partial \nu_1} := m_1(\|y\|_{L^2(\Omega)})\big(\nabla x(z), \nu\big)_{\mathbb{R}^N} = f_1(z) \qquad on\ \ \Gamma_2, \tag{4.3}$$

$$\begin{cases} \left|\dfrac{\partial x(z)}{\partial \nu_1}\right| \leq k_1(z) \\ -\dfrac{\partial x(z)}{\partial \nu_1} = k_1(z)\dfrac{x(z)}{|x(z)|} \ \ if\ x(z) \neq 0 \end{cases} \qquad \text{on}\ \ \Gamma_3, \tag{4.4}$$

and

$$-m_2(\|x\|_{L^2(\Omega)})\Delta y(z) + t(z, y(z)) = g_0(z) \qquad in\ \ \Omega, \tag{4.5}$$

$$y(z) = 0 \qquad on\ \ \Gamma_a, \tag{4.6}$$

$$\frac{\partial y(z)}{\partial \nu_2} := m_2(\|x\|_{L^2(\Omega)})\big(\nabla y(z), \nu\big)_{\mathbb{R}^N} = g_1(z) \qquad on\ \ \Gamma_b, \tag{4.7}$$

$$\begin{cases} \left|\dfrac{\partial y(z)}{\partial \nu_2}\right| \leq k_2(z) \\ -\dfrac{\partial y(z)}{\partial \nu_2} = k_2(z)\dfrac{y(z)}{|y(z)|} \ \ if\ y(z) \neq 0 \end{cases} \qquad on\ \ \Gamma_c. \tag{4.8}$$

Next, let us introduce two subspaces X and Y of $H^1(\Omega)$, which are defined by

$$X := \{x \in H^1(\Omega) \mid x(z) = 0 \text{ on } \Gamma_1\} \text{ and } Y := \{y \in H^1(\Omega) \mid y(z) = 0 \text{ on } \Gamma_a\}. \tag{4.9}$$

It is not difficult to prove that X endowed with the inner product and the corresponding norm

$$(x, v)_X = \int_\Omega (\nabla x(z), \nabla v(z))_{\mathbb{R}^N}\, dz \text{ and } \|x\|_X = \left(\int_\Omega |\nabla x(z)|^2\, dz\right)^{\frac{1}{2}}$$

for all $x, v \in X$, is a separable Hilbert space. Also, Y is a separable Hilbert space with the norm and inner product by

$$\|y\|_Y = \left(\int_\Omega |\nabla y(z)|^2\, dz\right)^{\frac{1}{2}} \text{ and } (y, w)_Y = \int_\Omega (\nabla y(z), \nabla w(z))_{\mathbb{R}^N}\, dz$$

for all $y, w \in Y$, respectively. Denote by $\gamma_1 \colon X \to L^2(\Omega)$ (resp., $\gamma_2 \colon Y \to L^2(\Omega)$) the embedding operator from X to $L^2(\Omega)$ (resp., the embedding operator from Y to $L^2(\Omega)$).

We make the following assumptions.

$\underline{H(s)}$: $s \colon \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $\theta \mapsto s(z, \theta)$ is monotone for a.e. $z \in \Omega$ and there exist $c_s \in L^2(\Omega)_+$ and $d_s > 0$ satisfying

$$|s(z, \theta)| \le c_s(z) + d_s|\theta| \text{ for all } \theta \in \mathbb{R} \text{ and a.e. } z \in \Omega.$$

$\underline{H(t)}$: $t \colon \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $\theta \mapsto t(z, \theta)$ is monotone for a.e. $z \in \Omega$ and there exist $c_t \in L^2(\Omega)_+$ and $d_t > 0$ satisfying

$$|t(z, \theta)| \le c_t(z) + d_t|\theta| \text{ for all } \theta \in \mathbb{R} \text{ and a.e. } z \in \Omega.$$

$\underline{H(4)}$: $f_0, g_0 \in L^2(\Omega)$, $f_1 \in L^2(\Gamma_2)$, $g_1 \in L^2(\Gamma_b)$, $k_1 \in L^2(\Gamma_3)_+$ and $k_2 \in L^2(\Gamma_c)_+$.

$\underline{H(m_1)}$: $m_1 \colon \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function such that there exist constants $0 < c_{m_1} < d_{m_1} < +\infty$ satisfying

$$c_{m_1} \le m_1(\theta) \le d_{m_1} \text{ for all } \theta \ge 0.$$

$\underline{H(m_2)}$: $m_2 \colon \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function such that there exist constants $0 < c_{m_2} < d_{m_2} < +\infty$ satisfying

$$c_{m_2} \le m_2(\theta) \le d_{m_2} \text{ for all } \theta \ge 0.$$

Let $(x, y) \in X \times Y$ be such that (4.1)–(4.4) hold. For any $v \in X$ fixed, we multiply (4.1) by $v - x$ and use Green's formula to find

$$m_1(\|y\|_{L^2(\Omega)}) \int_\Omega (\nabla x(z), \nabla v(z) - \nabla x(z))_{\mathbb{R}^N}\, dz + \int_\Omega s(z, x(z))(v(z) - x(z))\, dz$$
$$= \int_\Omega f_0(z)(v(z) - x(z))\, dz + \int_\Gamma \frac{\partial x(z)}{\partial \nu_1}(v(z) - x(z))\, d\Gamma.$$

Note that

$$\int_\Gamma \frac{\partial x(z)}{\partial \nu_1}(v(z) - x(z))\, d\Gamma = \int_{\Gamma_1} \frac{\partial x(z)}{\partial \nu_1}(v(z) - x(z))\, d\Gamma$$
$$+ \int_{\Gamma_2} \frac{\partial x(z)}{\partial \nu_1}(v(z) - x(z))\, d\Gamma + \int_{\Gamma_3} \frac{\partial x(z)}{\partial \nu_1}(v(z) - x(z))\, d\Gamma,$$

it follows from the boundary conditions (4.2) and (4.3) that

$$m_1(\|y\|_{L^2(\Omega)}) \int_\Omega (\nabla x(z), \nabla v(z) - \nabla x(z))_{\mathbb{R}^N}\, dz - \int_{\Gamma_3} \frac{\partial x(z)}{\partial \nu_1}(v(z) - x(z))\, d\Gamma$$
$$+ \int_\Omega s(z, x(z))(v(z) - x(z))\, dz = \int_\Omega f_0(z)(v(z) - x(z))\, dz$$
$$+ \int_{\Gamma_2} f_1(z)(v(z) - x(z))\, d\Gamma. \tag{4.10}$$

By virtue of the definition of convex subgradient, boundary condition (4.4) can be rewritten to the following inclusion form

$$-\frac{\partial x(z)}{\partial \nu_1} \in k_1(z)\partial_c |x(z)| \text{ for a.e. } x \in \Gamma_3,$$

where the term $\partial_c|\theta|$ is the convex subdifferential operator of the modulus function $\mathbb{R} \ni \theta \mapsto |\theta| \in \mathbb{R}_+$. Therefore, it has

$$-\int_{\Gamma_3} \frac{\partial x(z)}{\partial \nu_1}(v(z) - x(z))\, d\Gamma \le \int_{\Gamma_3} k_1(z)|v(z)|\, d\Gamma - \int_{\Gamma_3} k_1(z)|x(z)|\, d\Gamma. \tag{4.11}$$

Combining (4.10)–(4.11), we have

$$m_1(\|y\|_{L^2(\Omega)}) \int_\Omega (\nabla x(z), \nabla v(z) - \nabla x(z))_{\mathbb{R}^N}\, dz + \int_{\Gamma_3} k_1(z)|v(z)|\, d\Gamma$$
$$- \int_{\Gamma_3} k_1(z)|x(z)|\, d\Gamma + \int_\Omega s(z, x(z))(v(z) - x(z))\, dz$$
$$\ge \int_\Omega f_0(z)(v(z) - x(z))\, dz + \int_{\Gamma_2} f_1(z)(v(z) - x(z))\, d\Gamma \tag{4.12}$$

for all $v \in X$. Similarly, it also gets

$$\begin{aligned} & m_2(\|x\|_{L^2(\Omega)}) \int_\Omega (\nabla y(z), \nabla w(z) - \nabla y(z))_{\mathbb{R}^N}\ dz + \int_{\Gamma_c} k_2(z)|w(z)|\,d\Gamma \\ & \quad - \int_{\Gamma_c} k_2(z)|y(z)|\,d\Gamma + \int_\Omega t(z, y(z))(w(z) - y(z))\,dz \\ & \geq \int_\Omega g_0(z)(w(z) - y(z))\,dz + \int_{\Gamma_b} g_1(z)(w(z) - y(z))\,d\Gamma \end{aligned} \tag{4.13}$$

for all $w \in Y$.

Let us consider the functions $G\colon Y \times X \to X^*$ and $F\colon X \times Y \to Y^*$ defined by

$$\begin{aligned} \langle G(y, x), v\rangle_X := {} & m_1(\|y\|_{L^2(\Omega)}) \int_\Omega (\nabla x(z), \nabla v(z))_{\mathbb{R}^N}\ dz + \int_\Omega s(z, x(z))v(z)\,dz \\ & - \int_{\Gamma_2} f_1(z)v(z)\,d\Gamma \end{aligned} \tag{4.14}$$

and

$$\begin{aligned} \langle F(x, y), w\rangle_Y := {} & m_2(\|x\|_{L^2(\Omega)}) \int_\Omega (\nabla y(z), \nabla w(z))_{\mathbb{R}^N}\ dz + \int_\Omega t(z, y(z))w(z)\,dz \\ & - \int_{\Gamma_2} g_1(z)w(z)\,d\Gamma, \end{aligned} \tag{4.15}$$

for all $x, v \in X$ and $y, w \in Y$.

Taking account of (4.12) and (4.13), we get the variational formulation of Problem 18 as follows.

Problem 19 *Find functions $x \in X$ and $y \in Y$ such that*

$$\langle G(y, x), v - x\rangle_X + \varphi(v) - \varphi(x) \geq \langle f_0, v - x\rangle_X \tag{4.16}$$

for all $v \in X$, and

$$\langle F(x, y), w - y\rangle_Y + \phi(w) - \phi(y) \geq \langle g_0, w - y\rangle_Y \tag{4.17}$$

for all $w \in Y$, where the functions $\varphi\colon X \to \mathbb{R}$ and $\phi\colon Y \to \mathbb{R}$ are defined by

$$\varphi(v) := \int_{\Gamma_3} k_1(z)|v(z)|\,d\Gamma \text{ and } \phi(w) := \int_{\Gamma_c} k_2(z)|w(z)|\,d\Gamma \tag{4.18}$$

for all $v \in X$ and $w \in Y$, respectively.

We are now in a position to give the existence theorem for Problem 19.

Theorem 20 *Assume that $H(s)$, $H(t)$, $H(4)$, $H(m_1)$ and $H(m_2)$ hold. Then, Problem 19 admits a solution.*

Proof We apply Theorem 7 to prove the desired conclusion. For any $y \in Y$ fixed, let $x_1, x_2 \in X$ be arbitrary. Note that

$$\begin{aligned}&\langle G(y, x_1) - G(y, x_2), x_1 - x_2\rangle_X \\&\quad = m_1(\|y\|_{L^2(\Omega)}) \int_\Omega (\nabla(x_1(z) - x_2(z)), \nabla(x_1(z) - x_2(z)))_{\mathbb{R}^N}\, dz \\&\qquad + \int_\Omega (s(z, x_1(z)) - s(z, x_2(z)))(x_1(z) - x_2(z))\, dz,\end{aligned}$$

it follows from hypotheses $H(s)$ and $H(m_1)$ that $x \mapsto G(y, x)$ is monotone. Besides, the continuity of s implies that for each $y \in Y$ the function $x \mapsto G(y, x)$ is continuous too. So, condition $H(G)$(i) is valid. Let sequences $\{x_n\} \subset X$ and $\{y_n\} \subset Y$ be such that

$$x_n \xrightarrow{w} x \text{ in } X \text{ and } y_n \xrightarrow{w} y \text{ in } Y \text{ as } n \to \infty$$

for some $(x, y) \in X \times Y$. Recall that the embedding from Y into $L^2(\Omega)$ is compact, we have $y_n \to y$ and $x_n \to x$ in $L^2(\Omega)$ as $n \to \infty$. Then, we have

$$\begin{aligned}&\limsup_{n\to\infty} m_1(\|y_n\|_{L^2(\Omega)}) \int_\Omega (\nabla v(z), \nabla v(z) - \nabla x_n(z))_{\mathbb{R}^N}\, dz \\&\quad \le \limsup_{n\to\infty} [m_1(\|y_n\|_{L^2(\Omega)}) - m_1(\|y\|_{L^2(\Omega)})] \int_\Omega (\nabla v(z), \nabla v(z) - \nabla x_n(z))_{\mathbb{R}^N}\, dz \\&\qquad + \limsup_{n\to\infty} m_1(\|y\|_{L^2(\Omega)}) \int_\Omega (\nabla v(z), \nabla v(z) - \nabla x_n(z))_{\mathbb{R}^N}\, dz \\&\quad \le \limsup_{n\to\infty} |m_1(\|y_n\|_{L^2(\Omega)}) - m_1(\|y\|_{L^2(\Omega)})| \left| \int_\Omega (\nabla v(z), \nabla v(z) - \nabla x_n(z))_{\mathbb{R}^N}\, dz \right| \\&\qquad + \limsup_{n\to\infty} m_1(\|y\|_{L^2(\Omega)}) \int_\Omega (\nabla v(z), \nabla v(z) - \nabla x_n(z))_{\mathbb{R}^N}\, dz \\&\quad = m_1(\|y\|_{L^2(\Omega)}) \int_\Omega (\nabla v(z), \nabla v(z) - \nabla x(z))_{\mathbb{R}^N}\, dz.\end{aligned}$$

This combined with the continuity of s, the convergence $x_n \to x$ in $L^2(\Omega)$ and Lebesgue dominated convergence theorem implies

$$\limsup_{n\to\infty} \langle G(y_n, v), v - x_n\rangle_X \le \langle G(y, v), v - x\rangle_X,$$

namely $H(G)$(ii) is available.

For any $x \in X$, using $H(s)$ and $H(m_1)$ deduces

$$\begin{aligned}\langle G(y, x), x\rangle_X \ge{}& m_1(\|y\|_{L^2(\Omega)}) \int_\Omega (\nabla x(z), \nabla x(z))_{\mathbb{R}^N}\, dz + \int_\Omega s(z, 0) x(z)\, dz \\&+ \int_\Omega [s(z, x(z)) - s(z, 0)] x(z)\, dz - \int_{\Gamma_2} f_1(z) x(z)\, d\Gamma\end{aligned}$$

$$
\begin{aligned}
&\geq c_{m_1}\|x\|_X^2 - \|s(\cdot,0)\|_{L^2(\Omega)}\|x\|_{L^2(\Omega)} - \|f_1\|_{L^2(\Gamma_2)}\|x\|_{L^2(\Gamma)} \\
&\geq (c_{m_1}\|x\|_X - \|c_s\|_{L^2(\Omega)}\|\gamma_1\| - \|f_1\|_{L^2(\Gamma_2)}\widetilde{c}_1)\|x\|_X,
\end{aligned}
$$

where we have used Hölder inequality and $\widetilde{c}_1 > 0$ is such that

$$\|x\|_{L^2(\Gamma)} \leq \widetilde{c}_1\|x\|_X \text{ for all } x \in X.$$

Let us consider the function $r\colon \mathbb{R}_+ \to \mathbb{R}$ defined by

$$r(\theta) = c_{m_1}\theta - \|c_s\|_{L^2(\Omega)}\|\gamma_1\| - \|f_1\|_{L^2(\Omega)}\widetilde{c}_1 \text{ for all } \theta \geq 0.$$

It is obvious that r fulfills the conditions of $H(G)$(iii). This means that G satisfies hypothesis $H(G)$(iii).

For any $x \in X$ and $y \in Y$ fixed, we apply hypotheses $H(m_1)$ and $H(s)$ to find

$$
\begin{aligned}
\|G(y,x)\|_{X^*} &= \sup_{v\in X,\|v\|_X=1} \langle G(y,x), v\rangle_X \\
&\leq \sup_{v\in X,\|v\|_X=1} m_1(\|y\|_{L^2(\Omega)})\int_\Omega (\nabla x(z), \nabla v(z))_{\mathbb{R}^N}\, dz \\
&\quad + \sup_{v\in X,\|v\|_X=1}\int_\Omega |s(z,x(z))||v(z)|\, dz \\
&\quad + \sup_{v\in X,\|v\|_X=1}\int_{\Gamma_2} |f_1(z)||v(z)|\, d\Gamma \\
&\leq d_{m_1}\|x\|_X + \sup_{v\in X,\|v\|_X=1}\int_\Omega (c_s(z) + d_s|x(z)|)|v(z)|\, dz \\
&\quad + \sup_{v\in X,\|v\|_X=1}\int_{\Gamma_2} |f_1(z)||v(z)|\, d\Gamma.
\end{aligned}
$$

Employing Hölder inequality finds

$$\|G(y,x)\|_{X^*} \leq d_{m_1}\|x\|_X + (\|c_s\|_{L^2(\Omega)} + d_s\|\gamma_1\|\|x\|_X)\|\gamma_1\| + \|f_1\|_{L^2(\Gamma_2)}\widetilde{c}_1.$$

It is not difficult to see that condition $H(G)$(iv) is valid with the constant $c_G > 0$ defined by

$$c_G := d_{m_1} + (\|c_s\|_{L^2(\Omega)} + d_s\|\gamma_1\|)\|\gamma_1\| + \|f_1\|_{L^2(\Gamma_2)}\widetilde{c}_1.$$

Additionally, we can verify that F fulfills hypotheses $H(F)$ with the function $l\colon \mathbb{R}_+ \to \mathbb{R}$

$$l(\theta) = c_{m_2}\theta - \|c_t\|_{L^2(\Omega)}\|\gamma_2\| - \|g_1\|_{L^2(\Gamma_b)}\widetilde{c}_2 \text{ for all } \theta \geq 0,$$

and constant $c_F > 0$

$$c_F := d_{m_2} + (\|c_t\|_{L^2(\Omega)} + d_t\|\gamma_2\|)\|\gamma_2\| + \|g_1\|_{L^2(\Gamma_b)}\widetilde{c}_2,$$

where $\widetilde{c}_2 > 0$ is such that

$$\|y\|_{L^2(\Gamma)} \le \widetilde{c}_2 \|y\|_Y \text{ for all } y \in Y.$$

For each $v \in X$ and $w \in Y$, we utilize Hölder inequality, hypotheses $H(4)$, and the continuity of the embeddings of X to $L^2(\Gamma_3)$ and Y to $L^2(\Gamma_c)$ for getting

$$\varphi(v) = \int_{\Gamma_3} k_1(z)|v(z)|\, d\Gamma \le \|k_1\|_{L^2(\Gamma_3)} \|v\|_{L^2(\Gamma_3)} < +\infty,$$
$$\phi(w) = \int_{\Gamma_c} k_2(z)|w(z)|\, d\Gamma \le \|k_2\|_{L^2(\Gamma_c)} \|w\|_{L^2(\Gamma_c)} < +\infty.$$

But, from the definitions of φ and ϕ, we can see that φ and ϕ are both continuous and convex.

Set $K = X$ and $L = Y$. Therefore, all conditions of Theorem 7 are verified. Employing this theorem, we conclude that the solution set of Problem 19 is nonempty and weakly compact in $X \times Y$. □

Since X and Y are Hilbert spaces, so, the duality mappings of X and Y are the identity operators I_X in X and I_Y in Y, respectively. Let sequences $\{\varepsilon_n\}$ and $\{\delta_n\}$ satisfy (3.1). We, next, consider the following regularized problem corresponding to Problem 19.

Problem 21 *Find $(x_n, y_n) \in X \times Y$ such that*

$$\langle G(y_n, x_n) + \varepsilon_n I_X(x_n), v - x_n \rangle_X + \varphi(v) - \varphi(x_n) \ge \langle f_0, v - x_n \rangle_X \tag{4.19}$$

for all $v \in X$, and

$$\langle F(x_n, y_n) + \delta_n I_Y(y_n), w - y_n \rangle_Y + \phi(w) - \phi(y_n) \ge \langle g_0, w - y_n \rangle_Y \tag{4.20}$$

for all $v \in Y$.

We invoke Theorem 15 directly to obtain the following existence and convergence results.

Theorem 22 *Suppose that $H(s)$, $H(t)$, $H(4)$, $H(m_1)$, $H(m_2)$ and (3.1) are fulfilled. Then, we have*

(i) *for each $n \in \mathbb{N}$, Problem 21 has at least a solution $(x_n, y_n) \in X \times Y$;*
(ii) *for any sequence of solutions $\{(x_n, y_n)\}$ of Problem 21, there exists a subsequence of $\{(x_n, y_n)\}$, still denoted by the same way, such that*

$$(x_n, y_n) \to (x, y) \text{ in } X \times Y \text{ as } n \to \infty,$$

where $(x, y) \in X \times Y$ is a solution of Problem 19.

Let $x_0, y_0 \in L^2(\Omega)$. We end the subsection to consider the following optimal control problem.

Problem 23 *Find $(f^*, g^*) \in L^2(\Omega) \times L^2(\Omega)$ such that*

$$I(f^*, g^*) = \inf_{(f,g)\in L^2(\Omega)\times L^2(\Omega)} I(f, g),$$

where the cost function $I : U \times V \to \mathbb{R}$ is defined by

$$I(f, g) = \inf_{(x,y)\in\Gamma(f,g)} \left(\frac{\rho}{2} \int_\Omega |x(z) - x_0(z)|^2 \, dz + \frac{\theta}{2} \int_\Omega |y(z) - y_0(z)|^2 \, dz \right) + \|f\|_{L^2(\Omega)} + \|g\|_{L^2(\Omega)}. \tag{4.21}$$

Here, $\Gamma(f, g)$ is the solution set of Problem 19 associated with $(f, g) \in X^* \times Y^*$, $\rho > 0$, $\theta > 0$ are two regularized parameters, and $x_0, y_0 \in L^2(\Omega)$ are two given target profiles.

Theorem 24 *Suppose that $H(s)$, $H(t)$, $H(4)$, $H(m_1)$ and $H(m_2)$ are fulfilled. Then Problem 23 admits an optimal control pair.*

Proof Let $U = V = Z_1 = Z_2 = L^2(\Omega)$. It is obvious that the embeddings of U to X^* and V to Y^* are both continuous and compact. Set $h : U \times V \to \mathbb{R}$, $h(f, g) = \|f\|_{L^2(\Omega)} + \|g\|_{L^2(\Omega)}$. It is obvious that $h(f, g) \geq 0$ for all $(f, g) \in U \times V$, h is coercive on $U \times V$, and h is weakly lower semicontinuous on $U \times V$. We are now in a position to utilize Theorem 17 to conclude that Problem 23 admits an optimal control pair. □

4.2 An Elliptic Feedback Control System

This subsection is devoted to the investigation of an elliptic mixed boundary value system with distributed control in which the distributed control is described by a least energy equation which explicitly relies on the status variable.

Let Ω in $\mathbb{R}^N$ ($N \geq 2$) be a bounded domain such that its boundary $\Gamma = \partial\Omega$ is locally Lipschitz and is divided into three measurable and disjoint parts Γ_1, Γ_2, and Γ_3 with $meas(\Gamma_1) > 0$. Let X be the Hilbert space defined in (4.9). The elliptic feedback control system is formulated as follows.

Problem 25 *Find functions $x : \Omega \to \mathbb{R}$ and $y : \Omega \to \mathbb{R}$ such that*

$$-m_1 \Delta x(z) + s(z, x(z)) = y(z) \quad \text{in } \Omega, \tag{4.22}$$

$$x(z) = 0 \quad \text{on } \Gamma_1, \tag{4.23}$$

$$\frac{\partial x(z)}{\partial \nu_3} := m_1 \big(\nabla x(z), \nu\big)_{\mathbb{R}^N} = f_1(z) \quad \text{on } \Gamma_2, \tag{4.24}$$

$$-\frac{\partial x(z)}{\partial \nu_3} \in \partial_c \psi(x(z)) \quad \text{on } \Gamma_3, \tag{4.25}$$

where the control variable $y \in H_0^1(\Omega)$ satisfies the following least energy condition

$$P(x, y) = \inf_{w\in H_0^1(\Omega)} P(x, w) \text{ for all } x \in X, \tag{4.26}$$

and $P(x, y)$ is defined by

$$P(x, y) := \frac{1}{2\sqrt{1 + \|x\|_{L^2(\Omega)}}} \int_\Omega |\nabla y(z)|^2 \, dz + \int_\Omega h_0(z) y(z) \, dz. \tag{4.27}$$

$H(\psi)$: $\psi : \mathbb{R} \to \mathbb{R}$ is a convex and lower semicontinuous function.

Using a standard procedure, we obtain the variational formulation of Problem 25 as follows:

Problem 26 *Find functions $x \in X$ and $y \in Y := H_0^1(\Omega)$ such that*

$$\begin{aligned} m_1 \int_\Omega (\nabla x(z), \nabla(v(z) - x(z)))_{\mathbb{R}^N} \, dz &+ \int_{\Gamma_3} \psi(v(z)) \, d\Gamma - \int_{\Gamma_3} \psi(x(z)) \, d\Gamma \\ &+ \int_\Omega s(z, x(z))(v(z) - x(z)) \, dz \ge \int_\Omega y(z)(v(z) - x(z)) \, dz \\ &+ \int_{\Gamma_2} f_1(z)(v(z) - x(z)) \, d\Gamma \end{aligned}$$

for all $v \in X$, and

$$P(x, y) \le P(x, w) \text{ for all } w \in H_0^1(\Omega).$$

However, it is not difficult to show that Problem 26 is equivalent to the following one.

Problem 27 *Find functions $x \in X$ and $y \in Y$ such that*

$$\begin{aligned} m_1 \int_\Omega (\nabla x(z), \nabla(v(z) - x(z)))_{\mathbb{R}^N} \, dz &+ \int_{\Gamma_3} \psi(v(z)) \, d\Gamma - \int_{\Gamma_3} \psi(x(z)) \, d\Gamma \\ &+ \int_\Omega s(z, x(z))(v(z) - x(z)) \, dz \ge \int_\Omega y(z)(v(z) - x(z)) \, dz \\ &+ \int_{\Gamma_2} f_1(z)(v(z) - x(z)) \, d\Gamma \end{aligned}$$

for all $v \in X$, and

$$\frac{1}{2\sqrt{1 + \|x\|_{L^2(\Omega)}}} \int_\Omega (\nabla y(z), \nabla w(z))_{\mathbb{R}^N} \, dz + \int_\Omega h_0(z) w(z) dz = 0$$

for all $w \in H_0^1(\Omega)$.

Arguing as in the proof of Theorem 20, we have the following existence result for Problem 27.

Theorem 28 *Assume that $H(s)$, $H(\psi)$, $f_1 \in L^2(\Gamma_2)$, and $h_0 \in L^2(\Omega)$ hold. Then, Problem 27 admits a solution.*

5 Conclusion

In this paper, we have introduced and studied a new kind of coupled variational inequalities on Banach spaces. Using Kakutani-Ky Fan fixed point theorem combined with Minty method and the arguments of monotonicity, we delivered the results concerning existence and uniqueness of solution to CVI. Then, we established a stability result for CVI and considered an optimal control problem driven by CVI. Moreover, these theoretical results were applied to explore two complicated elliptic partial differential systems: a coupled elliptic mixed boundary value system with nonlocal effect and a multivalued boundary condition, and a feedback control problem involving a least energy condition with respect to the control variable.

In fact, problems of this type are encountered in transport optimization, Nash equilibrium problem of multiple players, contact mechanics problems, and related fields. In the future, we plan to apply the theoretical results established in the current paper to an Nash equilibrium problem of multiple players, and investigate coupled quasi-variational inequalities.

Acknowledgements This project has received funding from the NNSF of China Grant Nos. 12001478, 12026255, 12026256, 12001072, 11971084, 11991024 and 11991020, and the European Union's Horizon 2020 Research and Innovation Programme under the Marie Skłodowska-Curie grant agreement No. 823731 CONMECH, National Science Center of Poland under Preludium Project No. 2017/25/N/ST1 /00611, and the Startup Project of Doctor Scientific Research of Yulin Normal University No. G2020ZK07. It is also supported by Natural Science Foundation of Guangxi Grant Nos. 2021GXNSFFA196004 and 2020GXNSFBA297137, and the Ministry of Science and Higher Education of Republic of Poland under Grants Nos. 4004/GGPJII/H2020/2018/0 and 440328/PnH2/2019. The forth author is supported by the China Postdoctoral Science Foundation Project (No. 2019M653332), and the Chongqing Natural Science Foundation Project (No. cstc2019jcyj-bshX0092).

References

1. Barbu, V., Korman, P.: Analysis and Control of Nonlinear Infinite Dimensional Systems. Academic Press, Boston (1993)
2. Brezis, H.: Functional Analysis. Sobolev Spaces and Partial Differential Equations. Universitext, Springer, New York (2011)
3. Brézis, H.: Equations et inéquations non linéaires dans les espaces vectoriels en dualité. Ann. Inst. Fourier **18**, 115–175 (1968)
4. Chen, X., Wang, Z.: Differential variational inequality approach to dynamic games with shared constraints. Math. Program. **146**, 379–408 (2014)
5. Chen, X., Wang, Z.: Convergence of regularized time-stepping methods for differential variational inequalities. SIAM J. Optim. **23**, 1647–1671 (2013)
6. Cojocaru, M.C., Matei, A.: Well-posedness for a class of frictional contact models via mixed variational formulations. Nonlinear Anal. **47**, 127–141 (2019)

7. Denkowski, Z., Migórski, S., Papageorgiou, N.S.: An Introduction to Nonlinear Analysis: Theory. Kluwer Academic/Plenum Publishers, Boston, Dordrecht, London, New York (2003)
8. Denkowski, Z., Migórski, S., Papageorgiou, N.S.: An Introduction to Nonlinear Analysis: Applications. Kluwer Academic/Plenum Publishers, Boston, Dordrecht, London, New York (2003)
9. Duvaut, G., Lions, J.-L.: Inequalities in Mechanics and Physics. Springer, Berlin Heidelberg, New York (1976)
10. Facchinei, F., Pang, J.-S.: Finite-Dimensional Variational Inequalities and Complementarity Problems, Vols. I and II. Springer, New York, (2003)
11. Fukushima, M.: Equivalent differentiable optimization problems and descent methods for asymmetric variational inequality problems. Math. Program. **53**, 99–110 (1992)
12. Fukushima, M.: A relaxed projection method for variational inequalities. Math. Program. **35**, 58–70 (1986)
13. Gasiński, L., Papageorgiou, N.S.: Nonlinear Analysis. Series in Mathematical Analysis and Applications. Chapman & Hall/CRC, Boca Raton, FL (2006)
14. Giannessi, F.: Vector Variational Inequalities and Vector Equilibria. Kluwer Academic Publishers, Dordrecht, Boston, London (2000)
15. Glowinski, R., Lions, J.L., Trémoliéres, R.: Numerical Analysis of Variational Inequalities. North-Holland, Amsterdam (1981)
16. Han, W., Sofonea, M.: Quasistatic Contact Problems in Viscoelasticity and Viscoplasticity. American Mathematical Society, International Press (2002)
17. Hlaváček, I., Haslinger, J., Nečas, J., Lovíšek, J.: Solution of Variational Inequalities in Mechanics, Applied Mathematical Sciences, vol. 66. Springer, New York (1988)
18. Kinderlehrer, D., Stampacchia, G.: An Introduction to Variational Inequalities and Their Application. Academic Press, New York (1980)
19. Li, G.X., Yang, X.M.: Convexification method for bilevel programs with a nonconvex follower's problem. J. Optim. Theory Appl. **188**, 724–743 (2021)
20. Lions, J.-L., Stampacchia, G.: Variational inequalities. Commun. Pure Appl. Math. **20**, 493–519 (1969)
21. Liou, Y.C., Yang, X.Q., Yao, J.C.: Mathematical programs with vector optimization constraints. J. Optim. Theory Appl. **126**, 345–355 (2005)
22. Liu, Y., Xu, H., Lin, G.H.: Stability analysis of two-stage stochastic mathematical programs with complementarity constraints via NLP regularization. SIAM J. Optim. **21**(3), 669–705 (2011)
23. Liu, Z.H., Migórski, S., Zeng, S.D.: Partial differential variational inequalities involving nonlocal boundary conditions in Banach spaces. J. Diff. Eq. **263**, 3989–4006 (2017)
24. Liu, Z.H., Motreanu, D., Zeng, S.D.: Nonlinear evolutionary systems driven by mixed variational inequalities and its applications. Nonlinear Anal. **42**, 409–421 (2018)
25. Liu, Z.H., Motreanu, D., Zeng, S.D.: Nonlinear evolutionary systems driven by quasi-hemivariational inequalities. Math. Meth. Appl. Sci. **41**, 1214–1229 (2018)
26. Liu, Z.H., Motreanu, D., Zeng, S.D.: Generalized penality and regularization method for differential variational-hemivariational inequalities. SIAM J. Optim. **31**, 1158–1183 (2021)
27. Liu, Z.H., Motreanu, D., Zeng, S.D.: Positive solutions for nonlinear singular elliptic equations of p-Laplacian type with dependence on the gradient. Calc. Var. PDEs **58**, 29 (2019)
28. Liu, Z.H., Zeng, S.D., Motreanu, D.: Partial differential hemivariational inequalities. Adv. Nonlinear Anal. **7**, 571–586 (2018)
29. Liu, Z.H., Zeng, B.: Existence results for a class of hemivariational inequalities involving the stable (g, f, α)-quasimonotonicity. Topol. Methods Nonlinear Anal. **47**, 195–217 (2016)
30. Migórski, S., Ochal, A., Sofonea, M.: Nonlinear Inclusions and Hemivariational Inequalities. Models and Analysis of Contact Problems, Advances in Mechanics and Mathematics, 26, Springer, New York, (2013)
31. Migórski, S.: A class of history-dependent systems of evolution inclusions with applications. Nonlinear Anal. **59**, 103246 (2021)
32. Migórski, S., Zeng, S.D.: A class of differential hemivariational inequalities in Banach spaces. J. Global Optim. **72**, 761–779 (2018)
33. Migórski, S., Zeng, S.D.: Penalty and regularization method for variational-hemivariational inequalities with application to frictional contact. Z. Angew. Math. Mech. **98**, 1503–1520 (2018)
34. Migórski, S., Sofonea, M., Zeng, S.D.: Well-posedness of history-dependent sweeping processes. SIAM J. Math. Anal. **51**, 1082–1107 (2019)

35. Migórski, S., Khan, A.A., Zeng, S.: Inverse problems for nonlinear quasi-hemivariational inequalities with application to mixed boundary value problems. Inverse Probl. **36**(2), 024006 (2020)
36. Migórski, S., Khan, A.A., Zeng, S.: Inverse problems for nonlinear quasi-variational inequalities with an application to implicit obstacle problems of p-Laplacian type. Inverse Probl. **35**(3), 035004 (2019)
37. Mosco, U.: Convergence of convex sets and of solutions of variational inequalities. Adv. Math. **3**, 510–585 (1969)
38. Pang, J.S., Stewart, D.E.: Differential variational inequalities. Math. Program. **113**, 345–424 (2008)
39. Papageorgiou, N.S., Kyritsi, S.: Handbook of Applied Analysis. Springer, New York (2009)
40. Sofonea, M., Matei, A.: Mathematical Models in Contact Mechanics. Cambridge University Press, UK (2012)
41. Tang, G.J., Huang, N.J.: Existence theorems of the variational-hemivariational inequalities. J. Global Optim. **56**, 605–622 (2013)
42. Wang, X., Huang, N.J.: A class of differential vector variational inequalities in finite dimensional spaces. J. Optim. Theory Appl. **162**, 633–648 (2014)
43. Zeidler, E.: Nonlinear Functional Analysis and Applications. II A/B, Springer, New York (1990)
44. Zeng, S.D., Migórski, S., Liu, Z.H.: Well-posedness, optimal control and sensitivity analysis for a class of differential variational-hemivariational inequalities. SIAM J. Optim. **31**, 2829–2862 (2021)
45. Zeng, S.D., Liu, Z.H., Migórski, S.: A class of fractional differential hemivariational inequalities with application to contact problem. Z. Angew. Math. Phys. **69**, 23 (2018)

Authors and Affiliations

Jinjie Liu[1] · Xinmin Yang[1] · Shengda Zeng[2,3,4] · Yong Zhao[5,6]

✉ Shengda Zeng
zengshengda@163.com

Jinjie Liu
jinjie.liu@sjtu.edu.cn

Xinmin Yang
xmyang@cqnu.edu.cn

1 National Center for Applied Mathematics Chongqing, and School of Mathematics Science, Chongqing Normal University, Chongqing 401331, People's Republic of China

2 Guangxi Colleges and Universities Key Laboratory of Complex System Optimization and Big Data Processing, Yulin Normal University, Yulin 537000, People's Republic of China

3 Department of Mathematics, Nanjing University, Nanjing, Jiangsu 210093, P.R. China

4 Jagiellonian University in Krakow, Faculty of Mathematics and Computer Science, ul. Lojasiewicza 6, Nanjing 30-348 Krakow, Poland

5 College of Mathematics and Statistics, Chongqing Jiaotong University, Chongqing 400074, People's Republic of China

6 College of Mathematics and Statistics, Chongqing University, Chongqing 401331, China