Vector Optimization w.r.t. Relatively Solid Convex Cones in Real Linear Spaces

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Abstract

In vector optimization, it is of increasing interest to study problems where the image space (a real linear space) is preordered by a not necessarily solid (and not necessarily pointed) convex cone. It is well-known that there are many examples where the ordering cone of the image space has an empty (topological/algebraic) interior, for instance in optimal control, approximation theory, duality theory. Our aim is to consider Pareto-type solution concepts for such vector optimization problems based on the intrinsic core notion (a well-known generalized interiority notion). We propose a new Henig-type proper efficiency concept based on generalized dilating cones which are relatively solid (i.e., their intrinsic cores are nonempty). Using linear functionals from the dual cone of the ordering cone, we are able to characterize the sets of (weakly, properly) efficient solutions under certain generalized convexity assumptions. Toward this end, we employ separation theorems that are working in the considered setting.

Keywords Vector optimization · Relatively solid convex cone · Intrinsic core · (Weak) Pareto efficiency · Henig proper efficiency · Generalized dilating cones · Scalarization · Separation theorem

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Dedicated to Professor Franco Giannessi.

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1 Introduction

It is known that in vector optimization (see Jahn [34]) as well as in Image Space Analysis (ISA) in infinite dimensional linear spaces (see Giannessi [19,20] and references therein) difficulties may arise because of the possible non-solidness of ordering cones (for instance in the fields of optimal control, approximation theory, duality theory). Thus, it is of increasing interest to derive optimality conditions and duality results for such vector optimization problems using generalized interiority conditions (see, e.g., Adán and Novo [1–4], Bagdasar and Popovici [6], Bao and Mordukhovich [7], Borwein and Goebel [10], Borwein and Lewis [11], Grad [23,24], Grad and Pop [25], Khazayel et al. [36], Zălinescu [43,44], and Cuong et al. [14]). Such conditions can be formulated using the well-established generalized interiority notions given by quasi-interior, quasi-relative interior, algebraic interior (also known as core), relative algebraic interior (also known as intrinsic core, pseudo-relative interior or intrinsic relative interior). Moreover, it is known that for defining Pareto-type solution concepts of vector optimization problems, generalized interiority notions are also useful.

In recent works related to vector optimization in real linear spaces (see, e.g., Adán and Novo [1–4], Bao and Mordukhovich [7], Khazayel et al. [36], Novo and Zălinescu [40], Popovici [41], and Zhou, Yang and Peng [45]), the intrinsic core notion is studied in more detail. Having two real linear spaces X and E, a vector-valued objective function $f : X \to E$, a certain set of constraints $\Omega \subseteq X$, a convex (ordering) cone $K \subseteq E$ (with possibly empty algebraic interior), a vector optimization problem is defined by

$$f(x) \to \min \text{ w.r.t. } K$$
$$x \in \Omega.$$

For this problem, a useful solution concept is to say that a point $\bar{x} \in \Omega$ is optimal if

$$\{x \in \Omega \mid f(x) \in f(\bar{x}) - \text{icor } K\} = \emptyset, \tag{1}$$

where icor K denotes the intrinsic core of K. It is important to know that in finite dimensional real linear spaces the intrinsic core of any convex set (cone) is nonempty but core could be empty (for instance recession cones of polyhedral sets in $\mathbb{R}^n$ are convex cones which are not necessarily solid). Replacing icor K in (1) by any other (generalized) interior of K one can define other solution concepts. By involving an appropriate set $S \subseteq E \setminus \{0\}$ with icor $K \subseteq S$, one can define a stronger solution concept by replacing (1) by

$$\{x \in \Omega \mid f(x) \in f(\bar{x}) - S\} = \emptyset. \tag{2}$$

Notice that (2) implies (1). This leads to other solution concepts such as the well-known concepts of Pareto efficiency (i.e., $\bar{x}$ satisfies (2) for $S := K \setminus (-K)$) or proper

Pareto efficiency (i.e., $\bar{x}$ satisfies (2) with $S := \operatorname{cor} C$ for some generalized dilating cone C).

In order to derive theoretical (duality assertions) and computational (algorithms based on scalarization) results in vector optimization, one needs strict monotonicity properties concerning the scalarizing functional. This is the reason that the solution concepts of weak and proper efficiency (based on generalized interiors of the ordering cone) are of big importance. Using linear scalarization, it is known that the set of properly (respectively, weakly) efficient solutions can be completely characterized in the (generalized) convex case (e.g., if $f[\Omega] + K$ is convex) if the ordering cone is pointed (respectively, solid).

In particular, the notion of proper Pareto efficiency is very important, not only from a theoretical point of view, but also from a practical point of view. In the literature, several notions of proper efficiency have been proposed. The concept of proper efficiency dates back to the work by Kuhn and Tucker [37]. Geoffrion [18] proposed a very useful concept for multiobjective optimization problems (i.e., $E := \mathbb{R}^m$ is the m-dimensional Euclidean space and $K := \mathbb{R}^m_+$ is the natural ordering cone in $\mathbb{R}^m$) for which the solutions have a bounded trade-off (which decision makers could prefer in view of applications). Some well-known generalizations of the mentioned proper efficiency concepts are given by Benson [8], Borwein [9], Borwein and Zhuang [12], Hartley [29], Henig [30], and Hurwicz [33]. These concepts and corresponding generalizations are discussed, among others, by Durea, Florea and Strugariu [15], Eichfelder and Kasimbeyli [16], Gutiérrez et al. [28], Hernández, Jiménez and Novo [31], Jahn [34, Ch. 4], and Luc [38, Def. 2.1]. Moreover, an overview on different concepts including corresponding relationships is given in Khan, Tammer and Zălinescu [35, Sec. 2.4], and Guerraggio, Molno and Zaffaroni [27]. Furthermore, special cases for concepts of properly efficient elements in finite dimensional spaces are presented in Ansari, Köbis and Yao [5, Sec. 3.2.3], and Giorgi, Guerraggio and Thierfelder [21, Sec. 6.4].

To attack vector optimization problems, it is known that the Image Space Analysis approach by Giannessi [19] (see also [20] for some perspectives on vector optimization via ISA) is of great importance. In the ISA approach, one is constructing a certain convex cone using the original ordering cone (defining the solution concept) as well as the cone which is used to describe the constraints. So, the idea arises to consider generalized interiority notions within ISA. We will not focus on this ISA approach but like to highlight that the framework of this paper is useful for this field. Notice, also for the case $X := \mathbb{R}^n$, $E := \mathbb{R}^m$ and $K := \mathbb{R}^m_+$, the cone in the ISA approach is not solid if beside inequality constraints also equality constraints appear in the problem.

The outline of the article is as follows.

First, in Sect. 2 we recall important algebraic properties of convex sets and convex cones in linear spaces. In our main results, we will deal with relatively solid, convex cones, and for proving them, we will use separation techniques in linear spaces that are based on the intrinsic core notion (see [36] and Proposition 2.2).

In Sect. 3, we study vector optimization problems involving relatively solid, convex cones which are not necessarily pointed. We will concentrate in this section on the concept of Pareto efficiency as well as on the concept of weak Pareto efficiency (based on the intrinsic core notion). For the sets of solutions of the given vector optimization problem w.r.t. these concepts, we derive some useful properties and relationships.

Henig-type proper efficiency concepts based on certain families of generalized dilating cones play the main role in Sect. 4. Our proposed concept uses generalized dilating cones which are relatively solid and convex (see Definition 4.2). Since our concept is based on the intrinsic core notion, we are able to find (in the case that K is not solid) a better inner approximation of the set of Pareto efficient solutions in comparison with most of the known (Henig-type) proper efficiency concepts (see also Remark 4.5).

In Sect. 5, we present scalarization results for vector optimization problems. For the linear scalarization case, we are able to state representations for the sets of (weakly, properly) efficient solutions under certain generalized convexity assumptions.

The article concludes in Sect. 6 with a brief summary and an outlook to future work.

2 Preliminaries in Preordered Linear Spaces

Throughout the paper, let $E \neq \{0\}$ be a real linear space, and let E' be its algebraic dual space, which is given by

$$E' = \{x' : E \to \mathbb{R} \mid x' \text{ is linear}\}.$$

It is well-known that E can be endowed with the strongest locally convex topology τ_c, that is generated by the family of all the semi-norms defined on E (see Khan, Tammer and Zălinescu [35, Sec. 6.3]). In the literature, the topology τ_c is known as the convex core topology. According to [35, Prop. 6.3.1], the topological dual space of E, namely $(E, \tau_c)^*$, is exactly the algebraic dual space E'. In recent works (see, e.g., Khazayel et al. [36], Novo and Zălinescu [40]), the convex core topology τ_c is used to derive properties for algebraic interiority notions (such as core and intrinsic core).

2.1 Algebraic Interiority Notions

Let us define, for any two points x and $\overline{x}$ in E, the closed, the open, the half-open line segments by

$$[x, \overline{x}] := \{(1 - \lambda)x + \lambda\overline{x} \mid \lambda \in [0, 1]\}, \quad (x, \overline{x}) := \{(1 - \lambda)x + \lambda\overline{x} \mid \lambda \in (0, 1)\},$$
$$[x, \overline{x}) := \{(1 - \lambda)x + \lambda\overline{x} \mid \lambda \in [0, 1)\}, \quad (x, \overline{x}] := \{(1 - \lambda)x + \lambda\overline{x} \mid \lambda \in (0, 1]\}.$$

Consider any set $\Omega \subseteq E$. The smallest affine (respectively, linear) subspace of E containing Ω is denoted by aff Ω (respectively, span Ω). Two special subsets of Ω will be of interest (c.f. Holmes [32, pp. 7–8]):

- the algebraic interior (or the core) of Ω, which is given as

$$\text{cor } \Omega := \{x \in \Omega \mid \forall v \in E \ \exists \varepsilon > 0 : \ x + [0, \varepsilon] \cdot v \subseteq \Omega\},$$

- the relative algebraic interior (or the intrinsic core) of Ω, which is defined by

$$\operatorname{icor} \Omega := \{x \in \Omega \mid \forall\, v \in \operatorname{aff}(\Omega - \Omega)\, \exists\, \varepsilon > 0 : \ x + [0, \varepsilon] \cdot v \subseteq \Omega\}.$$

Notice, for any nonempty set $\Omega \subseteq E$, we have

$$\operatorname{cor} \Omega = \begin{cases} \operatorname{icor} \Omega & \text{if } \operatorname{aff} \Omega = E, \\ \emptyset & \text{otherwise}, \end{cases}$$

and if $\operatorname{icor} \Omega \neq \emptyset$,

$$\operatorname{cor} \Omega \neq \emptyset \iff \operatorname{aff} \Omega = E.$$

The algebraic closure of Ω is defined using all linearly accessible points of Ω (c.f. Holmes [32, p. 9]) as

$$\operatorname{acl} \Omega := \{x \in E \mid \exists\, \overline{x} \in \Omega : \ [\overline{x}, x) \subseteq \Omega\}.$$

For any $d \in E$, the vector closure of Ω in the direction d is denoted by

$$\operatorname{vcl}_d \Omega := \{x \in E \mid \forall\, \lambda > 0\, \exists\, t \in [0, \lambda] : \ x + td \in \Omega\}.$$

Then, the set

$$\operatorname{vcl} \Omega := \bigcup_{d \in E} \operatorname{vcl}_d \Omega$$

is exactly the vector closure of Ω in the sense of Adán and Novo [3, Def. 1]. It is well-known that

$$\operatorname{int}_{\tau_c} \Omega \subseteq \operatorname{rint}_{\tau_c} \Omega \subseteq \operatorname{icor} \Omega \subseteq \Omega \subseteq \operatorname{acl} \Omega \subseteq \operatorname{vcl} \Omega \subseteq \operatorname{cl}_{\tau_c} \Omega \subseteq \operatorname{aff} \Omega.$$

Notice that $\operatorname{cl}_{\tau_c} \Omega$, $\operatorname{int}_{\tau_c} \Omega$ and $\operatorname{rint}_{\tau_c} \Omega$ denotes the closure, the interior and the relative interior of Ω with respect to the convex core topology τ_c, respectively.

2.2 Convex Sets

As usual, a set $\Omega \subseteq E$ is said to be convex if $(x, \overline{x}) \subseteq \Omega$ for any $x, \overline{x} \in \Omega$. Having a convex set $\Omega \subseteq E$, it is known that

$$\operatorname{acl} \Omega = \operatorname{vcl} \Omega \subseteq \operatorname{cl}_{\tau_c} \Omega, \quad \operatorname{cor} \Omega = \operatorname{int}_{\tau_c} \Omega, \quad \operatorname{icor} \Omega = \operatorname{rint}_{\tau_c} \Omega,$$

where each of these sets is convex as well. If, in addition, Ω is relatively solid (i.e., icor $\Omega \neq \emptyset$), then we have

$$\bigcup_{\overline{x} \in \text{icor}\,\Omega,\, x \in \text{acl}\,\Omega} [\overline{x}, x) \subseteq \text{icor}\,\Omega = \text{icor}(\text{acl}\,\Omega) = \text{icor}(\text{cl}_{\tau_c}\,\Omega),$$

$$\text{vcl}\,\Omega = \text{acl}\,\Omega = \text{acl}(\text{icor}\,\Omega) = \text{acl}(\text{acl}\,\Omega) = \text{cl}_{\tau_c}(\text{icor}\,\Omega) = \text{cl}_{\tau_c}\,\Omega.$$

In contrast, if Ω is a not relatively solid, convex set (hence, E has infinite dimension), then it may happen that acl $\Omega \neq \text{cl}_{\tau_c}\,\Omega$ (see Novo and Zălinescu [43, Ex. 1.1]).

To prove our main scalarization results for vector optimization problems in Sect. 5, we will apply well-known separation results for convex sets in linear spaces.

Proposition 2.1 *Assume that Ω^1, $\Omega^2 \subseteq E$ are nonempty, convex sets, and Ω^1 is solid. Then, the following assertions are equivalent:*

1° $\Omega^2 \cap \text{cor}\,\Omega^1 = \emptyset$.
2° $\exists\, x' \in E', \alpha \in \mathbb{R}, \forall\, \omega^1 \in \text{cor}\,\Omega^1,\ \omega^2 \in \text{acl}\,\Omega^2 :\ x'(\omega^2) \leq \alpha < x'(\omega^1)$.

For two relatively solid, convex sets we have the following separation result, which is a consequence of the well-known support theorem by Holmes [32, p. 21] (see also Khazayel et al. [36, Cor. 2.24]).

Proposition 2.2 *Assume that Ω^1, $\Omega^2 \subseteq E$ are relatively solid, convex sets. Then, the following assertions are equivalent:*

1° $(\text{icor}\,\Omega^1) \cap (\text{icor}\,\Omega^2) = \emptyset$.
2° $\exists\, x' \in E', \alpha \in \mathbb{R}, \forall\, \omega^1 \in \text{icor}\,\Omega^1,\ \omega^2 \in \text{icor}\,\Omega^2 :\ 0 \leq \alpha < x'(\omega^1) - x'(\omega^2)$.

2.3 Convex Cones

In what follows, $\mathbb{R}_+$ denotes the set of nonnegative real numbers, while $\mathbb{P} := \mathbb{R}_{++}$ denotes the set of positive real numbers. Recall that a cone $K \subseteq E$ (i.e., $0 \in K = \mathbb{R}_+ \cdot K$) is convex if $K + K = K$; nontrivial if $\{0\} \neq K \neq E$; pointed if $\ell(K) := K \cap (-K) = \{0\}$. The set $\ell(K)$ is called the lineality space of K. Notice that $\ell(K) \subseteq K \subseteq \text{aff}\,K$, and K is a linear subspace of E if and only if $K = \ell(K)$.

In this paper, we assume that

$$K \subseteq E \text{ is a convex cone with } K \neq \ell(K). \tag{3}$$

Then, according to Khazayel et al. [36, Lem. 2.9], the following hold:

$$0 \notin \text{icor}\,K, \quad (\text{icor}\,K) \cap \ell(K) = \emptyset, \quad \text{icor}\,K \subsetneq K \subsetneq \text{aff}\,K.$$

Moreover, if K is relatively solid, then $\text{cl}_{\tau_c}\,K$ is not a linear subspace, i.e., $\text{cl}_{\tau_c}\,K \neq \ell(\text{cl}_{\tau_c}\,K)$. The following convex cone

$$K^+ := \{y' \in E' \mid \forall\, k \in K :\ y'(k) \geq 0\}$$

is called the (algebraic) dual cone of K. It is well-known that acl $K^+ = K^+ = (\text{acl } K)^+ = (\text{cl}_{\tau_c} K)^+$. Moreover, if K is relatively solid, then

$$\emptyset \neq K^+ \setminus \ell(K^+) = \{x' \in E' \mid \forall k \in \text{icor } K : \ x'(k) > 0\}$$
$$= \{x' \in K^+ \setminus \{0\} \mid \exists k \in \text{icor } K : \ x'(k) > 0\}.$$

If K (respectively, K^+) is solid, then K^+ (respectively, K) is pointed. Define

$$K^{\#} := \{y' \in E' \mid \forall k \in K \setminus \{0\} : \ y'(k) > 0\} \subseteq K^+.$$

It is obvious that $\mathbb{P} \cdot K^{\#} = K^{\#} = K^{\#} + K^{\#} = K^+ + K^{\#}$ (hence, $K^{\#}$ is convex). Furthermore, if $K^{\#} \neq \emptyset$, then K is pointed. In particular, the following set

$$K^{\&} := \{y' \in E' \mid \forall k \in K \setminus \ell(K) : \ y'(k) > 0\}$$

will be of special interest. Since $K \neq \ell(K)$, we have $K^{\&} \subseteq K^+ \setminus \ell(K^+)$. Obviously, $K^{\#} = \emptyset$ if $\ell(K) \neq \{0\}$; $K^{\#} = K^{\&}$ if $\ell(K) = \{0\}$. In addition, the following assertions are provided by Khazayel et al. [36, Th. 4.1, Cor. 4.5]:

Lemma 2.1 ([36, Th. 4.1, Cor. 4.5]) *Suppose that K is τ_c-closed and satisfies* (3)*, and K^+ is relatively solid. Then:*

1° $\emptyset \neq \text{icor } K^+ \subseteq K^{\&}$.
2° *If E has finite dimension, then* $\text{icor } K^+ = K^{\&}$.
3° K *is pointed* $\iff K^{\#} \supseteq K^{\&} \iff K^{\#} \neq \emptyset$.

Notice that the τ_c-closedness assumption concerning K in Lemma 2.1 cannot be omitted, as the example by Khazayel et al. [36, Ex. 4.3] shows.

Lemma 2.2 ([36, Lem. 2.9]) *Suppose that K satisfies* (3)*. Then, $Q := K \setminus \ell(K)$ is a nonempty, convex set and the following properties hold:*

$$\mathbb{P} \cdot Q = Q, \qquad\qquad\qquad K + Q = Q + Q = Q,$$
$$\text{aff } Q = Q - Q = K - K, \qquad \text{icor } Q = \text{icor } K,$$
$$\bigcup_{\overline{x} \in Q, x \in K} [\overline{x}, x) \subseteq Q, \qquad\qquad \text{cor } K \subseteq \text{icor } K \subseteq Q \subseteq K \subseteq \text{acl } Q.$$

Having a cone K that satisfies (3), we are also interested in the analysis of the subsets

$$Q_0 := (K \setminus \ell(K)) \cup \{0\} \quad \text{and} \quad P_0 := (\text{icor } K) \cup \{0\}.$$

The next two lemmata, which are direct consequences of the results by Khazayel et al. [36], show that the sets Q_0 and P_0 are actually nontrivial, pointed, convex cones.

Lemma 2.3 *Suppose that K satisfies* (3). *Then, $Q_0 := (K \setminus \ell(K)) \cup \{0\}$ is a nontrivial, pointed, convex cone, and the following properties hold:*

$$
\begin{aligned}
\mathbb{R}_+ \cdot Q_0 = Q_0 &= Q_0 + Q_0 \subseteq \mathrm{cl}_{\tau_c} \, Q_0 = \mathrm{cl}_{\tau_c} \, K, \\
\mathrm{aff} \, Q_0 = Q_0 - Q_0 &= K - K = \mathrm{aff} \, K, \\
\mathrm{icor} \, Q_0 = \mathrm{icor} \, K &= Q_0 + \mathrm{icor} \, K = K + \mathrm{icor} \, Q_0, \\
K^{\&} = (Q_0)^{\#} &= (Q_0)^{\&} \subseteq (Q_0)^{+} = K^{+}.
\end{aligned}
$$

Proof Directly follows from [36, Lem. 2.7, Lem. 2.9, Lem. 2.14]. □

Lemma 2.4 *Suppose that K is relatively solid and satisfies* (3). *Then, $P_0 := (\mathrm{icor}\, K) \cup \{0\}$ is a nontrivial, pointed, relatively solid, convex cone, and the following properties hold:*

$$
\begin{aligned}
\mathbb{R}_+ \cdot P_0 = P_0 &= P_0 + P_0 \subseteq \mathrm{acl}\, P_0 = \mathrm{acl}\, K, \\
\mathrm{aff}\, P_0 = P_0 - P_0 &= (\mathrm{icor}\, K) - \mathrm{icor}\, K = K - K = \mathrm{aff}\, K, \\
\mathrm{icor}\, P_0 = \mathrm{icor}\, K &= P_0 + \mathrm{icor}\, K = K + \mathrm{icor}\, P_0, \\
K^{+} \setminus \ell(K^{+}) = (P_0)^{\#} &= (P_0)^{\&} \subseteq (P_0)^{+} = K^{+}.
\end{aligned}
$$

Proof It is well-known that P_0 is a convex cone (hence, $\mathbb{R}_+ \cdot P_0 = P_0 = P_0 + P_0$). Clearly, $K \neq \ell(K)$ implies that P_0 is nontrivial. Since $\mathrm{icor}\, K \subseteq Q$, we get $P_0 \subseteq Q_0$. Because Q_0 is pointed (by Lemma 2.3), we conclude that P_0 is pointed as well. Applying Lemma 2.2 for P_0 in the role of K (hence, $Q = \mathrm{icor}\, K$ and $Q_0 = P_0$), we get $(\mathrm{icor}\, K) - \mathrm{icor}\, K = P_0 - P_0$, $\mathrm{icor}\, K = \mathrm{icor}(\mathrm{icor}\, K) = \mathrm{icor}\, P_0$ and $\mathrm{acl}\, K = \mathrm{acl}\, P_0 (= \mathrm{cl}_{\tau_c}\, P_0 = \mathrm{cl}_{\tau_c}\, K)$. Clearly, $K^{+} = (\mathrm{acl}\, K)^{+} = (\mathrm{acl}\, P_0)^{+} = (P_0)^{+}$. Finally, in view of [36, Lem. 2.7], we conclude $\mathrm{aff}\, P_0 = P_0 - P_0 = K - K = \mathrm{aff}\, K$, $\mathrm{icor}\, K = P_0 + \mathrm{icor}\, K$ and $\mathrm{icor}\, P_0 = K + \mathrm{icor}\, P_0$. By [36, Cor. 4.9], we get $K^{+} \setminus \ell(K^{+}) = (P_0)^{\#} = (P_0)^{\&}$ taking into account the pointedness of P_0. □

The following lemma will play a key role for deriving characterizations of solution sets of vector optimization problems under certain generalized convexity assumptions (see Sect. 5).

Lemma 2.5 *Suppose that K is relatively solid and satisfies* (3). *Assume $A \subseteq E$ is a nonempty set. Then, the following assertions hold:*

1° *For any $x \in A$, we have $x + \mathrm{icor}\, K = \mathrm{icor}(x + K) = \mathrm{icor}(x + \mathrm{icor}\, K)$.*
2° $A + \mathrm{icor}\, K \supseteq \mathrm{icor}(A + K)$.
3° *If $\mathrm{aff}(K - K) = \mathrm{aff}((A + K) - (A + K))$, then $A + \mathrm{icor}\, K \subseteq \mathrm{icor}(A + K)$.*
4° *If A is relatively solid and convex, then*

$$
\left[\bigcup_{k \in K} \mathrm{vcl}_k (\mathrm{icor}\, A) \right] + \mathrm{icor}\, K = (\mathrm{icor}\, A) + \mathrm{icor}\, K = \mathrm{icor}(A + K).
$$

5° *If $A + K$ is a relatively solid, convex set, then*

$$\left[\bigcup_{k \in K} \mathrm{vcl}_k(\mathrm{icor}(A + K))\right] + \mathrm{icor}\, K = \mathrm{icor}(A + K) + \mathrm{icor}\, K = \mathrm{icor}(A + K).$$

6° *If K is solid, then $A + \mathrm{cor}\, K = \mathrm{cor}(A + K) = \mathrm{cor}(A + \mathrm{cor}\, K)$.*

Proof First, notice that $\mathrm{icor}(\Omega^1 + \Omega^2) = (\mathrm{icor}\,\Omega^1) + \mathrm{icor}\,\Omega^2$ if $\Omega^1, \Omega^1 \subseteq E$ are relatively solid, convex sets (see Khan, Tammer and Zălinescu [35, Prop. 6.3.2], Novo and Zălinescu [40, Cor. 2.1]).

1° Since K, $\mathrm{icor}\, K$ and $\{x\}$ are relatively solid, convex sets, we have $x + \mathrm{icor}\, K = (\mathrm{icor}\,\{x\}) + \mathrm{icor}\, K = \mathrm{icor}(x + K)$ and $x + \mathrm{icor}\, K = (\mathrm{icor}\,\{x\}) + \mathrm{icor}(\mathrm{icor}\, K) = \mathrm{icor}(x + \mathrm{icor}\, K)$.

2° Take some $x \in \mathrm{icor}(A + K)$, and $k \in \mathrm{icor}\, K \subseteq \mathrm{aff}\, K \subseteq \mathrm{aff}((A + K) - (A + K))$. Then, there exists $\varepsilon > 0$ such that $x - \varepsilon k \in A + K$, hence $x \in A + K + \varepsilon k \subseteq A + K + \mathrm{icor}\, K = A + \mathrm{icor}\, K$.

3° $A + \mathrm{icor}\, K = \bigcup_{x \in A}(x + \mathrm{icor}\, K) \overset{1°}{=} \bigcup_{x \in A} \mathrm{icor}(x + K) \subseteq \mathrm{icor}(A + K)$, where $\mathrm{aff}(K - K) = \mathrm{aff}((A + K) - (A + K))$ is needed for the last inclusion.

4° Clearly, since $\mathrm{icor}\, A \subseteq \bigcup_{k \in K} \mathrm{vcl}_k(\mathrm{icor}\, A)$, we have $\mathrm{icor}(A + K) = (\mathrm{icor}\, A) + \mathrm{icor}\, K \subseteq \left[\bigcup_{k \in K} \mathrm{vcl}_k(\mathrm{icor}\, A)\right] + \mathrm{icor}\, K$. In order to prove the remaining inclusion, fix some $k \in K$, $x \in \mathrm{vcl}_k(\mathrm{icor}\, A)$ and $\bar{k} \in \mathrm{icor}\, K$. Then, there is $\varepsilon > 0$ such that $\bar{k} + [0, \varepsilon] \cdot (-k) \subseteq \mathrm{icor}\, K$, and there is $t \in [0, \varepsilon]$ such that $x + tk \in \mathrm{icor}\, A$. Thus, $x + \bar{k} = (x + tk) + (\bar{k} + t(-k)) \in (\mathrm{icor}\, A) + \mathrm{icor}\, K = \mathrm{icor}(A + K)$.

5° Follows from 4° (applied for $A + K$ in the role of A). Indeed,

$$\left[\bigcup_{k \in K} \mathrm{vcl}_k(\mathrm{icor}(A + K))\right] + \mathrm{icor}\, K = \mathrm{icor}(A + K) + \mathrm{icor}\, K$$

$$= \mathrm{icor}(A + K + K) = \mathrm{icor}(A + K).$$

6° Since $E = \mathrm{aff}(K - K) \subseteq \mathrm{aff}((A + K) - (A + K))$, by 2° and 3° we get $A + \mathrm{cor}\, K = A + \mathrm{icor}\, K = \mathrm{icor}(A + K) = \mathrm{cor}(A + K)$.

Clearly, $\mathrm{cor}(A + \mathrm{cor}\, K) \subseteq A + \mathrm{cor}\, K$. Now, take some $x \in A + \mathrm{cor}\, K$ and $v \in E$. Then, there are $a \in A$ and $k \in \mathrm{cor}\, K$ such that $x = a + k$. Moreover, using [35, Lem. 2.5 (4)], there is $\varepsilon > 0$ such that $k + [0, \varepsilon]v \subseteq \mathrm{cor}\, K$. Consequently, we get $x + [0, \varepsilon]v = a + k + [0, \varepsilon]v \subseteq A + \mathrm{cor}\, K$, hence $x \in \mathrm{cor}(A + \mathrm{cor}\, K)$.

$\square$

Remark 2.1 The reverse implication in Lemma 2.5 (3°) is false (for $A := E$ and any convex cone K with $\mathrm{aff}\, K \neq E$ we have $\mathrm{aff}(K - K) \neq E = \mathrm{aff}((A + K) - (A + K))$ and $A + \mathrm{icor}\, K = E = \mathrm{icor}(A + K)$).

3 Pareto Efficiency and Weak Pareto Efficiency in Vector Optimization

Given two real linear spaces X and E, a nonempty feasible set $\Omega \subseteq X$, and a vector-valued objective function $f : X \to E$, we consider the following vector optimization problem:

$$f(x) \to \min \text{ w.r.t. } K$$
$$x \in \Omega, \tag{P}$$

where the image space E is preordered by a cone K such that (3) is fulfilled. It is well-known that K induces on E a preorder relation $\leqq_K$ defined, for any two points $y, \overline{y} \in E$, by

$$y \leqq_K \overline{y} \quad :\Longleftrightarrow \quad y \in \overline{y} - K.$$

For notational convenience, we consider the binary relations $\leq_K^0$, $\leq_K$ and $<_K$ that are defined, for any two points $y, \overline{y} \in E$, by

$$y \leq_K^0 \overline{y} \quad :\Longleftrightarrow \quad y \in \overline{y} - K \setminus \{0\},$$
$$y \leq_K \overline{y} \quad :\Longleftrightarrow \quad y \in \overline{y} - K \setminus \ell(K),$$
$$y <_K \overline{y} \quad :\Longleftrightarrow \quad y \in \overline{y} - \text{icor } K.$$

One type of solutions of the problem (P) can be defined according to the next definition (see, e.g., Bagdasar and Popovici [6, Sec. 2.2], Jahn [34, Def. 4.1], Khazayel et al. [36, Sec. 5], and Luc [38, Def. 2.1]).

Definition 3.1 (*Pareto efficiency*) A point $\overline{x} \in \Omega$ is said to be a Pareto efficient solution if for any $x \in \Omega$ the condition $f(x) \leqq_K f(\overline{x})$ implies $f(\overline{x}) \leqq_K f(x)$. The set of all Pareto efficient solutions of (P) is denoted by

$$\text{Eff}(\Omega \mid f, K) := \{\overline{x} \in \Omega \mid \forall x \in \Omega : f(x) \leqq_K f(\overline{x}) \Rightarrow f(\overline{x}) \leqq_K f(x)\}.$$

The following representations of $\text{Eff}(\Omega \mid f, K)$ are well-known.

Lemma 3.1 *Suppose that K satisfies (3). The following assertions hold:*

1° $\text{Eff}(\Omega \mid f, K) = \{\overline{x} \in \Omega \mid \nexists x \in \Omega : f(x) \leq_K f(\overline{x})\}.$
2° *If K is pointed, then*
    $\text{Eff}(\Omega \mid f, K) = \{\overline{x} \in \Omega \mid \nexists x \in \Omega : f(x) \leq_K^0 f(\overline{x})\} =: \text{Eff}_0(\Omega \mid f, K).$

Remark 3.1 Some authors are also interested to compute solutions of the set $\text{Eff}_0(\Omega \mid f, K)$ from Lemma 3.1 (2°) when K is a (not necessarily pointed) convex cone (see, e.g., Bao and Mordukhovich [7, p. 302]). Notice that $\text{Eff}_0(\Omega \mid f, K) \subseteq \text{Eff}(\Omega \mid f, K)$ and $\text{Eff}(\Omega \mid f, Q_0) = \text{Eff}_0(\Omega \mid f, Q_0) = \text{Eff}(\Omega \mid f, K)$, where $Q_0 = (K \setminus \ell(K)) \cup \{0\}$ (see also Lemma 2.3).

Lemma 3.2 *Assume that $K_1, K_2 \subseteq E$ are convex cones with $K_1 \setminus \ell(K_1) \neq \emptyset \neq K_2 \setminus \ell(K_2)$. Then, the following assertions hold:*

1° *If $K_1 \setminus \ell(K_1) \subseteq K_2 \setminus \ell(K_2)$, then* $\mathrm{Eff}(\Omega \mid f, K_2) \subseteq \mathrm{Eff}(\Omega \mid f, K_1)$.

2° *If $(K_1 \setminus \ell(K_1)) \cap \ell(K_2) = \emptyset$, then* $\mathrm{Eff}(\Omega \mid f, K_2) \subseteq \mathrm{Eff}(\Omega \mid f, K_1 \cap K_2)$.

Proof 1° Clearly, if $x \in \mathrm{Eff}(\Omega \mid f, K_2)$ and $\emptyset \neq K_1 \setminus \ell(K_1) \subseteq K_2 \setminus \ell(K_2)$, then $f[\Omega] \cap (f(x) - K_1 \setminus \ell(K_1)) \subseteq f[\Omega] \cap (f(x) - K_2 \setminus \ell(K_2)) = \emptyset$, i.e., $x \in \mathrm{Eff}(\Omega \mid f, K_1)$.

2° Follows by 1° (applied for $K_1 \cap K_2$ in the role of K_1). Indeed, since

$$
\begin{aligned}
(K_1 \cap K_2) \setminus \ell(K_1 \cap K_2) &= [(K_1 \cap K_2) \setminus (\ell(K_1) \cup \ell(K_2))] \\
&\quad \cup [(\ell(K_1) \cap (K_2 \setminus \ell(K_2)))] \\
&\quad \cup [(K_1 \setminus \ell(K_1)) \cap \ell(K_2)],
\end{aligned}
$$

it is easy to check that $(K_1 \cap K_2) \setminus \ell(K_1 \cap K_2) \subseteq K_2 \setminus \ell(K_2)$ if and only if $(K_1 \setminus \ell(K_1)) \cap \ell(K_2) = \emptyset$. $\qquad \square$

A kind of weak solution concept for the vector optimization problem (P) will be given in the next definition where the intrinsic core of the convex cone K is used.

Definition 3.2 (*Weak Pareto efficiency*) A point $\overline{x} \in \Omega$ is said to be a weakly Pareto efficient solution if there is no $x \in \Omega$ such that $f(x) <_K f(\overline{x})$. The set of all weakly Pareto efficient solutions of (P) is denoted by

$$
\mathrm{WEff}(\Omega \mid f, K) := \{\overline{x} \in \Omega \mid \nexists x \in \Omega : f(x) <_K f(\overline{x})\}.
$$

Remark 3.2 The weak solution concept considered in Definition 3.2, which is based on the intrinsic core notion, is also studied by Adán and Novo [2, Def. 5], Bao and Mordukhovich [7, p. 303], Khazayel et al. [36, Sec. 5] and Zhou, Yang and Peng [45, Def. 4.1].

It is obvious that $\mathrm{Eff}(\Omega \mid f, K) \subseteq \mathrm{WEff}(\Omega \mid f, K)$, and if K is not relatively solid, then $\mathrm{WEff}(\Omega \mid f, K) = \Omega$. Moreover, since $\mathrm{icor}\, K = (\mathrm{icor}\, K) \setminus (-\mathrm{icor}\, K)$, one can easily check that

$$
\mathrm{WEff}(\Omega \mid f, K) = \{\overline{x} \in \Omega \mid \forall x \in \Omega : f(x) <_K f(\overline{x}) \Rightarrow f(\overline{x}) <_K f(x)\}.
$$

Next, we present some localization results for the image points of weakly Pareto efficient solutions:

Lemma 3.3 *Suppose that K is relatively solid and satisfies* (3). *Then, the following assertions hold:*

1° *If $K \subseteq \mathrm{aff}(f[\Omega] - f[\Omega])$, then $f[\mathrm{WEff}(\Omega \mid f, K)] \subseteq f[\Omega] \setminus \mathrm{icor}\, f[\Omega]$.*

2° *If $f[\Omega]$ is solid, then $f[\mathrm{WEff}(\Omega \mid f, K)] \subseteq f[\Omega] \setminus \mathrm{cor}\, f[\Omega]$.*

3° *$f[\mathrm{WEff}(\Omega \mid f, K)] \subseteq f[\Omega] \setminus \mathrm{icor}(f[\Omega] + K)$.*

4° *If K is solid, then $f[\mathrm{WEff}(\Omega \mid f, K)] \subseteq f[\Omega] \setminus \mathrm{cor}(f[\Omega] + K) = f[\Omega] \setminus (f[\Omega] + \mathrm{cor}\, K)$.*

Proof 1° Assume that $K \subseteq \mathrm{aff}(f[\Omega] - f[\Omega])$. Take some $x \in \mathrm{WEff}(\Omega \mid f, K)$. Obviously, $f(x) \in f[\Omega]$. On the contrary, assume that $f(x) \in \mathrm{icor}\, f[\Omega]$. Thus, for $v \in -\mathrm{icor}\, K \subseteq \mathrm{aff}(f[\Omega] - f[\Omega])$ there is $\varepsilon > 0$ such that $f(x) + \varepsilon v \in f[\Omega]$, i.e., $f(\bar{x}) = f(x) + \varepsilon v$ for some $\bar{x} \in \Omega$. Then, $f(x) - f(\bar{x}) = -\varepsilon v \in \varepsilon \cdot (\mathrm{icor}\, K) \subseteq \mathrm{icor}\, K$, hence $f(\bar{x}) \in f[\Omega] \cap (f(x) - \mathrm{icor}\, K)$, a contradiction to $x \in \mathrm{WEff}(\Omega \mid f, K)$.

2° If $f[\Omega]$ is solid, then $E = \mathrm{aff}(f[\Omega] - f[\Omega])$ and $\mathrm{icor}\, f[\Omega] = \mathrm{cor}\, f[\Omega]$, hence 2° is a direct consequence of 1°.

3° Take some $x \in \mathrm{WEff}(\Omega \mid f, K)$. Obviously, $f(x) \in f[\Omega]$. On the contrary, assume that $f(x) \in \mathrm{icor}(f[\Omega] + K)$. Thus, for $v \in -\mathrm{icor}\, K \subseteq \mathrm{aff}(K - K) \subseteq \mathrm{aff}((f[\Omega]+K)-(f[\Omega]+K))$ there is $\varepsilon > 0$ such that $f(x)+\varepsilon v \in f[\Omega]+K$, i.e., $f(x) + \varepsilon v = f(\bar{x}) + k$ for some $\bar{x} \in \Omega$ and some $k \in K$. Then, $f(x) - f(\bar{x}) = k - \varepsilon v \in K + \mathrm{icor}\, K = \mathrm{icor}\, K$, hence $f(\bar{x}) \in f[\Omega] \cap (f(x) - \mathrm{icor}\, K)$, a contradiction to $x \in \mathrm{WEff}(\Omega \mid f, K)$.

4° Is a direct consequence of 3° and Lemma 2.5 (6°). $\qquad\square$

Remark 3.3 Notice that the condition $K \subseteq \mathrm{aff}(f[\Omega] - f[\Omega])$ in Lemma 3.3 (1°) is not superfluous (since, for $K \neq \{0\}$ and $\Omega = \{x\}$, we have $f[\Omega] = \{f(x)\} = \mathrm{icor}\, f[\Omega]$ and $K \neq \{0\} = \mathrm{aff}(f[\Omega] - f[\Omega])$ but $f[\mathrm{WEff}(\Omega \mid f, K)] = \{f(x)\} \not\subseteq \emptyset = f[\Omega] \setminus \mathrm{icor}\, f[\Omega]$).

Lemma 3.4 *Suppose that K is relatively solid and satisfies* (3). *Consider $P_0 = (\mathrm{icor}\, K) \cup \{0\}$ (from Lemma 2.4). Then,*

$$\mathrm{Eff}(\Omega \mid f, K) \subseteq \mathrm{WEff}(\Omega \mid f, K) \quad \text{and} \quad \mathrm{Eff}(\Omega \mid f, P_0) = \mathrm{WEff}(\Omega \mid f, K).$$

Proof First, let us show the inclusion $\mathrm{Eff}(\Omega \mid f, K) \subseteq \mathrm{WEff}(\Omega \mid f, K)$. Take some $\bar{x} \in \mathrm{Eff}(\Omega \mid f, K)$. By Lemma 2.2, $\mathrm{icor}\, K \subseteq K \setminus \ell(K)$, and so, applying Lemma 3.1 (1°) we infer $f[\Omega] \cap (f(\bar{x}) - \mathrm{icor}\, K) \subseteq f[\Omega] \cap (f(\bar{x}) - K \setminus \ell(K)) = \emptyset$. Thus, $\bar{x} \in \mathrm{WEff}(\Omega \mid f, K)$.

Let us show the remaining equality $\mathrm{Eff}(\Omega \mid f, P_0) = \mathrm{WEff}(\Omega \mid f, K)$. Since K is a relatively solid, convex cone with $K \neq \ell(K)$ (hence, $0 \notin \mathrm{icor}\, K$), we know that P_0 is a relatively solid, pointed, convex cone as well. Thus, we conclude

$$\begin{aligned}
\mathrm{Eff}(\Omega \mid f, P_0) &= \{\bar{x} \in \Omega \mid f[\Omega] \cap (f(\bar{x}) - P_0 \setminus \ell(P_0)) = \emptyset\} \\
&= \{\bar{x} \in \Omega \mid f[\Omega] \cap (f(\bar{x}) - P_0 \setminus \{0\}) = \emptyset\} \\
&= \{\bar{x} \in \Omega \mid f[\Omega] \cap (f(\bar{x}) - \mathrm{icor}\, K) = \emptyset\} \\
&= \mathrm{WEff}(\Omega \mid f, K).
\end{aligned}$$

$\qquad\square$

Remark 3.4 Taking into account Remark 3.1, all known results which are related to the set $\mathrm{Eff}_0(\Omega \mid f, \bar{K})$ for a (pointed) convex cone $\bar{K}$ (however, without assuming closedness of $\bar{K}$) are useful to derive results for $\mathrm{Eff}(\Omega \mid f, K)$ and $\mathrm{WEff}(\Omega \mid f, K)$ since $\mathrm{Eff}_0(\Omega \mid f, Q_0) = \mathrm{Eff}(\Omega \mid f, K)$ and $\mathrm{Eff}(\Omega \mid f, P_0) = \mathrm{Eff}_0(\Omega \mid f, P_0) = \mathrm{WEff}(\Omega \mid f, K)$ for the (nontrivial, pointed) convex cones Q_0 and P_0 (see also

Lemmata 2.3 and 2.4). For instance, algorithms derived by Günther and Popovici [26] for discrete vector optimization problems involving a (not necessarily closed) nontrivial, pointed, convex cone K could be applied for Q_0 and P_0, respectively. In this way, assuming Ω has a finite number of elements, one can compute the sets $\mathrm{Eff}_0(\Omega \mid f, K)$, $\mathrm{Eff}(\Omega \mid f, K)$ and $\mathrm{WEff}(\Omega \mid f, K)$ by using the effective algorithms in [26].

Lemma 3.5 *Assume that $K_1, K_2 \subseteq E$ are convex cones. Then, the following assertions hold:*

1° *If* icor $K_1 \subseteq$ icor K_2, *then* $\mathrm{WEff}(\Omega \mid f, K_2) \subseteq \mathrm{WEff}(\Omega \mid f, K_1)$.
2° *If* (icor K_1) $\cap$ icor $K_2 \neq \emptyset$, *then*
   $\mathrm{WEff}(\Omega \mid f, K_1) \cup \mathrm{WEff}(\Omega \mid f, K_2) \subseteq \mathrm{WEff}(\Omega \mid f, K_1 \cap K_2)$.
3° *If* aff $K_1 \subseteq$ aff K_2 *and* $K_1 \cap (\mathrm{icor}\ K_2) \neq \emptyset$, *then* $\mathrm{aff}(K_1 \cap K_2) = \mathrm{aff}\ K_1$ *and*
   $\mathrm{WEff}(\Omega \mid f, K_1) \cup \mathrm{WEff}(\Omega \mid f, K_2) \subseteq \mathrm{WEff}(\Omega \mid f, K_1 \cap K_2)$.

Proof 1° Clearly, if $x \in \mathrm{WEff}(\Omega \mid f, K_2)$, then $f[\Omega] \cap (f(x) - \mathrm{icor}\ K_1) \subseteq f[\Omega] \cap (f(x) - \mathrm{icor}\ K_2) = \emptyset$, i.e., $x \in \mathrm{WEff}(\Omega \mid f, K_1)$.

2° Notice that $\mathrm{icor}(\Omega^1 \cap \Omega^2) = (\mathrm{icor}\ \Omega^1) \cap \mathrm{icor}\ \Omega^2$ if $\Omega^1, \Omega^1 \subseteq E$ are convex sets with $(\mathrm{icor}\ \Omega^1) \cap \mathrm{icor}\ \Omega^2 \neq \emptyset$ (see Novo and Zălinescu [40, Cor. 2.1]). Thus, for any $i = 1, 2$, we have $\mathrm{icor}(K_1 \cap K_2) = (\mathrm{icor}\ K_1) \cap \mathrm{icor}\ K_2 \subseteq \mathrm{icor}\ K_i$, and so, 2° follows easily by 1°.

3° Assume that aff $K_1 \subseteq$ aff K_2 and $K_1 \cap (\mathrm{icor}\ K_2) \neq \emptyset$. By Novo and Zălinescu [40, Lem. 2.1], we get $\mathrm{aff}(K_1 \cap K_2) = \mathrm{aff}\ K_1$ and $(\mathrm{icor}\ K_1) \cap \mathrm{icor}\ K_2 = \mathrm{icor}(K_1 \cap K_2)$, and so, 3° follows also easily by 1°. □

4 Henig-type Proper Efficiency in Vector Optimization

As usual for Henig-type proper efficiency concepts, (generalized) dilating cones for the cone K (which satisfies (3)) will play an important role in our work. More precisely, our considered proper efficiency concepts will mainly be based on two specific families of cones, namely $\mathcal{C}(K)$ and $\mathcal{D}(K)$, that we introduce in the next section.

4.1 Generalized Dilating Cones

Let us define two specific families of convex cones related to K,

$$\mathcal{C}(K) := \{C \subseteq E \mid C \text{ is a convex cone with } K \setminus \ell(K) \subseteq \mathrm{icor}\ C \text{ and } C \neq \ell(C)\}.$$

and

$$\mathcal{D}(K) := \{D \subseteq E \mid D \text{ is a nontrivial, convex cone with } K \setminus \ell(K) \subseteq \mathrm{cor}\ D\}.$$

It is easy to check that $\mathcal{D}(K) \subseteq \mathcal{C}(K)$. Moreover, $K \subseteq \mathrm{acl}(K \setminus \ell(K)) \subseteq \mathrm{acl}(\mathrm{icor}\ C) = \mathrm{acl}\ C$ for $C \in \mathcal{C}(K)$ as well as $K \subseteq \mathrm{acl}\ D$ for $D \in \mathcal{D}(K)$. In Khan, Tammer and Zălinescu [35, Def. 2.4.14] (applied for (E, τ_c)), the cones from the set $\mathcal{D}(K)$ are

called "generalized dilating cones." We will also use this name for the cones of the family $\mathcal{C}(K)$.

From Khan, Tammer and Zălinescu [35, Lem. 2.4.15], we derive the following result, which states some important relationships between the cone K and cones from the set $\mathcal{D}(K)$.

Lemma 4.1 *Suppose that K satisfies* (3). *Then, the following assertions hold:*

1° $\mathcal{D}(K) \neq \emptyset \iff K^{\&} \neq \emptyset$.
2° *If $D \in \mathcal{D}(K)$, then* $\mathrm{cl}_{\tau_c}(K \setminus \ell(K)) + \mathrm{cor}\, D = K \setminus \ell(K) + \mathrm{cor}\, D$.
3° *If $D \in \mathcal{D}(K)$, then* $K + \mathrm{cor}\, D = \mathrm{cor}\, D$ *and* $K + \mathrm{acl}\, D = \mathrm{acl}\, D$.
4° $K^{\&} = \bigcup_{D \in \mathcal{D}(K)} D^{+} \setminus \{0\}$.
5° $K \in \mathcal{D}(K) \iff K \setminus \ell(K) = \mathrm{cor}\, K \iff K = (\mathrm{cor}\, K) \cup \ell(K)$.
6° *If $K \in \mathcal{D}(K)$, then K is solid.*

Proof 1°–4° Can easily be derived from Khan, Tammer and Zălinescu [35, Lem. 2.4.15] taking into account $\mathrm{int}_{\tau_c} S = \mathrm{cor}\, S$ for any convex set $S \subseteq E$.

5° Since $K \neq \ell(K)$, we have $\mathrm{cor}\, K \subseteq K \setminus \ell(K)$ and $(\mathrm{cor}\, K) \cap \ell(K) = \emptyset$, hence $K \in \mathcal{D}(K) \iff \mathrm{cor}\, K = K \setminus \ell(K) \iff K = (\mathrm{cor}\, K) \cup \ell(K)$.

6° If $K \in \mathcal{D}(K)$, by 5°, we have $\mathrm{cor}\, K = K \setminus \ell(K) \neq \emptyset$. □

The next lemma, which includes also an intrinsic counterpart to Lemma 4.1, states relationships between the cone K and cones from $\mathcal{C}(K)$ and $\mathcal{D}(K)$.

Lemma 4.2 *Suppose that K satisfies* (3). *Then, the following assertions hold:*

1° $\mathcal{C}(K) \neq \emptyset \iff K^{\&} \neq \emptyset$.
2° *If $C \in \mathcal{C}(K)$, then* $\mathrm{acl}(K \setminus \ell(K)) + \mathrm{icor}\, C = K \setminus \ell(K) + \mathrm{icor}\, C$.
3° *If $C \in \mathcal{C}(K)$, then* $K + \mathrm{icor}\, C = \mathrm{icor}\, C$ *and* $K + \mathrm{acl}\, C = \mathrm{acl}\, C$.
4° $K^{\&} = \bigcup_{C \in \mathcal{C}(K)} C^{+} \setminus \ell(C^{+})$.
5° $K \in \mathcal{C}(K) \iff K \setminus \ell(K) = \mathrm{icor}\, K \iff K = (\mathrm{icor}\, K) \cup \ell(K)$.
6° *If $K \in \mathcal{C}(K)$, then K is relatively solid.*
7° *If K is solid, then $\mathcal{C}(K) = \mathcal{D}(K)$.*

Proof For notational convenience, let us define $Q := K \setminus \ell(K)$.

1° If $\mathcal{C}(K) \neq \emptyset$, then there is $C \in \mathcal{C}(K)$ (in particular, $C \neq \ell(C)$ and $\mathrm{icor}\, C \neq \emptyset$) such that $\emptyset \neq C^{+} \setminus \ell(C^{+}) = ((\mathrm{icor}\, C) \cup \{0\})^{\#} \subseteq (Q \cup \{0\})^{\#} = K^{\&}$.
   If $K^{\&} \neq \emptyset$, then by Lemma 4.1 (1°) we get $\emptyset \neq \mathcal{D}(K) \subseteq \mathcal{C}(K)$.

2° Since $Q \subseteq \mathrm{acl}\, Q$, the inclusion "$\supseteq$" is clear. In order to show "$\subseteq$", take some $x \in (\mathrm{acl}\, Q) + \mathrm{icor}\, C$. There exist $k \in \mathrm{acl}\, Q$ and $c \in \mathrm{icor}\, C$ such that $x = k + c$. For the case $k \in Q$, the inclusion holds. Now, assume that $k \in (\mathrm{acl}\, Q) \setminus Q$. Consequently, there is $\overline{k} \in Q$ such that $[\overline{k}, k) \subseteq Q$. Define $v := k - \overline{k}$. Since $Q \subseteq C$, we have $k, \overline{k} \in \mathrm{acl}\, Q \subseteq \mathrm{acl}\, C$. Hence, (by [36, Lem. 2.7]), we get $v \in \mathrm{acl}\, C - \mathrm{acl}\, C = C - C = \mathrm{aff}(C - C)$. Thus, there is $\varepsilon > 0$ such that $c + [0, \varepsilon]v \subseteq C$. Using [35, Lem. 2.5 (4)], there is $\delta \in (0, \min\{1, \varepsilon\})$ such that $c + [0, \delta]v \subseteq \mathrm{icor}\, C$. Finally, we get

$$x = k + c = (k - \delta v) + (c + \delta v) = ((1 - \delta)k + \delta \overline{k}) + (c + \delta v)$$
$$\in (\overline{k}, k) + \mathrm{icor}\, C \subseteq Q + \mathrm{icor}\, C.$$

3° Using similar ideas as in the proof of [35, Lem. 2.4.15], one gets

$$
\begin{aligned}
\operatorname{icor} C &\subseteq K + \operatorname{icor} C & &(\text{since } 0 \in K) \\
&\subseteq \operatorname{acl} Q + \operatorname{icor} C & &(\text{in view of Lemma } 2.2) \\
&= Q + \operatorname{icor} C & &(\text{in view of } 2°) \\
&\subseteq \operatorname{icor} C + \operatorname{icor} C & &(\text{since } C \in \mathcal{C}(K)) \\
&\subseteq \operatorname{icor} C,
\end{aligned}
$$

hence $Q + \operatorname{icor} C = \operatorname{icor} C$.

Because $0 \in K \subseteq \operatorname{acl} C$, we get $K + \operatorname{acl} C = \operatorname{acl} C$.

4° The result is obvious if $K^{\&} = \emptyset$ (see part 1°). Otherwise, since $D \in \mathcal{D}(K) \subseteq \mathcal{C}(K)$ is solid, hence D^{+} is pointed (i.e., $\ell(D^{+}) = \{0\}$), we get

$$
K^{\&} = \bigcup_{D \in \mathcal{D}(K)} D^{+} \setminus \{0\} \subseteq \bigcup_{C \in \mathcal{C}(K)} C^{+} \setminus \ell(C^{+})
$$

taking into account Lemma 4.1 (4°).

For any $C \in \mathcal{C}(K)$, we have $K^{\&} \supseteq C^{+} \setminus \ell(C^{+})$, as the proof of 1° shows.

5° Since $K \neq \ell(K)$, we have $\operatorname{icor} K \subseteq Q$ and $(\operatorname{icor} K) \cap \ell(K) = \emptyset$, hence $K \in \mathcal{C}(K) \iff \operatorname{icor} K = Q \iff K = (\operatorname{icor} K) \cup \ell(K)$.

6° If $K \in \mathcal{C}(K)$, by 5°, we have $\operatorname{icor} K = Q \neq \emptyset$.

7° Suppose that K is solid. $\mathcal{D}(K) \subseteq \mathcal{C}(K)$ is clear. Take some $C \in \mathcal{C}(K)$. First, notice that C is a nontrivial, convex cone. Because $\operatorname{cor} K \subseteq Q \subseteq \operatorname{icor} C$, we get $\operatorname{icor} C = \operatorname{cor} C \neq \emptyset$. Thus, we conclude that $C \in \mathcal{D}(K)$.   □

Remark 4.1 By Lemma 2.1 (1°), Lemma 4.1 (1°) and Lemma 4.2 (1°), for any τ_c-closed, convex cone $K \subseteq E$ with a relatively solid dual cone K^{+}, we have $K^{\&} \neq \emptyset$, and so $\mathcal{C}(K) \supseteq \mathcal{D}(K) \neq \emptyset$.

4.2 Henig Proper Efficiency

In the following, we study a well-known Henig-type proper efficiency concept.

Definition 4.1 (*Proper efficiency in the sense of Henig*)

A point $x \in \Omega$ is said to be a classical Henig properly efficient solution if there is a nontrivial, convex cone $D \subseteq E$ with $K \setminus \ell(K) \subseteq \operatorname{cor} D$ (i.e., $D \in \mathcal{D}(K)$) such that $x \in \operatorname{Eff}(\Omega \mid f, D)$. The set of all classical Henig properly efficient solutions of (P) is denoted by $\operatorname{PEff}_c(\Omega \mid f, K)$.

Remark 4.2 Consider a real linear topological space (E, τ). The original concept of proper efficiency proposed by Henig in [30] is formulated with the family of dilating cones

$$
\mathcal{D}_{\operatorname{He}}(K) := \{D \subseteq E \mid D \text{ is a convex cone with } K \setminus \{0\} \subseteq \operatorname{int}_{\tau} D\}.
$$

According to Henig [30, Def. 2.1], a point $x \in \Omega$ is properly efficient if there is $D \in \mathcal{D}_{He}(K)$ such that $x \in \text{Eff}_0(\Omega \mid f, D)$. El Maghri and Laghdir [17], Khan, Tammer and Zălinescu [35, Def. 2.4.13], and Luc [38, Def. 2.1] are studying the concept from Definition 4.1 with the family of dilating cones

$$\mathcal{D}_{KTZ}(K) := \{D \subseteq E \mid D \text{ is a nontrivial, convex cone with } K \setminus \ell(K) \subseteq \text{int}_\tau D\}$$

in the role of $\mathcal{D}(K)$. Notice that $\mathcal{D}_{He}(K) \setminus \{E\} \subseteq \mathcal{D}_{KTZ}(K) \subseteq \mathcal{D}(K) \subseteq \mathcal{C}(K)$. If E is endowed with the convex core topology τ_c, then $\mathcal{D}_{KTZ}(K) = \mathcal{D}(K)$, and if further K is pointed, then $\mathcal{D}_{He}(K) \setminus \{E\} = \mathcal{D}_{KTZ}(K) = \mathcal{D}(K)$.

Lemma 4.3 ([35, Sec. 2.4]) *Suppose that K satisfies* (3)*. Then, the following assertions hold:*

$1°$ $\text{PEff}_c(\Omega \mid f, K) \subseteq \text{Eff}(\Omega \mid f, K)$.
$2°$ $\text{PEff}_c(\Omega \mid f, K) = \bigcup_{D \in \mathcal{D}(K)} \text{Eff}(\Omega \mid f, D) = \bigcup_{D \in \mathcal{D}(K)} \text{WEff}(\Omega \mid f, D)$
    $= \bigcup_{D \in \mathcal{D}(K)} \text{WEff}(\Omega \mid f, \text{acl } D)$.
$3°$ *If $K \in \mathcal{D}(K)$, then* $\text{PEff}_c(\Omega \mid f, K) = \text{Eff}(\Omega \mid f, K) = \text{WEff}(\Omega \mid f, K)$.

4.3 An Extension of Henig Proper Efficiency

In the following, we will propose an extension of the concept of proper efficiency in the sense of Henig [30]. To our knowledge, it is a extended approach to use the family $\mathcal{C}(K)$ of generalized dilating cones of K in order to define a new Henig-type proper efficiency concept.

Definition 4.2 (*Extended proper efficiency in the sense of Henig*)
    A point $x \in \Omega$ is said to be a Henig properly efficient solution if there is a convex cone $C \subseteq E$ with $K \setminus \ell(K) \subseteq \text{icor } C$ and $C \neq \ell(C)$ (i.e., $C \in \mathcal{C}(K)$) such that $x \in \text{Eff}(\Omega \mid f, C)$. The set of all Henig properly efficient solutions of (P) is denoted by $\text{PEff}(\Omega \mid f, K)$.

Remark 4.3 Zhou, Yang and Peng [45, Def. 4.2] introduced a similar Henig-type proper efficiency concept based on the family of generalized dilating cones

$$\mathcal{C}_{ZYP}(K) := \{C \subseteq E \mid C \text{ is a nontrivial, pointed, convex cone with } K \setminus \{0\} \subseteq \text{icor } C\},$$

where the authors assume that K is a nontrivial, pointed, convex cone. Hence, this family $\mathcal{C}_{ZYP}(K)$ is always contained in the family $\mathcal{C}(K)$ (however, notice that $C \in \mathcal{C}(K)$ may not be pointed). In Remark 4.7, we will take a closer look on the relationships between our concept from Definition 4.2 and the concept proposed by Zhou, Yang and Peng [45, Def. 4.2].

First properties for the set of Henig properly efficient solutions (in the sense of Definition 4.2) are studied in the following lemma.

Lemma 4.4 *Suppose that K satisfies* (3)*. Then, the following assertions hold:*

$1°$ $\mathrm{PEff}_c(\Omega \mid f, K) \subseteq \mathrm{PEff}(\Omega \mid f, K) \subseteq \mathrm{Eff}(\Omega \mid f, K)$.

$2°$ *If $\mathcal{C}(K) = \mathcal{D}(K)$, then $\mathrm{PEff}_c(\Omega \mid f, K) = \mathrm{PEff}(\Omega \mid f, K)$.*

$3°$ *If $\mathcal{C}(K) = \emptyset$ ($\iff \mathcal{D}(K) = \emptyset \iff K^{\&} = \emptyset$), then $\mathrm{PEff}_c(\Omega \mid f, K) =$ $\mathrm{PEff}(\Omega \mid f, K) = \emptyset$.*

$4°$ $\mathrm{PEff}(\Omega \mid f, K) = \bigcup_{C \in \mathcal{C}(K)} \mathrm{Eff}(\Omega \mid f, C) = \bigcup_{C \in \mathcal{C}(K)} \mathrm{WEff}(\Omega \mid f, C)$.

$5°$ *If $K \in \mathcal{C}(K)$, then $\mathrm{PEff}(\Omega \mid f, K) = \mathrm{Eff}(\Omega \mid f, K) = \mathrm{WEff}(\Omega \mid f, K)$.*

Proof $1°$ Since $\mathcal{D}(K) \subseteq \mathcal{C}(K)$, we have $\mathrm{PEff}_c(\Omega \mid f, K) \subseteq \mathrm{PEff}(\Omega \mid f, K)$. Now, let $x \in \mathrm{PEff}(\Omega \mid f, K)$. Hence, there is $C \in \mathcal{C}(K)$ with $x \in \mathrm{Eff}(\Omega \mid f, C)$. In view of Lemma 2.2, we have $\emptyset \neq K \setminus \ell(K) \subseteq \mathrm{icor}\, C \subseteq C \setminus \ell(C)$. Thus, by Lemma 3.2 $(1°)$, we conclude $x \in \mathrm{Eff}(\Omega \mid f, C) \subseteq \mathrm{Eff}(\Omega \mid f, K)$.

$2°$ This assertion is obvious.

$3°$ Follows by the definition of $\mathrm{PEff}(\Omega \mid f, K)$ taking into account Lemmata 4.1 and 4.2 .

$4°$ By the definition of $\mathrm{PEff}(\Omega \mid f, K)$, and since $\mathrm{Eff}(\Omega \mid f, C) \subseteq \mathrm{WEff}(\Omega \mid f, C)$ for any $C \in \mathcal{C}(K)$ (by Lemma 3.4), we have

$$\mathrm{PEff}(\Omega \mid f, K) = \bigcup_{C \in \mathcal{C}(K)} \mathrm{Eff}(\Omega \mid f, C) \subseteq \bigcup_{C \in \mathcal{C}(K)} \mathrm{WEff}(\Omega \mid f, C). \quad (4)$$

Fix an arbitrarily $\bar{C} \in \mathcal{C}(K)$ and define $\tilde{C} := (\mathrm{icor}\, \bar{C}) \cup \{0\}$. By Lemma 2.4 (applied for $\bar{C}$ in the role of K), $\tilde{C}$ is a convex cone with $\emptyset \neq K \setminus \ell(K) \subseteq \mathrm{icor}\, \bar{C} = \mathrm{icor}\, \tilde{C}$ and $\tilde{C} \neq \{0\} = \ell(\tilde{C})$, hence $\tilde{C} \in \mathcal{C}(K)$. Lemma 3.4 yields

$$\mathrm{WEff}(\Omega \mid f, \bar{C}) = \mathrm{Eff}(\Omega \mid f, \tilde{C}) \subseteq \bigcup_{C \in \mathcal{C}(K)} \mathrm{Eff}(\Omega \mid f, C). \quad (5)$$

Combining (4) and (5), we conclude $4°$.

$5°$ Suppose that $K \in \mathcal{C}(K)$. By Lemma 4.2 $(5°)$, we get $K \setminus \ell(K) = \mathrm{icor}\, K \neq \emptyset$, and so $\mathrm{Eff}(\Omega \mid f, K) = \mathrm{WEff}(\Omega \mid f, K)$. Moreover, by $1°$, we have $\mathrm{PEff}(\Omega \mid f, K) \subseteq \mathrm{Eff}(\Omega \mid f, K)$, and by $4°$ and the fact that $K \in \mathcal{C}(K)$,

$$\mathrm{Eff}(\Omega \mid f, K) \subseteq \bigcup_{C \in \mathcal{C}(K)} \mathrm{Eff}(\Omega \mid f, C) = \mathrm{PEff}(\Omega \mid f, K).$$

$\square$

Remark 4.4 In view of Lemma 4.2 $(5°)$, for any not relatively solid, convex cone K we have $K \notin \mathcal{C}(K)$.

Remark 4.5 Lemma 4.4 $(1°)$ motivates our proper efficiency concept given in Definition 4.2. It is important to know that any inclusion stated in Lemma 4.4 $(1°)$ can be strict. Indeed, if K is not solid, we will see in Example 5.3 that $\mathrm{PEff}_c(\Omega \mid f, K) = \mathrm{PEff}(\Omega \mid f, K)$ does not hold in general (even in the case that $f[\Omega] + K$ is a solid, convex set). More precisely, in Example 5.3, we consider a relatively solid (but not solid), convex cone K which satisfies $K \notin \mathcal{C}(K)$ and

$$\mathrm{PEff}_c(\Omega \mid f, K) \subsetneq \mathrm{PEff}(\Omega \mid f, K) \subsetneq \mathrm{Eff}(\Omega \mid f, K) = \mathrm{WEff}(\Omega \mid f, K).$$

Lemma 4.5 *Suppose that K satisfies* (3). *Then,*

$$\text{PEff}(\Omega \mid f, K) = \bigcup_{C \in \overline{\mathcal{C}}(K)} \text{WEff}(\Omega \mid f, C) = \bigcup_{C \in \mathcal{C}_0(K)} \text{WEff}(\Omega \mid f, C)$$

$$= \bigcup_{C \in \mathcal{C}_0(K)} \text{Eff}(\Omega \mid f, C),$$

where

$\overline{\mathcal{C}}(K) := \{C \subseteq E \mid C \text{ is a convex cone with } K \setminus \ell(K) \subseteq \text{icor } C \text{ and acl } C = C \neq \ell(C)\}$,
$\mathcal{C}_0(K) := \{C \subseteq E \mid C \text{ is a nontrivial, pointed, convex cone with } K \setminus \ell(K) \subseteq \text{icor } C\}$.

Proof Clearly, $\overline{\mathcal{C}}(K) \subseteq \mathcal{C}(K)$ and $\mathcal{C}_0(K) \subseteq \mathcal{C}(K)$, hence

$$\bigcup_{C \in \overline{\mathcal{C}}(K) \cup \mathcal{C}_0(K)} \text{WEff}(\Omega \mid f, C) \subseteq \bigcup_{C \in \mathcal{C}(K)} \text{WEff}(\Omega \mid f, C) = \text{PEff}(\Omega \mid f, K)$$

in view of Lemma 4.4 (4°). Take some $C \in \mathcal{C}(K)$, i.e., C is a relatively solid, convex cone with $\emptyset \neq K \setminus \ell(K) \subseteq \text{icor } C$ and $C \neq \ell(C)$. Define $\bar{C} := (C \setminus \ell(C)) \cup \{0\}$ and $\tilde{C} := (\text{icor } C) \cup \{0\}$. By Lemmas 2.3 and 2.4 (applied for the convex cone C), we get $\text{icor } C = \text{icor } \bar{C} = \text{icor } \tilde{C} = \text{icor}(\text{acl } C)$. Notice that the convex cones $\bar{C}$ and $\tilde{C}$ are relatively solid, nontrivial and pointed. Since $C(\neq \ell(C))$ is relatively solid, the convex cone acl C satisfies $\text{acl } C = \text{cl}_{\tau_c} C \neq \ell(\text{cl}_{\tau_c} C) = \ell(\text{acl } C)$. Hence, for $\bar{C}, \tilde{C} \in \mathcal{C}_0(K)$ and acl $C \in \overline{\mathcal{C}}(K)$ we have

$$\text{WEff}(\Omega \mid f, C) = \text{WEff}(\Omega \mid f, \bar{C}) = \text{WEff}(\Omega \mid f, \tilde{C}) = \text{WEff}(\Omega \mid f, \text{acl } C).$$

Thus, we derive the first two equalities.

Observing that $\text{Eff}(\Omega \mid f, C) \subseteq \text{WEff}(\Omega \mid f, C) = \text{Eff}(\Omega \mid f, \tilde{C})$ for any $C \in \mathcal{C}_0(K)$, we get the last equality.   □

Remark 4.6 Notice that we have

$$K^{\&} \neq \emptyset \iff \mathcal{D}(K) \neq \emptyset \iff \mathcal{C}(K) \neq \emptyset \iff \mathcal{C}_0(K) \neq \emptyset \iff \overline{\mathcal{C}}(K) \neq \emptyset.$$

Remark 4.7 Consider the pointed, convex cone $Q_0 = (K \setminus \ell(K)) \cup \{0\}$ from Lemma 2.3. For the family $\mathcal{C}_{\text{ZYP}}(Q_0)$ (in the sense of Zhou, Yang and Peng [45], see Remark 4.3), we have $\mathcal{C}_{\text{ZYP}}(Q_0) = \mathcal{C}_0(K) \supseteq \mathcal{C}_{\text{ZYP}}(K)$, hence (by Lemma 4.5),

$$\text{PEff}(\Omega \mid f, K) = \bigcup_{C \in \mathcal{C}_0(K)} \text{Eff}(\Omega \mid f, C)$$

$$= \bigcup_{C \in \mathcal{C}_{\text{ZYP}}(Q_0)} \text{Eff}_0(\Omega \mid f, C) \quad (=: \text{PEff}_{\text{ZYP}}(\Omega \mid f, Q_0))$$

$$\supseteq \bigcup_{C \in \mathcal{C}_{\mathrm{ZYP}}(K)} \mathrm{Eff}_0(\Omega \mid f, C) \quad (=: \mathrm{PEff}_{\mathrm{ZYP}}(\Omega \mid f, K)).$$

If K is pointed, then $Q_0 = K$, $\mathcal{C}_0(K) = \mathcal{C}_{\mathrm{ZYP}}(K)$ and $\mathrm{PEff}(\Omega \mid f, K) = \mathrm{PEff}_{\mathrm{ZYP}}(\Omega \mid f, K)$, hence solutions according to Definition 4.2 are exactly the same solutions according to [45, Def. 4.2]. Notice that all results derived in our paper (for not necessarily pointed, convex cones) are useful for the concepts considered in [45].

Lemma 4.6 *Assume that $K_1, K_2 \subseteq E$ are convex cones with $K_1 \setminus \ell(K_1) \neq \emptyset \neq K_2 \setminus \ell(K_2)$. Then, the following assertions hold:*

$1°$ *If $K_1 \setminus \ell(K_1) \subseteq K_2 \setminus \ell(K_2)$, then $\mathrm{PEff}(\Omega \mid f, K_2) \subseteq \mathrm{PEff}(\Omega \mid f, K_1)$.*
$2°$ *If $(K_1 \cap K_2) \setminus \ell(K_1 \cap K_2) \neq \emptyset$ and $(K_1 \setminus \ell(K_1)) \cap \ell(K_2) = \emptyset$, then $\mathrm{PEff}(\Omega \mid f, K_2) \subseteq \mathrm{PEff}(\Omega \mid f, K_1 \cap K_2)$.*

Proof $1°$ Take some $x \in \mathrm{PEff}(\Omega \mid f, K_2)$. Then, there is $C \in \mathcal{C}(K_2)$ such that $x \in \mathrm{Eff}(\Omega \mid f, C)$. Because $K_1 \setminus \ell(K_1) \subseteq K_2 \setminus \ell(K_2) \subseteq \mathrm{icor}\, C$, we have $C \in \mathcal{C}(K_1)$, hence $x \in \mathrm{PEff}(\Omega \mid f, K_1)$.
$2°$ Directly follows by $1°$ (applied for $K_1 \cap K_2$ in the role of K_1). Notice that $(K_1 \cap K_2) \setminus \ell(K_1 \cap K_2) \subseteq K_2 \setminus \ell(K_2)$ is equivalent to $(K_1 \setminus \ell(K_1)) \cap \ell(K_2) = \emptyset$. □

5 Scalarization Results

Let two real linear spaces X and E, a nonempty feasible set $\Omega \subseteq X$, and a vector-valued objective function $f : X \to E$ be given. Beside the vector optimization problem (P) from Sect. 3, we consider the following scalar optimization problem

$$\begin{cases} (\varphi \circ f)(x) \to \min \\ x \in \Omega, \end{cases} \tag{P_φ}$$

where $\varphi : E \to \mathbb{R}$ is a real-valued function. Clearly, to get useful relationships between the problems (P) and (P_φ) one needs to impose certain properties on φ. By solving the scalar problem (P_φ) (with a specific function φ), one can also get some knowledge about the original vector problem (P). Applying such a strategy is called scalarization method in the literature of vector optimization. Within such methods, the function φ is called scalarization function. For more details, we refer also the reader to the standard books of vector/set optimization by Boţ, Grad and Wanka [13], Grad [23], Jahn [34], Khan, Tammer and Zălinescu [35], and Luc [38].

In this paper, we like to analyze the relationships between solutions of (P_φ) and the (weakly, properly) efficient solutions of (P) for the case that φ satisfies certain monotonicity properties. For doing this, we recall monotonicity concepts for the function φ (c.f. Jahn [34, Def. 5.1]).

Given binary relations $\sim_E \in \{\leqq_K, \leqq_K^0, \leq_K, <_K\}$ and $\sim_\mathbb{R} \in \{<, \leq\}$, a function $\varphi : E \to \mathbb{R}$ is said to be $(\sim_E, \sim_\mathbb{R})$-increasing if

$$\forall y, \overline{y} \in E, \ y \sim_E \overline{y} : \ \varphi(y) \sim_\mathbb{R} \varphi(\overline{y}).$$

As an immediate consequence, any $(\leq^0_K, <)$-increasing function is $(\leqq_K, \leq)$-increasing, while any $(\leq_K, <)$-increasing function is $(<_K, <)$-increasing.

Lemma 5.1 ([36, Lem. 5.5]) *Suppose that K satisfies (3). Then, the following assertions hold:*

1° *Any $x' \in K^+$ is $(\leqq_K, \leq)$-increasing.*
2° *Assume that K is relatively solid. Any $x' \in K^+ \setminus \ell(K^+)$ is $(<_K, <)$-increasing.*
3° *Any $x' \in K^\&$ is $(\leq_K, <)$-increasing.*
4° *Any $x' \in K^\#$ is $(\leq^0_K, <)$-increasing.*

Next, we present some scalarization results for the vector optimization problem (P) by using increasing scalarization functions (see also Khazayel et al. [36, Lem. 5.6]).

Lemma 5.2 *Consider a real-valued function $\varphi : E \to \mathbb{R}$. Then, the following assertions hold:*

1° *If φ is $(<_K, <)$-increasing, then $\operatorname{argmin}_{x \in \Omega} (\varphi \circ f)(x) \subseteq \operatorname{WEff}(\Omega \mid f, K)$.*
2° *If φ is $(\leq_K, <)$-increasing, then $\operatorname{argmin}_{x \in \Omega} (\varphi \circ f)(x) \subseteq \operatorname{Eff}(\Omega \mid f, K)$.*
3° *If φ is $(\leqq_K, \leq)$-increasing, and $\operatorname{argmin}_{x \in \Omega} (\varphi \circ f)(x) = \{\overline{x}\}$ for some $\overline{x} \in \Omega$, then $\overline{x} \in \operatorname{Eff}(\Omega \mid f, K)$.*
4° *If φ is $(<_C, <)$-increasing or $(\leq_C, <)$-increasing for some $C \in \mathcal{C}(K)$, then $\operatorname{argmin}_{x \in \Omega} (\varphi \circ f)(x) \subseteq \operatorname{PEff}(\Omega \mid f, K)$.*
5° *If φ is $(<_D, <)$-increasing or $(\leq_D, <)$-increasing for some $D \in \mathcal{D}(K)$, then $\operatorname{argmin}_{x \in \Omega} (\varphi \circ f)(x) \subseteq \operatorname{PEff}_c(\Omega \mid f, K)$.*

Proof Assertions $1° - 3°$ are given in [36, Lem. 5.6], while assertion 4° is a consequence of 1° and 2° (applied for $C \in \mathcal{C}(K)$ in the role of K) taking into account $\operatorname{PEff}(\Omega \mid f, K) = \bigcup_{C \in \mathcal{C}(K)} \operatorname{Eff}(\Omega \mid f, C) = \bigcup_{C \in \mathcal{C}(K)} \operatorname{WEff}(\Omega \mid f, C)$.

The proof of 5° is similar to the proof of 4°.   □

Remark 5.1 Notice that the convex cone K considered in Lemma 5.2 is neither assumed to be pointed nor solid, in contrast to the known results by Jahn [34, Lem. 5.14 and 5.24].

For the linear scalarization case, we derive the following result:

Theorem 5.1 *Suppose that K satisfies (3). Then:*

1° *For any $x' \in K^+ \setminus \ell(K^+)$, we have $\operatorname{argmin}_{x \in \Omega} (x' \circ f)(x) \subseteq \operatorname{WEff}(\Omega \mid f, K)$.*
2° *For any $x' \in K^\&$, we have $\operatorname{argmin}_{x \in \Omega} (x' \circ f)(x) \subseteq \operatorname{PEff}_c(\Omega \mid f, K) \subseteq \operatorname{PEff}(\Omega \mid f, K)$.*
3° *For any $x' \in K^+$ with $\operatorname{argmin}_{x \in \Omega} (x' \circ f)(x) = \{\overline{x}\}$ for some $\overline{x} \in \Omega$, we have $\overline{x} \in \operatorname{Eff}(\Omega \mid f, K)$.*
4° *Assume that K is pointed. For any $x' \in K^\#$, we have $\operatorname{argmin}_{x \in \Omega} (x' \circ f)(x) \subseteq \operatorname{PEff}_c(\Omega \mid f, K) \subseteq \operatorname{PEff}(\Omega \mid f, K)$.*

Proof Assertions 1° and 3° are given in Khazayel et al. [36, Th. 5.7]. Let us prove assertion 2°. Take some $x' \in K^\&$. Consider the nontrivial, convex cone $\bar{D} := \{y \in E \mid x'(y) \geq 0\}$. Because $x' \in K^\&$, we have $K \setminus \ell(K) \subseteq \{y \in E \mid x'(y) > 0\} = \operatorname{cor} \bar{D}$,

and so $\bar{D} \in \mathcal{D}(K)$. For any $y, \bar{y} \in E$ with $y <_{\bar{D}} \bar{y}$, we have $\bar{y} - y \in \mathrm{cor}\, \bar{D}$, hence $x'(\bar{y} - y) > 0$, or equivalently, $x'(\bar{y}) > x'(y)$. We conclude that x' is $(<_{\bar{D}}, <)$-increasing. Thus, the conclusion follows by Lemma 5.2 (5°).

When K is pointed, then $K^{\&} = K^{\#}$, and so we get 4° by applying 2°. $\qquad\square$

Remark 5.2 Notice, $\mathrm{cor}\, K^+ \subseteq K^{\#}$, and if K is τ_c-closed, then $\mathrm{icor}\, K^+ \subseteq K^{\&}$.

To derive representations for the sets $\mathrm{WEff}(\Omega \mid f, K)$, $\mathrm{PEff}_c(\Omega \mid f, K)$ and $\mathrm{PEff}(\Omega \mid f, K)$ using linear scalarization, we need some well-known generalized convexity concepts. The vector function $f : X \to E$ is called

- K-convex on the convex set $\Omega \subseteq X$ if, for any $x, \bar{x} \in \Omega$ and $\lambda \in (0, 1)$, we have $f(\lambda x + (1 - \lambda)\bar{x}) \in \lambda f(x) + (1 - \lambda)f(\bar{x}) - K$.
- K-convexlike on $\Omega \subseteq X$ if $f[\Omega] + K$ is a convex set.

Lemma 5.3 *Consider $f : X \to E$ and $\Omega \subseteq X$. The following assertions hold:*

1° *If f is K-convex on Ω, then f is K-convexlike on Ω.*
2° *If f is K-convexlike on Ω, then f is $(\mathrm{acl}\, C)$-convexlike on Ω for any $C \in \mathcal{C}(K)$.*
3° *If f is K-convex on Ω, then f is $(\mathrm{acl}\, C)$-convex on Ω for any $C \in \mathcal{C}(K)$.*

Proof 1° This fact is well-known.
2° In view of Lemma 4.2 (3°), we have $K + \mathrm{acl}\, C = \mathrm{acl}\, C$, hence $f[\Omega] + \mathrm{acl}\, C = (f[\Omega] + K) + \mathrm{acl}\, C$. Thus, $f[\Omega] + \mathrm{acl}\, C$ is convex whenever $f[\Omega] + K$ is convex.
3° Since $K \subseteq \mathrm{acl}\, C$, the $(\mathrm{acl}\, C)$-convexity of f is obvious if f is K-convex. $\qquad\square$

For the case that K is nontrivial and solid, the following result is well-known (see, e.g., Boţ, Grad and Wanka [13, Cor. 2.4.26] and Jahn [34, Cor. 5.29]):

Proposition 5.1 *Assume that K is a nontrivial, solid, convex cone in E. If f is K-convexlike on Ω, then*

$$\mathrm{WEff}(\Omega \mid f, K) = \bigcup_{x' \in K^+ \setminus \{0\}} \mathrm{argmin}_{x \in \Omega}\, (x' \circ f)(x).$$

Remark 5.3 Consider any nontrivial, convex cone K with $\mathrm{int}_{\tau}\, K \neq \emptyset$ in a real linear topological space (E, τ). Let E^* be the topological dual space of E, and K^* be the topological dual cone (w.r.t. τ) of K. It is known that $\mathrm{cor}\, K = \mathrm{int}_{\tau}\, K$ (see Holmes [32, p. 59]), and whenever f is K-convexlike on Ω we have

$$\mathrm{WEff}(\Omega \mid f, K) = \bigcup_{x^* \in K^* \setminus \{0\}} \mathrm{argmin}_{x \in \Omega}\, (x^* \circ f)(x). \qquad (6)$$

Indeed, the inclusion "$\supseteq$" in (6) follows directly by Proposition 5.1 taking into account that $K^* \subseteq K^+$. The proof of the inclusion "$\subseteq$" in (6) is similar to the proof in Jahn [34, Th. 5.13] by using the well-known separation result in real linear topological spaces (see, e.g., Jahn [34, Th. 3.16] and Zălinescu [43, Th. 1.1.3]). We mention also a remark by Boţ, Grad and Wanka [13, Rem. 2.4.11] related to the validity of (6) in real linear topological spaces.

Notice, under the K-convexity of f also Luc [38, Th. 4.2.10] and El Maghri and Laghdir [17, Th. 3.1] stated the representation (6).

Next, we present a counterpart to Proposition 5.1 for the case that K is relatively solid but not necessarily solid.

Theorem 5.2 *Suppose that K is relatively solid and satisfies (3). In addition, assume that the function f is K-convexlike on Ω, and $f[\Omega] + K$ is relatively solid. Then, the following assertions hold:*

1°

$$\mathrm{WEff}(\Omega \mid f, K) \subseteq \bigcup_{x' \in K^+ \setminus \{0\}} \mathrm{argmin}_{x \in \Omega}\, (x' \circ f)(x).$$

2° *If K^+ is pointed, then*

$$\mathrm{WEff}(\Omega \mid f, K) = \bigcup_{x' \in K^+ \setminus \{0\}} \mathrm{argmin}_{x \in \Omega}\, (x' \circ f)(x).$$

3° *If $\bar{x} \in \mathrm{WEff}(\Omega \mid f, K)$ and $f(\bar{x}) + \mathrm{icor}\, K \subseteq \mathrm{icor}(f[\Omega] + K)$, then*

$$\bar{x} \in \bigcup_{x' \in K^+ \setminus \ell(K^+)} \mathrm{argmin}_{x \in \Omega}\, (x' \circ f)(x).$$

4° *If $f[\mathrm{WEff}(\Omega \mid f, K)] + \mathrm{icor}\, K \subseteq \mathrm{icor}(f[\Omega] + K)$, then*

$$\mathrm{WEff}(\Omega \mid f, K) = \bigcup_{x' \in K^+ \setminus \ell(K^+)} \mathrm{argmin}_{x \in \Omega}\, (x' \circ f)(x).$$

Proof Notice that assertion 2° is a direct consequence of Theorem 5.1 (1°) and assertion 1°. Let us show assertion 1°.

Consider $\bar{x} \in \mathrm{WEff}(\Omega \mid f, K)$, i.e., $f[\Omega] \cap (f(\bar{x}) - \mathrm{icor}\, K) = \emptyset$. More precisely, we have $(f[\Omega] + K) \cap (f(\bar{x}) - \mathrm{icor}\, K) = \emptyset$. Indeed, assuming $f(x) + k \in f(\bar{x}) - \mathrm{icor}\, K$ for some $x \in \Omega$ and $k \in K$, we get $f(x) \in f(\bar{x}) - \mathrm{icor}\, K - K = f(\bar{x}) - \mathrm{icor}\, K$, a contradiction.

Now, it is easy to see that the sets $f[\Omega] + K$ and $f(\bar{x}) - K$ are nonempty, relatively solid and convex, taking into account $\mathrm{icor}(f[\Omega] + K) \neq \emptyset$ and $\mathrm{icor}(f(\bar{x}) - K) = f(\bar{x}) - \mathrm{icor}\, K \neq \emptyset$ (by Lemma 2.5). By the separation result stated in Proposition 2.2, there exist $x' \in E' \setminus \{0\}$ and $\alpha \in \mathbb{R}$ such that

$$0 \leq \alpha < x'(y) - x'(f(\bar{x}) - c) \tag{7}$$

for all $y \in \mathrm{icor}(f[\Omega] + K)$ and $c \in \mathrm{icor}\, K$. It is easy to check that $x'(c) \geq 0$ for all $c \in \mathrm{icor}\, K$ (hence, also $x' \in K^+ \setminus \{0\}$). Indeed, on the contrary assume that $x'(c) < 0$ for some $c \in \mathrm{icor}\, K$. Notice that $\lambda c \in \mathrm{icor}\, K$ for all $\lambda > 0$, and $x'(\lambda c) = \lambda x'(c) \to -\infty$ for $\lambda \to +\infty$. Hence, we have

$$0 \leq \alpha < x'(y) - x'(f(\bar{x})) + x'(\lambda c) \to -\infty \tag{8}$$

for $\lambda \to +\infty$, a contradiction.

By (7), we also get

$$0 \le x'(f(x) + k) - x'(f(\overline{x}) - c) \tag{9}$$

for all $k, c \in K$ and $x \in \Omega$. Finally, letting $k = 0$ and $c = 0$ in (9), it follows $x'(f(\overline{x})) \le x'(f(x))$ for all $x \in \Omega$, which actually means that $\overline{x} \in \mathrm{argmin}_{x \in \Omega} (x' \circ f)(x)$. The proof of 1° is complete.

In order to show 3°, assume that $f(\bar{x}) + \mathrm{icor}\, K \subseteq \mathrm{icor}(f[\Omega] + K)$. Following the lines in the proof of 1°, from (7) we get

$$0 \le \alpha < x'(f(\overline{x}) + k) - x'(f(\overline{x}) - c) = x'(k + c) \tag{10}$$

for all $k, c \in \mathrm{icor}\, K$. Take any $\tilde{k} \in \mathrm{icor}\, K$. Letting $k := \frac{1}{2}\tilde{k}$ and $c := \frac{1}{2}\tilde{k}$, we get $0 < x'(k + c) = x'(\tilde{k})$. We conclude that $x' \in K^+ \setminus \ell(K^+)$. Thus, 3° is valid.

The inclusion "$\subseteq$" in 4° follows by 3°, while "$\supseteq$" in 4° is provided by Theorem 5.1 (1°).   □

Remark 5.4 Assume that $K \ne E$ is solid (hence, relatively solid and $K \ne \ell(K)$). Then, K^+ is pointed and $f[\Omega] + \mathrm{icor}\, K = f[\Omega] + \mathrm{cor}\, K = \mathrm{cor}(f[\Omega] + K) = \mathrm{icor}(f[\Omega] + K)$ (by Lemma 2.5), which also shows the (relative) solidness of $f[\Omega] + K$. Thus, we recover the well-known result from Proposition 5.1.

Remark 5.5 By Lemma 3.3 (2°), we have $f[\mathrm{WEff}(\Omega \mid f, K)] \subseteq f[\Omega] \setminus \mathrm{icor}(f[\Omega] + K)$. Thus, in the case that $f[\Omega] + K$ is relatively solid and convex, the well-known rule

$$\mathrm{icor}(f[\Omega] + K) + \mathrm{icor}\, K = \mathrm{icor}(f[\Omega] + K + K) = \mathrm{icor}(f[\Omega] + K)$$

cannot directly be used to guarantee the condition $f(x) + \mathrm{icor}\, K \subseteq \mathrm{icor}(f[\Omega] + K)$ for $x \in \mathrm{WEff}(\Omega \mid f, K)$ (as needed in Theorem 5.2 (3°)). However, in view of Lemma 2.5 (5°), this condition is fulfilled if

$$f(x) \in \mathrm{vcl}_k(\mathrm{icor}(f[\Omega] + K)) \quad \text{for some } k \in K.$$

The mentioned inclusion $f[\mathrm{WEff}(\Omega \mid f, K)] + \mathrm{icor}\, K \subseteq \mathrm{icor}(f[\Omega] + K)$ in Theorem 5.2 (4°) (respectively, the assumption $f(\bar{x}) + \mathrm{icor}\, K \subseteq \mathrm{icor}(f[\Omega] + K)$ in Theorem 5.2 (3°)) is not superfluous, as the following example shows:

Example 5.1 Let the linear space $X := E := \mathbb{R}^2$ be endowed with the Euclidean norm and consider the identity function $f = \mathrm{id}_X : X \to X$. Define Ω as the closed unit ball (denoted by $\overline{B}_2$) in the normed space X, and K as the relatively solid (not solid), convex cone $\{0\} \times \mathbb{R}_+$. The dual cone $K^+ = \mathbb{R} \times \mathbb{R}_+$ is a solid, convex cone with $\ell(K^+) = \mathbb{R} \times \{0\}$ (hence, K^+ is not pointed). Moreover, $f[\Omega] + K = \overline{B}_2 + (\{0\} \times \mathbb{R}_+)$ is a solid (hence, relatively solid), convex set. It is easy to check that

$$\mathrm{WEff}(\Omega \mid f, K) = \{x \in \mathbb{R} \times (-\mathbb{R}_+) \mid ||x||_2 = 1\},$$

$$\bigcup_{x' \in K^+ \setminus \ell(K^+)} \mathrm{argmin}_{x \in \Omega} \, (x' \circ f)(x) = \{x \in \mathbb{R} \times (-\mathbb{P}) \mid ||x||_2 = 1\},$$

$$\bigcup_{x' \in \ell(K^+) \setminus \{0\}} \mathrm{argmin}_{x \in \Omega} \, (x' \circ f)(x) = \{x^1 := (-1, 0), x^2 := (1, 0)\},$$

and so

$$\mathrm{WEff}(\Omega \mid f, K) = \bigcup_{x' \in K^+ \setminus \{0\}} \mathrm{argmin}_{x \in \Omega} \, (x' \circ f)(x)$$

$$\supsetneq \bigcup_{x' \in K^+ \setminus \ell(K^+)} \mathrm{argmin}_{x \in \Omega} \, (x' \circ f)(x).$$

Notice that $f(x) + \mathrm{icor}\, K \subseteq \mathrm{icor}(f[\Omega] + K)$ for all $x \in \overline{B}_2 \setminus \{x^1, x^2\}$ but $f(x^i) + \mathrm{icor}\, K \not\subseteq \mathrm{icor}(f[\Omega] + K)$ for $i = 1, 2$. We conclude that $f[\mathrm{WEff}(\Omega \mid f, K)] + \mathrm{icor}\, K \not\subseteq \mathrm{icor}(f[\Omega] + K)$. Moreover, this example shows that the conclusion in Theorem 5.2 (2°) does not imply the pointedness of K^+.

The conclusion in Theorem 5.2 (4°) does not imply the inclusion $f[\mathrm{WEff}(\Omega \mid f, K)] + \mathrm{icor}\, K \subseteq \mathrm{icor}(f[\Omega] + K)$ in general, as the following example shows:

Example 5.2 Consider the linear space $X := E := \mathbb{R}^2$ with the maximum norm. Let the function f, the convex cone K and its dual cone K^+ be given as in Example 5.1. Define Ω as the closed unit ball (denoted by $\overline{B}_\infty$) in the normed space X. Then, $f[\Omega] + K = \overline{B}_\infty + (\{0\} \times \mathbb{R}_+)$ is a solid (hence relatively solid), convex set. Obviously, we have

$$\bigcup_{x' \in K^+ \setminus \ell(K^+)} \mathrm{argmin}_{x \in \Omega} \, (x' \circ f)(x) = [-1, 1] \times \{-1\} = \mathrm{WEff}(\Omega \mid f, K),$$

$$\bigcup_{x' \in \ell(K^+) \setminus \{0\}} \mathrm{argmin}_{x \in \Omega} \, (x' \circ f)(x) = (\{-1\} \times [-1, 1]) \cup (\{1\} \times [-1, 1]),$$

and so

$$\mathrm{WEff}(\Omega \mid f, K) = \bigcup_{x' \in K^+ \setminus \ell(K^+)} \mathrm{argmin}_{x \in \Omega} \, (x' \circ f)(x)$$

$$\subsetneq \bigcup_{x' \in K^+ \setminus \{0\}} \mathrm{argmin}_{x \in \Omega} \, (x' \circ f)(x).$$

Notice that for $x^1 := (-1, -1), x^2 := (1, -1) \in \mathrm{WEff}(\Omega \mid f, K)$ we have $f(x^i) + \mathrm{icor}\, K \not\subseteq \mathrm{icor}(f[\Omega] + K)$, $i = 1, 2$, and so $f[\mathrm{WEff}(\Omega \mid f, K)] + \mathrm{icor}\, K \not\subseteq \mathrm{icor}(f[\Omega] + K)$. From this example, we can also deduce that the inclusion in Theorem 5.2 (1°) does not need to be an equality in general. Hence, the pointedness assumption concerning K^+ in Theorem 5.2 (2°) is a not superfluous condition.

Remark 5.6 Adán and Novo [2, Th. 2] stated sufficient conditions (involving a generalized convexlikeness assumption on f) for the inclusion

$$\text{WEff}(\Omega \mid f, K) \subseteq \bigcup_{x' \in S} \text{argmin}_{x \in \Omega} (x' \circ f)(x),$$

where

$$S := \{x' \in K^+ \setminus \{0\} \mid \exists x \in \text{WEff}(\Omega \mid f, K),$$
$$y \in (\text{icor } K) \cup (f[\Omega] - f(x)) : \ x'(y) \neq 0\} \subseteq K^+ \setminus \{0\}.$$

If $\text{WEff}(\Omega \mid f, K) \neq \emptyset$, then

$$K^+ \setminus \ell(K^+) = \{x' \in K^+ \setminus \{0\} \mid \exists k \in \text{icor } K : \ x'(k) > 0\} \subseteq S,$$

and if further K^+ is pointed, then $K^+ \setminus \ell(K^+) = S = K^+ \setminus \{0\}$. Notice that in both Examples 5.1 and 5.2 we have $K^+ \setminus \ell(K^+) \subsetneq S = K^+ \setminus \{0\}$. In Example 5.1, we have $\text{WEff}(\Omega \mid f, K) = \bigcup_{x' \in K^+ \setminus \{0\}} \text{argmin}_{x \in \Omega} (x' \circ f)(x) =: A$, while in Example 5.2, the set $\text{WEff}(\Omega \mid f, K)$ is a proper subset of A, where A contains two additional line segments.

Theorem 5.3 *Suppose that K satisfies* (3)*, and f is K-convexlike on Ω. If either K and $f[\Omega] + K$ are relatively solid (e.g., if E has finite dimension) or $f[\Omega] + K$ is solid, then*

$$\text{Eff}(\Omega \mid f, K) \subseteq \bigcup_{x' \in K^+ \setminus \{0\}} \text{argmin}_{x \in \Omega} (x' \circ f)(x).$$

Proof If K and $f[\Omega] + K$ are relatively solid, then the conclusion directly follows by Theorem 5.2 (1°) taking into account $\text{Eff}(\Omega \mid f, K) \subseteq \text{WEff}(\Omega \mid f, K)$.

Now, assume that $f[\Omega] + K$ is solid. Define $Q := K \setminus \ell(K)$. Take some $\bar{x} \in \text{Eff}(\Omega \mid f, K)$, i.e., $f[\Omega] \cap (f(\bar{x}) - Q) = \emptyset$. Then, since $Q + K = Q$ by Lemma 2.2, it is easy to check that $(f[\Omega] + K) \cap (f(\bar{x}) - Q) = \emptyset$. Applying Proposition 2.1 for $\Omega^1 := f[\Omega] + K$ and $\Omega^2 := f(\bar{x}) - Q$, there exist $x' \in E' \setminus \{0\}$ and $\alpha \in \mathbb{R}$ such that

$$x'(f(\bar{x}) - k^2) \leq \alpha \leq x'(f(x) + k^1) \tag{11}$$

for all $x \in \Omega$, $k^1 \in K$ and $k^2 \in K \subseteq \text{acl } Q$ (notice that $f(\bar{x}) - \text{acl } Q \subseteq \text{acl}(f(\bar{x}) - Q)$). Since K is a cone, one easily gets $x' \in K^+ \setminus \{0\}$. By (11) for any $x \in \Omega$ and $k^1 := k^2 := 0$, we conclude $\bar{x} \in \text{argmin}_{x \in \Omega} (x' \circ f)(x)$. $\square$

Remark 5.7 Notice that in Theorem 5.3, no pointedness assumption concerning K is formulated. In the well-known result by Boţ, Grad and Wanka [13, Th. 2.4.21] and Jahn [34, Th. 5.4], the conclusion of Theorem 5.3 is stated under the assumptions that f is K-convexlike on Ω, $f[\Omega] + K$ is solid, and K is a nontrivial, pointed, convex cone.

Proposition 5.2 *Consider a real linear topological space* (E, τ), *and suppose that* K *satisfies* (3). *If* f *is a* K-*convexlike function on* Ω, *then*

$$\mathrm{PEff}_{\mathrm{KTZ}}(\Omega \mid f, K) := \bigcup_{D \in \mathcal{D}_{\mathrm{KTZ}}(K)} \mathrm{Eff}(\Omega \mid f, D) = \bigcup_{x' \in K^{\&} \cap E^*} \mathrm{argmin}_{x \in \Omega} (x' \circ f)(x).$$

Proof Assume that f is a K-convexlike function on Ω. Then, for any $D \in \mathcal{D}_{\mathrm{KTZ}}(K)$, the function f is $(\mathrm{cl}_{\tau} D)$-convexlike on Ω (the proof is similar to Lemma 5.3), and so (by Remark 5.3),

$$\mathrm{WEff}(\Omega \mid f, \mathrm{cl}_{\tau} D) = \bigcup_{x^* \in (\mathrm{cl}_{\tau} D)^* \setminus \{0\}} \mathrm{argmin}_{x \in \Omega} (x^* \circ f)(x).$$

Moreover, similar to Lemma 4.1 (4°), one has

$$K^{\&} \cap E^* = \bigcup_{D \in \mathcal{D}_{\mathrm{KTZ}}(K)} D^* \setminus \{0\}$$

by Khan, Tammer and Zălinescu [35, Lem. 2.4.15]. Consequently, we conclude

$$
\begin{aligned}
\mathrm{PEff}_{\mathrm{KTZ}}(\Omega \mid f, K) &= \bigcup_{D \in \mathcal{D}_{\mathrm{KTZ}}(K)} \mathrm{Eff}(\Omega \mid f, D) \\
&= \bigcup_{D \in \mathcal{D}_{\mathrm{KTZ}}(K)} \mathrm{WEff}(\Omega \mid f, D) \\
&= \bigcup_{D \in \mathcal{D}_{\mathrm{KTZ}}(K)} \mathrm{WEff}(\Omega \mid f, \mathrm{cl}_{\tau} D) \\
&= \bigcup_{D \in \mathcal{D}_{\mathrm{KTZ}}(K)} \bigcup_{x^* \in (\mathrm{cl}_{\tau} D)^* \setminus \{0\}} \mathrm{argmin}_{x \in \Omega} (x^* \circ f)(x) \\
&= \bigcup_{D \in \mathcal{D}_{\mathrm{KTZ}}(K)} \bigcup_{x^* \in D^* \setminus \{0\}} \mathrm{argmin}_{x \in \Omega} (x^* \circ f)(x) \\
&= \bigcup_{x^* \in K^{\&} \cap E^*} \mathrm{argmin}_{x \in \Omega} (x^* \circ f)(x).
\end{aligned}
$$

$\square$

Remark 5.8 The representation given in Theorem 5.2 was established by

- Luc [38, Th. 4.2.11] for the case that E is a reflexive real linear topological space, K is a convex cone, and f is a K-convex function;
- Makarov and Rachkovski [39, Th. 3.2] for the case that E is a separated real linear topological space, K is a nontrivial, pointed, closed, convex cone with $K^{\#} \cap E^* \neq \emptyset$, and f is a K-convexlike function;
- El Maghri and Laghdir [17, Th. 3.1] for the case that E is a separated real linear topological space, K is a nontrivial, pointed, convex cone, and f is a K-convex function.

Our proof of the representation given in Proposition 5.2 shows that no pointedness assumption related to K is needed.

The next theorem studies properties of the set of classical Henig properly efficient solutions of (P) (in the sense of Definition 4.1) where the cone K is not necessarily assumed to be pointed.

Theorem 5.4 *Suppose that K satisfies (3). Assume that the function f is K-convexlike on Ω. Then:*

1°

$$\mathrm{PEff}_c(\Omega \mid f, K) = \bigcup_{x' \in K^\&} \mathrm{argmin}_{x \in \Omega} (x' \circ f)(x).$$

2° *If E has finite dimension and K is τ_c-closed, then*

$$\mathrm{PEff}_c(\Omega \mid f, K) = \bigcup_{x' \in \mathrm{icor}\, K^+} \mathrm{argmin}_{x \in \Omega} (x' \circ f)(x).$$

Proof If E is endowed with the convex core topology τ_c, then we have $\mathrm{PEff}_c(\Omega \mid f, K) = \mathrm{PEff}_{\mathrm{KTZ}}(\Omega \mid f, K)$ and $K^\& \cap E^* = K^\&$. Hence, by applying Proposition 5.2 for (E, τ_c) we get assertion 1°.

Assertion 2° is a direct consequence of assertion 1° taking into account Lemma 2.1 (2°). $\qquad\square$

Remark 5.9 Adán and Novo [3, Sec. 4] stated linear scalarization results for specific types of proper efficiency concepts (in the sense of Benson, Borwein and Hurwicz, respectively). Notice that in [3, Cor. 4.1] a solidness assumption on the dual cone K^+ (hence, K is pointed and $K^\# = K^\&$) is assumed (due to the application of a separation result in [3, Th. 2.2]; see also Novo and Zălinescu [40, Cor. 2.3]).

Finally, we study properties of the set of Henig properly efficient solutions of (P) where the cone K is not necessarily assumed to be pointed.

Theorem 5.5 *Suppose that K is relatively solid and satisfies (3). In addition, assume that the function f is K-convexlike on Ω. Then:*

1° *If $\bar{x} \in \mathrm{PEff}(\Omega \mid f, K)$ and $f(\bar{x}) + \mathrm{icor}\, K \subseteq \mathrm{icor}(f[\Omega] + K)$, then*

$$\bar{x} \in \bigcup_{x' \in K^\&} \mathrm{argmin}_{x \in \Omega} (x' \circ f)(x).$$

2° *If $f[\mathrm{PEff}(\Omega \mid f, K)] + \mathrm{icor}\, K \subseteq \mathrm{icor}(f[\Omega] + K)$, then*

$$\mathrm{PEff}(\Omega \mid f, K) = \bigcup_{x' \in K^\&} \mathrm{argmin}_{x \in \Omega} (x' \circ f)(x).$$

3° *If E has finite dimension, K is τ_c-closed, and $f[\mathrm{PEff}(\Omega \mid f, K)] + \mathrm{icor}\, K \subseteq$*
   $\mathrm{icor}(f[\Omega] + K)$, then

$$\mathrm{PEff}(\Omega \mid f, K) = \bigcup_{x' \in \mathrm{icor}\, K^+} \mathrm{argmin}_{x \in \Omega}\, (x' \circ f)(x).$$

Proof Let us show assertion 1°. Consider $\overline{x} \in \mathrm{PEff}(\Omega \mid f, K)$. In view of Lemma 4.4 (4°), there exists a convex cone $C \subseteq Y$ with $K \setminus \ell(K) \subseteq \mathrm{icor}\, C$ and $C \neq \ell(C)$ (i.e., $C \in \mathcal{C}(K)$) such that $\overline{x} \in \mathrm{WEff}(\Omega \mid f, C)$, which means $f[\Omega] \cap (f(\overline{x}) - \mathrm{icor}\, C) = \emptyset$. Now, we actually have $(f[\Omega] + K) \cap (f(\overline{x}) - \mathrm{icor}\, C) = \emptyset$. Indeed, assuming $f(x) + k \in f(\overline{x}) - \mathrm{icor}\, C$ for some $x \in \Omega$ and $k \in K$, we get $f(x) \in f(\overline{x}) - \mathrm{icor}\, C - K = f(\overline{x}) - \mathrm{icor}\, C$ by Lemma 4.2 (3°), a contradiction.

Now, it is easy to see that the sets $f[\Omega] + K$ and $f(\overline{x}) - C$ are nonempty, relatively solid and convex, taking into account $\mathrm{icor}(f[\Omega] + K) \neq \emptyset$ and $\mathrm{icor}(f(\overline{x}) - C) = f(\overline{x}) - \mathrm{icor}\, C \supseteq f(\overline{x}) - (K \setminus \ell(K)) \neq \emptyset$ (by Lemma 2.5). By the separation result stated in Proposition 2.2, there exist $x' \in E' \setminus \{0\}$ and $\alpha \in \mathbb{R}$ such that

$$0 \leq \alpha < x'(y) - x'(f(\overline{x}) - c) \tag{12}$$

for all $y \in \mathrm{icor}(f[\Omega] + K)$ and $c \in \mathrm{icor}\, C$.

Assume that $f(\overline{x}) + \mathrm{icor}\, K \subseteq \mathrm{icor}(f[\Omega] + K)$. From (12), we get

$$0 \leq \alpha < x'(f(\overline{x}) + k) - x'(f(\overline{x}) - c) = x'(k + c) \tag{13}$$

for all $k \in \mathrm{icor}\, K$ and $c \in \mathrm{icor}\, C$. Take any $\tilde{k} \in K \setminus \ell(K)$ and $\bar{k} \in \mathrm{icor}\, K$. Since $\mathrm{icor}\, K \subseteq K \setminus \ell(K) \subseteq \mathrm{icor}\, C \subseteq C$, we have $\tilde{k} \in \mathrm{icor}\, C$ and $-\bar{k} \in -\mathrm{icor}\, K \subseteq -C \subseteq \mathrm{aff}(C - C)$. Thus, there exists $\varepsilon > 0$ such that $\tilde{k} + \varepsilon(-\bar{k}) \in \mathrm{icor}\, C$. Consequently, by (13) for $k := \varepsilon\bar{k} \in \mathrm{icor}\, K$, $c := \tilde{k} - \varepsilon\bar{k}$, we get

$$0 \leq \alpha < x'(k + c) = x'(\varepsilon\bar{k} + \tilde{k} - \varepsilon\bar{k}) = x'(\tilde{k}). \tag{14}$$

Since (14) holds for any $\tilde{k} \in K \setminus \ell(K)$, we conclude $x' \in K^{\&}$.

By (12), we also get

$$0 \leq x'(f(x) + k) - x'(f(\overline{x}) - c) \tag{15}$$

for all $k \in K$, $c \in C$ and $x \in \Omega$. Finally, letting $k = 0$ and $c = 0$ in (15), it follows $x'(f(\overline{x})) \leq x'(f(x))$ for all $x \in \Omega$, which means that $\overline{x} \in \mathrm{argmin}_{x \in \Omega}\, (x' \circ f)(x)$. The proof of 1° is complete.

The inclusion "$\subseteq$" in 2° follows by 1°, while "$\supseteq$" in 2° is provided by Theorem 5.1 (2°).

Assertion 3° is a direct consequence of assertion 2° taking into account Lemma 2.1 (2°). $\qquad\square$

The mentioned inclusion $f[\mathrm{PEff}(\Omega \mid f, K)] + \mathrm{icor}\, K \subseteq \mathrm{icor}(f[\Omega] + K)$ in Theorem 5.5 (2°, 3°) (respectively, the assumption $f(\overline{x}) + \mathrm{icor}\, K \subseteq \mathrm{icor}(f[\Omega] + K)$

in Theorem 5.5 (1°)) is not superfluous, as Example 5.3 shows. Moreover, in this Example 5.3, we will see that

$$\text{PEff}_c(\Omega \mid f, K) \subsetneqq \text{PEff}(\Omega \mid f, K) \subsetneqq \text{Eff}(\Omega \mid f, K)$$

may hold when K is a relatively solid (but not solid), convex cone with $K \notin \mathcal{C}(K)$ and $K \neq \ell(K)$.

Example 5.3 Let the linear space $X := E := \mathbb{R}^3$ be endowed with the Euclidean norm and consider the identity function $f = \text{id}_X : X \to X$. Define Ω as the closed unit ball (denoted by $\overline{B}_2$) in the normed space X, and K as the relatively solid (not solid), pointed, algebraically closed (respectively, τ_c-closed), convex cone $\mathbb{R}_+ \times \{0\} \times \mathbb{R}_+$. Then, $f[\Omega] + K = \overline{B}_2 + (\mathbb{R}_+ \times \{0\} \times \mathbb{R}_+)$ is a solid (hence, relatively solid), convex set. The dual cone of K is given by $K^+ = \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+$, which is a convex cone with $\text{cor } K^+ = \mathbb{P} \times \mathbb{R} \times \mathbb{P} \neq \emptyset$ and $\ell(K^+) = \{0\} \times \mathbb{R} \times \{0\} \supsetneqq \{(0, 0, 0)\}$ (hence, K^+ is solid but not pointed). By Lemma 2.1 (2°), we have $\text{cor } K^+ = \text{icor } K^+ = K^\& = K^\#$. Thus, it is easy to check that

$$K^+ \setminus \ell(K^+) = (\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+) \setminus (\{0\} \times \mathbb{R} \times \{0\}) \supsetneqq \mathbb{P} \times \mathbb{R} \times \mathbb{P} = \text{cor } K^+ = K^\# = K^\&.$$

Because $\overline{B}_2$ is solid, applying Lemma 3.3 (1°) yields

$$\begin{aligned}
\text{PEff}_c(\Omega \mid f, K) &\subseteq \text{PEff}(\Omega \mid f, K) \subseteq \text{Eff}(\Omega \mid f, K) \subseteq \text{WEff}(\Omega \mid f, K) \\
&\subseteq \{x \in \mathbb{R}^3 \mid ||x||_2 = 1\}.
\end{aligned}$$

Since $K \supsetneqq (\mathbb{P} \times \{0\} \times \mathbb{P}) \cup \{(0, 0, 0)\} = (\text{icor } K) \cup \ell(K)$, we have $K \notin \mathcal{C}(K)$ by Lemma 4.2 (5°). By Theorems 5.1 (1°), 5.2 (1°) and 5.4 (1°), and taking into account $K^\& \subseteq K^+ \setminus \ell(K^+)$, we get

$$\bigcup_{x' \in K^+ \setminus \{0\}} \text{argmin}_{x \in \Omega} (x' \circ f)(x) \supseteq \text{WEff}(\Omega \mid f, K)$$

$$\supseteq \bigcup_{x' \in K^+ \setminus \ell(K^+)} \text{argmin}_{x \in \Omega} (x' \circ f)(x)$$

$$\supseteq \bigcup_{x' \in K^\&} \text{argmin}_{x \in \Omega} (x' \circ f)(x)$$

$$= \text{PEff}_c(\Omega \mid f, K).$$

Define $x^1 := (0, -1, 0)$ and $x^2 := (0, 1, 0)$. Since, for any $x' \in \mathbb{R}^3 \setminus \{(0, 0, 0)\}$ with $||x'||_2 = 1$ we have $-1 = \min_{x \in \overline{B}_2} \langle x', x \rangle$, one can check that

$$\bigcup_{x' \in K^+ \setminus \{0\}} \text{argmin}_{x \in \Omega} (x' \circ f)(x) = \bigcup_{x' \in (\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+) \setminus \{(0,0,0)\}} \text{argmin}_{x \in \overline{B}_2} \langle x', x \rangle$$

$$= \bigcup_{\substack{x' \in (\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+) \setminus \{(0,0,0)\}, \\ ||x'||_2 = 1}} \{x \in \overline{B}_2 \mid ||x||_2 \cos(\theta) = -1 \ (\theta \text{ is the angle between } x' \text{ and } x)\}$$

$$= \{x \in (-\mathbb{R}_+) \times \mathbb{R} \times (-\mathbb{R}_+) \mid ||x||_2 = 1\} =: A^1,$$

and

$$\bigcup_{x' \in K^+ \setminus \ell(K^+)} \mathrm{argmin}_{x \in \Omega} \, (x' \circ f)(x) = \bigcup_{x' \in (\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+) \setminus (\{0\} \times \mathbb{R} \times \{0\})} \mathrm{argmin}_{x \in \overline{B}_2} \langle x', x \rangle$$

$$= \{x \in (-\mathbb{R}_+) \times \mathbb{R} \times (-\mathbb{R}_+) \mid ||x||_2 = 1\} \setminus \{x^1, x^2\} =: A^2,$$

as well as

$$\bigcup_{x' \in K^{\&}} \mathrm{argmin}_{x \in \Omega} \, (x' \circ f)(x) = \bigcup_{x' \in \mathbb{P} \times \mathbb{R} \times \mathbb{P}} \mathrm{argmin}_{x \in \overline{B}_2} \langle x', x \rangle$$

$$= \{x \in (-\mathbb{P}) \times \mathbb{R} \times (-\mathbb{P}) \mid ||x||_2 = 1\} =: A^3.$$

We directly conclude that $\mathrm{PEff}_c(\Omega \mid f, K) = A^3$ and $x^1, x^2 \notin \mathrm{PEff}_c(\Omega \mid f, K)$. Moreover, one can check that $x^1, x^2 \in \mathrm{PEff}(\Omega \mid f, K)$. For doing this, define the generalized dilating cone

$$C := \{(y_1, y_2, y_3) \in \mathbb{R}^3 \mid y_2 = 0 \ \wedge \ y_1 + y_3 \geq 0\}.$$

Clearly, C is a convex cone with $C \neq \ell(C)$ and

$$K \setminus \ell(K) = (\mathbb{R}_+ \times \{0\} \times \mathbb{R}_+) \setminus \{(0, 0, 0)\}$$
$$\subseteq \{(y_1, y_2, y_3) \in \mathbb{R}^3 \mid y_2 = 0 \ \wedge \ y_1 + y_3 > 0\} = \mathrm{icor} \, C,$$

hence $C \in \mathcal{C}(K)$. Now, observe that

$$f[\Omega] \cap (f(x^i) - \mathrm{icor} \, C) = \overline{B}_2 \cap (x^i - \mathrm{icor} \, C) = \emptyset. \tag{16}$$

Indeed, take any $y \in x^i - \mathrm{icor} \, C$, i.e., there is $c \in \mathrm{icor} \, C$ with $y = x^i - c$. Then, since $\langle c, x^i \rangle = 0$ and $c \neq (0, 0, 0)$ we get

$$||y||_2^2 = \langle y, y \rangle = \langle x^i, x^i \rangle - 2 \langle c, x^i \rangle + \langle c, c \rangle = ||x^i||_2^2 + ||c||_2^2 = 1 + ||c||_2^2 > 1,$$

which means that $y \notin \overline{B}_2$. Thus, (16) holds true.

Since $x^1, x^2 \in \mathrm{PEff}(\Omega \mid f, K) \subseteq \mathrm{WEff}(\Omega \mid f, K)$ and $A^1 \supseteq \mathrm{WEff}(\Omega \mid f, K) \supseteq A^2$, we also get $A^1 \supseteq \mathrm{WEff}(\Omega \mid f, K) \supseteq A^2 \cup \{x^1, x^2\} = A^1$, hence $\mathrm{WEff}(\Omega \mid f, K) = A^1$. Applying Lemma 2.5 (4°) for $A := \overline{B}_2$, we conclude

$$\left[\bigcup_{k \in K} \mathrm{vcl}_k(\mathrm{cor} \, \overline{B}_2) \right] + \mathrm{icor} \, K = (\mathrm{cor} \, \overline{B}_2) + \mathrm{icor} \, K = \mathrm{cor}(\overline{B}_2 + K). \tag{17}$$

Now, the following inclusion holds true

$$A^2 \subseteq \bigcup_{k \in K} \operatorname{vcl}_k(\operatorname{cor} \overline{B}_2). \tag{18}$$

Indeed, take some $x = (x_1, x_2, x_3) \in A^2 = \{x \in (-\mathbb{R}_+) \times \mathbb{R} \times (-\mathbb{R}_+) \mid ||x||_2 = 1\} \setminus \{x^1, x^2\}$ and $\lambda > 0$. For $x_1 < 0$, we choose $k := (1, 0, 0)$; otherwise, for $x_1 = 0$ (hence $x_3 < 0$), we choose $k := (0, 0, 1)$. Then, $k \in K = \mathbb{R}_+ \times \{0\} \times \mathbb{R}_+$, and so $x + tk \in ([x_1, 0] \times \{x_2\} \times [x_3, 0]) \setminus \{x\}$ for some $t > 0$. W.l.o.g. one can assume that $t \in (0, \lambda]$. Clearly, we have $||x + tk||_2 < ||x||_2 = 1$, hence $x + tk \in \operatorname{cor} \overline{B}_2$. Thus, $x \in \operatorname{vcl}_k(\operatorname{cor} \overline{B}_2)$.

Combining (17) and (18), we conclude $f(x) + \operatorname{icor} K \subseteq \operatorname{icor}(f[\Omega] + K)$ for all $x \in A^2$.

Since $x^1, x^2 \in \operatorname{PEff}(\Omega \mid f, K) \setminus A^3$, from Theorem 5.5 (1°), we can deduce that $f(x^i) + \operatorname{icor} K \not\subseteq \operatorname{icor}(f[\Omega] + K)$ for all $i = 1, 2$, and so $x^i \notin \bigcup_{k \in K} \operatorname{vcl}_k(\operatorname{cor} \overline{B}_2)$ for all $i = 1, 2$, in view of (17). This shows also that $f[\operatorname{PEff}(\Omega \mid f, K)] + \operatorname{icor} K \not\subseteq \operatorname{icor}(f[\Omega] + K)$.

As already observed above, $\operatorname{PEff}(\Omega \mid f, K) \subseteq \operatorname{WEff}(\Omega \mid f, K) = A^1$, hence we get $\operatorname{PEff}(\Omega \mid f, K) \setminus \{x^1, x^2\} \subseteq A^2$. Thus, as a consequence of Theorem 5.5 (1°), Theorem 5.1 (2°), formulae (17) and (18), we have

$$\operatorname{PEff}(\Omega \mid f, K) \setminus \{x^1, x^2\} \subseteq A^3 \subseteq \operatorname{PEff}(\Omega \mid f, K).$$

Recalling that $x^1, x^2 \in \operatorname{PEff}(\Omega \mid f, K)$, we conclude

$$\operatorname{PEff}(\Omega \mid f, K) \subseteq A^3 \cup \{x^1, x^2\} \subseteq \operatorname{PEff}(\Omega \mid f, K) \cup \{x^1, x^2\} = \operatorname{PEff}(\Omega \mid f, K),$$

hence $\operatorname{PEff}(\Omega \mid f, K) = A^3 \cup \{x^1, x^2\}$.

Finally, let us show that $\operatorname{Eff}(\Omega \mid f, K) = \operatorname{WEff}(\Omega \mid f, K)$. Clearly, $\operatorname{Eff}(\Omega \mid f, K) \subseteq \operatorname{WEff}(\Omega \mid f, K)$. Take some $x \in \operatorname{WEff}(\Omega \mid f, K) = A^1$. We prove that

$$f[\Omega] \cap (f(x) - K \setminus \ell(K)) = \overline{B}_2 \cap (x - K \setminus \ell(K)) = \emptyset, \tag{19}$$

i.e., $x \in \operatorname{Eff}(\Omega \mid f, K)$. Take an arbitrarily $y \in x - K \setminus \ell(K)$. Since $x \in A^1$, we have $x = (x_1, x_2, x_3) \in (-\mathbb{R}_+) \times \mathbb{R} \times (-\mathbb{R}_+)$ and $||x||_2 = 1$. Moreover, $-K \setminus \ell(K) = (((-\mathbb{R}_+) \times \{0\} \times (-\mathbb{R}_+)) \setminus \{(0, 0, 0)\}$. Thus, we conclude

$$y = (y_1, y_2, y_3) \in x - K \setminus \ell(K) \subseteq ((x_1 - \mathbb{R}_+) \times \{x_2\} \times (x_3 - \mathbb{R}_+)) \setminus \{x\}.$$

Now, it is easy to check that $|y_1| \geq |x_1|$, $y_2 = x_2$ and $|y_3| \geq |x_3|$, where at least one inequality is strict. Hence, $1 = ||x||_2 < ||y||_2$, which means that $y \notin \overline{B}_2$. The proof of (19) is complete.

Observing that $A^3 \subsetneq A^3 \cup \{x^1, x^2\} \subsetneq A^1$, we conclude

$$\operatorname{PEff}_c(\Omega \mid f, K) \subsetneq \operatorname{PEff}(\Omega \mid f, K) \subsetneq \operatorname{Eff}(\Omega \mid f, K) = \operatorname{WEff}(\Omega \mid f, K).$$

The conclusion in Theorem 5.5 (2°) does not imply the inclusion $f[\mathrm{PEff}(\Omega \mid f, K)] + \mathrm{icor}\, K \subseteq \mathrm{icor}(f[\Omega] + K)$ in general, as the following example shows:

Example 5.4 Consider the data from Example 5.2. It is easy to check that

$$K^+ \setminus \ell(K^+) = \mathrm{icor}\, K^+ = \mathrm{cor}\, K^+ = \mathbb{R} \times \mathbb{P} = K^\& = K^\#.$$

As mentioned in Example 5.2,

$$\mathrm{WEff}(\Omega \mid f, K) = [-1, 1] \times \{-1\} = \bigcup_{x' \in K^+ \setminus \ell(K^+)} \mathrm{argmin}_{x \in \Omega}\, (x' \circ f)(x).$$

By Theorem 5.4 (1°),

$$\mathrm{PEff}_c(\Omega \mid f, K) = \bigcup_{x' \in K^\&} \mathrm{argmin}_{x \in \Omega}\, (x' \circ f)(x),$$

hence $\mathrm{WEff}(\Omega \mid f, K) = \mathrm{PEff}_c(\Omega \mid f, K)$. This shows also $\mathrm{PEff}_c(\Omega \mid f, K) = \mathrm{PEff}(\Omega \mid f, K) = \mathrm{Eff}(\Omega \mid f, K) = \mathrm{WEff}(\Omega \mid f, K) = [-1, 1] \times \{-1\}$. In particular, we have $\{x^1, x^2\} \subseteq \mathrm{PEff}(\Omega \mid f, K)$, where x^1, x^2 are given as in Example 5.2. As already mentioned in Example 5.2, $f(x^i) + \mathrm{icor}\, K \nsubseteq \mathrm{icor}(f[\Omega] + K), i = 1, 2$, hence $f[\mathrm{PEff}(\Omega \mid f, K)] + \mathrm{icor}\, K \nsubseteq \mathrm{icor}(f[\Omega] + K)$. Figure 1 illustrates Examples 5.2 and 5.4 .

6 Conclusions

In this paper, we studied vector optimization problems involving not necessarily pointed and not necessarily solid, convex cones in real linear spaces. Based on the "intrinsic core" interiority notion (a well-known generalized interiority notion), we defined our solution concepts (proper/weak efficiency) for such vector optimization problems.

On the one hand, we proposed a Henig-type proper efficiency solution concept based on generalized dilating convex cones which have nonempty intrinsic cores (but cores could be empty). Notice that any convex cone has a nonempty intrinsic core in finite dimension; however, this property may fail in infinite dimension. We showed certain useful properties of the new solution concept, pointed out that the set of solutions w.r.t. this concept is always between the set of classical Henig properly efficient solutions and the set of Pareto efficient solutions and showed an example where all these three sets do not coincide, i.e., $\mathrm{PEff}_c(\Omega \mid f, K) \subsetneqq \mathrm{PEff}(\Omega \mid f, K) \subsetneqq \mathrm{Eff}(\Omega \mid f, K)$.

On the other hand, using linear functionals from the dual cone of the ordering cone K, we were able to characterize the sets $\mathrm{WEff}(\Omega \mid f, K)$, $\mathrm{PEff}_c(\Omega \mid f, K)$ and $\mathrm{PEff}(\Omega \mid f, K)$ under the assumption $f[\Omega] + K$ is convex. The analysis of the rule $A + \mathrm{icor}\, K \subseteq \mathrm{icor}(A + K)$ for $A \subseteq f[\Omega]$ (see also Lemma 2.5) was essential for deriving the representations of the sets $\mathrm{WEff}(\Omega \mid f, K)$ and $\mathrm{PEff}(\Omega \mid f, K)$.
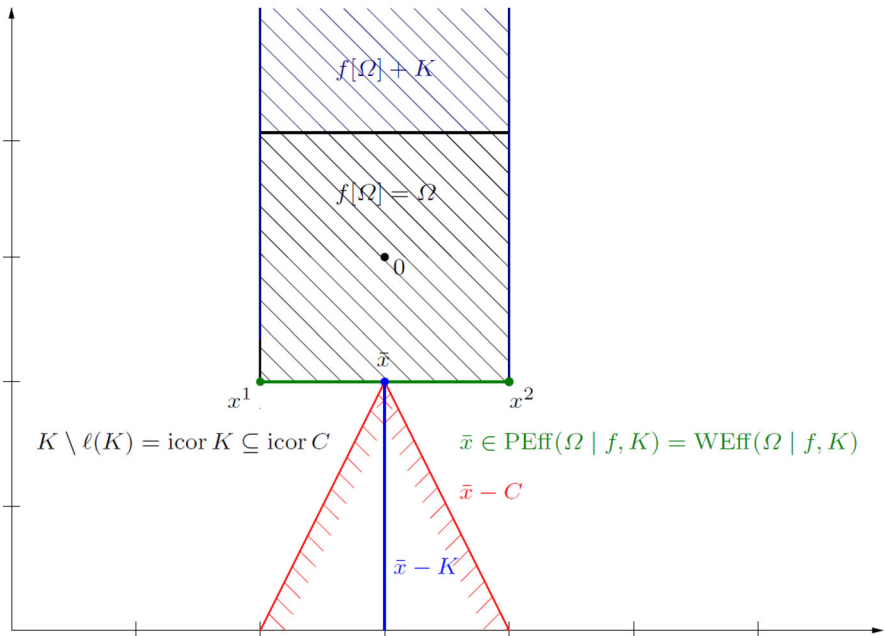
Fig. 1 Illustration of Examples 5.2 and 5.4

In forthcoming works, we aim to extend the scalarization results derived in Sect. 5. One idea could be to involve further generalized convexity concepts, and another idea could be to consider nonlinear scalarization techniques (for instance based on the so-called Gerstewitz function, see [35, Sec. 5.2] and [42]). In this context, Arrow–Barankin–Blackwell type theorems for our considered Henig-type proper efficiency concepts are of interest (see, e.g., [22]).

The investigation of the Image Space Analysis (ISA) approach (in the sense of Giannessi [19,20]) involving a relatively solid (but not necessarily solid), convex cone could be a further interesting task.

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