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Efficient Proximal Mapping Computation for Low-Rank Inducing Norms

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Abstract
Low-rank inducing unitarily invariant norms have been introduced to convexify problems with a low-rank/sparsity constraint. The most well-known member of this family is the so-called nuclear norm. To solve optimization problems involving such norms with proximal splitting methods, efficient ways of evaluating the proximal mapping of the low-rank inducing norms are needed. This is known for the nuclear norm, but not for most other members of the low-rank inducing family. This work supplies a framework that reduces the proximal mapping evaluation into a nested binary search, in which each iteration requires the solution of a much simpler problem. The simpler problem can often be solved analytically as demonstrated for the so-called low-rank inducing Frobenius and spectral norms. The framework also allows to compute the proximal mapping of increasing convex functions composed with these norms as well as projections onto their epigraphs.

Keywords Low-rank optimization · Low-rank inducing norms · Proximal splitting · Regularization · Matrix completion

Mathematics Subject Classification 90C06 · 90C25 · 90C26 · 15A83 · 90C30 · 90C59 · 90C90

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1 Introduction

Non-convex optimization problems with a rank or cardinality constraint appear in many data driven areas such as machine learning, image analysis and multivariate linear regression [6,8,13,26,40] as well as areas within control such as system identification, model reduction, low-order controller design, and low-complexity modeling [2,15,18,35,47]. Besides the low-rank constraint, these problems are often convex and therefore one of the most common techniques for solving such problems is to convexify them by using regularizers or taking convex envelopes [8,15,17,19]. A promising class of such regularizers and convex envelopes is the class of so-called unitarily invariant low-rank inducing norms [17], i.e., convex envelopes of unitarily invariant norms whose domain is restricted to matrices with prescribed bounded rank. As many common loss functions, e.g., squared distance in the Frobenius norm, contain terms of unitarily invariant norms, these norms have the attractive feature to exactly convexify, i.e., the convexified problem in terms of the low-rank inducing norm coincides on with the original at all low-rank matrices of prescribed bounded rank. Therefore, if the convexified problem has a low-rank solution, it is guaranteed to be a solution to the non-convex one.

Although low-rank inducing norms often admit a representation as semi-definite programs (SDP) [17], proximal splitting algorithms [9] are often used for large-scale problems, where standard interior-point method solvers have too costly iterations [39]. The main objective of this work is to efficiently compute the needed proximal mappings of low-rank inducing norms that are composed with increasing convex functions. To this end, we develop a generic nested binary search algorithm, which in each iteration solves a simple problem. While for well-known low-rank inducing norms such as the nuclear norm [38] and the low-rank inducing Frobenius norm [1,14,29,30], our algorithm will recover the same efficiency, for other norms such as the low-rank inducing spectral norm [46], our approach improves the computational complexity significantly, especially in the vector-valued case. Finally, [45] proposes a non-analytic approach for an extended class of not necessarily unitarily invariant low-rank inducing norms (see [27]). This approach, however, depends on the complexity and convergence rates of other optimization algorithms.

The paper is organized as follows. We start by introducing some preliminaries on norms and convex optimization. Subsequently, a formal definition of the class of low-rank inducing norms, including their application to rank constrained optimization problems is outlined. Then, we discuss and derive our main results, the binary search framework and outline an algorithm for evaluating their epigraph projections. For the low-rank inducing Frobenius and spectral norms, we make these computations explicit and arrive at implementable algorithms for which the computational cost is analyzed. Subsequently, a case study is performed in order to illustrate the performance of our algorithm through proximal splitting. Finally, we draw a conclusion and point the reader to our freely available implementations of these algorithms in MATLAB and Python.

2 Preliminaries

The set of reals is denoted by $\mathbb{R}$, the set of real vectors by $\mathbb{R}^n$, the set of vectors with nonnegative entries by $\mathbb{R}^n_{\geq 0}$ and the set of real matrices by $\mathbb{R}^{n\times m}$. In the remainder of the paper, we assume without loss of generality that $n \leq m$. The singular value decomposition of $X \in \mathbb{R}^{n\times m}$ is denoted by $X = \sum_{i=1}^n \sigma_i(X) u_i v_i^\mathsf{T}$ with non-increasingly ordered singular values $\sigma_1(X) \geq \cdots \geq \sigma_n(X)$ (counted with multiplicity). The corresponding vector of all singular values is given by $\sigma(X) := (\sigma_1(X), \ldots, \sigma_n(X))$. For all $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, we define the ℓ_p norms with $1 \leq p < \infty$ by $\ell_p(x) := \left(\sum_{i=1}^q |x_i|^p\right)^{\frac{1}{p}}$ and $\ell_\infty(x) := \max_i |x_i|$, where $|\cdot|$ denotes the absolute value.

A matrix norm $\|\cdot\| : \mathbb{R}^{n\times m} \to \mathbb{R}_{\geq 0}$ is called *unitarily invariant* if for all unitary matrices $U \in \mathbb{R}^{n\times n}$ and $V \in \mathbb{R}^{m\times m}$ and all $X \in \mathbb{R}^{n\times m}$ it holds that $\|UXV\| = \|X\|$. Equivalently, unitary invariance can be characterized by *symmetric gauge functions* (see e.g., [25, Theorem 7.4.7.2]):

Definition 1 A function $g : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is a symmetric gauge function if

i. g is a norm.
ii. $\forall x \in \mathbb{R}^n : g(|x|) = g(x)$, where $|x|$ denotes the element-wise absolute value.
iii. $g(Px) = g(x)$ for all permutation matrices $P \in \mathbb{R}^{n\times n}$ and all $x \in \mathbb{R}^n$.

Proposition 1 *The norm $\|\cdot\| : \mathbb{R}^{n\times m} \to \mathbb{R}_{\geq 0}$ is unitarily invariant if and only if $\|\cdot\| = g(\sigma_1(\cdot), \ldots, \sigma_n(\cdot))$, where g is a symmetric gauge function.*

Throughout this work, we use the notation $\|X\|_g := g(\sigma(X))$. For $X, Y \in \mathbb{R}^{n\times m}$ the *Frobenius inner product* is defined as $\langle X, Y\rangle := \sum_{i=1}^m \sum_{j=n}^n x_{ij} y_{ij} = \mathrm{trace}(X^\mathsf{T} Y)$ with *Frobenius norm* $\|X\|_{\ell_2} := \ell_2(\sigma(X)) = \sqrt{\langle X, X\rangle}$. The *nuclear norm* and the *spectral norm* are given by $\|\cdot\|_{\ell_1} := \ell_1(\sigma(\cdot))$ and $\|\cdot\|_{\ell_\infty} := \ell_\infty(\sigma(\cdot)) = \sigma_1(\cdot)$. The *dual norm* to $\|\cdot\|_g$ is defined as

$$\|\cdot\|_{g^D} := \max_{\|X\|_g \leq 1} \langle \cdot, X\rangle =: g^D(\sigma_1(\cdot), \ldots, \sigma_n(\cdot)). \tag{1}$$

Dual norms inherit the unitary invariance as well as the duality relationship for ℓ_p norms, i.e., $g = \ell_p$ implies $g^D = \ell_q$ with $p, q \in [1, \infty]$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$. We will also make use of truncated dual gauge functions. Let $y \in \mathbb{R}^n$, $r \in \{1, \ldots, n\}$, and $g^D : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$. The *truncated dual gauge function* is then defined as

$$g_r^D(y) := g^D(\mathrm{sort}(y)_1, \ldots, \mathrm{sort}(y)_r, 0, \ldots, 0), \tag{2}$$

where $\mathrm{sort} : \mathbb{R}^n \to \mathbb{R}^n$ denotes sorting in descending order.

Next, we introduce some standard notation and results from convex optimization [5,41]. For $f : \mathbb{R}^{n\times m} \to \mathbb{R} \cup \{\infty\}$, we denote by $\mathrm{dom}(f)$ and $\mathrm{epi}(f)$ the *effective domain* and *epigraph* of f, respectively. Its *subdifferential* at $X \in \mathbb{R}^{n\times m}$ is written as $\partial f(X)$. In particular, by [24, Example VI.3.1]

$$\partial \|X\|_g = \{G \in \mathbb{R}^{n\times m} : \langle G, X\rangle = \|X\|_g,\ \|G\|_{g^D} = 1\}. \tag{3}$$

Further, f is said to be *proper* if $\mathrm{dom} f \neq \emptyset$ and *closed* if $\mathrm{epi}(f)$ is a closed set. The *conjugate (dual) function* of f is denoted by f^* and $f^{**} := (f^*)^*$ is called the *biconjugate function* or *convex envelope* of f. For $f : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$, we say that f *increasing* if $x \leq y \;\Rightarrow\; f(x) \leq f(y)$ for all $x, y \in \mathrm{dom}(f)$ and if there exist $x, y \in \mathbb{R}$ such that $x < y$ and $f(x) < f(y)$. Moreover, its *monotone conjugate* is defined as [41] $f^+(y) := \sup_{x \geq 0} [xy - f(x)]$ for all $y \in \mathbb{R}$. The *0-infinity indicator (or characteristic)* function of a set $\mathcal{S} \subset \mathbb{R}^{n \times m}$ is denoted by $\chi_{\mathcal{S}}$, which we also use for the indicator function of the set of matrices with at most rank r, i.e., $\chi_{\mathrm{rank}(\cdot) \leq r}$. For any $Z \in \mathbb{R}^{n \times m}$, the *proximal mapping* of a closed, proper and convex function $f : \mathbb{R}^{n \times m} \to \mathbb{R} \cup \{\infty\}$ is defined as

$$\mathrm{prox}_{\gamma f}(Z) := \operatorname*{argmin}_X \left(f(X) + \frac{1}{2\gamma} \|X - Z\|_{\ell_2}^2 \right). \tag{4}$$

In particular, $\mathrm{prox}_{\gamma \chi_{\mathcal{C}}}(Z)$ coincides with the unique Euclidean projection

$$\Pi_{\mathcal{C}}(Z) := \operatorname*{argmin}_{X \in \mathcal{C}} \|X - Z\|_{\ell_2}$$

onto $\mathcal{C}$ for any closed, non-empty, convex set $\mathcal{C} \subset \mathbb{R}^{n \times m}$. Moreover, by the *extended Moreau decomposition* it holds for all $f : \mathbb{R}^{n \times m} \to \mathbb{R} \cup \{\infty\}$, $Z \in \mathbb{R}^{n \times m}$ and $\gamma > 0$ that (see [4, Theorem 6.29])

$$\mathrm{prox}_{\gamma f}(Z) = Z - \gamma \mathrm{prox}_{\gamma^{-1} f^*}(\gamma^{-1} Z). \tag{5}$$

Finally, we denote *compositions* of two functions f and g by $(f \circ g)(\cdot) := f(g(\cdot))$.

3 Low-Rank Inducing Norms

This section introduces the family of unitarily invariant *low-rank inducing norms*, which has been discussed in [17]. Besides recapping some elementary properties, we briefly motivate the usefulness of these norms as convex envelopes or additive regularizers in optimization problems to promote low-rank solutions.

Low-rank inducing norms are defined as the dual norm of a *rank constrained dual norm*

$$\|Y\|_{g^D, r} := \max_{\substack{\mathrm{rank}(X) \leq r \\ \|X\|_g \leq 1}} \langle X, Y \rangle. \tag{6}$$

This means that the *low-rank inducing norms* corresponding to $\|\cdot\|_g$ are

$$\|X\|_{g, r*} := \max_{\|Y\|_{g^D, r} \leq 1} \langle Y, X \rangle. \tag{7}$$

For $r = n$, the rank constraint in (6) is redundant and $\|\cdot\|_g \equiv \|\cdot\|_{g, r*}$. Some important properties of these norms are summarized next [17].

Lemma 1 *Let $X, Y \in \mathbb{R}^{n\times m}$, $r \in \mathbb{N}$ be such that $1 \le r \le n$, and $g : \mathbb{R}^n \to \mathbb{R}_{\ge 0}$ be a symmetric gauge function. Then $\|\cdot\|_{g^D,r}$ is a unitarily invariant norm with*

$$\|Y\|_{g^D,r} = g_r^D(\sigma(Y)), \tag{8}$$

where g_r^D is defined in (2). Its dual norm $\|\cdot\|_{g,r*}$ satisfies

$$\|\cdot\|_{g,r*} = (\|\cdot\|_g + \chi_{\mathrm{rank}(\cdot)\le r}(\cdot))^{**}. \tag{9}$$

In this work, we especially consider the so-called *low-rank inducing Frobenius and spectral norms*, i.e., the cases when $g = \ell_2$ and $g = \ell_\infty$. Since $\ell_2^D = \ell_2$ and $\ell_\infty^D = \ell_1$, it follows from (8) that $\|X\|_{\ell_2,r*} := \max_{\|Y\|_{\ell_2,r}\le 1}$ with $\|Y\|_{\ell_2^D,r} := \sqrt{\sum_{i=1}^r \sigma_i^2(Y)}$ and $\|X\|_{\ell_\infty,r*} := \max_{\|Y\|_{\ell_1,r}\le 1}\langle Y, X\rangle$ with $\|Y\|_{\ell_1,r} = \sum_{i=1}^r \sigma_i(Y)$.

The following motivates the main interest in low-rank inducing norms (see [16,17,19] for details).

Proposition 2 *Assume that $f_0 : \mathbb{R}^{n\times m} \to \mathbb{R}\cup\{\infty\}$ is a proper closed convex function, and that $r \in \mathbb{N}$ is such that $1 \le r \le \min\{m, n\}$. Let $f_1 : \mathbb{R}_{\ge 0} \to \mathbb{R}\cup\{\infty\}$ be an increasing, proper closed convex function, and let $\theta > 0$. Then*

$$(f_1 \circ \|\cdot\|_{g,r*})^* = f_1^+(\|\cdot\|_{g^D,r}) \tag{10}$$

and

$$\inf_{\substack{X\in\mathbb{R}^{n\times m}\\ \mathrm{rank}(X)\le r}} \left[f_0(X) + \theta f_1(\|X\|_g)\right] \ge -\inf_{D\in\mathbb{R}^{n\times m}} \left[f_0^*(D) + \theta f^+(\theta^{-1}\|D\|_{g^D,r})\right] \tag{11}$$

$$= \inf_{X\in\mathbb{R}^{n\times m}} \left[f_0(X) + \theta f_1(\|X\|_{g,r*})\right]. \tag{12}$$

If $X^\star$ solves (12) such that $\mathrm{rank}(X^\star) \le r$, then equality holds and $X^\star$ is also a solution to the problem on the left of (11).

In other words, Proposition 2 shows that low-rank inducing norms can be used both as additive regularizers and direct convex envelopes to find (approximate) solutions to

$$\begin{aligned} \underset{X}{\text{minimize}} \quad & L(X) \\ \text{subject to} \quad & \mathrm{rank}(X) \le r. \end{aligned} \tag{13}$$

For regularization as in [15,42], we set $f_0 = L$ and choose a suitable f_1 and θ to find an approximate solution. In the second case, when L can be split into $L = f_0 + f_1(\|\cdot\|_g)$ as in Proposition 2, then

$$\min_{X\in\mathbb{R}^{n\times m}} \left[f_0(X) + f_1(\|X\|_{g,r*})\right] \tag{14}$$

may return an (exact) solution to (13).

4 Proximal Mappings

For problems of small dimensions, it is often convenient to solve (14) through semi-definite programming (SDP). However, conventional SDP solvers are typically based on interior-point methods (see [39]) with an iteration cost that grows unfavorably with the problem dimension. For large-scale problems, proximal splitting methods can be used [4,9]. To efficiently solve (14), proximal splitting methods require efficient computation of the proximal mapping of $f_1(\|\cdot\|_{g,r*})$.

In this section, we present our main results on developing a *nested binary search framework* for computing this proximal mapping for simple choices of f_1, efficiently. Explicit and implementable steps for these computations will be shown for the common cases $f_1 = (\cdot)$ and $f_1 = (\cdot)^2$ with $g = \ell_2$ and $g = \ell_\infty$ [3,17,19,37]. In Sect. 4.3, the computational complexity of our generic algorithm as well as these particular cases is derived. In cases where f_1 is not simple, we can write (14) as

$$\min_{t\in\mathbb{R},\ X\in\mathbb{R}^{n\times m}} f_0(X) + f_1(t) + \chi_{\mathrm{epi}(\|\cdot\|_{g,r*})}(X,t), \tag{15}$$

where $\chi_{\mathrm{epi}(\|\cdot\|_{g,r*})}$ is the indicator function of the epigraph to $\|\cdot\|_{g,r*}$. Then a consensus formulation for proximal splitting methods (see [9]) requires an evaluation of the proximal mappings for f_1 and $\chi_{\mathrm{epi}(\|\cdot\|_{g,r*})}$. Since f_1 is one-dimensional, convex, proper and increasing, its proximal mapping is fast to evaluate. We will see as part of our complexity analysis in Sect. 4.3 that computing $\mathrm{prox}_{\chi_{\mathrm{epi}(\|\cdot\|_{g,r*})}}$, $\mathrm{prox}_{\|\cdot\|_{g,r*}}$ and $\mathrm{prox}_{\|\cdot\|^2_{g,r*}}$, i.e., $f_1 = (\cdot)$ and $f_1 = (\cdot)^2$, is equally costly.

Note that in contrast to $\|\cdot\|_{g,r*}$, its dual norm $\|\cdot\|_{g^D,r}$ is explicitly known by its definition (8), which is why we derive our search framework for

$$\mathrm{prox}_{\gamma^{-1}f_1^+(\|\cdot\|_{g^D,r})}(Z) \quad \text{and} \quad \Pi_{-\mathrm{epi}(\|\cdot\|_{g^D,r})}(Z, z_v), \tag{16}$$

with

$$-\mathrm{epi}(\|\cdot\|_{g^D,r}) := \{(Y,-w) : \|Y\|_{g^D,r} \le w\}$$

which by (5) and (10) yields the sought proximal mappings

$$\mathrm{prox}_{\gamma f_1(\|\cdot\|_{g,r*})}(Z) = Z - \gamma\,\mathrm{prox}_{\gamma^{-1}f_1^+(\|\cdot\|_{g^D,r})}(\gamma^{-1}Z) \tag{17a}$$

$$\mathrm{prox}_{\chi_{\mathrm{epi}(\|\cdot\|_{g,r*})}}(Z,z_v) = \Pi_{\mathrm{epi}(\|\cdot\|_{g,r*})}(Z,z_v) = (Z,z_v) - \Pi_{-\mathrm{epi}(\|\cdot\|_{g^D,r})}(Z,z_v). \tag{17b}$$

4.1 Search Framework

Next, we present our main result, which shows that (16), and hence Eqs. (17a) and (17b), can be computed by a nested parameter search. Since the computations of (16)

Table 1 Example choices for f and γ in (18) for the computation of (16) and thus Eqs. (17a) and (17b). $\chi_{\|\cdot\|_{g^D}\leq\gamma}$ stands for the indicator function of the set $\{X : \|X\|_{g^D} \leq \gamma\}$

Solution $(Y^\star, w^\star)$ to (18)	$f(w)$	γ	Via Eqs.(17a) and (17b)
$(Y^\star, -w^\star) = \Pi_{-\text{epi}(\|\cdot\|_{g^D,r})}(Z, z_v)$	$\frac{1}{2}(w+z_v)^2$	1	$\Pi_{\text{epi}(\|\cdot\|_{g,r*})}(Z, z_v)$
$Y^\star = \text{prox}_{\frac{\gamma}{2}\|\cdot\|^2_{g^D,r}}(Z)$	$\frac{w^2\chi_{[0,\infty)}(w)}{2}$	> 0	$\text{prox}_{\frac{\gamma}{2}\|\cdot\|^2_{g,r*}}(Z)$
$Y^\star = \text{prox}_{\chi_{\|\cdot\|_{g^D,r}\leq\gamma}}(Z)$	$\chi_{[0,\gamma]}(w)$	> 0	$\text{prox}_{\gamma\|\cdot\|_{g,r*}}(Z)$

can be compactly unified as

$$\begin{aligned} \underset{Y,w}{\text{minimize}} \quad & f(w) + \frac{\gamma}{2}\|Y - Z\|^2_{\ell_2} \\ \text{subject to} \quad & w \geq \|Y\|_{g^D,r}\,, \;\; Y \in \mathbb{R}^{n\times m}, \end{aligned} \tag{18}$$

where f is closed, proper and convex, our results are stated for all such problems. Table 1 summaries common choices for f and its relationship to Eqs. (17a) and (17b) via (16).

Before we state the main theorem on how to solve (18) with a nested binary search method, we outline the steps that give rise to this algorithm. It is well-known that the solution $Y^\star$ to (18) and Z have a simultaneous SVD [31,36] and, therefore, only the singular values of $Y^\star$ need to be computed. Let $y_i = \sigma_i(Y)$ and $z_i = \sigma_i(Z)$, then it follows that (18) reduces to the vector-valued problem

$$\begin{aligned} \underset{y,w}{\text{minimize}} \quad & f(w) + \frac{\gamma}{2}\sum_{i=1}^{n}(y_i - z_i)^2 \\ \text{subject to} \quad & w \geq g_r^D(y), \;\; y \in \mathbb{R}^n. \end{aligned} \tag{19}$$

Since $z \in \mathbb{R}^n_{\geq 0}$ is monotonically decreasing, it can be shown that the minimizer of (19) is nonnegative. The problem is, therefore, equivalent to solving

$$\begin{aligned} \underset{y,w}{\text{minimize}} \quad & f(w) + \frac{\gamma}{2}\sum_{i=1}^{n}(y_i - z_i)^2 \\ \text{subject to} \quad & w \geq g^D(y_1, \ldots, y_r, 0, \ldots, 0), \;\; y \in \mathbb{R}^n, \\ & y_1 \geq \cdots \geq y_n. \end{aligned} \tag{20}$$

Since only the r first elements in y are included in the norm constraint, the solution may have a chain of equalities around y_r, i.e., there exist integers $t \geq 1$ and $s \geq 0$

such that

$$\begin{aligned} \underset{y,w}{\text{minimize}} \quad & f(w) + \frac{\gamma}{2}\sum_{i=1}^{n}(y_i - z_i)^2 \\ \text{subject to} \quad & w \geq g^D(y_1, \ldots, y_r, 0, \ldots, 0), \ y \in \mathbb{R}^n, \\ & y_1 \geq \cdots \geq y_{r-t+2} > y_{r-t+1} = \cdots = y_r \\ & \qquad = \cdots = y_{r+s} > y_{r+s+1} \geq \cdots \geq y_n. \end{aligned} \tag{21}$$

The base case $t = 1$ and $s = 0$ implies that $y_{r-1} > y_r > y_{r+1}$, i.e., the chain has length one. Thus, if we can solve (21) for an arbitrary, but fixed pair (t, s), an optimal (t^*, s^*) could be determined by comparison with all pairs. As this would be very inefficient, the proposed search rules are devised to find (t^*, s^*) by only considering a few pairs.

To state these rules, we need to introduce the *truncated gauge function* of g^D as

$$g^D_{r,s,t}(x) := g^D\Big((Tx)_1, \ldots, (Tx)_{r-t}, \underbrace{(Tx)_{r-t+1}, \ldots, (Tx)_{r-t+1}}_{t \text{ times}}, 0, \ldots, 0\Big),$$

where $x \in \mathbb{R}^n$ and the truncation operator $T : \mathbb{R}^n \to \mathbb{R}^{r-t+1}$ is defined for all $1 \leq r \leq n$ and $(t, s) \in \{1, \ldots, r\} \times \{0, \ldots, n-r\}$ as

$$(Tx)_i := \begin{cases} \operatorname{sort}(x)_i, & \text{if } 1 \leq i \leq r-t, \\ \dfrac{\sum_{i=r-t+1}^{r+s} \operatorname{sort}(x)_i}{\sqrt{t+s}}, & \text{if } i = r-t+1. \end{cases} \tag{22}$$

Note that $g^D_{r,s,t}$ is indeed a gauge function with dual gauge function [23, Lemma 2.2.2])

$$g_{r,s,t}(x) := g\Big((Tx)_1, \ldots, (Tx)_{r-t}, \underbrace{\tfrac{(Tx)_{r-t+1}(s+t)}{t}, \ldots, \tfrac{(Tx)_{r-t+1}(s+t)}{t}}_{t \text{ times}}, 0, \ldots, 0\Big).$$

For the special case $(t, s) = (1, 0)$, it reduces to g^D_r in (2). We are now ready to state our main theorem.

Theorem 1 *Let $Z = \sum_{i=1}^{n} \sigma_i(Z) u_i v_i^T \in \mathbb{R}^{n\times m}$, $\gamma > 0$, $1 \leq r \leq n$, $g : \mathbb{R}^n \to \mathbb{R}$ be a gauge function, and $f : \mathbb{R} \to \mathbb{R}$ be proper, closed and convex. For each $(t, s) \in \{1, \ldots, r\} \times \{0, \ldots, n-r\}$ let $(y^{(t,s)}, w^{(t,s)}) \in \mathbb{R}^{n+1}$ be defined as*

$$y_i^{(t,s)} := \begin{cases} \tilde{y}_i, & \text{if } 1 \leq i \leq r-t, \\ \dfrac{\tilde{y}_i}{\sqrt{t+s}}, & \text{if } r-t+1 \leq i \leq r+s, \\ \sigma_i(Z) & \text{if } i \geq r+s+1. \end{cases} \tag{23a}$$

$$w^{(t,s)} := \tilde{w} \tag{23b}$$

where $(\tilde{y}, \tilde{w}) \in \mathbb{R}^{r-t+2}$ *solves*

$$\begin{aligned} \underset{\tilde{y},\tilde{w}}{\text{minimize}} \quad & f(\tilde{w}) + \frac{\gamma}{2}\sum_{i=1}^{r-t+1}(\tilde{y}_i - \tilde{z}_i)^2 \\ \text{subject to} \quad & \tilde{w} \geq g^D_{r,s,t}(\tilde{y}),\ \tilde{y} \in \mathbb{R}^{r-t+1}, \end{aligned} \tag{24}$$

and $\tilde{z} := T\sigma(Z)$ *is given by* (22). *Then* $(Y^\star, w^\star) = (\sum_{i=1}^n y_i^{(t^\star,s^\star)} u_i v_i^T, w^{(t^\star,s^\star)})$ *is the solution to* (18), *where*

$$t^\star := \min\left\{\{t : y_{r-t}^{(t,s_t^\star)} > y_{r-t+1}^{(t,s_t^\star)}\} \cup \{r\}\right\} \tag{25a}$$

$$\begin{aligned} s_t^\star &:= \min\left\{\{s : y_{r+s}^{(t,s)} > y_{r+s+1}^{(t,s)}\} \cup \{n-r\}\right\} \\ s^\star &:= s^\star_{t^\star} \end{aligned} \tag{25b}$$

In particular, $(t^\star, s^\star)$ *can be found by a nested binary search over* t *and* s *with the following rules for increasing/decreasing* t *and* s*:*

I. $y_{r-t}^{(t,s_t^\star)} \geq y_{r-t+1}^{(t,s_t^\star)}$ *for all* $t \geq t^\star$.
II. $y_{r-t}^{(t,s_t^\star)} \leq y_{r-t+1}^{(t,s_t^\star)}$ *for all* $t < t^\star$.
III. *If* $t < t^\star$ *and* $y_{r-t}^{(t,s_t^\star)} = y_{r-t+1}^{(t,s_t^\star)}$ *then* $\left(y^{(t,s_t^\star)}, w^{(t,s_t^\star)}\right) = \left(y^{(t^\star,s^\star)}, w^{(t^\star,s^\star)}\right)$.
IV. $y_{r+s}^{(t,s)} \geq y_{r+s+1}^{(t,s)}$ *for all* $s \geq s_t^\star$.
V. $y_{r+s}^{(t,s)} \leq y_{r+s+1}^{(t,s)}$ *for all* $s < s_t^\star$.
VI. *If* $s < s_t^\star$ *and* $y_{r+s}^{(t,s)} = y_{r+s+1}^{(t,s)}$ *then* $\left(y^{(t,s)}, w^{(t,s)}\right) = \left(y^{(t,s_t^\star)}, w^{(t,s_t^\star)}\right)$.

A few words on Theorem 1 may be helpful. The first part simply makes explicit that $(y^{(t,s)}, w^{(t,s)})$ in Eqs. (23a) and (23b) is the solution of (21) with fixed t and s, i.e., it solves

$$\begin{aligned} \underset{y,w}{\text{minimize}} \quad & f(w) + \frac{\gamma}{2}\sum_{i=1}^{n}(y_i - z_i)^2 \\ \text{subject to} \quad & w \geq g^D_r(y),\ y \in \mathbb{R}^n, \\ & y_{r-t+1} = \cdots = y_{r+s}, \end{aligned} \tag{26}$$

via the solution of the lower-dimensional problem (24). For fixed t in (21), the search rules for s (Items IV. to VI.) can be used to find an optimal $s = s_t^\star$ that minimizes the cost in (21) among all choices of s that fulfil the constraint $y_{r+s}^{(t,s)} \geq y_{r+s+1}^{(t,s)} \geq \cdots \geq y_n^{(t,s)}$. Since $y_i^{(t,s)} = z_i$ for $i \geq r+s+1$ by (23a), it suffices to check that $y_{r+s}^{(t,s)} \geq y_{r+s+1}^{(t,s)}$, where by (25b) $s_t^\star$ is the smallest of such s. Similarly, the search rules for finding an optimal $t = t^\star$ minimize the cost in (21) among all choices $(t, s) = (t, s_t^\star)$ that do not violate the constraint $y_{r-t}^{(t,s_t^\star)} \geq y_{r-t+1}^{(t,s_t^\star)}$. Using nested binary search (see [28]) over s (inner loop) and t (outer loop), an optimal (t^*, s^*) can be found efficiently under the assumption that (24) has an efficiently computable solution for all choices (t, s). For more details, see the derivation of the proof to Theorem 1 in Appendix 2 and our explicit implementation for determining $\Pi_{-\mathrm{epi}(\|\cdot\|_{g^D,r})}(Z, z_v)$ in Algorithm 1.

Algorithm 1 Binary search for determining $\Pi_{-(\mathrm{epi}(\|\cdot\|_{g^D,r})}(Z, z_v)$

Input: Let $Z \in \mathbb{R}^{n\times m}$, $z_v \in \mathbb{R}$ and $r \in \{1, \dots, n\}$.
Compute SVD $Z = \sum_{i=1}^n \sigma_i(Z) u_i v_i^\top$ and let $z = \sigma(Z)$.
//Let $f = \frac{1}{2}(w + z_v)^2$ and $\gamma = 1$.
//Find $(t^\star, s^\star)$ in Theorem 1 through binary search over (t, s).
Set $t_{\min} = 1$, $t_{\max} = r$, and $t = \lfloor \frac{t_{\min}+t_{\max}}{2} \rfloor$
//Binary search over t to find $t^\star$: each iteration solves (21) for fixed t with optimal $s = s_t^\star$ and updates t based on the search rules for feasibility and optimality in Theorem 1
while $t_{\min} \neq t_{\max}$ **do**
  Set $s_{\min} = 0$, $s_{\max} = n - r$, and $s = \lfloor \frac{s_{\min}+s_{\max}}{2} \rfloor$
//Binary search over s to find $s_t^\star$: each iteration solves (21) for fixed (t, s) and updates s based on the search rules for feasibility and optimality in Theorem 1
  while $s_{\min} \neq s_{\max}$ **do**
    Determine $(y_{r+s}^{(t,s)}, y_{r+s+1}^{(t,s)})$ in eqs.(23a) and (23b) via solving (24) (see Propositions 3 and 4 for the the cases $g = \ell_2$ and $g = \ell_1$, respectively).
    if $y_{r+s}^{(t,s)} < y_{r+s+1}^{(t,s)}$ **then**
      $s_{\min} = s + 1$
    else
      $s_{\max} = s$
    end if
  end while
  Set $s_t^\star = s_{\min}$
  Determine $(y_{r-t}^{(t,s_t^\star)}, y_{r-t+1}^{(t,s_t^\star)})$ in eqs.(23a) and (23b) via solving (24) (see Propositions 3 and 4 for the the cases $g = \ell_2$ and $g = \ell_1$, respectively).
  if $y_{r-t}^{(t,s)} < y_{r-t+1}^{(t,s)}$ **then**
    $t_{\min} = t + 1$
  else
    $t_{\max} = t$
  end if
end while
Set $t^\star = t_{\min}$ and p.r.n. binary search for $s^\star = s_{t^\star}^\star$
Output: $(Y^\star, w^\star) = (\sum_{i=1}^n y_i^{(t^\star,s^\star)} u_i v_i^\top, -w^{(t^\star,s^\star)})$ with $(y^{(t^\star,s^\star)}, w^{(t^\star,s^\star)})$ given by eqs. (23a) and (23b).

4.2 Low-Rank Inducing Frobenius and Spectral Norms

Next, we exemplify solutions to (24) for the instances in Table 1 with $g = \ell_2$ and $g = \ell_\infty$. A general result on the solvability of (24) is given in Appendix 3.

In particular, we will discuss solutions to (24) for all $g = \tau\ell_2$ and $g = \tau\ell_\infty$, $\tau > 0$, because this enables us to handle the first two cases in Table 1 simultaneously through the identity

$$\operatorname{prox}_{\frac{\tau^2}{2}\|\cdot\|^2_{g,r*}}(Z) = Z - \operatorname{prox}_{\frac{1}{2}\|\cdot\|^2_{\frac{g^D}{\tau},r}}(Z) = Z - \Pi_Y\left(\Pi_{-\mathrm{epi}(\|\cdot\|_{\frac{g^D}{\tau},r})}(Z, 0)\right),$$

where $\Pi_Y(Y, w) := Y$ and $\tau > 0$. It is easy to adjust these computations for the third case, because $\operatorname{prox}_{\tau\|\cdot\|_{g,r*}}(Z) = \operatorname{prox}_{\|\cdot\|_{\tau g,r*}}(Z)$.

Proposition 3 *Let $Z = \sum_{i=1}^n \sigma_i(Z) u_i v_i^T \in \mathbb{R}^{n\times m}$, $g = \tau\ell_2$ with $\tau > 0$, $1 \le r \le n$, $\gamma = 1$, $z_v \in \mathbb{R}$ and $\tilde{z} := T\sigma(z)$. Then, $\Pi_{-\mathrm{epi}(\|\cdot\|_{g^D,r})}(Z, z_v)$ can be computed via Theorem 1 with $f(w) = \frac{1}{2}(w + z_v)^2$, where the solution to (24) is characterized by one of the following three distinct cases:*

$$(\tilde{y}, \tilde{w}) = (\tilde{z}, z_v) \iff \sqrt{\sum_{i=1}^{r-t} \tilde{z}_i^2 + \frac{t}{s+t}\tilde{z}_{r-t+1}^2} \le -\tau z_v, \tag{27a}$$

$$(\tilde{y}, \tilde{w}) = 0 \iff \sqrt{\sum_{i=1}^{r-t} \tilde{y}_i^2 + \frac{s+t}{t}\tilde{y}_{r-t+1}^2} \le \frac{z_v}{\tau} \tag{27b}$$

and otherwise

$$\tilde{y}_i = \frac{\tilde{z}_i}{1 + \frac{\mu}{\tau^2\tilde{w}}}, \quad 1 \le i \le r - t \tag{27c}$$

$$\tilde{y}_{r-t+1} = \frac{\tilde{z}_{r-t+1}}{1 + \frac{\mu t}{\tau^2\tilde{w}(s+t)}} \tag{27d}$$

$$\tilde{w} = \mu - z_v \tag{27e}$$

where the unique $\mu \ge 0$ is a solution to the fourth-order polynomial

$$\left[\left(\tilde{w}\tau + \frac{\mu}{\tau}\right)^2 - c_1\right]\left[(t+s)\tau\tilde{w} + \frac{\mu}{\tau}t\right]^2 - tc_2^2\left(\tilde{w}\tau + \frac{\mu}{\tau}\right)^2 = 0 \tag{27f}$$

$c_1 := \sum_{i=1}^{r-t} \tilde{z}_i^2$ and $c_2 := \sqrt{t+s}\,\tilde{z}_{r-t+1}$.

Similarly, $\operatorname{prox}_{\chi_{\|\cdot\|_{\tau g^D,r}\le\gamma}}(Z)$ can be determined by setting $f(w) = \chi_{[0,\gamma]}(w)$, where it suffices to consider the two cases: (27a) with $z_v = -1$, and Eqs. (27c), (27d) and (27f) with $\tilde{w} = 1$.

Proposition 4 *Let* $Z = \sum_{i=1}^{n} \sigma_i(Z) u_i v_i^T \in \mathbb{R}^{n\times m}$, $g = \tau \ell_\infty$ *with* $\tau > 0$, $1 \le r \le n$, $\gamma = 1$, $z_v \in \mathbb{R}$ *and* $\tilde{z} := T\sigma(z)$. *Further, let*

$$\hat{z} := \left(\tilde{z}_1, \ldots, \tilde{z}_j, \frac{t}{\sqrt{(t+s)}} \tilde{z}_{r-t+1}, \tilde{z}_{j+1}, \ldots, \tilde{z}_{r-t}\right) \in \mathbb{R}^{r-t+1},$$

$$\alpha := \left(\underbrace{1, \ldots, 1}_{\text{length } j}, \frac{t^2}{(t+s)}, 1, \ldots, 1\right) \in \mathbb{R}^{r-t+1}.$$

where j *is chosen such that*

$$\tilde{z}_j > \tfrac{\sqrt{(t+s)}}{t} \tilde{z}_{r-t+1} \ge \tilde{z}_{j+1} \quad \text{or} \quad \tilde{z}_{r-t} \ge \tfrac{\sqrt{(t+s)}}{t} \tilde{z}_{r-t+1}. \tag{28}$$

Then, $\Pi_{-\mathrm{epi}(\|\cdot\|_{g^D,r})}(Z, z_v)$ *can be computed via Theorem 1 with* $f(w) = \frac{1}{2}(w + z_v)^2$, *where the solution to (24) is characterized by one of the following three distinct cases:*

$$(\tilde{y}, \tilde{w}) = (\tilde{z}, zv) \iff \sum_{i=1}^{r-t} |\tilde{z}_i| + \frac{t}{\sqrt{t+s}} |\tilde{z}_{r-t+1}| \le -\tau z_v \tag{29a}$$

$$(\tilde{y}, \tilde{w}) = 0 \iff \max\left(\tilde{z}_1, \frac{\sqrt{t+s}}{t} \tilde{z}_{r-t+1}\right) \le \frac{z_v}{\tau} \tag{29b}$$

and otherwise

$$\tilde{y}_i = \max\left(\tilde{z}_i - \tfrac{\mu}{\tau}, 0\right), \ 1 \le i \le r-t, \tag{29c}$$

$$\tilde{y}_{r-t+1} = \max\left(\tilde{z}_{r-t+1} - \tfrac{t\mu}{\sqrt{(t+s)}\tau}, 0\right), \tag{29d}$$

$$\tilde{w} = \mu - z_v \tag{29e}$$

where $\mu = \hat{\mu}_{k^\star}$ *with* $\hat{\mu}_k = \frac{z_v + \sum_{i=1}^{k} \hat{z}_i}{1 + \sum_{i=1}^{k} \alpha_i}$ *and* $k^\star$ *can be identified by a search over* k *with the following rules for increasing/decreasing* k:

I. $k^\star = \max\{k : \hat{z}_k - \alpha_k \hat{\mu}_k \ge 0\}$
II. $\hat{z}_k - \alpha_k \hat{\mu}_k \ge 0$ *for all* $k \le k^\star$
III. $\hat{z}_k - \alpha_k \hat{\mu}_k < 0$ *for all* $k > k^\star$

Similarly, $\mathrm{prox}_{\chi_{\|\cdot\|_{\tau g^D,r} \le \gamma}}(Z)$ *can be determined by setting* $f(w) = \chi_{[0,\gamma]}(w)$, *where it suffices to consider the two cases: (29a) with* $z_v = -1$ *and Eqs. (29c) and (29d), where* $\mu = \hat{\mu}_{k^\star}$ *can be found with the search rules from above and* $\hat{\mu}_k = \frac{\sum_{i=1}^{k} \hat{z}_i}{\sum_{i=1}^{k} \alpha_i}$.

Propositions 3 and 4 are proved in Appendixces 4 and 6, respectively, and implementations for the user are available for MATLAB and Python at [20,21].

Table 2 Complexity of computing $\Pi_{\mathrm{epi}(\|\cdot\|_{\ell_2,r*})}(Z, z_v)$ and $\Pi_{\mathrm{epi}(\|\cdot\|_{\ell_1,r*})}(Z, z_v)$ given a pre-computed SVD of Z in comparison with others

Source	$\Pi_{\mathrm{epi}(\|\cdot\|_{\ell_2,r*})}(Z, z_v)$	$\Pi_{\mathrm{epi}(\|\cdot\|_{\ell_1,r*})}(Z, z_v)$
[14,29]	$\mathcal{O}(\log(r)\log(n-r+1)+n)$	n/a
[46]	n/a	$\mathcal{O}(r(n-r+1))$
This work	$\mathcal{O}(\log(r)\log(n-r+1)+n)$	$\mathcal{O}(r\log(r)\log(n-r+1)+n)$

4.3 Computational Complexity

In the following, we evaluate the computational complexity, i.e., counting all flops (see [44]) of the discussed approaches for computing

$$\mathrm{prox}_{\chi_{\mathrm{epi}(\|\cdot\|_{g,r*})}}(Z, z_v) = \Pi_{\mathrm{epi}(\|\cdot\|_{g,r*})}(Z, z_v) = (Z, z_v) - \Pi_{-\mathrm{epi}(\|\cdot\|_{g^D,r})}(Z, z_v).$$

Since the same analysis also applies to the other cases discussed in Table 1, this will allow us to compare our approach to existing methods. Our evaluation starts with a discussion of Algorithm 1 for a general gauge function, followed by an explicit discussion for the cases of $g = \ell_2$ and $g = \ell_\infty$ in Sects. 4.3.1 and 4.3.2 of which a summary is given in Table 2.

In order to apply the binary search rules in Theorem 1, we only need to determine $(y^{(t,s)}_{r-t}, y^{(t,s)}_{r-t+1}, y^{(t,s)}_{r+s}, y^{(t,s)}_{r+s+1})$, whose computational cost we assume to be bounded by $C(n,r)$. Then, the complexity of Algorithm 1 is the sum of:

1. SVD for Z providing all $\sigma_i(Z)$ and $u_i v_i^\top$ such that $Z = \sum_{i=1}^n \sigma_i(Z) u_i v_i^\top$ (see [44]): $\mathcal{O}(mn^2)$.
2. Binary search rules (see [28]) in Theorem 1 for t and s:

$$\mathcal{O}(C(n,r)\log(r)\log(n-r))$$

3. Determine the final full solution: $\mathcal{O}(n)$.
4. Compute $\mathrm{prox}_{\chi_{\mathrm{epi}(\|\cdot\|_{g,r*})}}(Z, z_v)$ from $\Pi_{-\mathrm{epi}(\|\cdot\|_{g^D,r})}(Z, z_v)$: $\mathcal{O}(n)$

In practise, the first cost may be significantly reduced by employing sparse SVD solvers (see e.g., [32,34]). In particular, for the vector-valued case, this corresponds to a simple sorting of the entries. The second cost is determined by the coordinate transformation (23a), i.e.,

$$(y^{(t,s)}_{r-t}, y^{(t,s)}_{r-t+1}, y^{(t,s)}_{r+s}, y^{(t,s)}_{r+s+1}) = \left(\tilde{y}_{r-t}, \frac{1}{\sqrt{s+t}}\tilde{y}_{r-t+1}, \frac{1}{\sqrt{s+t}}\tilde{y}_{r-t+1}, \sigma_{r+s+1}(Z)\right)$$

and therefore the cost for $C(n,r)$ equals the cost $\tilde{C}(n,r)$ for solving (24) to find $(\tilde{y}_{r-t}, \tilde{y}_{r-t+1})$. To compute the full solution $y^{(t^\star,s^\star)}$, once an optimal pair $(t^\star, s^\star)$ is found, the cost for these pre- and post-computing steps is at most $\mathcal{O}(n)$. Finally, computing $\mathrm{prox}_{\chi_{\mathrm{epi}(\|\cdot\|_{g,r*})}}(Z, z_v)$ from $\Pi_{-\mathrm{epi}(\|\cdot\|_{g^D,r})}(Z, z_v)$ only contributes an additional $n+1$ subtractions.

Remark 1 The cost for computing $\tilde{z}_{r-t+1}$ is given by the cost for knowing $\sum_{i=r+1}^{r+s} \sigma_i(Z)$ (for $s > 0$) and $\sum_{i=r-t+1}^{r} \sigma_i(Z)$. Both sums could be computed a priori for all t and s through incremental summation with cost $\mathcal{O}(n)$. However, in practice it may be cheaper to store and re-use the intermediate sums, when deriving $\sum_{i=r-t+1}^{r} z_i$ and $\sum_{i=r+1}^{r+s} z_i$. This means we only need to compute additional intermediate sums whenever t and s get increased within the binary search.

4.3.1 Low-Rank Inducing Frobenius Norms

In order to determine the computational cost $\tilde{C}(n, r)$ for $g = \ell_2$, we need to analyze the complexity of the three cases in Proposition 3. All cases require the evaluation of $\sum_{i=1}^{r-t} \tilde{z}_i^2 = \sum_{i=1}^{r-t} \sigma_i^2(Z)$ as either part of the inequalities Eqs. (27a) and (27b) or as coefficients in polynomial (27f). These sums can be computed once for all $t \in \{1, \dots, r\}$ with cost $\mathcal{O}(r)$. Then testing Eqs. (27a) and (27b) as well as solving the fourth-order polynomial (27f) are of cost $\mathcal{O}(1)$. Our generic approach, therefore, recovers in this special case the same complexity as the algorithms in [14,29] (see Table 2).

4.3.2 Low-Rank Inducing Spectral Norms

As in the previous case, determining $\tilde{C}(n, r)$ for $g = \ell_\infty$ requires us to compute the complexity of the three cases in Proposition 4. The cases Eqs. (29a) and (29b) require the evaluation of $\sum_{i=1}^{r-t} \tilde{z}_i = \sum_{i=1}^{r-t} \sigma_i(Z)$. This can be done once for all $t \in \{1, \dots, r\}$ with cost $\mathcal{O}(r)$, and verifying the corresponding inequalities is then of complexity $\mathcal{O}(1)$.

Determining μ in the third case of Proposition 4 requires:

a) Find j in (28): $\mathcal{O}(\log(r - t + 1))$, because $\tilde{z}_1 \geq \cdots \geq \tilde{z}_{r-t}$.
b) Determine $\mu_{k^\star} = \mu$ through binary search: $\mathcal{O}(r - t + 1)$, because $\sum_{i=1}^{r-t+1} \hat{z}_i$ may need to be computed.

Thus, $\tilde{C}(n, r)$ is dominated by the complexity of determining μ, which by the preceding analysis is at most $\mathcal{O}(r)$. Compared to [46], our approach reduces the overall cost significantly (see Table 2), which is especially important for the corresponding vector-valued problem.

5 Case Study: Matrix Completion

In the following, we will see how the binary search parameters (t, s, k) from Algorithm 1 and Proposition 4 evolve when solving an optimization problem with a proximal splitting method. We consider the convexified low-rank matrix completion problem (see, e.g., [7,8,17] for motivation and examples)

$$\begin{aligned} \underset{M}{\text{minimize}} \quad & \|M\|_{\ell_\infty, r*} \\ \text{subject to} \quad & m_{ij} = n_{ij}, \ (i, j) \in \mathcal{I} \end{aligned} \tag{30}$$

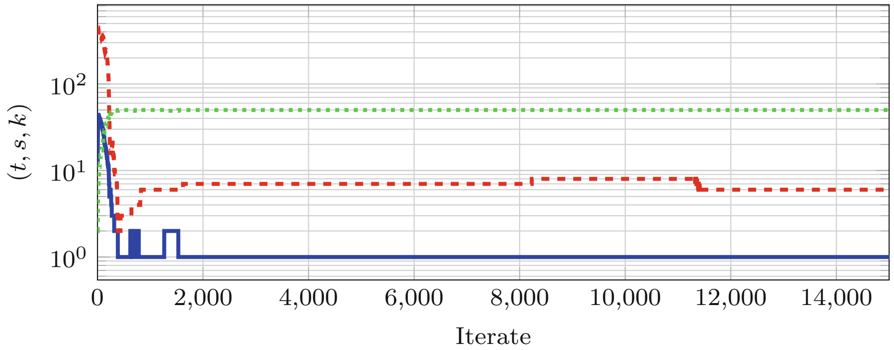

Fig. 1 Parameter path of (t, s, k) from Algorithm 1 and Proposition 4 when computing $\mathrm{prox}_{\|\cdot\|_{\ell_\infty,r*}}$ within the Douglas–Rachford iterations (32). There are no values for the first iterate, because $\mathrm{prox}_{\|\cdot\|_{\ell_\infty,r*}}(Z_0) = 0$ and the iterations are stopped when $\|X_i - Y_i\|_F \leq 10^{-8}$. Strict inequalities in Algorithm 1 are determined to be valid if the corresponding nonnegative difference is above the relative threshold $10^{-12}\sum_{i=1}^r \sigma_i(Z)$. The local plateauing after relatively few iterations suggests to use (t, s, k) from the previous iterations as an initial guess for the current iteration as well as to employ sparse SVD solvers in order to save computational time

with $r = 50$, $\mathcal{I} := \{n_{ij} : n_{ij} > 0\}$ and $N = \sum_{i=1}^r u_i u_i^\mathsf{T}$ being defined through the SVD of

$$H := \begin{pmatrix} 1 & 1 \cdots 1 & 1 \\ 1 & & 0 \\ \vdots & \iddots & \vdots \\ 1 & \iddots & 0 \\ 1 & 0 \cdots 0 & 0 \end{pmatrix} = \sum_{i=1}^{500} \sigma_i(H) u_i u_i^\mathsf{T} \in \mathbb{R}^{500\times 500}. \tag{31}$$

Note that a smaller version of this example has been solved successfully in [17] by using an SDP-solver, but this larger example is far out of the scope of typical SDP-solvers [39,43]. Therefore, we apply the following Douglas–Rachford splitting scheme (see [9,11,33]):

$$X_i = \mathrm{prox}_{\|\cdot\|_{\ell_\infty,r*}}(Z_{i-1}), \quad Y_i = \Pi_{\mathcal{L}}(2X_i - Z_{i-1}), \quad Z_i = Z_{i-1} + Y_i - X_i \tag{32}$$

with $\mathcal{L} := \{X \in \mathbb{R}^{500\times 500} : x_{ij} = n_{ij},\ (i, j) \in \mathcal{I}\}$, $Z_0 = 0$ and $\lim_{i\to\infty} X_i = \lim_{i\to\infty} Y_i$ being a solution to (30). By the construction of N, it can be shown that $\lim_{i\to\infty} X_i = N$ (see [17]).

The parameter path of (t, s, k) for computing X_i is shown in Fig. 1. We observe that as X_i approaches N, the values of t, s and k start plateauing. Thus, by using the values from one iterate in the subsequent iterate, the practical computational cost may reduce significantly. Finally, after the initial transient, the variance of each parameter is small compared to the overall 500 singular values. As a result, sparse SVD algorithms, which only compute a small predefined number of largest singular values (see, e.g., [32,34]), can be effectively applied. This emphasizing that our complexity analysis is important to both, vector- and matrix-valued problems.

6 Conclusion

This work presents a binary search framework for computing the proximal mappings of all unitarily invariant low-rank inducing norms and their epigraph projections. In particular, complete algorithms for the low-rank inducing Frobenius and spectral norms are presented. Our framework unifies and extends the known proximal mapping computations in the following sense: (i) So far, only proximal mappings for the squared low-rank inducing Frobenius norm [14] and the (non-squared) low-rank inducing spectral norm [46] have been derived. This framework is independent of the particular unitary invariant norm and its composition with a convex increasing function. (ii) Excluding the cost for an SVD, i.e., the cost for the analogous vector-valued problem, we recover the same complexity for the squared low-rank inducing Frobenius norm as in [14,29], but significantly decrease the complexity for the (non-squared) low-rank inducing spectral norm. Further, we show that these costs also transfer to compositions with simple functions.

Finally, in our case study we have seen that within a proximal splitting method, the computational cost of our proximal mappings may be reduced to approximately linear cost, besides the singular value decomposition, after a small number of iterations and is therefore roughly the same as in case of the nuclear norm/spectral norm. Further, our example also demonstrates that sparse singular value decomposition (see e.g., [32,34]) can be effectively applied, underlining the importance of our analysis even for the matrix case. Implementations for the low-rank inducing Frobenius and spectral norms are available for MATLAB and Python at [20,21].

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Lemmas, Proofs and Additional Discussion

Search Rules

Lemma 2 *Let f be proper, closed and convex, $z_1 \geq \cdots \geq z_n \geq 0$ and $\left(y^{(t)}, w^{(t)}\right)$ denote the t-dependent solution to*

$$\begin{array}{ll} \underset{y,w}{\text{minimize}} & f(w) + \frac{\gamma}{2}\sum_{i=1}^{n}(y_i - z_i)^2 \\ \text{subject to} & w \geq g_r^D(y),\ y \in \mathbb{R}^n, \\ & y_{r-t+1} = \cdots = y_r \geq \cdots \geq y_n. \end{array} \tag{33}$$

where $1 \leq t \leq r$. *Then there exists* $t^\star$ *such that* $\left(y^{(t^\star)}, w^{(t^\star)}\right)$ *is the solution to*

$$\begin{array}{ll} \underset{y,w}{\text{minimize}} & f(w) + \frac{\gamma}{2}\sum_{i=1}^{n}(y_i - z_i)^2 \\ \text{subject to} & w \geq g_r^D(y),\ y \in \mathbb{R}^n, \\ & y_1 \geq \cdots \geq y_n \geq 0, \end{array} \tag{34}$$

with $y^{(t^\star)}_{r-t^\star} > y^{(t^\star)}_{r-t^\star+1}$ *and* $y^{(t^\star)}_{r-t^\star} = y^{(t^\star)}_{r-t^\star+1}$ *if* $t^\star = r$. *Further,*

i. $t^\star = \min\left\{\{t : y^{(t)}_{r-t} > y^{(t)}_{r-t+1}\} \cup \{r\}\right\}$.
ii. If $y^{(t')}_{r-t'} \geq y^{(t')}_{r-t'+1}$ *then* $y^{(t)}_{r-t} \geq y^{(t)}_{r-t+1}$ *for all* $t \geq t'$.
iii. If $y^{(t')}_{r-t'} < y^{(t')}_{r-t'+1}$ *then* $y^{(t)}_{r-t} < y^{(t)}_{r-t+1}$ *for all* $t \leq t'$.

In particular, $t^\star$ *can be found by a search over* t, *where* t *is increased/decreased according to the following rules:*

I $y^{(t)}_{r-t} \geq y^{(t)}_{r-t+1}$ *for all* $t \geq t^\star$.
II $y^{(t)}_{r-t} \leq y^{(t)}_{r-t+1}$ *for all* $t < t^\star$.
III If $t < t^\star$ *and* $y^{(t)}_{r-t} = y^{(t)}_{r-t+1}$ *then* $\left(y^{(t)}, w^{(t)}\right) = \left(y^{(t^\star)}, w^{(t^\star)}\right)$.

Proof First we show the equivalence between Eqs. (34) and (33). To this end note that it is not necessary to explicitly restrict y to be nonnegative. The unique solution $(y^\star, w^\star)$ to (34) fulfills $0 \leq y_i^\star \leq z_i$ for $1 \leq i \leq n$. The upper bound holds, because otherwise by [25, Theorem 7.4.8.4] $g_r^D(\bar{y}) \leq g_r^D(y^\star)$ with $\bar{y}_i^\star := \min\{z_i, y_i^\star\}$ and thus $\bar{y}^\star$ is a feasible solution to (34) with smaller cost. Similarly, the lower bound holds, because otherwise $\bar{y}^\star$ with $\bar{y}_i^\star = \max\{0, y_i^\star\}$ is a feasible solution to (34) with smaller cost by Definition 1 (ii). Then there exists $t^\star$ such that $y^\star_{r-t^\star} > y^\star_{r-t^\star+1} = \cdots = y^\star_r$ where $t^\star = r$ if $y_1^\star = y_r^\star$, which implies that $y_{r-t^\star} \geq y_{r-t^\star+1}$ is assumed to be inactive and therefore can be removed from (34). Thus, also the constraints $y_1 \geq \cdots \geq y_{r-t^\star}$ can be removed, because the cost function and the sorting of z ensures that the solution will always fulfill them. This yields solving (34) reduces to finding $t^\star$ such that (33) solves (34).

Next, we characterize $t^\star$ in terms of solution to (33). In the following, we let $p(t)$ denote the optimal cost of (33) as a function of t. Since adding constraints cannot reduce the optimal cost, p is a nondecreasing function.

Item i.: By the same reasoning that led to the equivalence between Eqs. (34) and (33), it holds that $y_1^{(t)} \geq \cdots \geq y^{(t)}_{r-t}$, $1 \leq t \leq r$, which is why the set $\{t : y^{(t)}_{r-t} >$

$y^{(t)}_{r-t+1}\} \cup \{r\}$ contains all t for which the solution of (33) is feasible for (34). Since p is nondecreasing and $\left(y^{(t^\star)}, w^{(t^\star)}\right)$ is unique, the first claim follows.

Item ii.: The second claim is proven by contradiction. Let $(y^{(t')}, w^{(t')})$ be such that $y^{(t')}_{r-t'} \geq y^{(t')}_{r-t'+1}$. Further assume that $y^{(t'+1)}_{r-t'-1} < y^{(t'+1)}_{r-t'}$. In the following, we construct another solution $(\tilde{y}, \tilde{w}) \in \mathbb{R}^{q+1}$ to (33) with $t = t' + 1$, which has a cost that is no larger than $p(t' + 1)$. However, (33) has a unique solution due to strong convexity of the cost function. This yields the desired contradiction. The contradicting solution is constructed as a convex combination $\tilde{w} = (1 - \alpha)w^{(t'+1)} + \alpha w^{(t')}$ with $\alpha \in (0, 1]$ and a partially sorted convex combination of $y^{(t')}$ and $y^{(t'+1)}$ with the same α. Let $\hat{y} := (1 - \alpha)y^{(t'+1)} + \alpha y^{(t')}$ and let

$$\tilde{y} := (\mathrm{sort}(\hat{y}_1, \ldots, \hat{y}_{r-t'-2}, \hat{y}_{r-t'}), \hat{y}_{r-t'-1}, \hat{y}_{r-t'+1}, \ldots, \hat{y}_q)$$

be the partially sorted convex combination. To select α, we note that by assumption, $y^{(t')}_{r-t'-1} \geq y^{(t')}_{r-t'} \geq y^{(t')}_{r-t'+1}$ and $y^{(t'+1)}_{r-t'-1} < y^{(t'+1)}_{r-t'} = y^{(t'+1)}_{r-t'+1}$. Therefore, there exists an $\alpha \in (0, 1]$ such that

$$\begin{aligned}\tilde{y}_{r-t'} = \hat{y}_{r-t'-1} &= (1 - \alpha)y^{(t'+1)}_{r-t'-1} + \alpha y^{(t')}_{r-t'-1} \\ &= (1 - \alpha)y^{(t'+1)}_{r-t'+1} + \alpha y^{(t')}_{r-t'+1} = \hat{y}_{r-t'+1} = \tilde{y}_{r-t'+1}.\end{aligned}$$

Since $y^{(t')}_{r-t'+1} = \cdots = y^{(t')}_r$ and $y^{(t'+1)}_{r-t'-1} = \cdots = y^{(t'+1)}_r$, it follows that $\tilde{y}_{r-t'} = \cdots = \tilde{y}_r$. Furthermore, the construction of $\tilde{y}$ as well as the sorting yield that $\tilde{y}_r \geq \cdots \geq \tilde{y}_q$ and $\tilde{y}_1 \geq \cdots \geq \tilde{y}_{r-t'-1}$, which is why $\tilde{y}$ satisfies the chain of inequalities in (33) for $t = t' + 1$.

It remains to show that $\tilde{y}$ satisfies the epigraph constraint and that the cost is not higher than $p(t' + 1)$. These properties are already fulfilled for $\hat{y}$ being a convex combination of two feasible points with costs $p(t')$ and $p(t' + 1)$, respectively, where $p(t') \leq p(t' + 1)$. Therefore, it is left to show that the sorting involved in $\tilde{y}$ maintains these properties. First, we show that sorting of any sub-vector in $\hat{y}$ does not increase the cost. Suppose that $z_i \geq z_j$, $\hat{y}_i \leq \hat{y}_j$, i.e., $\hat{y}$ is not sorted the same way as z. Then

$$\begin{aligned}\frac{1}{2}\left((z_i - \hat{y}_i)^2 + (z_j - \hat{y}_j)^2\right) &= (z_i - z_j)(\hat{y}_j - \hat{y}_i) + \frac{1}{2}\left((z_i - \hat{y}_j)^2 + (z_j - \hat{y}_i)^2\right) \\ &\geq \frac{1}{2}\left((z_i - \hat{y}_j)^2 + (z_j - \hat{y}_i)^2\right),\end{aligned}$$

and thus the cost is not increased by sorting $\hat{y}$ or any sub-vector of it. Further, a permutation of the first r elements of $\hat{y}$ does not influence the epigraph constraint, because $g^D_r(\hat{y})$ is permutation invariant by definition.

Next notice that $\tilde{y}$ is obtained from $\hat{y}$ by first swapping $\hat{y}_{r-t'-1}$ and $\hat{y}_{r-t'}$. From the choice of α, we conclude that

$$\hat{y}_{r-t'} = (1 - \alpha)y^{(t'+1)}_{r-t'} + \alpha y^{(t')}_{r-t'} \geq (1 - \alpha)y^{(t'+1)}_{r-t'+1} + \alpha y^{(t')}_{r-t'+1} = \hat{y}_{r-t'+1} = \hat{y}_{r-t'-1}.$$

Thus, this swap is a sorting which does neither increase the cost, nor does it violate the epigraph constraint. Analogously, sorting the first $r-t'$ elements of the resulting vector to obtain $\tilde{y}$ has the same effect and therefore we receive the desired contradiction.

Item iii.: Suppose that there exist t and t' with $t' > t$ such that $y^{(t')}_{r-t'} < y^{(t')}_{r-t'+1}$ and $y^{(t)}_{r-t} \geq y^{(t)}_{r-t+1}$. Then Item ii. shows that $y^{(t')}_{r-t'} \geq y^{(t')}_{r-t'+1}$, which is a contradiction.

Items I. to III.: The statements follow immediately from Items i. to iii.

Lemma 3 *Let f and z be as in Lemma 2 and $\left(y^{(t,s)}, w^{(t,s)}\right)$ denote the (t,s)-dependent solution to*

$$\begin{aligned} \underset{y,w}{\text{minimize}} \quad & f(w) + \frac{\gamma}{2}\sum_{i=1}^{n}(y_i - z_i)^2 \\ \text{subject to} \quad & w \geq g_r^D(y),\ \ y \in \mathbb{R}^n, \\ & y_{r-t+1} = \cdots = y_{r+s}. \end{aligned} \tag{35}$$

where $0 \leq s \leq r-n$ and t is fixed within $1 \leq t \leq r$. Then, there exists $s^\star$ such that $\left(y^{(t,s^\star)}, w^{(t,s^\star)}\right)$ is the solution to (33) with $y^{(t,s^\star)}_{r+s^\star} > y^{(t,s^\star)}_{r+s^\star+1}$ and $y^{(t,s^\star)}_{r+s^\star} = y^{(t)}_{r+s^\star+1}$ if $s^\star = n-r$. Further,

i $s^\star = \min\left\{\{s : y^{(t,s^\star)}_{r+s^\star} > y^{(t,s^\star)}_{r+s^\star+1}\} \cup \{n-r\}\right\}$.

ii If $y^{(t,s')}_{r+s'} \geq y^{(t,s')}_{r+s'+1}$ *then* $y^{(t,s)}_{r+s} \geq y^{(t,s)}_{r+s+1}$ *for all* $s \geq s'$.

iii If $y^{(t,s')}_{r+s'} < y^{(t,s')}_{r+s'+1}$ *then* $y^{(t,s)}_{r+s} < y^{(t,s)}_{r+s+1}$ *for all* $s \leq s'$.

In particular, $s^\star$ can be found by a search over s, where s is increased/decreased according to the following rules:

I $y^{(t,s)}_{r+s} \geq y^{(t,s)}_{r+s+1}$ *for all* $s \geq s^\star$.

II $y^{(t,s)}_{r+s} \leq y^{(t,s)}_{r+s+1}$ *for all* $s < s^\star$.

III If $s < s^\star$ *and* $y^{(t,s)}_{r+s} = y^{(t,s)}_{r+s+1}$ *then* $\left(y^{(t,s)}, w^{(t,s)}\right) = \left(y^{(t,s^\star)}, w^{(t,s^\star)}\right)$.

The proof of Lemma 3 goes analogously to the proof of Lemma 2 and is therefore omitted.

Lemma 4 *Let f and z be as in Lemma 2, $1 \leq t \leq r$ and $0 \leq s \leq n-r$. Moreover, let $\tilde{z} := Tz \in \mathbb{R}^{r-t+1}$ be defined by (22) and be $(\tilde{y}^{(t,s)}, w^{(t,s)})$ the (t,s)-dependent solution to*

$$\begin{aligned} \underset{\tilde{y},w}{\text{minimize}} \quad & f(w) + \frac{\gamma}{2}\sum_{i=1}^{r-t+1}(\tilde{y}_i - \tilde{z}_i)^2 \\ \text{subject to} \quad & w \geq g_{r,s,t}^D(\tilde{y}),\ \ \tilde{y} \in \mathbb{R}^{r-t+1} \end{aligned} \tag{36}$$

Then $(y^{(t,s)}, w^{(t,s)})$ is a solution to (35), where

$$y_i^{(t,s)} := \begin{cases} \tilde{y}_i^{(t,s)}, & \text{if } 1 \leq i \leq r-t, \\ \dfrac{\tilde{y}_i^{(t,s)}}{\sqrt{t+s}}, & \text{if } r-t+1 \leq i \leq r+s, \\ z_i, & \text{if } i \geq r+s+1. \end{cases} \tag{37}$$

Proof Letting $\tilde{y} \in \mathbb{R}^{r-t+1}$ be defined as

$$\tilde{y}_i = \begin{cases} y_i, & \text{if } 1 \leq i \leq r-t, \\ \sqrt{t+s}\, y_{r-t+1}, & \text{if } i = r-t+1, \end{cases} \tag{38}$$

and notice that

$$\sum_{i=r-t+1}^{r+s} (y_r - z_i)^2 = (\tilde{y}_{r-t+1} - \tilde{z}_{r-t+1})^2 + \sum_{i=r-t+1}^{r+s} z_i^2 - \left(\frac{1}{\sqrt{t+s}} \sum_{i=r-t+1}^{r+s} z_i \right)^2,$$

yields the reduced dimensional problem (36).

Proof to Theorem 1

By Lemma 2, (34) can be solved by performing a search over the t-dependent solutions to (33), where by Lemma 3 these solutions can be determined for each t by a search over the s-dependent solutions to (35). In order to solve (35), we apply Lemma 4 to reduce (35) to solving (24) in Theorem 1. Hence, the remainder of the theorem is a direct application of Lemmas 3 and 2 and thus a nested search with the stated rules succeeds in finding $(t^\star, s^\star)$.

General Solution to (24)

In every step of the binary search (24) must be solved. Provided a very mild constraint qualification holds (which it does for our functions of interest), the solution will fall into one of three cases, depending on f and the singular values of Z. The different cases are described in the following.

Proposition 5 *Suppose that there exits $(\bar{y}, \bar{w})$ such that $\bar{w} \in \mathrm{relint}(\mathrm{dom} f)$ and $\bar{w} > g^D_{r,s,t}(\bar{y})$. Then $(\tilde{y}, \tilde{w})$ is a solution to (24) if and only if one of the following cases applies:*

$$\textit{Case 1: } \tilde{y} = \tilde{z} \iff \tilde{w} = \underset{w}{\mathit{argmin}}\, f \textit{ and } \tilde{w} \geq g^D_{r,s,t}(\tilde{z}) \tag{C1}$$

$$\textit{Case 2: } (\tilde{y}, \tilde{w}) = 0 \iff g_{r,s,t}(\tilde{z}) \leq \frac{\mu}{\gamma} \textit{ and } \mu \in \partial f(0) \tag{C2}$$

$$\textit{Case 3: } \frac{\gamma}{\mu}(\tilde{z} - \tilde{y}) \in \partial g^D_{r,s,t}(\tilde{y}) \;\; \mu \in \partial f(\tilde{w}) \cap \mathbb{R}_{\geq 0} \textit{ and } \tilde{w} = g^D_{r,s,t}(\tilde{y}) \tag{C3}$$

where $\tilde{z} := T\sigma(Z)$ is given by (22).

Proof A solution $(\tilde{y}, \tilde{w})$ to (24) fulfills $0 \in \partial(f(\tilde{w}) + \frac{\gamma}{2}\|\tilde{y} - \tilde{z}\|^2 + \chi_{\mathrm{epi}(g^D_{r,s,t})}(\tilde{y}, \tilde{w}))$ by [24, Theorem VI.2.2.1], which under the assumed constraint qualification is equivalent to

$$0 \in \begin{pmatrix} \gamma(\tilde{y} - \tilde{z}) \\ \partial f(\tilde{w}) \end{pmatrix} + \mathcal{N}_{\mathrm{epi}(g^D_{r,s,t})}(\tilde{y}, \tilde{w}) \tag{40}$$

where $\mathcal{N}$ denotes the normal cone to $\mathrm{epi}(g^D_{r,s,t})$ and the summation is understood set-wise. Then by [24, Proposition VI.1.3.1]

$$\mathcal{N}_{\mathrm{epi}(g^D_{r,s,t})}(\tilde{y}, \tilde{w}) = \begin{cases} \{(\mu G, -\mu) : G \in \partial g^D_{r,s,t}(\tilde{y}),\ \mu \geq 0\} & \text{if } \tilde{w} = g^D_{r,s,t}(\tilde{y}) \\ \{0\} & \text{if } (\tilde{y}, \tilde{w}) \in \mathrm{int}(\mathrm{epi}(g^D_{r,s,t})) \end{cases} \tag{41}$$

which is why we need to distinguish the cases $\tilde{y} = \tilde{z}$ and $\tilde{w} = g^D_{r,s,t}(\tilde{y})$. Thus, the proof follows by invoking (3).

Remark 2 In the epigraph case with $f(x) = \frac{1}{2}(w + z_v)^2$ and $\gamma = 1$, (C1) corresponds to that $(z, -z_v)$ is in the cone given by the epigraph of $g^D_{r,s,t}$, (C2) corresponds to that (z, z_v) is in the cone given by the epigraph of the dual gauge function $g_{r,s,t}$, and (C3) covers the remaining cases.

The problem of solving (18) therefore reduces to checking Eqs. (C1), (C2) and (C3) within the nested binary search, which has been made explicit for $g^D = \ell_2$ in Appendix 4 and $g^D = \ell_1$ in Appendix 6.

Proof to Proposition 3

For $\tau > 0$ and a gauge function $\tilde{g}$ it holds that $g = \tau\tilde{g}$ is gauge function with $g^D = \frac{\tilde{g}}{\tau}$. Setting $\gamma = 1$ and $f(w) = \frac{1}{2}(w + z_v)$ in Theorem 1, Eqs. (C1), (C2), and (C3) in Proposition 5 then become

$$(\tilde{y}, \tilde{w}) = (\tilde{z}, zv) \iff -\tau z_v \geq \tilde{g}^D_{r,s,t}(\tilde{z}) \tag{42a}$$

$$(\tilde{y}, \tilde{w}) = 0 \iff \tilde{g}_{r,s,t}(\tilde{z}) \leq \frac{z_v}{\tau} \tag{42b}$$

$$\frac{\tau}{\mu}(\tilde{z} - \tilde{y}) \in \partial\tilde{g}^D_{r,s,t}(\tilde{y}), \quad \mu = \tilde{w} + z_v \geq 0 \text{ and } \tau\tilde{w} = \tilde{g}^D_{r,s,t}(\tilde{y}). \tag{42c}$$

For our particular case $\tilde{g} = \ell_2$, it follows immediately that Eqs. (42a) and (42b) correspond to Eqs. (27a) and (27b). Furthermore, by taking the gradient of $g^D_{r,s,t}$, (42c) becomes Eqs. (27c), (27e) and (27d) with the constraints $\mu \geq 0$ and $\tau\tilde{w} = g^D_{r,s,t}(\tilde{y})$. Thus, it is left to compute $\mu \geq 0$. Plugging Eqs. (27c), (27e) and (27d) into

$\tau^2\tilde{w}^2 = g^D_{r,s,t}(\tilde{y})^2$ and making some rearrangements yields

$$1 = \frac{\sum_{i=1}^{r-t} \tilde{z}_i^2}{\left(\tilde{w}\tau + \frac{\mu}{\tau}\right)^2} + \frac{t}{s+t} \frac{\tilde{z}^2_{r-t+1}}{\left(\tilde{w}\tau + \frac{\mu t}{(s+t)\tau}\right)^2}.$$

Then defining $c_1 := \sum_{i=1}^{r-t} \tilde{z}_i^2$ and $c_2 := \sqrt{t+s}\tilde{z}_{r-t+1}$, this can be rewritten as the fourth-order polynomial equation (27f) which can be solved explicitly for unique $\mu \geq 0$ after the substitution (27e) is performed. This proves the first part of Proposition 3. For $f(w) = \chi_{[0,\gamma]}(w)$, Eqs. (C1), (C2) and (C3) are

$$\tilde{y} = \tilde{z} \iff \tau \geq \tilde{g}^D_{r,s,t}(\tilde{z}) \tag{43a}$$

$$\frac{\gamma}{\mu}(\tilde{z} - \tilde{y}) \in \partial \tilde{g}^D_{r,s,t}(\tilde{y}), \quad \mu \geq 0 \text{ and } \tau = \tilde{g}^D_{r,s,t}(\tilde{y}). \tag{43b}$$

Note that (C2) is redundant here, because it coincides with (43a). Hence, for $g = \ell_2$ (43a) becomes (27a) with $z_v = -1$ and (43b) is equivalent to Eqs. (27f), (27c) and (27d) with $\tilde{w} = 1$.

Break Point Search

Lemma 5 *Let $(\tilde{z}, z_v)$ fulfill neither of Eqs. (29a) and (29b), and $\hat{z}$ and α be as in Proposition 4. Further, let $\mu^\star$ be the solution to $\sum_{i=1}^{r-t+1} \max(\hat{z}_i - \alpha_i\mu, 0) + z_v - \mu = 0$ and $\hat{\mu}_k$ be the solution to $\left(\sum_{i=1}^{k} \hat{z}_i - \alpha_i\mu\right) + z_v - \mu = 0$, i.e., $\hat{\mu}_k = \frac{z_v + \sum_{i=1}^k \hat{z}_i}{1+\sum_{i=1}^k \alpha_i}$. Then there exists $k^\star \in \{1, \ldots r-t+1\}$ such that $\hat{z}_{k^\star} - \alpha_{k^\star}\mu^\star \geq 0$, $\hat{z}_i - \alpha_i\mu^\star < 0$ for all $i > k^\star$ and*

i. $\hat{\mu}_{k^\star} = \mu^\star$.
ii. $k^\star = \max\{k : \hat{z}_k - \alpha_k\hat{\mu}_k \geq 0\}$.
iii. If $\hat{z}_k - \alpha_k\hat{\mu}_k \geq 0$, then $\hat{z}_i - \alpha_i\hat{\mu}_i \geq 0$ for all $i \leq k$.
vi. If $\hat{z}_k - \alpha_k\hat{\mu}_k < 0$, then $\hat{z}_i - \alpha_i\hat{\mu}_i < 0$ for all $i \geq k$.

In particular,

I. $\hat{z}_k - \alpha_k\hat{\mu}_k \geq 0$ for all $k \leq k^\star$.
II. $\hat{z}_k - \alpha_k\hat{\mu}_k < 0$ for all $k > k^\star$.

Proof We first show some results needed to prove Items ii. and iii. Let $g_k(\mu) := \sum_{i=1}^k \max(\hat{z}_i - \alpha_i\mu, 0) + z_v - \mu$, and let μ_k be the unique solution to the equation $g_k(\mu) = 0$. Since all g_i are strictly decreasing in μ and $g_k(\mu) = g_{k-1}(\mu) + \max(\hat{z}_k - \alpha_k\mu, 0) \geq g_{k-1}(\mu)$, we have

a. $\mu_{k-1} \leq \mu_k$.
b. $\hat{z}_k - \alpha_k\mu_k \leq 0 \iff g_{k-1}(\mu_k) = g_k(\mu_k) = 0 \iff \mu_{k-1} = \mu_k$.

Moreover, the break point sorting in $\hat{z}$ implies that if l and μ are such that $\hat{z}_l - \alpha_l \mu \geq 0$, then also $\hat{z}_i - \alpha_i \mu \geq 0$ for all $i \leq l$. Thus,

$$\hat{z}_k - \alpha_k \mu \geq 0 \;\Leftrightarrow\; \sum_{i=1}^{k} \max(\hat{z}_i - \alpha_i \mu, 0) + z_v - \mu = \left(\sum_{i=1}^{k} \hat{z}_i - \alpha_i \mu\right) + z_v - \mu.$$

In conjunction with the uniqueness of μ_k, this implies that

c. $\hat{z}_k - \alpha_k \mu_k \geq 0$ or $\hat{z}_k - \alpha_k \hat{\mu}_k \geq 0 \;\Leftrightarrow\; \hat{\mu}_k = \mu_k$.

Item i.: This has already been proven in the discussion before Lemma 5.

Item ii.: By the definition of $k^\star$ and Item i. it holds that

$$\hat{z}_{k^\star} - \alpha_{k^\star} \hat{\mu}_{k^\star} \geq 0 \qquad \text{and} \qquad \hat{z}_i - \alpha_i \hat{\mu}_{k^\star} < 0 \;\text{ for all }\; i > k^\star. \tag{44}$$

Thus, by Item c. $\hat{\mu}_{k^\star} = \mu^\star = \mu_{k^\star}$ and $\hat{z}_i - \alpha_i \mu_i < 0$ for all $i > k^\star$. Then Item b. implies that $\hat{\mu}_{k^\star} = \mu^\star = \mu_{r-t+1} = \mu_{r-t} = \cdots = \mu_{k^\star}$. Therefore, if there exists $k > k^\star$ with $\hat{z}_k - \alpha_k \hat{\mu}_k \geq 0$, it will hold by Item c. that $\hat{\mu}_k = \mu_k = \hat{\mu}_{k^\star}$, which contradicts (44), because $0 \leq \hat{z}_k - \alpha_k \hat{\mu}_k = \hat{z}_k - \alpha_k \hat{\mu}_{k^\star} < 0$. This proves that $k^\star = \max\{k : \hat{z}_k - \alpha_k \hat{\mu}_k \geq 0\}$.

Item iii.: Assume that $\hat{z}_k - \alpha_k \hat{\mu}_k \geq 0$. Then, by the break point sorting it holds that $\hat{z}_{k-1} - \alpha_{k-1} \hat{\mu}_k \geq 0$ and by Items a. and c. that $\hat{\mu}_k = \mu_k \geq \mu_{k-1}$. Thus, we conclude that

$$0 \leq \hat{z}_{k-1} - \alpha_{k-1} \hat{\mu}_k = \hat{z}_{k-1} - \alpha_{k-1} \mu_k \leq \hat{z}_{k-1} - \alpha_{k-1} \mu_{k-1} = \hat{z}_{k-1} - \alpha_{k-1} \hat{\mu}_{k-1},$$

where the last equality follows again by Item c. The other indices follow inductively.

Item iv.: Let on the contrary k be such that $\hat{z}_k - \alpha_k \hat{\mu}_k < 0$, but with $i \in \{k, \ldots, r - t + 1\}$ such that $\hat{z}_i - \alpha_i \hat{\mu}_i \geq 0$. Then, by Item iii., $\hat{z}_k - \alpha_k \hat{\mu}_k \geq 0$, which is a contradiction.

Items I. and II.: Follow immediately from Items ii. to iv.

Proof to Proposition 4

Analogous to showing Proposition 3, Eqs. (42a) and (42b) correspond to Eqs. (C1) and (C2) in Proposition 5, which translate for $\tilde{g} = \ell_\infty$ to

$$(\tilde{y}^\star, w^\star) = (\tilde{z}, zv) \iff \sum_{i=1}^{r-t} |\tilde{z}_i| + \frac{t}{\sqrt{t+s}} |\tilde{z}_{r-t+1}| \leq -\tau z_v$$
$$(\tilde{y}^\star, w^\star) = 0 \iff \max\left(|\tilde{z}_1|, \ldots, |\tilde{z}_{r-t-2}|, \frac{\sqrt{t+s}}{t} |\tilde{z}_{r-t+1}|\right) \leq \frac{z_v}{\gamma}$$

Since $\tilde{z}$ is nonnegative and decreasingly sorted, the second case simplifies to (29b). For (42c), we need to note that $\tilde{y} \in \mathbb{R}_{\geq 0}^{r-t+1}$ and therefore the conditions for $\tilde{y}_i = 0$

and $\tilde{y}_i > 0$ become

$$\tilde{y}_i = 0 \Leftrightarrow \tilde{z}_i \in \left[0, \tfrac{\mu}{\tau}\right], \qquad \tilde{y}_i > 0 \Leftrightarrow \tilde{y}_i = \tilde{z}_i - \tfrac{\mu}{\tau}$$

for all $i \in \{1, \ldots, r-t\}$. These equivalences also hold for $\tilde{y}_{r-t+1}$ with μ multiplied by $t/\sqrt{s+t}$. Therefore, Eqs. (29c), (29d) and (29e) follow together with the constraints $\tau \tilde{w} = \tilde{g}^D_{r,s,t}(\tilde{y})$ and $\mu \geq 0$. Then, plugging Eqs. (29c) and (29d) into $\tau w^\star = \tilde{g}^D_{r,s,t}(\tilde{y})$ yields

$$\begin{aligned} 0 &= \frac{1}{\tau}\sum_{i=1}^{r-t} |\tilde{y}_i| + \frac{t}{\sqrt{t+s}}|\tilde{y}_{r-t+1}| - \tilde{w} \\ &= \sum_{i=1}^{r-t} \max\left(\frac{\tilde{z}_i}{\gamma} - \frac{\mu}{\gamma^2}, 0\right) + \max\left(\frac{t}{\sqrt{t+s}\gamma}\tilde{z}_{r-t+1} - \frac{t^2\mu}{(t+s)\gamma^2}, 0\right) + z_v - \mu. \end{aligned} \tag{45a}$$

which determines the unique solution to $\mu \geq 0$. We solve the equation by using a so-called *break point searching algorithm*, as it has been done for similar problems in [10,12,22].

In our case, the break points are given by the smallest values of μ for which each max expressions as function of μ becomes zero, i.e., $\left(\gamma\tilde{z}_1, \ldots, \gamma\tilde{z}_{r-t}, \frac{\gamma\sqrt{s+t}}{t}\tilde{z}_{r-t+1}\right)$. Then we define $\hat{z} := \frac{1}{\gamma}\left(\tilde{z}_1, \ldots, \tilde{z}_j, \dfrac{t}{\sqrt{(t+s)}}\tilde{z}_{r-t+1}, \tilde{z}_{j+1}, \ldots, \tilde{z}_{r-t}\right)$, to be the vector that sorts $\frac{1}{\gamma}\left(\tilde{z}_1, \ldots, \tilde{z}_{r-t}, \frac{t}{\sqrt{t+s}}\tilde{z}_{r-t+1}\right)$ by decreasing break points, i.e., j fulfills

$$\tilde{z}_j > \tfrac{\sqrt{(t+s)}}{t}\tilde{z}_{r-t+1} \geq \tilde{z}_{j+1} \quad \text{or} \quad \tilde{z}_{r-t} \geq \tfrac{\sqrt{(t+s)}}{t}\tilde{z}_{r-t+1}. \tag{46a}$$

Therefore, (45a) can be equivalently written as

$$\sum_{i=1}^{r-t+1} \max(\hat{z}_i - \alpha_i\mu, 0) + z_v - \mu = 0 \tag{46b}$$

with $\alpha = \frac{1}{\gamma^2}\left(1, \ldots, 1, \dfrac{t^2}{(t+s)}, 1, \ldots, 1\right)$. Hence, there exists an index $k^\star \in \{1, \ldots, r-t+1\}$ such that the unique solution $\mu \geq 0$ to (46b) fulfills

$$\hat{z}_{k^\star} - \alpha_{k^\star}\mu \geq 0 \quad \text{and} \quad \hat{z}_i - \alpha_i\mu < 0 \ \text{ for all } \ i > k^\star, \tag{46c}$$

which is why μ can be determined as

$$\mu = \frac{z_v + \sum_{i=1}^{k^\star}\hat{z}_i}{1 + \sum_{i=1}^{k^\star}\alpha_i}. \tag{46d}$$

Consequently, computing μ equals a search for $k^\star \in \{1, \ldots, r - t + 1\}$ for which (46d) satisfies (46c). This can be done with the search rules in Lemma 5.

Finally, if $f(w) = \chi_{[0,\gamma]}(w)$, then Eqs. (C1), (C2) and (C3) are given by Eqs. (43a) and (43b). For $\tilde{g} = \ell_\infty$, this corresponds to (29a) with $z_v = -1$, and Eqs. (29c) and (29d) with the constraint that $\sum_{i=1}^{r-t+1} \max(\hat{z}_i - \alpha_i \mu, 0) = \tau$, respectively. Therefore, $\hat{\mu}_k = \frac{\sum_{i=1}^{k} \hat{z}_i}{\sum_{i=1}^{k} \alpha_i}$, $\mu = \dfrac{\sum_{i=1}^{k^\star} \hat{z}_i}{\sum_{i=1}^{k^\star} \alpha_i}$ and it is readily seen that $k^\star$ obeys the same rules as in Lemma 5.

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