Journal of Optimization Theory and Applications (2021) 190:339–342
https://doi.org/10.1007/s10957-021-01855-6

CORRECTION

Correction to: Kurdyka–Łojasiewicz Property of Zero-Norm Composite Functions

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Received: 27 January 2021 / Accepted: 2 April 2021 / Published online: 6 May 2021

Correction to:
Journal of Optimization Theory and Applications (2021) 188:94–112
https://doi.org/10.1007/s10957-020-01779-7

1 Introduction

In our paper [6], there is a gap for the statement of Proposition 3.2 and Remark 3.2. In addition, on line 14 of page 110, the set $[x' \in C,\ \mathrm{supp}(x') = J,\ x' \to \overline{x},\ x' \neq \overline{x}]$ may be empty. In this erratum, we provide the correct statements for Proposition 3.2 and Remark 3.2 and update the proof of Proposition 3.2.

2 Corrected Result

First, we give the correct statement of [6, Proposition 3.2 & Remark 3.2].

Proposition 2.1 ([6, Proposition 3.2] corrected) **(i)** *When* $h(\cdot) = \nu\|\cdot\|_0$ *for a constant* $\nu > 0$*, if* $\psi : \mathbb{R}^p \to]-\infty, +\infty]$ *is a proper closed piecewise linear regular function,*

Communicated by Boris S. Mordukhovich.

The original article can be found online at https://doi.org/10.1007/s10957-020-01779-7.

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then for any $\overline{x} \in \mathrm{dom}\,\psi$,

$$\widehat{\partial}(\psi+h)(\overline{x}) = \partial\psi(\overline{x})+\partial h(\overline{x}) = \partial(\psi+h)(\overline{x}) \tag{1}$$
$$\partial^{\infty}(\psi+h)(\overline{x}) = [\partial\psi(\overline{x})+\partial h(\overline{x})]^{\infty}; \tag{2}$$

if ψ *is an indicator function of some closed convex set* $C \subseteq \mathbb{R}^p$, *then for any* $\overline{x} \in C$ *such that* $\{x \in \mathbb{R}^p \mid x_i = 0 \text{ for } i \notin \mathrm{supp}(\overline{x})\} \cap \mathrm{ri}(C) \neq \emptyset$, *it holds that*

$$\widehat{\partial}(\psi+h)(\overline{x}) = \partial\psi(\overline{x})+\partial h(\overline{x}) = \partial(\psi+h)(\overline{x}) = \partial^{\infty}(\psi+h)(\overline{x}) = [\widehat{\partial}(\psi+h)(\overline{x})]^{\infty}.$$

(ii) *When* $h = \delta_{\Omega}$, *the indicator function of* $\Omega := \{x \in \mathbb{R}^p : \|x\|_0 \leq \kappa\}$ *for an integer* $\kappa > 0$, *the results of part (i) hold at any* $\overline{x} \in \mathrm{dom}\psi$ *with* $\|\overline{x}\|_0 = \kappa$, *and at any* $\overline{x} \in \mathrm{dom}\psi$ *with* $\|\overline{x}\|_0 < \kappa$ *it holds that* $\partial(\psi+h)(\overline{x}) \subseteq \partial\psi(x) + \partial h(\overline{x})$.

Remark 2.1 ([6, Remark 3.2] corrected) When ψ is a locally Lipschitz regular function, the first part of Proposition 2.1 still holds by [5, Theorem 9.13(b) & Corollary 10.9], and now equality (2) is also given in [2, Proposition 1.107(iii)] and [3, Prop. 1.29].

In what follows, we provide the proof of Proposition 2.1.

The proof of Proposition 2.1: First, we consider that ψ is a proper closed piecewise linear regular function. Fix any $\overline{x} \in \mathrm{dom}\,\psi$. Notice that epi$\psi$ and epi$h$ are the union of finitely many polyhedral sets. By combining [4, Proposition 1] and [1, Section 3.2], it then follows that

$$\partial(\psi+h)(\overline{x}) \subseteq \partial\psi(\overline{x}) + \partial h(\overline{x}) \;\text{ and }\; \partial^{\infty}(\psi+h)(\overline{x}) \subseteq \partial^{\infty}\psi(\overline{x}) + \partial^{\infty}h(\overline{x}). \tag{3}$$

From the first inclusion, $\partial(\psi+h)(\overline{x}) \supseteq \widehat{\partial}(\psi+h)(\overline{x}) \supseteq \widehat{\partial}\psi(\overline{x}) + \widehat{\partial}h(\overline{x})$, and the regularity of ψ and h, we obtain the equalities in (1). When $\partial\psi(\overline{x}) = \emptyset$, obviously, the equalities in (2) hold. So, it suffices to consider the case where $\partial\psi(\overline{x}) \neq \emptyset$. From the second inclusion in (3), it follows that

$$[\partial^{\infty}(\psi+h)(\overline{x})]^{\circ} \supseteq [\partial^{\infty}\psi(\overline{x}) + \partial^{\infty}h(\overline{x})]^{\circ} = [\partial^{\infty}\psi(\overline{x})]^{\circ} \cap [\partial^{\infty}h(\overline{x})]^{\circ}$$

where K° denotes the negative polar of a cone K. By combining this inclusion with [5, Exercise 8.23] and the second equality of (1), for any $w \in \mathbb{R}^p$ we have

$$\begin{aligned}\widehat{d}(\psi+h)(\overline{x})(w) &\leq \widehat{d}\psi(\overline{x})(w) + \widehat{d}h(\overline{x})(w) = d\psi(\overline{x})(w) + dh(\overline{x})(w)\\ &\leq d(\psi+h)(\overline{x})(w) \leq \widehat{d}(\psi+h)(\overline{x})(w)\end{aligned}$$

where $\widehat{d}h(\overline{x})$ and $dh(\overline{x})$, respectively, denote the regular subderivative and the subderivative of $\psi+h$ at $\overline{x}$, the equality is due to the regularity of ψ and h, and the second inequality is using [5, Corollary 10.9]. By [5, Corollary 8.19], this shows that $\psi+h$ is regular, and hence $\partial^{\infty}(\psi+h)(\overline{x}) = [\widehat{\partial}(\psi+h)(\overline{x})]^{\infty} = [\partial\psi(\overline{x}) + \partial h(\overline{x})]^{\infty}$. Thus, we obtain the first part.

Next we consider the case $\psi = \delta_C$. Fix any $\overline{x} \in C$ with $\mathrm{ri}(C) \cap L_{\overline{x}} \neq \emptyset$, where $L_{\overline{x}} := \{x \in \mathbb{R}^p \mid x_i = 0 \text{ for } i \notin \mathrm{supp}(\overline{x})\}$. Let $J = \mathrm{supp}(\overline{x})$. We first argue

$$\widehat{\partial}(\delta_C + h)(\overline{x}) \subseteq \partial \delta_{C \cap L_{\overline{x}}}(\overline{x}). \tag{4}$$

If there exists $\varepsilon > 0$ such that $[\mathbb{B}(\overline{x}, \varepsilon)\backslash\{\overline{x}\}] \cap [C \cap L_{\overline{x}}] = \emptyset$, then $\partial \delta_{C \cap L_{\overline{x}}}(\overline{x}) = \mathcal{N}_{C \cap L_{\overline{x}}}(\overline{x}) = \mathbb{R}^p$, and the inclusion in (4) clearly holds. So, it suffices to consider that for any $\varepsilon > 0$, $[\mathbb{B}(\overline{x}, \varepsilon)\backslash\{\overline{x}\}] \cap [C \cap L_{\overline{x}}] \neq \emptyset$. Pick any $v \in \widehat{\partial}(\delta_C + h)(\overline{x})$. By the definition of regular subgradient, it follows that

$$\begin{aligned}
0 &\le \liminf_{\overline{x} \neq x' \to \overline{x}} \frac{h(x') + \delta_C(x') - h(\overline{x}) - \delta_C(\overline{x}) - \langle v, x' - \overline{x}\rangle}{\|x' - \overline{x}\|} \\
&\le \liminf_{\substack{\overline{x} \neq x' \xrightarrow[C]{} \overline{x} \\ \mathrm{supp}(x') = J}} \frac{h(x') - h(\overline{x}) - \langle v, x' - \overline{x}\rangle}{\|x' - \overline{x}\|} = \liminf_{\substack{\overline{x} \neq x' \xrightarrow[C]{} \overline{x} \\ \mathrm{supp}(x') = J}} \frac{-\langle v, x' - \overline{x}\rangle}{\|x' - \overline{x}\|} \\
&= \liminf_{\substack{\overline{x} \neq x' \xrightarrow[C]{} \overline{x} \\ \mathrm{supp}(x') = J}} \frac{\delta_{C \cap L_{\overline{x}}}(x') - \delta_{C \cap L_{\overline{x}}}(\overline{x}) - \langle v, x' - \overline{x}\rangle}{\|x' - \overline{x}\|},
\end{aligned}$$

which implies that $v \in \widehat{\partial}\delta_{C \cap L_{\overline{x}}}(\overline{x}) = \partial \delta_{C \cap L_{\overline{x}}}(\overline{x})$. The inclusion in (4) holds. By combining (4) with [5, Corollary 10.9] and $\partial h(\overline{x}) = \mathcal{N}_{L_{\overline{x}}}(\overline{x})$, we have

$$\begin{aligned}
\partial \delta_C(\overline{x}) + \partial h(\overline{x}) = \widehat{\partial}\delta_C(\overline{x}) + \widehat{\partial} h(\overline{x}) &\subseteq \widehat{\partial}(\delta_C + h)(\overline{x}) \subseteq \partial(\delta_C + \delta_{L_{\overline{x}}})(\overline{x}) \\
&= \partial \delta_C(\overline{x}) + \partial \delta_{L_{\overline{x}}}(\overline{x}) = \partial \delta_C(\overline{x}) + \partial h(\overline{x}).
\end{aligned} \tag{5}$$

where the second equality is due to $\mathrm{ri} C \cap L_{\overline{x}} \neq \emptyset$. In fact, from the above arguments, we conclude that for all $x \in C$ with $\mathrm{ri}(C) \cap L_x \neq \emptyset$,

$$\widehat{\partial}(\delta_C + h)(x) = \partial \delta_C(x) + \partial h(x) = \partial \delta_{C \cap L_x}(x). \tag{6}$$

Now we argue that $\partial(\delta_C + h)(\overline{x}) \subseteq \partial \delta_C(\overline{x}) + \partial h(\overline{x})$. To this end, pick any $v \in \partial(\delta_C + h)(\overline{x})$. Then, there exist sequences $x^k \xrightarrow[\delta_C + h]{} \overline{x}$ and $v^k \in \widehat{\partial}(\delta_C + h)(x^k)$ with $v^k \to v$ as $k \to \infty$. Since $\delta_C(x^k) + h(x^k) \to \delta_C(\overline{x}) + h(\overline{x})$, we must have $x^k \in C$ and $h(x^k) \to h(\overline{x})$ for all k large enough. The latter, along with $\mathrm{supp}(x^k) \supseteq J$, implies that $\mathrm{supp}(x^k) = J$ for all sufficiently large k. By invoking (6), for all sufficiently large k, $v^k \in \partial \delta_C(x^k) + \partial h(x^k)$. By passing to the limit $k \to \infty$ and using $h(x^k) \to h(\overline{x})$, we obtain $v \in \partial \delta_C(\overline{x}) + \partial h(\overline{x})$. By the arbitrariness of v in $\partial(\delta_C + h)(\overline{x})$, the stated inclusion follows. In particular, together with $\partial(\delta_C + h)(\overline{x}) \supseteq \widehat{\partial}(\delta_C + h)(\overline{x}) = \partial \delta_C(\overline{x}) + \partial h(\overline{x})$ and (5),

$$\widehat{\partial}(\delta_C + h)(\overline{x}) = \partial(\delta_C + h)(\overline{x}) = \mathcal{N}_C(\overline{x}) + \partial h(\overline{x}) = \partial \delta_{C \cap L_{\overline{x}}}(\overline{x}). \tag{7}$$

Next we argue $\partial^\infty(\delta_C + h)(\overline{x}) = \partial^\infty \delta_{C \cap L_{\overline{x}}}(\overline{x})$. Pick any $u \in \partial^\infty(\delta_C + h)(\overline{x})$. Then, there exist sequences $x^k \xrightarrow[\delta_C + h]{} \overline{x}$ and $u^k \in \widehat{\partial}(\delta_C + h)(x^k)$ with $\lambda_k u^k \to u$ for some

$\lambda_k \downarrow 0$ as $k \to \infty$. By following the same arguments as above, $\mathrm{supp}(x^k) = J$ for all k large enough. Together with (6) and $u^k \in \widehat{\partial}(\delta_C + h)(x^k)$, we have $u^k \in \widehat{\partial}\delta_{C\cap L_{\overline{x}}}(x^k)$ for all k large enough. Notice that $x^k \xrightarrow[C\cap L_{\overline{x}}]{} \overline{x}$. So, $u \in \partial^\infty(\delta_C + \delta_{L_{\overline{x}}})(\overline{x})$ and $\partial^\infty(\delta_C + h)(\overline{x}) \subseteq \partial^\infty\delta_{C\cap L_{\overline{x}}}(\overline{x})$. Conversely, pick any $u \in \partial^\infty\delta_{C\cap L_{\overline{x}}}(\overline{x})$. There exist $x^k \xrightarrow[C\cap L_{\overline{x}}]{} \overline{x}$ and $u^k \in \widehat{\partial}\delta_{C\cap L_{\overline{x}}}(x^k)$ with $\lambda_k u^k \to u$ for some $\lambda_k \downarrow 0$ as $k \to \infty$. Clearly, $(\delta_C + h)(x^k) \to (\delta_C + h)(\overline{x})$. Also, from (6) and $u^k \in \widehat{\partial}\delta_{C\cap L_{\overline{x}}}(x^k)$, we have $u^k \in \widehat{\partial}(\delta_C + h)(x^k)$. So, $u \in \partial^\infty(\delta_C + h)(\overline{x})$, and $\partial^\infty(\delta_C + h)(\overline{x}) \supseteq \partial^\infty\delta_{C\cap L_{\overline{x}}}(\overline{x})$. The stated equality follows. From [5, Exercise 8.14 & Proposition 8.12], $\partial\delta_{C\cap L_{\overline{x}}}(\overline{x}) = \partial^\infty\delta_{C\cap L_{\overline{x}}}(\overline{x}) = [\widehat{\partial}\delta_{C\cap L_{\overline{x}}}(\overline{x})]^\infty$. Thus,

$$\partial\delta_{C\cap L_{\overline{x}}}(\overline{x}) = \partial^\infty\delta_{C\cap L_{\overline{x}}}(\overline{x}) = [\widehat{\partial}\delta_{C\cap L_{\overline{x}}}(\overline{x})]^\infty = \partial^\infty(\delta_C + h)(\overline{x}).$$

Together with (7), we obtain the conclusion for $h(\cdot) = \nu\|\cdot\|_0$. By following the same arguments as above, one may obtain the second part. □

3 Conclusion

Since [6, Proposition 3.2] is only used to check [6, Assumption 4.1(iii)], the results in [6, Section 4] are all correct by Proposition 2.1.

Acknowledgements Funding was provided by National Natural Science Foundation of China (Grant Nos. 11971177 and 11701186).

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