Dynamic Monopolistic Competition

A Steady-State Analysis

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Abstract

I study a dynamic variant of the Dixit–Stiglitz (Am Econ Rev 67(3), 1977) model of monopolistic competition by introducing price stickiness à la Fershtman and Kamien (Econometrica 55(5), 1987). The analysis is restricted to bounded quantity and price paths that fulfill the necessary conditions for an open-loop Nash equilibrium. I show that there exists a symmetric steady state and that its stability depends on the degree of product differentiation. When moving from complements to perfect substitutes, the steady state is either a locally asymptotically unstable (spiral) source, a stable (spiral) sink or a saddle point. I further apply the Hopf bifurcation theorem and prove the existence of limit cycles, when passing from a stable to an unstable steady state. Lastly, I provide a numerical example and show that there exists a stable limit cycle.

Keywords Monopolistic competition · Differential games · Hopf Bifurcation · Limit cycles

Mathematics Subject Classification 91A10 · 91A23 · 37G15

1 Introduction

Ever since the seminal duopoly article of Fershtman and Kamien [10] there has been a lasting interest in oligopolies with sticky price adjustment. Dockner [5] generalizes the duopoly to an n -firm oligopoly, and Cellinia and Lamberini [3] consider product differentiation à la Singh and Vives [19]. Recently, Wiszniewska-Matyszek et al. [21] departed from a steady-state analysis and derived analytical results on the actual adjustment process over time. The com-

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mon denominator of all those aforementioned articles is the assumption of a linear demand function and the focus on saddle point equilibria. The linear demand function is convenient, because linear-quadratic differential games are analytically tractable.

In the present article, however, I consider a utility function with constant elasticity of substitution (CES) between the goods and thus derive a highly nonlinear demand system. The idea dates back to Dixit and Stiglitz [4] and captures product differentiation from the perspective of the consumer. That is, for the limit case of perfect complements, the indifference curves are L-shaped, and for perfect substitutes, they are linear. The model thus generalizes the differential game of Lambertini [14], who considers perfect substitutes and a hyperbolic demand function.

Assuming that price and quantity paths are bounded, I show that open-loop strategies—which fulfill the necessary conditions for a Nash equilibrium—yield a unique symmetric steady state. Also, introducing product differentiation adds complexity. If the adjustment speed is sufficiently low, then depending on the degree of product differentiation (bifurcation parameter), the steady state is either a (spiral) source, a (spiral) sink or a saddle point.

I also find that there exists a stable limit cycle. That is, the equilibrium path converges to a closed orbit and cycles periodically around the steady state. The model thus provides a rationale for cobweb [6] dynamics in a continuous time world. Generally, complex dynamic economic systems are well known in the literature [1,7,8]. I am only aware, however, of two differential game models [9,20] in which limit cycles exist and, to the best of my knowledge, there is not a single dynamic competition model that allows for complex dynamics.

2 Model

Let $N = \{1, 2, \dots, n\}$ denote the set of firms. As the focus of the paper is on competition with differentiated products, I would naturally assume that there are at least two firms in the market.

Assumption 1 There are at least two firms $n \geq 2$.

There is one consumer, who demands $\mathbf{q} = (q_j)_{j \in N} \in \mathbb{R}_+^n$ quantities of the n differentiated goods. Her endowment is $y \in \mathbb{R}_{++}$ such that the budget set reads $B(\mathbf{p}) = \{\mathbf{q} : \sum_{j \in N} p_j q_j \leq y\}$ with prices $\mathbf{p} = (p_j)_{j \in N} \in \mathbb{R}_+^n$. Preferences are captured by the following utility function:

$$u(\mathbf{q}) = \left[\sum_{j \in N} q_j^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}}$$

where $\sigma \in]0, \infty[\setminus \{1\}$ expresses the constant elasticity of substitution between the goods. For the limit cases, one gets familiar utility functions

$$u(\mathbf{q}) = \begin{cases} \min_{j \in N} \{q_j\} & \text{for } \sigma \rightarrow 0, \\ \prod_{j \in N} q_j & \text{for } \sigma \rightarrow 1, \\ \sum_{j \in N} q_j & \text{for } \sigma \rightarrow \infty. \end{cases}$$

It is convenient to define $r = (\sigma - 1)/\sigma \in]-\infty, 1[\setminus \{0\}$. The consumer solves

$$\max_{\mathbf{q} \in B(\mathbf{p})} u(\mathbf{q})$$

such that the demand functions are given by $\mathbf{q}^d(\mathbf{p}) \in \arg \max_{\mathbf{q} \in B(\mathbf{p})} u(\mathbf{q})$. I would need to solve, however, for the inverse demand function $p_i^d = (q_i^d)^{-1}$, because I consider quantity competition.

Proposition 2.1 *The inverse demand function is given by*

$$p_i^d(\mathbf{q}) = \frac{y q_i^{r-1}}{\sum_{j \in N} q_j^r}.$$

Proof If $u(\cdot)$ was quasi-concave, then the necessary conditions for maximizing $u(\cdot)$ were also sufficient. Therefore, I first show that $u(\cdot)$ is quasi-concave.

Lemma 2.1 *The utility function $u(\cdot)$ is quasi-concave.*

Proof Let $h(\mathbf{q}) = \sum_{j \in N} q_j^r$ and $w(x) = x^{\frac{1}{r}}$. Note that

$$\begin{aligned} \frac{\partial^2 h(\mathbf{q})}{\partial q_i \partial q_j} &= \begin{cases} r(r-1)q_i^{r-2} & \text{for } j = i \\ 0 & \text{for } j \in N \setminus \{i\} \end{cases} \\ \frac{\partial^2 -h(\mathbf{q})}{\partial q_i \partial q_j} &= \begin{cases} -r(r-1)q_i^{r-2} & \text{for } j = i \\ 0 & \text{for } j \in N \setminus \{i\} \end{cases} \end{aligned}$$

such that the Hessians of $h(\mathbf{q})$ and $-h(\mathbf{q})$ are negative definite for $r \in]0, 1[$ and $r < 0$, respectively. Therefore, the functions $h(\mathbf{q})$ and $-h(\mathbf{q})$ are strictly concave for $r \in]0, 1[$ and $r < 0$, respectively. Further note that $w(x)$ and $w(-x)$ are increasing functions for $r \in]0, 1[$ and $r < 0$, respectively. Therefore, $u(\mathbf{q}) = w(h(\mathbf{q})) = w(-(-h(\mathbf{q})))$ is a monotone increasing transformation of a strictly concave function, which implies the quasi-concavity of $u(\cdot)$. $\square$

Next we consider the necessary conditions for maximizing $u(\cdot)$ and solve for a point that satisfies them. Setting up the Lagrangian yields

$$L(\mathbf{p}, \mathbf{q}, \mu) = u(\mathbf{q}) + \mu \left(y - \sum_{j \in N} p_j q_j \right).$$

The $n + 1$ first-order conditions for the optimal demand of q_i read

$$\frac{\partial L(\mathbf{q}, \mathbf{p}, \mu)}{\partial q_i} = u(\mathbf{q})^{1-r} q_i^{r-1} - \mu p_i = 0 \quad (i \in N), \quad (1)$$

$$\frac{\partial L(\mathbf{q}, \mathbf{p}, \mu)}{\partial \mu} = y - \sum_{j \in N} p_j q_j = 0. \quad (2)$$

From (1) and (2), I can derive the inverse demand function. Solve (1) for μp_i and put it in relation with any μp_j for $j \in N$

$$\frac{q_j^{r-1}}{q_i^{r-1}} = \frac{p_j}{p_i}.$$

Isolating p_j yields

$$\frac{q_j^{r-1}}{q_i^{r-1}} p_i = p_j.$$

Now, multiply both sides with q_j

$$\frac{q_j^r}{q_i^{r-1}} p_i = p_j q_j,$$

sum over $j \in N$ and make use of (2)

$$\frac{\sum_{j \in N} q_j^r}{q_i^{r-1}} p_i = y.$$

Eventually, one derives the inverse demand function

$$p_i^d(\mathbf{q}) = \frac{y q_i^{r-1}}{\sum_{j \in N} q_j^r}.$$

□

2.1 Static Game

In the static game, I assume that each firm sets quantities in order to maximize profits and then I solve for the symmetric Nash equilibrium. The action space of each firm $i \in N$ is given by $Q_i = \mathbb{R}_+$ and the jointly admissible action space by $Q = \left(\times_{j \in N} Q_j\right) \setminus \{0\}$. I do not allow $\mathbf{q} = (0, \dots, 0)$, because $p_i^d(0, \dots, 0)$ is not defined. Each firm sets $q_i \in Q_i$ in order to maximize profits $\pi_i : Q \rightarrow \mathbb{R}$

$$\pi_i(\mathbf{q}) = p_i^d(\mathbf{q})q_i - \frac{c}{2}q_i^2, \quad (3)$$

where $c > 0$ denotes a cost parameter. The static game is then given by the triplet $\Gamma = (N, (Q_j)_{j \in N}, (\pi_j)_{j \in N})$. It is useful to collect the actions of all agents but i 's in $\mathbf{q}_{-i} = (q_j)_{j \in N \setminus \{i\}}$ and denote with slight abuse of notation $\mathbf{q} = (q_i, \mathbf{q}_{-i})$. Now each firm $i \in N$ maximizes profits over choosing quantities q_i . A Nash equilibrium is then defined as an action profile $\mathbf{q}^* \in Q$ such that each $i \in N$ is best off by playing q_i^* , given all the other agents $j \in N \setminus \{i\}$ play their respective equilibrium action q_j^* .

Definition 2.1 The action profile $\mathbf{q}^* \in Q$ is a Nash equilibrium for the game Γ , if the following inequalities hold for all firms $i \in N$:

$$\pi_i(\mathbf{q}^*) \geq \pi_i(q_i, \mathbf{q}_{-i}^*) \quad \forall q_i \in Q_i.$$

Proposition 2.2 For $r > 0$ there exists a unique interior symmetric equilibrium at $\mathbf{q}^* = (q^*, \dots, q^*) \in \mathbb{R}_{++}^n$ given by

$$q^* = \frac{1}{n} \sqrt{\frac{ry(n-1)}{c}}.$$

For $r < 0$, the solution is not defined.

Proof Necessary Conditions Taking the first-order condition of $\pi_i(\mathbf{q})$ with respect to q_i yields

$$\frac{\partial \pi_i(\mathbf{q})}{\partial q_i} = \frac{\partial p_i^d(\mathbf{q})}{\partial q_i} q_i + p_i^d(\mathbf{q}) - cq_i = 0.$$

The partial derivative of the inverse demand function is given by

$$\frac{\partial p_i^d(\mathbf{q})}{\partial q_i} = y \frac{(r-1)q_i^{r-2} \sum_{j \in N} q_j^r - r q_i^{2(r-1)}}{\left(\sum_{j \in N} q_j^r\right)^2}.$$

In the symmetric equilibrium $\mathbf{q}^* = (q^*, \dots, q^*)$, this expression becomes

$$\frac{\partial p_i^d(\mathbf{q}^*)}{\partial q_i} = \frac{y(r(n-1) - n)}{q^{*2} n^2}. \quad (4)$$

The first-order condition then reads

$$\frac{\partial \pi_i(q^*)}{\partial q_i} = \frac{y(r(n-1)-n)}{q^{*2}n^2}q^* + \frac{y}{nq^*} - cq^* = 0.$$

Solving for q^* eventually yields

$$q^* = \pm \frac{1}{n} \sqrt{\frac{yr(n-1)}{c}}.$$

As only the positive root satisfies nonnegative quantities $q^* \geq 0$, there exists a unique equilibrium. The jointly admissibility constraint requires $q^* > 0$ such that $r > 0$ is necessary for a well-defined equilibrium.

Sufficient Conditions As I am considering the symmetric equilibrium, I fix $i = 1$ as a representative firm and denote profits by

$$\pi(q, \tilde{q}) = \pi_1(q, \tilde{q}, \dots, \tilde{q}) = \frac{yq^r}{q^r + (n-1)\tilde{q}^r} - \frac{c}{2}q^2.$$

Hefti [12, Thm. 1] states that there exists a unique symmetric equilibrium if

$$\frac{\partial^2 \pi(q^*, q^*)}{\partial q^2} < 0.$$

Consider the second partial derivative

$$\begin{aligned} & \frac{\partial^2 \pi(q, \tilde{q})}{\partial q^2} \\ &= \frac{yr(n-1)q^{r-2}\tilde{q}^r(q^r + (n-1)\tilde{q}^r)[(r-1)(q^r + (n-1)\tilde{q}^r) - 2rq^r]}{(q^r + (n-1)\tilde{q}^r)^4} - c. \end{aligned}$$

At $(q, \tilde{q}) = (q^*, q^*)$ this expression becomes

$$\frac{\partial^2 \pi(q^*, q^*)}{\partial q^2} = \frac{c}{n}((n-2)r - 2n).$$

As $c, n > 0$, the term $(n-2)r - 2n$ defines the sign. With $n \geq 2$ the expression is obviously negative for all $r < 0$. Further note that $n-2 < 2n$ such that $(n-2)r < 2n$ for all $r \in]0, 1[$ and therefore $\partial^2 \pi(q^*, q^*)/\partial q^2 < 0$ for all degrees of substitutability $r \in]-\infty, 1[\setminus\{0\}$. $\square$

For further reference, I also state the equilibrium price and profits

$$\begin{aligned} p^* &= p_i^d(q^*) = \sqrt{\frac{cy}{r(n-1)}}, \\ \pi^* &= \pi_i(q^*) = \frac{y}{n} \left(1 - \frac{r(n-1)}{2n} \right). \end{aligned}$$

In principle the individually rationality constraint $\pi^* \geq 0$ gives rise to an upper bound on the elasticity parameter $r < 2n/(n-1)$. As $r < 1$ by assumption and $1 < 2n/(n-1)$ for all $n \geq 2$, this bound never binds.

In the Singh and Vives [19] framework goods are characterized as complements, independent or substitutes with respect to the cross price elasticity of demand

$$\varepsilon_{ij}(\mathbf{p}) = \frac{\partial q_i^d(\mathbf{p})}{\partial p_j} \frac{p_j}{q_i^d(\mathbf{p})} \quad (i \neq j).$$

The goods are complements (independent/substitutes) if $\varepsilon_{ij}(\mathbf{p}) < 0$ ($=/ >$). Here

$$q_i^d(\mathbf{p}) = \frac{y p_i^{\frac{1}{r-1}}}{\sum_{j \in N} p_j^{\frac{r}{r-1}}}$$

denotes the demand function for good $i \in N$ which can be directly derived by inverting the system of inverse demand functions $p_i^d(\mathbf{q})$. At a symmetric point $\mathbf{p}^* = (p^*, \dots, p^*)$ the elasticity $\varepsilon_{ij}(\mathbf{p})$ becomes

$$\varepsilon_{ij}(\mathbf{p}^*) = \frac{r}{n(1-r)} \begin{cases} > 0 & \text{for } r \in]0, 1[, \\ = 0 & \text{for } r = 0, \\ < 0 & \text{for } r < 0. \end{cases}$$

In the static game we thus require for an interior solution that the goods are substitutes.

2.2 Dynamic Game

In the dynamic game I assume that prices adjust gradually over time. The adjustment process is governed by the following differential equation:

$$\frac{dp_i(t)}{dt} = f(p_i(t), \mathbf{q}(t)) = s \left[p_i^d(\mathbf{q}(t)) - p_i(t) \right]$$

where $s > 0$ denotes the speed of adjustment. The idea of sluggish adjustment in a duopoly differential game was first introduced by Simaan and Takayama [17,18]. It captures inherent market dynamics in the sense that prices decrease if current supply is relatively large and increase if current supply is relatively small. Let us denote by $P_i \subseteq \mathbb{R}_+$ the set of admissible prices. Only in the steady state $p_i(t) = p_i^d(\mathbf{q}(t))$ holds so I need to redefine profits of each firm with respect to the current price $p_i(t)$

$$\Pi_i(p_i(t), q_i(t)) = p_i(t)q_i(t) - \frac{c}{2}q_i(t)^2.$$

Each firm now chooses a quantity path $q_i : [0, \infty[\rightarrow Q_i$. Let us denote by $\mathcal{Q}_i$ the set of admissible open-loop strategies for each firm $i \in N$

$$\mathcal{Q}_i = \{q_i : [0, \infty[\rightarrow Q_i \mid q_i(\cdot) \text{ is measurable on } [0, \infty[\}.$$

For a given initial condition $p_i^0 = p_i(0) \in P_i$ a strategy profile $\mathbf{q}(\cdot) \in \times_{j \in N} \mathcal{Q}_j$ determines the price path

$$\rho_i(t; p_i^0, \mathbf{q}(\cdot)) = p_i(t) = p_i^0 + \int_0^t f(p_i(s), \mathbf{q}(s)) ds.$$

The payoff functional of each firm is now given by the discounted stream of payoffs

$$J_i(p_i^0, \mathbf{q}(\cdot)) = \int_0^\infty e^{-\varrho t} \Pi_i(\rho_i(t; p_i^0, \mathbf{q}(\cdot)), q_i(t)) dt,$$

where $\varrho > 0$ denotes the time preference rate. Let $\mathcal{Q} \subset \times_{j \in N} \mathcal{Q}_j$ denote the set of jointly admissible strategies, such that each price path $p_i(\cdot)$ stays in the state space $p_i(t) \in P_i$ for all $i \in N$ and $t \geq 0$ and payoffs are finite $\max_{j \in N} |J_j(p_j^0, \mathbf{q}(\cdot))| < \infty$. Let $\mathbf{p}^0 = (p_j^0)_{j \in N} \in P = \times_{j \in N} P_j$ denote a tuple of initial conditions. The differential game is then described by a triplet

$$\Gamma(\mathbf{p}^0) = (N, (\mathcal{Q}_j)_{j \in N}, (J_j(p_j^0, \cdot))_{j \in N}).$$

The definition of a Nash equilibrium is straightforward.

Definition 2.2 The strategy profile $\bar{\mathbf{q}}(\cdot) \in \mathcal{Q}$ is a Nash equilibrium for the game $\Gamma(\mathbf{p}^0)$, if the following inequalities hold for all firms $i \in N$ and initial conditions $\mathbf{p}^0 \in P$:

$$J_i(p_i^0, \bar{\mathbf{q}}(\cdot)) \geq J_i(p_i^0, q_i(\cdot), \bar{\mathbf{q}}_{-i}(\cdot)) \quad \forall q_i(\cdot) \in \mathcal{Q}_i.$$

The maximum principle [2, cf. Ch. 14] provides necessary conditions for an equilibrium.

Theorem 2.1 Let $\lambda_i = (\lambda_i^j)_{j \in N}$. For each $i \in N$ define the Hamiltonian $H_i : P \times Q \times \mathbb{R}^n \rightarrow \mathbb{R}$ by

$$H_i(\mathbf{p}, \mathbf{q}, \lambda_i) = \Pi_i(p_i, q_i) + \sum_{j \in N} \lambda_i^j f(p_j, \mathbf{q}).$$

Let $\bar{\mathbf{q}}(\cdot) \in \mathcal{Q}$ be a given strategy profile and denote by $\bar{\mathbf{p}}(\cdot) = (p_j(\cdot; p_j^0, \bar{\mathbf{q}}(\cdot)))_{j \in N}$ the n price paths that where induced by $\bar{\mathbf{q}}(\cdot)$. For $\bar{\mathbf{q}}(\cdot)$ to be an equilibrium, it is necessary that there exist n^2 absolutely continuous functions $\lambda_i : [0, \infty[\rightarrow \mathbb{R}^n$, $i \in N$, such that

for all $t \in [0, \infty[$ and $i \in N$ the following conditions hold:

$$\bar{q}_i(t) \in \arg \max_{q_i \in Q_i} H_i(\bar{\mathbf{p}}(t), q_i, \bar{\mathbf{q}}_{-i}(t), \boldsymbol{\lambda}_i(t)), \quad (5)$$

$$\frac{d\lambda_i^j(t)}{dt} = \varrho \lambda_i^j(t) - \frac{\partial H_i(\bar{\mathbf{p}}(t), \bar{\mathbf{q}}(t), \boldsymbol{\lambda}_i(t))}{\partial p_j} \quad (j \in N), \quad (6)$$

$$\lim_{t \rightarrow \infty} e^{-\varrho t} H_i(\bar{\mathbf{p}}(t), \bar{\mathbf{q}}(t), \boldsymbol{\lambda}_i(t)) = 0. \quad (7)$$

If the trajectories $(\bar{\mathbf{p}}(\cdot), \bar{\mathbf{q}}(\cdot), (\boldsymbol{\lambda}_i(\cdot))_{i \in N})$ are bounded, then the transversality condition (7) is fulfilled. As it is common in the literature, I will focus on bounded solutions. The condition is thus satisfied if the time paths either converge to a steady state or periodically cycle around a steady state. Let us therefore define a closed orbit.

Definition 2.3 Let $(p, q) : [0, \infty[\rightarrow \mathbb{R}^2$ denote a two-dimensional trajectory. For $t \geq 0$ and $(p(t), q(t)) \in \mathbb{R}^2$, a *closed orbit* $\mathcal{O}(p(t), q(t)) \subset \mathbb{R}^2$ describes periodical trajectories

$$\mathcal{O}(p(t), q(t)) = \{(p(\tau), q(\tau))_{\tau \in [t, \infty[} \mid \exists T > t : (p(T), q(T)) = (p(t), q(t))\}.$$

Proposition 2.3 Assume that $q_i(\cdot)$, $i \in N$, is differentiable on $[0, \infty[$. If the degree of substitutability is sufficiently large $r > \underline{r} = -\varrho n / s(n-1)$, then there exists a unique symmetric interior steady state at $(p, q) = (p_\infty, q_\infty)$ with

$$p_\infty = \sqrt{\frac{cy(\varrho + s)}{\varrho n + sr(n-1)}},$$

$$q_\infty = \frac{1}{n} \sqrt{\frac{y(\varrho n + sr(n-1))}{c(\varrho + s)}}.$$

Further, a symmetric interior solution $q_i(t) = \bar{q}(t) > 0$ for all $i \in N$ and $t \geq 0$ needs to solve the following pair of differential equations

$$\frac{d\bar{p}(t)}{dt} = s \left(\frac{y}{n\bar{q}(t)} - \bar{p}(t) \right), \quad (8)$$

$$\frac{d\bar{q}(t)}{dt} = \frac{-\bar{q}(t)^2 c(s + \varrho) + \bar{q}(t)(2s + \varrho)\bar{p}(t) + \frac{sy}{n^2}(r(n-1) - 2n)}{2\bar{p}(t) - 3c\bar{q}(t)}. \quad (9)$$

Proof Necessary Conditions For the sake of readability, I will drop from now on the time argument t whenever no confusion arises. Define the Hamiltonian for $i \in N$

$$H_i(\mathbf{p}, \mathbf{q}, \boldsymbol{\lambda}_i) = p_i q_i - \frac{c}{2} q_i^2 + \sum_{j \in N} \lambda_i^j s [p_j^d(\mathbf{q}) - p_j].$$

The adjoint equations read

$$\frac{d\lambda_i^j}{dt} = \varrho \lambda_i^j - \frac{\partial H_i(\mathbf{p}, \mathbf{q}, \boldsymbol{\lambda}_i)}{\partial p_j} = (\varrho + s)\lambda_i^j - \mathbf{1}_{\{i\}}(j)q_i$$

As already mentioned, I look for bounded solutions of $(\lambda_i(\cdot))_{i \in N}$ such that $|\lambda_i^j(t)| < \infty$ for all $t \geq 0$ and $(i, j) \in N \times N$. As the time preference rate $\varrho > 0$ and the adjustment speed $s > 0$ are positive, the solution of the adjoint equation $\lambda_i^j(t) = \lambda_i^j(0)e^{t(\varrho+s)}$, $i \neq j$, is unbounded for all $\lambda_i^j(0) \neq 0$. I can thus instantly solve for $\lambda_i^j(t) = 0$ for all $j \in N \setminus \{i\}$. Now consider the first-order condition for an interior solution

$$\frac{\partial H_i(\mathbf{p}, \mathbf{q}, \boldsymbol{\lambda}_i)}{\partial q_i} = p_i - cq_i + \lambda_i^i s \frac{\partial p_i^d(\mathbf{q})}{\partial q_i} = 0.$$

I solve for a symmetric equilibrium in the sense that prices and quantities are the same among the firms over time, such that $\bar{\mathbf{q}} = (\bar{q}, \dots, \bar{q})$ and $\bar{\mathbf{p}} = (\bar{p}, \dots, \bar{p})$ and $\lambda_i^i = \bar{\lambda}$. Under consideration of (4) the first-order condition becomes

$$\frac{\partial H_i(\bar{\mathbf{p}}, \bar{\mathbf{q}}, \boldsymbol{\lambda}_i)}{\partial q_i} = \bar{p} - c\bar{q} + \bar{\lambda}s \frac{y(r(n-1) - n)}{\bar{q}^2 n^2} = 0. \quad (10)$$

Differentiating the first-order condition over time yields a law of motion for the quantity over time

$$\begin{aligned} 0 &= \frac{d \left[\frac{\partial H_i(\bar{\mathbf{p}}, \bar{\mathbf{q}}, \boldsymbol{\lambda}_i)}{\partial q_i} \right]}{dt} = \frac{d\bar{p}}{dt} - c \frac{d\bar{q}}{dt} \\ &\quad + \frac{sy(r(n-1) - n)}{n^2} \left[\frac{\frac{d\bar{\lambda}}{dt} \bar{q}^2 - \bar{\lambda} 2\bar{q} \frac{d\bar{q}}{dt}}{\bar{q}^4} \right]. \end{aligned} \quad (11)$$

Solving (10) for $\bar{\lambda}$

$$\bar{\lambda} = -\frac{\bar{q}^2 n^2 (\bar{p} - c\bar{q})}{sy(r(n-1) - n)} \quad (12)$$

and substituting (12) as well as

$$\begin{aligned} \frac{d\bar{p}}{dt} &= f(\bar{p}, \bar{q}) = s \left(\frac{y}{n\bar{q}} - \bar{p} \right), \\ \frac{d\bar{\lambda}}{dt} &= (\varrho + s)\bar{\lambda} - \bar{q}, \end{aligned}$$

into (11) yields for $d\bar{q}/dt$

$$\frac{d\bar{q}}{dt} = g(\bar{p}, \bar{q}) = \frac{-\bar{q}^2 c(s + \varrho) + \bar{q}(2s + \varrho)\bar{p} + \frac{sy}{n^2}(r(n-1) - 2n)}{2\bar{p} - 3c\bar{q}}.$$

Note that (p_∞, q_∞) solves $f(p_\infty, q_\infty) = 0 = g(p_\infty, q_\infty)$. For positive prices and quantities $(p_\infty, q_\infty) \in \mathbb{R}_{++}^2$, I require $r > \underline{r} = -\varrho n/s(n-1)$.

Sufficient Conditions If $H_i(\mathbf{p}, \mathbf{q}, \lambda_i)$ is concave in $(\mathbf{p}, \mathbf{q}_i)$ for any tuple $(\mathbf{q}_{-i}, \lambda_i) \in (Q \setminus Q_i) \times \mathbb{R}^n$ and for all $(\mathbf{p}, \mathbf{q}_i) \in P \times Q_i$, then the necessary conditions are sufficient [18, 20]. Due to the highly nonlinear system, I cannot prove that the sufficient conditions are satisfied. To the contrary, it is straightforward to show that along the symmetric trajectory with $(\mathbf{p}, \mathbf{q}) = (\bar{p}, \bar{q})$ and $\lambda_i^j = 0$ for all $j \in N \setminus \{i\}$ and $\lambda_i^i = \bar{\lambda}$, the Hamiltonian H_i is not concave. Fix

$$H(\bar{p}, \bar{q}, \bar{\lambda}) = H_i(\bar{p}, \bar{q}, 0, \dots, \bar{\lambda}, 0, \dots) = \bar{p}\bar{q} - \frac{c}{2}\bar{q}^2 + \bar{\lambda}s \left(\frac{y}{n\bar{q}} - \bar{p} \right).$$

Note that H was concave, if

$$\begin{aligned} \frac{\partial^2 H(\bar{p}, \bar{q}, \bar{\lambda})}{\partial \bar{p} \partial \bar{p}} &\leq 0, \\ \frac{\partial^2 H(\bar{p}, \bar{q}, \bar{\lambda})}{\partial \bar{q} \partial \bar{q}} &\leq 0, \\ \frac{\partial^2 H(\bar{p}, \bar{q}, \bar{\lambda})}{\partial \bar{p} \partial \bar{p}} \frac{\partial^2 H(\bar{p}, \bar{q}, \bar{\lambda})}{\partial \bar{q} \partial \bar{q}} - \left(\frac{\partial^2 H(\bar{p}, \bar{q}, \bar{\lambda})}{\partial \bar{p} \partial \bar{q}} \right)^2 &\geq 0. \end{aligned}$$

As

$$\frac{\partial^2 H(\bar{p}, \bar{q}, \bar{\lambda})}{\partial \bar{p} \partial \bar{p}} = 0 \quad \text{and} \quad \frac{\partial^2 H(\bar{p}, \bar{q}, \bar{\lambda})}{\partial \bar{p} \partial \bar{q}} = 1$$

the third condition is violated. $\square$

There are two apparent differences between the static equilibrium and the steady state of the dynamic game. In the dynamic game, there exists an interior steady state for the optimal control problem of the monopolist $n = 1$ and the lower bound on the degree of substitutability is negative $\underline{r} < 0$ for all triplets $(n, \varrho, s) \in \mathbb{N} \times \mathbb{R}_{++}^2$, such that complementary goods are also captured. The following corollary summarizes the differences.

Corollary 2.1 Let Π_∞ denote the steady-state profit given by

$$\Pi_\infty = \Pi_i(\bar{p}, \bar{q}) = \frac{y}{n} \left(1 - \frac{\varrho n + sr(n-1)}{2n(\varrho + s)} \right).$$

For $r \in]0, 1[$ I find that the prices and profits (quantities) are lower (higher) in the dynamic game compared to the static game

$$\begin{aligned}\frac{p_\infty}{p^*} &= \sqrt{\frac{r(\varrho + s)(n-1)}{\varrho n + rs(n-1)}} < 1, \\ \frac{q_\infty}{q^*} &= \sqrt{\frac{\varrho n + rs(n-1)}{r(\varrho + s)(n-1)}} > 1, \\ \frac{\Pi_\infty}{\pi^*} &= \frac{n(\varrho + 2s) - sr(n-1)}{(\varrho + s)(2n - r(n-1))} < 1.\end{aligned}$$

Further, the consumer surplus is larger in the dynamic game

$$\Delta CS = \int_{p_\infty}^{\infty} \frac{y}{np} dp - \int_{p^*}^{\infty} \frac{y}{np} dp = \frac{y}{n} \ln \left(\frac{p^*}{p_\infty} \right) > 0.$$

For the limit cases of instant price adjustment ($s \rightarrow \infty$) or farsightedness ($\varrho \rightarrow 0$), the steady-state prices, quantities and profits of the static and dynamic game coincide

$$\lim_{s \rightarrow \infty} q_\infty = \lim_{\varrho \rightarrow 0} q_\infty = q^*.$$

If prices were about to instantly adjust with respect to a change in quantities or if the firms do not discount future profits, then the static equilibrium approximates the steady state of the dynamic game. For reference I also note that the competitive outcome of “prices equal marginal costs” corresponds to two different limit cases. Let $q^\ddagger = \sqrt{y/cn}$ solve $p^d(q^\ddagger, \dots, q^\ddagger) = y/nq^\ddagger = cq^\ddagger$. One should note that the competitive output exceeds the steady state as well as the Cournot equilibrium output $q^\ddagger > q_\infty > q^*$. For the limit cases of no price adjustment ($s \rightarrow 0$) or myopia ($\varrho \rightarrow \infty$), the steady-state prices, quantities and profits of perfect competition and the dynamic game coincide

$$\lim_{s \rightarrow 0} q_\infty = \lim_{\varrho \rightarrow \infty} q_\infty = q^\ddagger.$$

If prices do not react to an increase in production ($s \rightarrow 0$) or the firms are myopic ($\varrho \rightarrow \infty$), then the firms increase the supply to increase profits. These results are all in line with Fershtman and Kamien [10, Prop. 1]. I would also like to highlight that, due to the differentiated products, the case of perfect competition cannot be approximated by solely considering an increase in the number of firms. The price equals marginal costs rule which requires

$$p_\infty = cq^\ddagger \implies \frac{n}{n-1} = r.$$

As $n/(n-1) > 1 > r$, the equation is only true for the double limit of perfect substitutes ($r \rightarrow 1$) and an increase in firms ($n \rightarrow \infty$).

The real novelty of the article is the introduction of product differentiation by means of the CES utility function. I thus state the comparative statics with respect to r .

Proposition 2.4 *The steady-state quantity increases in the degree of product differentiation, whereas the price and profits decrease.*

Proof Taking the partial derivative of q_∞ , p_∞ and Π_∞ with respect to r yields the result.

$$\begin{aligned}\frac{dq_\infty}{dr} &= \frac{s(n-1)}{2n} \sqrt{\frac{y}{c(\varrho+s)(\varrho n + sr(n-1))}} > 0 \quad \forall r > \underline{r} \\ \frac{dp_\infty}{dr} &= -\frac{s(n-1)}{2(\varrho n + sr(n-1))} \sqrt{\frac{cy(\varrho+s)}{\varrho n + sr(n-1)}} < 0 \quad \forall r > \underline{r} \\ \frac{d\Pi_\infty}{dr} &= -\frac{ys(n-1)}{2n^2(\varrho+s)} < 0\end{aligned}\quad \square$$

Next I study the local stability around the steady state. For the general theory on comparative statics and local stability of symmetric open-loop equilibria, one should consult Ling and Caputo [15]. In order to discriminate between different steady-state types, I make the following assumption.

Assumption 2 The speed of adjustment is sufficiently low compared to the time preference rate $s < \varrho/2$.

Proposition 2.5 *Define the following thresholds:*

$$\bar{r} = \frac{n(2s - \varrho)}{3s(n-1)}, \quad r_{\text{crit}} = \frac{\varrho n(2s - \varrho)}{s(2\varrho - s)(n-1)}.$$

Note that $\underline{r} < r_{\text{crit}} < \bar{r} < 1$ for all $n \geq 2$, $\varrho > 0$ and $s < \varrho/2$. The steady state is an unstable (spiral) source for $r \in]\underline{r}, r_{\text{crit}}[$, a stable (spiral) sink for $r \in]r_{\text{crit}}, \bar{r}[$ and a saddle point for $r \in]\bar{r}, 1[$.

Proof I analyze the local stability of the steady state by means of the sign of the trace and the determinant of the following Jacobian:

$$J(\bar{p}, \bar{q}) = \begin{bmatrix} \frac{\partial f(\bar{p}, \bar{q})}{\partial \bar{p}} & \frac{\partial f(\bar{p}, \bar{q})}{\partial \bar{q}} \\ \frac{\partial g(\bar{p}, \bar{q})}{\partial \bar{p}} & \frac{\partial g(\bar{p}, \bar{q})}{\partial \bar{q}} \end{bmatrix}.$$

The respective partial derivatives are given by

$$\begin{aligned}\frac{\partial f(\bar{p}, \bar{q})}{\partial \bar{p}} &= -s, \\ \frac{\partial f(\bar{p}, \bar{q})}{\partial \bar{q}} &= -\frac{sy}{n\bar{q}^2}, \\ \frac{\partial g(\bar{p}, \bar{q})}{\partial \bar{p}} &= \frac{-\bar{q}^2 n^2 c(4s + \varrho) - 2sy(r(n-1) - 2n)}{n^2(2\bar{p} - 3c\bar{q})^2}, \\ \frac{\partial g(\bar{p}, \bar{q})}{\partial \bar{q}} &= \frac{n^2 c(s + \varrho)(3\bar{q}c - 4\bar{p})\bar{q} + 2n^2(2s + \varrho)\bar{p}^2 + 3csy(r(n-1) - 2n)}{n^2(2\bar{p} - 3c\bar{q})^2}.\end{aligned}$$

Evaluating the determinant and trace of the Jacobian at (p_∞, q_∞) yields

$$\begin{aligned}D_\infty(r) &= \det(J(p_\infty, q_\infty)) = \frac{2s(\varrho + s)(\varrho n + sr(n-1))}{n(2s - \varrho) - 3rs(n-1)}, \\ T_\infty(r) &= \text{tr}(J(p_\infty, q_\infty)) = \frac{\varrho n(2s - \varrho) - rs(2\varrho - s)(n-1)}{n(2s - \varrho) - 3rs(n-1)}.\end{aligned}$$

I added r as an explicit argument to the determinant and trace, because it is the bifurcation parameter. That is, I consider qualitative changes of the steady state with respect to the elasticity of substitution.

The steady state is a saddle point if $D_\infty(r) < 0$. As the numerator is positive for all $r > \underline{r}$, the sign of $D_\infty(r)$ solely depends on the denominator and it follows

$$D_\infty(r) < 0 \quad \text{for } r > \bar{r} = \frac{n(2s - \varrho)}{s(n-1)}.$$

Note that $\underline{r} < \bar{r}$ for all $(n, \varrho, s) \in \mathbb{N} \times \mathbb{R}_{++}^2$. As $r < 1$ is bounded from above, I require $\bar{r} < 1$ and thus $s(n-3) + \varrho n > 0$. This equation is true for all $n \geq 3$ and for $n = 2$ one gets the additional bound $s < 2\varrho$.

Now I consider the case $D_\infty(r) > 0$. As $D_\infty(r) > 0$, the denominator of $T_\infty(r)$ is positive. For a locally stable steady state I require $T_\infty(r) < 0$ and thus

$$T_\infty(r) < 0 \quad \text{for } r > r_{\text{crit}} = \frac{\varrho n(2s - \varrho)}{s(2\varrho - s)(n-1)}. \quad \square$$

Proposition 2.6 *For $2\varrho \neq s$ there exists a limit cycle.*

Proof It is rather straightforward to show the existence of limit cycles by applying the Hopf bifurcation theorem [11, cf. Thm. 3.4.2]. There exists a limit cycle if (i) $T_\infty(r_{\text{crit}}) = 0$ and (ii) $T'(r_{\text{crit}}) \neq 0$. I already established (i) in the previous proof. Checking the derivative also establishes (ii)

Fig. 1 Phase diagram

$$T'(r_{\text{crit}}) = \frac{s(n-1)(2\varrho-s)^2}{n(2s^2 + \varrho(s-\varrho))} \neq 0 \quad \forall 2\varrho \neq s. \quad \square$$

3 Example

In order to show the existence of a stable limit cycle, I consider a numerical example with parameters $(n, c, s, \varrho, y, r) = (2, 1, 0.25, 1, 1, -17/7)$ and initial conditions $(p_0, q_0) = (p_\infty, 1.05 \cdot q_\infty)$. In the left panel of Fig. 1 I illustrate a stable limit cycle in the price-quantity space $\{(p, q)\} \subseteq \mathbb{R}_+^2$. Prices and quantities are spiraling away from the steady state and eventually converge to a limit cycle.¹

Consider the right panel of Fig. 1. The steady state is the intersection of the $\dot{p} = 0$ and $\dot{q} = 0$ loci. One can differentiate between four regions I–IV. The dynamics are explainable via the standard cobweb model. In region I high supply (q relative large) causes a decrease in prices $\dot{p} < 0$, which causes a decrease in supply $\dot{q} < 0$. In region II a low price (p relatively small) causes a decrease in supply $\dot{q} < 0$, which causes an increase in the price level $\dot{p} > 0$. In region III low supply (q relatively small) causes an increase in prices $\dot{p} > 0$, which causes an increase in supply $\dot{q} > 0$. In region IV a high price (p relatively large) causes an increase in supply $\dot{q} > 0$, which causes a decrease in prices $\dot{p} < 0$.

4 Conclusions

I have introduced a model of dynamic monopolistic competition by means of a differential game. Assuming that the strategy space is restricted to bounded quantity time paths, then there exists a unique symmetric equilibrium under open-loop strategies that fulfill the necessary conditions for a Nash equilibrium. Using the degree of product differentiation as the bifurcation parameter, it has been shown that the stability properties depend on the nature of the product. If the goods are rather substitutes, then

¹ In Appendix A, I provide the Python code to produce the limit cycle.

the equilibrium is a saddle point and can be reached via a unique pair of monotonous price and quantity time paths. The more complementary the goods become, the more complex is the behavior around the steady state. For intermediate cases, the steady state is locally asymptotically stable, whereas it becomes unstable for rather complementary goods. When passing from a stable to an unstable equilibrium, a Hopf bifurcation occurs such that limit cycles emerge. Existence of a stable limit cycle has been established via a numerical example.

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Appendix: Python Code for Figure 1

```
import numpy as np
from scipy.integrate import odeint
import matplotlib.pyplot as plt

s = 1/4; n = 2; c = 1; y = 1; d = 1;
r = d*n*(2*s-d)/(s*(2*d-s)*(n-1))-1/7

# steady state
qss = ((y*(d*n+s*r*(n-1)))/(c*(d+s)))**(1/2)/n
pss = y/(n*qss)

# function that returns dp/dt and dq/dt
def model(z,t):
    dpdt = s*(y/(n*z[1]) - z[0])
    dqdt = ((-z[1]**2*c*(s+d) + z[1]*z[0]*(2*s+d)
             + s*y*(r*(n-1)-n**2)/n**2)
            /(2*z[0]-3*c*z[1]))
    dzdt = [dpdt, dqdt]
    return dzdt

# initial condition
z0 = [pss, 1.05*qss]
```



```
# time points
t = np.linspace(0, 200, 10001)

# solve ODE
z = odeint(model, z0, t)

# plot
plt.plot(z[:,0], z[:,1], color='k')
plt.xlabel('p')
plt.ylabel('q')
plt.show()
```

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