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FORUM

Note on "Oligopoly with Hyperbolic Demand: A Differential Game Approach"

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Abstract
We correct the local stability analysis of the two-player case of Lambertini (J Optim Theory Appl 145(1):108–119, 2010). Due to a sign error, the steady state was mischaracterized as being stable, while it is unstable.

1 Introduction

Lambertini [1] considers a dynamic oligopoly with hyperbolic demand. He solves for the symmetric equilibrium under open-loop strategies and characterizes the stability of the steady state. For the duopoly (two firms) case, there occurred a sign error that led to wrong conclusions. Given that the speed of price adjustment was sufficiently large, the steady state is mischaracterized as being locally asymptotically stable, while it is in fact unstable.

2 Correction

There are $N \in \mathbb{N}$ firms that produce a homogenous good over an infinite time horizon $t \in [0, \infty[$. Firm $i \in \{1, 2, \ldots, N\}$ produces $q_i : [0, \infty[\to \mathbb{R}_+$ quantities of the good. The price of the good is $p(t) \geq 0$, and its evolution over time follows

$$\dot{p}(t) := \frac{\mathrm{d}p(t)}{\mathrm{d}t} = s\left(\frac{a}{\sum_{i=1}^{N} q_i(t)} - p(t)\right)$$

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where $s > 0$ denotes the speed of adjustment and $a > 0$ is the income of a representative consumer. Each firm maximizes its discounted profits and thus faces the following dynamic optimization problem:

$$\max_{(q_i(t))_{t\geq 0}} \int_0^\infty e^{-\rho t} \left[p(t)q_i(t) - cq_i(t)^2 \right] \mathrm{d}t$$

where $c > 0$ denotes a production cost parameter and $\rho > 0$ is the common time preference rate. Applying the maximum principle yields a symmetric equilibrium candidate ($q_i(t) = q(t)$ for all i) that is characterized by the following system of differential equations:

$$\begin{aligned}
\dot{p}(t) &= f(p(t), q(t)) \\
&= s\left(\frac{a}{Nq(t)} - p(t)\right), \\
\dot{q}(t) &= g(p(t), q(t)) \\
&= \frac{1}{2p(t) - 6cq(t)} \left[-2cq(t)^2(s+\rho) + p(t)q(t)(2s+\rho) - \frac{as}{N^2}(N+1) \right].
\end{aligned}$$

Lambertini [1, Prop. 3.1] identifies a steady state $(\overline{p}, \overline{q}) \in \mathbb{R}_+^2$ with $f(\overline{p}, \overline{q}) = 0 = g(\overline{p}, \overline{q})$ and claims:

Proposition 2.1 *If $N \geq 3$, the steady state identified by*

$$\overline{p} := \sqrt{\frac{2ac(s+\rho)}{\rho N + s(N-1)}}, \quad \overline{q} := \frac{1}{N}\sqrt{\frac{a(\rho N + s(N-1))}{2c(s+\rho)}}$$

is a saddle point for all admissible values of s. If $N \in \{1, 2\}$, then:

- *For $N = 1$, $(\overline{p}, \overline{q})$ is (i) a saddle point for all $s \in]0, \rho/2[$ and (ii) an unstable focus for all $s > \rho/2$.*
- *For $N = 2$, $(\overline{p}, \overline{q})$ is (i) a saddle point for all $s \in]0, 2\rho[$, (ii) a stable node for all $s \in]2\rho, 2.226\rho[$ and (iii) a stable focus for all $s > 2.226\rho$.*

The cases $N = 1$ and $N \geq 3$ are correct. For $N = 2$ there occurred a sign error and the steady state is not stable but unstable. The proposition needs to be corrected in the following sense.

Proposition 2.1 *(corrected) If $N \geq 3$, the steady state identified by*

$$\overline{p} := \sqrt{\frac{2ac(s+\rho)}{\rho N + s(N-1)}}, \quad \overline{q} := \frac{1}{N}\sqrt{\frac{a(\rho N + s(N-1))}{2c(s+\rho)}}$$

is a saddle point for all admissible values of s. If $N \in \{1, 2\}$, then:

- *For $N = 1$, $(\overline{p}, \overline{q})$ is (i) a saddle point for all $s \in]0, \rho/2[$ and (ii) an unstable focus for all $s > \rho/2$.*

– *For* $N = 2$, $(\overline{p}, \overline{q})$ *is (i) a saddle point for all* $s \in]0, 2\rho[$, *(ii) an* unstable *node for all* $s \in]2\rho, 2.226\rho[$ *and (iii) an* unstable *focus for all* $s > 2.226\rho$.

Proof As the cases $N = 1$ and $N \geq 3$ are correct, we fix $N = 2$. The Jacobian of (f, g) reads

$$\mathrm{J}(p, q) = \begin{bmatrix} \frac{\partial f(p,q)}{\partial p} & \frac{\partial f(p,q)}{\partial q} \\ \frac{\partial g(p,q)}{\partial p} & \frac{\partial g(p,q)}{\partial q} \end{bmatrix}$$

where the partial derivatives are given by

$$\begin{aligned} \frac{\partial f(p,q)}{\partial p} &= -s, \\ \frac{\partial f(p,q)}{\partial q} &= -\frac{as}{2q^2}, \\ \frac{\partial g(p,q)}{\partial p} &= \frac{1}{8(p-3cq)^2}\left[-q^2 4c(4s+\rho) + 3sa\right], \\ \frac{\partial g(p,q)}{\partial q} &= \frac{1}{8(p-3cq)^2}\left[8c(\rho+s)(3cq-2p)q + 4p^2(\rho+2s) - 9csy\right]. \end{aligned}$$

Case $s \in]0, 2\rho[$ Let us consider the determinant of the Jacobian evaluated at the steady state:

$$\begin{aligned} \overline{\det} :=& \frac{\partial f(\overline{p},\overline{q})}{\partial p}\frac{\partial g(\overline{p},\overline{q})}{\partial q} - \frac{\partial f(\overline{p},\overline{q})}{\partial q}\frac{\partial g(\overline{p},\overline{q})}{\partial p} \\ =& \frac{2s(\rho+s)(2\rho+s)}{s-2\rho}. \end{aligned}$$

As $\overline{\det} < 0$ for all $s \in]0, 2\rho[$, the steady state is a saddle point.

Case $s \in]2\rho, \infty[$ For $s > 2\rho$ the determinant $\overline{\det}$ is positive. In order to determine the stability, we further investigate the trace of the Jacobian evaluated at the steady state:

$$\begin{aligned} \overline{\mathrm{tr}} :=& \frac{\partial f(\overline{p},\overline{q})}{\partial p} + \frac{\partial g(\overline{p},\overline{q})}{\partial q} \\ =& \frac{s^2 + 2\rho s - 2\rho^2}{s - 2\rho}. \end{aligned}$$

The denominator is positive for $s > 2\rho$ and the trace thus positive for all $s > \rho\left(\sqrt{3}-1\right)$. Since $2 > \sqrt{3}-1$ the trace is positive for all $s > 2\rho$ and the steady state *unstable*. We further investigate whether the eigenvalues $\lambda_{1,2}$ of $\mathrm{J}(\overline{p}, \overline{q})$ are complex or real in order to discriminate between a node and a focus. Let us first recall that the eigenvalues are given by

$$\lambda_{1,2} = \frac{1}{2}\left(\overline{\mathrm{tr}} \pm \sqrt{\overline{\mathrm{tr}}^2 - 4\overline{\det}}\right).$$

Depending on the sign of the discriminant

$$\begin{aligned}\overline{\mathrm{disc}} :=& \overline{\mathrm{tr}}^2 - 4\overline{\det}\\ =& \frac{4\rho^2 + 24\rho^3 s + 32\rho^2 s^2 - 4\rho s^3 - 7s^4}{(s-2\rho)^2}\\ =:& \frac{h(\rho, s)}{(s-2\rho)^2},\end{aligned}$$

the steady state is either a node or a focus. Note that $h(\rho, 2\rho) = 36\rho^4 > 0$ and $\lim_{s\to\infty} h(\rho, s) \to -\infty < 0$ such that there exists $\bar{s} > 2\rho$ with $h(\rho, \bar{s}) = 0$. The equation $h(\rho, s) = 0$ has four solutions

$$\bar{s}_{1,2} = \frac{\rho}{7}\left(-1 - 5\sqrt{2} \pm \sqrt{65 - 18\sqrt{2}}\right) \approx \rho(-0.2547, -2.0514),$$
$$\bar{s}_{3,4} = \frac{\rho}{7}\left(-1 + 5\sqrt{2} \pm \sqrt{65 + 18\sqrt{2}}\right) \approx \rho(+2.2260, -0.4914).$$

Only $\bar{s}_3$ satisfies $s > 2\rho$ and we thus deduce

$$\overline{\mathrm{disc}} \begin{cases} > 0 & \text{for } s \in]2\rho, 2.226\rho[, \\ < 0 & \text{for } s \in]2.226\rho, \infty[. \end{cases}$$

□

3 Conclusions

This manuscript provides a correction for a special case of [1, Prop. 3.1]. With these modifications, the local stability of the steady state changes. While it was characterized as being locally asymptotically stable, it is actually unstable.

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Reference

1. Lambertini, L.: Oligopoly with hyperbolic demand: a differential game approach. J. Optim. Theory Appl. **145**(1), 108–119 (2010)