

Minimax Theorems for Extended Real-Valued Abstract Convex–Concave Functions

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Abstract In this paper, we provide sufficient and necessary conditions for the minimax equality for extended real-valued abstract convex–concave functions. As an application, we get sufficient and necessary conditions for the minimax equality for extended real-valued convex–concave functions.

Keywords Abstract convexity · Extended Φ -convex functions · Minimax theorems

Mathematics Subject Classification 32F17 · 49J52 · 49K27 · 49K35 · 52A01

1 Introduction

Abstract convexity is a tool for global optimization. Abstract convex functions are defined as pointwise suprema of a given class of functions. This concept was first introduced by Moreau [1] and developed by Dolecki and Kurcyusz [2]. The monographs of Rubinov [3] and Rolewicz–Pallaschke [4] are devoted to this topic and discuss basic constructions of abstract convexity, like abstract conjugacy and abstract subdifferential, together with applications in optimization and numbers of important examples. In the present paper, we provide minimax theorems for functions, which are abstract convex with respect to one variable and concave with respect to the second variable.

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The first minimax theorem was proved by von Neumann in [5]. Since then there have been many generalizations of the original result, an exhaustive survey is given, e.g., in [6]; see also [7]. To the best of our knowledge, the existing minimax theorems are restricted to real-valued functions. Extended real-valued functions appear in standard constructions in variational analysis. It is a natural question what are the conditions under which the minimax equality for such functions holds. In this paper, we use the tools developed in [8] and [9] to provide minimax theorems for extended real-valued abstract convex–concave functions.

2 Preliminaries

We provide sufficient and necessary conditions for the minimax equality

$$\sup_{y \in Y} \inf_{x \in X} a(x, y) = \inf_{x \in X} \sup_{y \in Y} a(x, y),$$

where X is nonempty set, Y is a real vector space and $a : X \times Y \rightarrow \bar{\mathbb{R}} := \mathbb{R} \cup \{\pm \infty\}$ is an extended real-valued Φ -convex (abstract convex) function with respect to x and concave with respect to y .

Now, we recall some definitions related to Φ -convexity. For any $f, g : X \rightarrow \bar{\mathbb{R}}$, $f \leq g$ if and only if $f(x) \leq g(x)$ for all $x \in X$.

Let Φ be defined as follows

$$\Phi := \{\ell + c, \ell \in L, c \in \mathbb{R}\},$$

where L is an arbitrary class of functions $\ell : X \rightarrow \bar{\mathbb{R}}$. The class L is called a *set of abstract linear functions*. The class Φ is called a *set of abstract affine functions* if it is stable by adding constants. Note that, if L is the set of real-valued linear functions defined on X , then Φ is the set of real-valued affine functions defined on X .

The set $\text{supp } f \subset \Phi$, defined as

$$\text{supp } f := \{\varphi \in \Phi : \varphi \leq f\}$$

is called the *support* of f with respect to Φ . A function $f : X \rightarrow \bar{\mathbb{R}}$ is called Φ -convex if

$$\forall x \in X \quad f(x) = \sup\{\varphi(x) : \varphi \in \text{supp } f\}.$$

A function $f : X \rightarrow \bar{\mathbb{R}}$ is *proper* if its effective domain is nonempty, i.e.,

$$\text{dom } f := \{x \in X : f(x) < +\infty\} \neq \emptyset$$

and $f(x) > -\infty$ for all $x \in X$. The set $\text{epi } f := \{(x, r) \in X \times \mathbb{R} : r \geq f(x)\}$ is called the *epigraph* of f .

Let us note that a Φ -convex function, taking the value $-\infty$, has an empty support or must contain in its support functions admitting $-\infty$. In the case where, Φ contains

only real-valued functions $\varphi : X \rightarrow \mathbb{R}$, the Φ -convex function $f : X \rightarrow \bar{\mathbb{R}}$ is proper, if and only if $\text{supp } f \neq \emptyset$ and $\text{dom } f \neq \emptyset$.

For any $\varphi : X \rightarrow \mathbb{R}$ and $Z \subset X$ we define the strict lower level set of φ at the level $\alpha \in \mathbb{R}$ as

$$[\varphi < \alpha]_Z := \{x \in Z : \varphi(x) < \alpha\},$$

if $Z = X$, then we use the notation $[\varphi < \alpha]_X := [\varphi < \alpha]$.

Minimax theorems for functions $a : X \times Y \rightarrow \mathbb{R} \cup \{+\infty\}$, where for each $y \in Y$ the function $a(\cdot, y) : X \rightarrow \mathbb{R} \cup \{+\infty\}$ is Φ -convex, are based on the following intersection property introduced in [8] and investigated in [10] and [9].

Definition 2.1 Let $\varphi_1, \varphi_2 : X \rightarrow \mathbb{R}$ be any two functions and $\alpha \in \mathbb{R}$. The functions φ_1 and φ_2 are said to have *the intersection property on X at the level $\alpha \in \mathbb{R}$* iff for every $t \in [0, 1]$

$$[t\varphi_1 + (1-t)\varphi_2 < \alpha] \cap [\varphi_1 < \alpha] = \emptyset \quad \text{or} \quad [t\varphi_1 + (1-t)\varphi_2 < \alpha] \cap [\varphi_2 < \alpha] = \emptyset. \tag{1}$$

Remark 2.1 Functions $\varphi_1, \varphi_2 : X \rightarrow \mathbb{R}$, which belong to the class

$$\Phi_{lsc} := \{\varphi : X \rightarrow \mathbb{R}, \varphi(x) = -a\|x\|^2 + \langle \ell, x \rangle + c, \quad x \in X, \ell \in X^*, a \geq 0, c \in \mathbb{R}\},$$

where X is a normed space, and X^* is a topological dual to X , have the intersection property on X at the level α if and only if

$$[\varphi_1 < \alpha] \cap [\varphi_2 < \alpha] = \emptyset.$$

For more results along this line see [9].

Let us note that by Rubinov’s theorem ([3], Example 6.6) a proper lower semicontinuous function $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ defined on Hilbert space X is Φ_{lsc} -convex if there exists $\bar{\varphi} \in \Phi_{lsc}$ such that $\bar{\varphi} < f$.

The following theorem has been proved in [10].

Theorem 2.1 (Theorem 3.3.3 of [10]) *Let X be a set and Y be a real vector space. Let Φ be a class of real-valued functions $\varphi : X \rightarrow \mathbb{R}$ which contains the constant functions. Let $a : X \times Y \rightarrow \mathbb{R} \cup \{+\infty\}$ be a function such that*

- for any $y \in Y$ the function $a(\cdot, y) : X \rightarrow \mathbb{R} \cup \{+\infty\}$ is Φ -convex on X ,
- for any $x \in X$ the function $a(x, \cdot) : Y \rightarrow \mathbb{R} \cup \{+\infty\}$ is concave on Y .

The following conditions are equivalent:

- (i) *for every $\alpha \in \mathbb{R}$, $\alpha < \inf_{x \in X} \sup_{y \in Y} a(x, y)$, there exist $y_1, y_2 \in Y$ and $\varphi_1 \in \text{supp } a(\cdot, y_1)$, $\varphi_2 \in \text{supp } a(\cdot, y_2)$ such that the intersection property holds for φ_1, φ_2 on X at the level α .*
- (ii) $\sup_{y \in Y} \inf_{x \in X} a(x, y) = \inf_{x \in X} \sup_{y \in Y} a(x, y)$.

The proof of this fact will be given as a corollary of Theorem 5.1.

3 Generalized Intersection Property

Throughout this paper, we use the following convention

$$+\infty + (-\infty) = -\infty + (+\infty) = +\infty, \quad 0 \cdot (+\infty) = 0, \quad 0 \cdot (-\infty) = 0,$$

$$\forall t \in]0, 1[: t \cdot (+\infty) = +\infty, \quad t \cdot (-\infty) = -\infty.$$

In [11], the above addition was called inf-addition (the authors considered also sup-addition).

To derive necessary and sufficient conditions for the minimax equality for extended real-valued Φ -convex–concave functions we use the *generalized intersection property* introduced below.

Definition 3.1 Let $\varphi_1, \varphi_2 : X \rightarrow \bar{\mathbb{R}}$ be any two functions and $\alpha \in \mathbb{R}$. The functions φ_1 and φ_2 are said to have the *generalized intersection property on X at the level $\alpha \in \mathbb{R}$* iff (2) hold:

$$\forall t \in [0, 1] : [t\varphi_1 + (1-t)\varphi_2 < \alpha] \cap [\varphi_1 < \alpha] = \emptyset \text{ or } [t\varphi_1 + (1-t)\varphi_2 < \alpha] \cap [\varphi_2 < \alpha] = \emptyset \quad (2)$$

and (3) or (4) or (5) hold:

$$[\varphi_1 < \alpha] = \emptyset \tag{3}$$

$$[\varphi_2 < \alpha] = \emptyset \tag{4}$$

$$\exists t \in]0, 1[: [\varphi_1 < \alpha] \cap [t\varphi_1 + (1-t)\varphi_2 < \alpha] = \emptyset \quad (a)$$

$$\exists t \in]0, 1[: [\varphi_2 < \alpha] \cap [t\varphi_1 + (1-t)\varphi_2 < \alpha] = \emptyset \quad (b) \tag{5}$$

Proposition 3.1 Let φ_1, φ_2 admit only finite values, i.e., $\varphi_1, \varphi_2 : X \rightarrow \mathbb{R}$. Then, φ_1, φ_2 have the *generalized intersection property on X at the level α* if and only if φ_1, φ_2 have the *intersection property on X at the level α* .

Proof Assume that φ_1, φ_2 have the intersection property on X at the level α .

It is obvious that (2) holds. Then, the following situations may occur:

1. $[\varphi_1 < \alpha] = \emptyset$, hence (3) holds.
2. $[\varphi_2 < \alpha] = \emptyset$, hence (4) holds.
3. $[\varphi_1 < \alpha] \neq \emptyset$ and $[\varphi_2 < \alpha] \neq \emptyset$. Then, for any $\bar{x} \in [\varphi_1 < \alpha]$, we have $0 \leq F_1(\bar{x}) := \frac{\alpha - \varphi_2(\bar{x})}{\varphi_1(\bar{x}) - \varphi_2(\bar{x})} < 1$ and for any $\bar{t}, F_1(\bar{x}) < \bar{t} < 1$ we get

$$[\varphi_2 < \alpha] \cap [\bar{t}\varphi_1 + (1-\bar{t})\varphi_2 < \alpha] = \emptyset,$$

and (5)b holds. Analogously, for any $\bar{x} \in [\varphi_2 < \alpha]$, we have $0 < F_2(\bar{x}) := \frac{\alpha - \varphi_2(\bar{x})}{\varphi_1(\bar{x}) - \varphi_2(\bar{x})} \leq 1$, and for any $\bar{t}, 0 < \bar{t} < F_2(\bar{x})$ we get

$$[\varphi_1 < \alpha] \cap [\bar{t}\varphi_1 + (1-\bar{t})\varphi_2 < \alpha] = \emptyset,$$

and (5)a holds. □

Remark 3.1 (a) If there exists $\bar{x} \in X$ such that $\varphi_1(\bar{x}) = -\infty$ and $\varphi_2(\bar{x}) < +\infty$, then $[\varphi_1 < \alpha] \neq \emptyset$ for all $\alpha \in \mathbb{R}$ and

$$[t\varphi_1 + (1 - t)\varphi_2 < \alpha] \cap [\varphi_1 < \alpha] \neq \emptyset \text{ for } t \in]0, 1[.$$

If there exists $\tilde{x} \in X$ such that $\varphi_2(\tilde{x}) = -\infty$ and $\varphi_1(\tilde{x}) < +\infty$, then $[\varphi_2 < \alpha] \neq \emptyset$ for all $\alpha \in \mathbb{R}$ and

$$[t\varphi_1 + (1 - t)\varphi_2 < \alpha] \cap [\varphi_2 < \alpha] \neq \emptyset \text{ for } t \in [0, 1[.$$

Hence, (2) does not hold.

(b) If $[t\varphi_1 + (1-t)\varphi_2 < \alpha] = \emptyset$ for every $t \in]0, 1[$ or $[\varphi_1 < \alpha] = \emptyset$ and $[\varphi_2 < \alpha] = \emptyset$, then φ_1, φ_2 have the generalized intersection property on X at the level α .

Below we give examples related to generalized intersection property.

Example 3.1 1. Let $\varphi_1, \varphi_2 : \mathbb{R} \rightarrow \bar{\mathbb{R}}$ be defined as follows

$$\varphi_1(x) = \begin{cases} -\infty & \text{if, } x < a_1, \\ c_1 & \text{if, } x = a_1, \\ +\infty & \text{if, } x > a_1, \end{cases}$$

$$\varphi_2(x) = \begin{cases} -\infty & \text{if, } x > a_2, \\ c_2 & \text{if, } x = a_2, \\ +\infty & \text{if, } x < a_2, \end{cases}$$

where $a_1, a_2, c_1, c_2 \in \mathbb{R}, a_1 < a_2$. It is easy to see that functions φ_1, φ_2 have the generalized intersection property at every level $\alpha \in \mathbb{R}$.

2. Let $\varphi_1, \varphi_2 : \mathbb{R} \rightarrow \bar{\mathbb{R}}$ be defined as follows

$$\varphi_1(x) = \begin{cases} -\infty & \text{if, } x < a_1, \\ c_1 & \text{if, } x = a_1, \\ +\infty & \text{if, } x > a_1, \end{cases}$$

$$\varphi_2(x) = \begin{cases} -\infty & \text{if, } x > a_2, \\ c_2 & \text{if, } x = a_2, \\ +\infty & \text{if, } x < a_2, \end{cases}$$

where $a_1, a_2, c_1, c_2 \in \mathbb{R}, a_1 > a_2$. It is easy to see that for φ_1, φ_2 condition (5) does not hold, thus functions φ_1, φ_2 do not have the generalized intersection property at any level α .

4 An Auxiliary Lemma

Now we prove a lemma which is crucial for our results.

Lemma 4.1 *Let $\varphi_1, \varphi_2 : X \rightarrow \bar{\mathbb{R}}$ and $\alpha \in \mathbb{R}$. The functions φ_1, φ_2 have the generalized intersection property on X at the level α if and only if there exists $t_0 \in [0, 1]$ such that*

$$t_0\varphi_1(x) + (1 - t_0)\varphi_2(x) \geq \alpha \text{ for all } x \in X. \tag{6}$$

Proof Assume that φ_1, φ_2 have the generalized intersection property on X at the level α . If $[\varphi_1 < \alpha] = \emptyset$, then $\varphi_1(x) \geq \alpha$ for all $x \in X$, thus, in view of convention adopted,

(6) holds for $t_0 = 1$. If $[\varphi_2 < \alpha] = \emptyset$, then $\varphi_2(x) \geq \alpha$ for $x \in X$, thus (6) holds for $t_0 = 0$.

Assume now that $[\varphi_1 < \alpha]$ and $[\varphi_2 < \alpha]$ are nonempty and define the sets T_1, T_2 ,

$$T_1 := \{t \in]0, 1[: [\varphi_1 < \alpha] \cap [t\varphi_1 + (1 - t)\varphi_2 < \alpha] \neq \emptyset\},$$

$$T_2 := \{t \in]0, 1[: [\varphi_2 < \alpha] \cap [t\varphi_1 + (1 - t)\varphi_2 < \alpha] \neq \emptyset\}.$$

We show that $]0, 1[\setminus (T_1 \cup T_2) \neq \emptyset$. Consider the following cases:

1. $T_1 = \emptyset$. By (5)b, $T_2 \neq]0, 1[$.
2. $T_2 = \emptyset$. By (5)a, $T_1 \neq]0, 1[$.
3. $T_1 \neq \emptyset, T_2 \neq \emptyset$. We need to show that there exists $t_0 \in]0, 1[\setminus (T_1 \cup T_2)$. Observe that $T_1 \cap T_2 = \emptyset$. We show that T_1 and T_2 are open subsets of $]0, 1[$. To show that T_1 is open, take any $\bar{t} \in T_1$, i.e., there exists $\bar{x} \in X$ such that

$$\varphi_1(\bar{x}) < \alpha \quad \text{and} \quad \bar{t}\varphi_1(\bar{x}) + (1 - \bar{t})\varphi_2(\bar{x}) < \alpha.$$

Clearly, it must be $\varphi_1(\bar{x}) < +\infty$ and $\varphi_2(\bar{x}) < +\infty$. By (5)a, it cannot be $\varphi_1(\bar{x}) = -\infty$. Hence, $-\infty < \varphi_1(\bar{x}) < +\infty$. By (2), it must be $\varphi_2(\bar{x}) > -\infty$ since otherwise $[\varphi_1 < \alpha] \cap [\varphi_2 < \alpha] \neq \emptyset$. Let

$$\varepsilon := \frac{\alpha - (\bar{t}\varphi_1(\bar{x}) + (1 - \bar{t})\varphi_2(\bar{x}))}{\varphi_2(\bar{x}) - \varphi_1(\bar{x})} > 0.$$

Then, for every $t \in [\bar{t} - \varepsilon, \bar{t} + \varepsilon]$, we have $t \in T_1$.

To show that T_2 is open, take any $\bar{t} \in T_2$, i.e., there exists $\bar{x} \in X$ such that

$$\varphi_2(\bar{x}) < \alpha \quad \text{and} \quad \bar{t}\varphi_1(\bar{x}) + (1 - \bar{t})\varphi_2(\bar{x}) < \alpha.$$

Clearly, $\varphi_1(\bar{x})$ and $\varphi_2(\bar{x})$ are finite. Let

$$\varepsilon := \frac{\alpha - (\bar{t}\varphi_1(\bar{x}) + (1 - \bar{t})\varphi_2(\bar{x}))}{\varphi_1(\bar{x}) - \varphi_2(\bar{x})} > 0$$

then for every $t \in [\bar{t} - \varepsilon, \bar{t} + \varepsilon]$ we have $t \in T_2$.

Since T_1, T_2 are nonempty and disjoint, and we showed that they are open, we get $]0, 1[\setminus (T_1 \cup T_2) \neq \emptyset$.

For all $t_0 \in]0, 1[\setminus (T_1 \cup T_2)$ we have

$$t_0\varphi_1(x) + (1 - t_0)\varphi_2(x) \geq \alpha \quad \text{for all } x \in X.$$

To show the converse implication, assume now that (6) holds. If $t_0 = 0$, then $\varphi_2(x) \geq \alpha$, for all $x \in X$, and conditions (2) and (4) of Definition 3.1 are fulfilled. If $t_0 = 1$, then $\varphi_1(x) \geq \alpha$, for all $x \in X$, and conditions (2) and (3) of Definition 3.1 hold.

Consider now the case where (6) holds with $t_0 \in]0, 1[$. It is easy to see that (5) is true for t_0 .

To complete the proof we need to show that condition (2) holds. Observe first that, by (6), it must be $[\varphi_1 < \alpha] \cap [\varphi_2 < \alpha] = \emptyset$, since otherwise there exists $\bar{x} \in [\varphi_1 < \alpha] \cap [\varphi_2 < \alpha]$ such that $t_0\varphi_1(\bar{x}) + (1 - t_0)\varphi_2(\bar{x}) \geq \alpha$.

By contradiction, suppose that $[\varphi_1 < \alpha] \cap [\varphi_2 < \alpha] = \emptyset$ and (2) does not hold, i.e., there exist $t_1 \in]0, 1[$ and $\bar{x}, \tilde{x} \in X$ such that

$$\begin{aligned}\bar{x} &\in [\varphi_1 < \alpha] \cap [t_1\varphi_1 + (1 - t_1)\varphi_2 < \alpha] \\ \tilde{x} &\in [\varphi_2 < \alpha] \cap [t_1\varphi_1 + (1 - t_1)\varphi_2 < \alpha].\end{aligned}\quad (7)$$

Then

$$\varphi_1(\bar{x}) < \alpha, \quad \varphi_2(\bar{x}) \geq \alpha, \quad \varphi_2(\tilde{x}) < \alpha, \quad \varphi_1(\tilde{x}) \geq \alpha.$$

Now we show that $\varphi_1(\bar{x})$, $\varphi_2(\bar{x})$, $\varphi_1(\tilde{x})$ and $\varphi_2(\tilde{x})$ are finite. Clearly $\varphi_1(\bar{x}) < +\infty$, $\varphi_1(\tilde{x}) > -\infty$ and $\varphi_2(\tilde{x}) < +\infty$, $\varphi_2(\bar{x}) > -\infty$. Moreover, by (7), it must be $\varphi_1(\tilde{x}) < +\infty$ and $\varphi_2(\bar{x}) < +\infty$; consequently, by (6) it must be $\varphi_1(\bar{x}) > -\infty$ and $\varphi_2(\tilde{x}) > -\infty$. By (7),

$$\frac{\alpha - \varphi_2(\bar{x})}{\varphi_1(\bar{x}) - \varphi_2(\bar{x})} < t_1 < \frac{\alpha - \varphi_2(\tilde{x})}{\varphi_1(\tilde{x}) - \varphi_2(\tilde{x})},$$

which contradicts (6). \square

5 Main Results

Now we are in a position to prove necessary and sufficient conditions for the minimax equality for extended real-valued Φ -convex–concave functions.

We say that a function $f : X \rightarrow \bar{\mathbb{R}}$ is concave in the sense of Ky Fan [12] if for any $x_1, x_2 \in X$ and $t \in [0, 1]$ there exists $x_0 \in X$ such that $f(x_0) \geq tf(x_1) + (1 - t)f(x_2)$.

Theorem 5.1 *Let X be a set and Y be a real vector space. Let $a : X \times Y \rightarrow \bar{\mathbb{R}}$,*

- $\forall y \in Y, a(\cdot, y) : X \rightarrow \bar{\mathbb{R}}$ is Φ -convex on X ,
- $\forall x \in X, a(x, \cdot) : Y \rightarrow \bar{\mathbb{R}}$ is concave on Y .

If

- (i) *for every $\alpha \in \mathbb{R}$, $\alpha < \inf_{x \in X} \sup_{y \in Y} a(x, y)$, there exist $y_1, y_2 \in Y$ and $\varphi_1 \in \text{supp } a(\cdot, y_1)$, $\varphi_2 \in \text{supp } a(\cdot, y_2)$ such that the generalized intersection property holds for φ_1, φ_2 on X at the level α .*

then

- (ii) $\sup_{y \in Y} \inf_{x \in X} a(x, y) = \inf_{x \in X} \sup_{y \in Y} a(x, y)$.

Moreover, if Φ contains all constant functions, then (ii) \Rightarrow (i).

Proof (i) \Rightarrow (ii) If $\inf_{x \in X} \sup_{y \in Y} a(x, y) = -\infty$, then by the fact that the inequality

$$\sup_{y \in Y} \inf_{x \in X} a(x, y) \leq \inf_{x \in X} \sup_{y \in Y} a(x, y)$$

always holds, we get the required conclusion.

Let $\alpha < \inf_{x \in X} \sup_{y \in Y} a(x, y)$. By Lemma 4.1, there exists $t_0 \in [0, 1]$ such that

$$t_0 \varphi_1(x) + (1 - t_0) \varphi_2(x) \geq \alpha \quad \text{for all } x \in X, \tag{8}$$

where $\varphi_1 \in \text{supp } a(\cdot, y_1)$, $\varphi_2 \in \text{supp } a(\cdot, y_2)$, $y_1, y_2 \in Y$. Hence

$$t_0 a(x, y_1) + (1 - t_0) a(x, y_2) \geq \alpha \quad \text{for all } x \in X. \tag{9}$$

By concavity of $a(x, \cdot)$ and by (9), there exists $y_0 \in Y$ such that

$$a(x, y_0) \geq \alpha \quad \text{for all } x \in X.$$

Thus

$$\sup_{y \in Y} \inf_{x \in X} a(x, y) \geq \alpha,$$

for all $\alpha < \inf_{x \in X} \sup_{y \in Y} a(x, y)$. Hence, we get

$$\sup_{y \in Y} \inf_{x \in X} a(x, y) \geq \inf_{x \in X} \sup_{y \in Y} a(x, y).$$

(ii) \Rightarrow (i) We need only to consider the case $\inf_{x \in X} \sup_{y \in Y} a(x, y) > -\infty$. Let $\alpha < \inf_{x \in X} \sup_{y \in Y} a(x, y)$. By the equality, $\sup_{y \in Y} \inf_{x \in X} a(x, y) = \inf_{x \in X} \sup_{y \in Y} a(x, y)$, we get

$$\sup_{y \in Y} \inf_{x \in X} a(x, y) > \alpha.$$

So, there exists $\bar{y} \in Y$ such that

$$a(x, \bar{y}) \geq \alpha \quad \text{for all } x \in X.$$

Thus, the function $\bar{\varphi} := \alpha$ belongs to the support set $\text{supp } a(\cdot, \bar{y})$. By the fact that $[\bar{\varphi} < \alpha] = \emptyset$, we get

$$[t\bar{\varphi} + (1 - t)\varphi < \alpha] \cap [\bar{\varphi} < \alpha] = \emptyset \quad \text{for all } \varphi \in \Phi.$$

Then, for all $\varphi \in \Phi$, the functions $\bar{\varphi}$ and φ have the generalized intersection property on X at the level α . □

Proof of Theorem 2.1 The proof follows directly from Remark 3.1a and Theorem 5.1. □

Let us note that in the proof of Theorem 5.1 the roles of $a(\cdot, y)$ and $a(x, \cdot)$ are not symmetric, i.e., one cannot get the conclusion of Theorem 5.1 under the assumption that $a(x, \cdot)$ is Ψ -concave for a certain class Ψ .

By examining the proof of Theorem 5.1, we see that the fact, that functions $a(\cdot, y)$ are pointwise suprema of functions from Φ is not used. What is needed in the proof is that, for $\alpha < \inf_{x \in X} \sup_{y \in Y} a(x, y)$ there exist $y_1, y_2 \in Y$ and any functions $\varphi_1, \varphi_2 : X \rightarrow \bar{\mathbb{R}}, \varphi_1 \leq a(\cdot, y_1), \varphi_2 \leq a(\cdot, y_2)$, satisfying the generalized intersection property on X at the level α .

This allows to formulate the following theorem.

Theorem 5.2 *Let X be a set and Y be a real vector space.*

Let $a : X \times Y \rightarrow \bar{\mathbb{R}}$ be such that:

- for any $x \in X$ the function $a(x, \cdot) : Y \rightarrow \bar{\mathbb{R}}$ is concave on Y .*

The following conditions are equivalent:

- (i) *for every $\alpha \in \mathbb{R}, \alpha < \inf_{x \in X} \sup_{y \in Y} a(x, y)$, there exist $y_1, y_2 \in Y$ and $\varphi_1 \in \text{supp } a(\cdot, y_1), \varphi_2 \in \text{supp } a(\cdot, y_2)$ such that the generalized intersection property holds for φ_1, φ_2 on X at the level α .*
- (ii) $\sup_{y \in Y} \inf_{x \in X} a(x, y) = \inf_{x \in X} \sup_{y \in Y} a(x, y)$.

Proof The same as the proof of Theorem 5.1. □

Remark 5.1 Let X, Y, Φ and $a(\cdot, \cdot)$ be as in Theorem 5.1.

1. If there exists $\bar{y} \in Y$ such that $a(\cdot, \bar{y}) \equiv +\infty$, then $\inf_{x \in X} a(x, \bar{y}) = +\infty$, hence $\sup_{y \in Y} \inf_{x \in X} a(x, y) = +\infty$. On the other hand $\sup_{y \in Y} a(x, y) = +\infty$, for every $x \in X$, so $\inf_{x \in X} \sup_{y \in Y} a(x, y) = +\infty$ and minimax equality always holds for function $a(\cdot, \cdot)$.
2. If there exists $\bar{x} \in X$ such that $a(\bar{x}, \cdot) \equiv -\infty$, then $\inf_{x \in X} \sup_{y \in Y} a(x, y) = -\infty$, and condition (i) of Theorem 5.1 always holds.

In Theorem 5.1 we can change the assumptions on function $a(\cdot, \cdot)$ symmetrically. It is possible to assume that $a(\cdot, \cdot)$ is convex as a function of X and Φ -concave as a function of y , i.e., is equal to the pointwise infimum of all functions φ greater than or equal to $a(\cdot, \cdot)$. In such case condition (i) of Theorem 5.1 has to hold for all $\alpha > \sup_{y \in Y} \inf_{x \in X} a(x, y)$ and definition of generalized intersection property with opposite inequalities.

6 Φ -convexity of Convex Functions

In this section, we present a class Φ such that all convex functions defined on \mathbb{R}^n are Φ -convex (Theorem 6.1). This class is used in next section to provide minimax theorem for convex–concave functions.

We recall that $x = (x^1, \dots, x^n)^T \in \mathbb{R}^n$ is said to be “lexicographically less” than $y = (y^1, \dots, y^n)^T \in \mathbb{R}^n$, denoted $x <_L y$, if $x \neq y$ and for $k = \min\{i \in \{1, \dots, n\} \mid x_i \neq y_i\}$ we have $x_k < y_k$.

For $k \in \{0, 1, \dots, n\}$, we denote by $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^k)$ the set of all linear mappings $u : \mathbb{R}^n \rightarrow \mathbb{R}^k$.

Following [13], for any $u \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^k)$, with $\text{rank } u = k$, $z \in \mathbb{R}^k$, $x^* \in (\mathbb{R}^n)^*$ and $d \in \mathbb{R}$, we define $\varphi_{u,z,x^*,d} : \mathbb{R}^n \rightarrow \bar{\mathbb{R}}$ by

$$\varphi_{u,z,x^*,d}(x) = \begin{cases} -\infty & \text{if, } u(x) <_L z, \\ \langle x^*, x \rangle + d & \text{if, } u(x) = z, \\ +\infty & \text{if, } u(x) >_L z, \end{cases} \quad (\Delta)$$

and let $\bar{\Phi} = \{\varphi_{u,z,x^*,d}\}$ be a set of all functions $\varphi_{u,z,x^*,d}$ defined above.

A function $f : \mathbb{R}^n \rightarrow \bar{\mathbb{R}}$ is convex if epigraph of f is a convex set in \mathbb{R}^n .

Theorem 6.1 (Theorem 2.1, [13]) *Let $f : \mathbb{R}^n \rightarrow \bar{\mathbb{R}}$. The following statements are equivalent*

- (i) f is convex
- (ii) f is $\bar{\Phi}$ -convex.

In the example, below we investigate functions $\varphi_1, \varphi_2 : \mathbb{R} \rightarrow \bar{\mathbb{R}}$, $\varphi_1, \varphi_2 \in \bar{\Phi}$ which have the generalized intersection property on \mathbb{R} at the given level α .

Example 6.1 Let $\varphi_{u_1,z_1,x_1^*,d_1}, \varphi_{u_2,z_2,x_2^*,d_2} : \mathbb{R} \rightarrow \bar{\mathbb{R}}$, $\varphi_1 := \varphi_{u_1,z_1,x_1^*,d_1}$,

$\varphi_2 := \varphi_{u_2,z_2,x_2^*,d_2}$, $\varphi_1, \varphi_2 \in \bar{\Phi}$, $\alpha \in \mathbb{R}$. In this setting we have $k = 0$ or $k = 1$, then either $u_1 \equiv 0$ or u_1 is linear function from \mathbb{R} to \mathbb{R} .

1. $u_1 \equiv 0$.
 - (a) If $z_1 > 0$, then $\varphi_1 \equiv -\infty$. φ_1 and φ_2 have the generalized intersection property if and only if $u_2 \equiv 0$ and $z_2 < 0$, i.e., $\varphi_2 \equiv +\infty$.
 - (b) If $z_1 < 0$, then $\varphi_1 \equiv +\infty$. Thus, φ_1 and φ_2 have the generalized intersection property for every u_2, z_2, x_2^*, d_2 .
 - (c) If $z_1 = 0$, then φ_1 is affine function. Thus, φ_1 and φ_2 have the generalized intersection property if and only if $u_2 \equiv 0$, $z_2 = 0$ and x_1^*, d_1, x_2^*, d_2 are such that $[\varphi_1 < \alpha] \cap [\varphi_2 < \alpha] = \emptyset$.
2. $u_1(x) = a_1x, a_1 \neq 0$ then (Δ) takes the form

$$\varphi_1(x) = \begin{cases} -\infty & \text{if, } a_1x < z_1, \\ \langle x_1^*, x \rangle + d_1 & \text{if, } a_1x = z_1, \\ +\infty & \text{if, } a_1x > z_1. \end{cases} \quad (\Delta\Delta)$$

We have the following possible situations

- (a) $u_2 \equiv 0$, then φ_1, φ_2 have the generalized intersection property if and only if $z_2 < 0$, i.e., $\varphi_2 \equiv +\infty$.
- (b) $u_2(x) = a_2x, a_2 \neq 0, a_1 > 0$, then φ_1, φ_2 have the generalized intersection property if and only if $a_2 < 0$ and $z_1 < z_2$.
- (c) $u_2(x) = a_2x, a_2 \neq 0, a_1 < 0$, then φ_1, φ_2 have the generalized intersection property if and only if $a_2 > 0$ and $z_1 > z_2$.

7 Minimax Theorems for Convex–Concave Functions

Taking into account Theorem 6.1, we can formulate the minimax theorem for convex–concave functions.

Theorem 7.1 *Let Y be a real vector space. Let $a : \mathbb{R}^n \times Y \rightarrow \bar{\mathbb{R}}$ be such that*

- for any $y \in Y$ the function $a(\cdot, y) : \mathbb{R}^n \rightarrow \bar{\mathbb{R}}$ is convex on \mathbb{R}^n ,
- for any $x \in \mathbb{R}^n$ the function $a(x, \cdot) : Y \rightarrow \bar{\mathbb{R}}$ is concave on Y .

The following conditions are equivalent:

- (i) for every $\alpha \in \mathbb{R}, \alpha < \inf_{x \in \mathbb{R}^n} \sup_{y \in Y} a(x, y)$, there exist $y_1, y_2 \in Y$ and $\varphi_1 \in \text{supp } a(\cdot, y_1), \varphi_2 \in \text{supp } a(\cdot, y_2)$ such that the generalized intersection property holds for φ_1, φ_2 on \mathbb{R}^n at the level α .
- (ii) $\sup_{y \in Y} \inf_{x \in \mathbb{R}^n} a(x, y) = \inf_{x \in \mathbb{R}^n} \sup_{y \in Y} a(x, y)$.

Proof Follows from Theorems 6.1 and 5.1. □

The following remark is based on Example 6.1.

Remark 7.1 Let $X, Y = \mathbb{R}$, and function $a(\cdot, \cdot)$ be as in Theorem 7.1.

1. If there exists $\bar{y} \in \mathbb{R}$ such that $a(\cdot, \bar{y}) \equiv +\infty$, then from Remark 5.1 we get that minimax equality holds,
2. If for every $y \in \mathbb{R}$ and every $x \in \mathbb{R}$ we have $a(x, y) < +\infty$, then for every $\varphi \in \text{supp } a(\cdot, y)$ and every $x \in \mathbb{R}$ we have $\varphi(x) < +\infty$. Thus, for given level $\alpha \in \mathbb{R}$ there exist $y_1, y_2 \in \mathbb{R}$ and $\varphi_1 \in \text{supp } a(\cdot, y_1), \varphi_2 \in \text{supp } a(\cdot, y_2)$ such that the generalized intersection property holds for φ_1, φ_2 on \mathbb{R} at the level α if and only if φ_1, φ_2 are affine functions and $[\varphi_1 < \alpha] \cap [\varphi_2 < \alpha] = \emptyset$.
3. If there exist $\bar{y} \in \mathbb{R}$ and $\bar{x} \in \mathbb{R}$ such that $a(\bar{x}, \bar{y}) = +\infty$ and $\tilde{x} \in X$ such that $a(\tilde{x}, \bar{y}) < +\infty$, then all situations from Example 6.1 may occur.
4. If function $a(\cdot, \cdot)$ takes value $-\infty$, then all situations from Example 6.1 may occur.

We can formulate the following corollary of Theorem 7.1.

Corollary 7.1 *Let $a : \mathbb{R} \times \mathbb{R} \rightarrow \bar{\mathbb{R}}$ be such that*

- for every $y \in \mathbb{R}$ and every $x \in \mathbb{R} a(x, y) < +\infty$.
- for any $y \in \mathbb{R}$ the function $a(\cdot, y) : \mathbb{R} \rightarrow \bar{\mathbb{R}}$ is convex on \mathbb{R} ,
- for any $x \in \mathbb{R}$ the function $a(x, \cdot) : \mathbb{R} \rightarrow \bar{\mathbb{R}}$ is concave on \mathbb{R} .

The following conditions are equivalent:

- (i) for every $\alpha \in \mathbb{R}, \alpha < \inf_{x \in \mathbb{R}} \sup_{y \in \mathbb{R}} a(x, y)$, there exist $y_1, y_2 \in \mathbb{R}$ and $\varphi_1 \in \text{supp } a(\cdot, y_1), \varphi_2 \in \text{supp } a(\cdot, y_2)$ such that φ_1, φ_2 are affine functions and $[\varphi_1 < \alpha] \cap [\varphi_2 < \alpha] = \emptyset$.
- (ii) $\sup_{y \in \mathbb{R}} \inf_{x \in \mathbb{R}} a(x, y) = \inf_{x \in \mathbb{R}} \sup_{y \in \mathbb{R}} a(x, y)$.

Now, we present an example of improper convex–concave function $a(\cdot, \cdot)$, which does not possess the generalized intersection property at any level α .

Example 7.1 Let $X, Y = \mathbb{R}$ and $a : \mathbb{R} \times \mathbb{R} \rightarrow \bar{\mathbb{R}}$.

$$a(x, y) = \begin{cases} +\infty & \text{if, } x \geq 0, y < 0 \text{ or } x < 0, y \geq 0, \\ -\infty & \text{if, } x \geq 0, y \geq 0 \text{ or } x < 0, y < 0, \end{cases}$$

then $\inf_{x \in \mathbb{R}} \sup_{y \in \mathbb{R}} a(x, y) = +\infty$ and $\sup_{y \in \mathbb{R}} \inf_{x \in \mathbb{R}} a(x, y) = -\infty$. Then, for every $\alpha \in \mathbb{R}$ the generalized intersection property does not hold. Indeed, for $\bar{y} \geq 0$ function $a(\cdot, \bar{y})$ takes the form

$$a(x, \bar{y}) = \begin{cases} +\infty & \text{if, } x < 0, \\ -\infty & \text{if, } x \geq 0, \end{cases}$$

and if $\varphi \in \text{supp } a(\cdot, \bar{y})$, then $\varphi \equiv -\infty$.

For $\bar{y} < 0$ function $a(\cdot, \bar{y})$ takes the form

$$a(x, \bar{y}) = \begin{cases} +\infty & \text{if, } x \geq 0, \\ -\infty & \text{if, } x < 0, \end{cases}$$

and if $\varphi \in \text{supp } a(\cdot, \bar{y})$, then

$$\varphi(x) = \begin{cases} -\infty & \text{if, } x < 0, \\ c & \text{if, } x = 0, \\ +\infty & \text{if, } x > 0, \end{cases}$$

where $c \in \mathbb{R}$.

8 Conclusions

We provide minimax theorems for extended real-valued Φ -convex–concave functions (or symmetrically for convex- Φ -concave functions). A distinguished feature of these results is that we do not need any topological structure on the spaces involved. On the other hand, the results obtained are applicable to functions being pointwise suprema of abstract affine functions. In particular, we obtain minimax theorems for extended real-valued convex–concave functions which do not have to be proper or lower (upper) semicontinuous.

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