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Approximate Euclidean Steiner Trees

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Abstract An approximate Steiner tree is a Steiner tree on a given set of terminals in Euclidean space such that the angles at the Steiner points are within a specified error from 120°. This notion arises in numerical approximations of minimum Steiner trees. We investigate the worst-case relative error of the length of an approximate Steiner tree compared to the shortest tree with the same topology. It has been conjectured that this relative error is at most linear in the maximum error at the angles, independent of the number of terminals. We verify this conjecture for the two-dimensional case as long as the maximum angle error is sufficiently small in terms of the number of terminals. In the two-dimensional case we derive a lower bound for the relative error in length. This bound is linear in terms of the maximum angle error when the angle error is sufficiently small in terms of the number of terminals. We find improved estimates of the relative error in length for larger values of the maximum angle error and calculate exact values in the plane for three and four terminals.

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1 Introduction

The Euclidean Steiner problem asks for a tree of shortest total length that interconnects a given collection of points or *terminals* in Euclidean space. For example, to interconnect the four vertices of a square in the plane, a shortest tree contains two further points apart from the four terminals (Fig. 1). Such a shortest tree is called a *minimum Steiner tree* on the given collection of terminals, and the additional points are called *Steiner points*. The Steiner problem is well studied, especially in the plane. An overview of the extensive literature on this problem can be found in the monographs of Hwang, Richards and Winter [1], Cieslik [2], Prömel and Steger [3], and the recent Brazil and Zachariasen [4]. For more on the history of the problem, see Boltyanski, Martini, and Soltan [5] and the recent Brazil, Graham, Thomas, and Zachariasen [6].

It is well known that a minimum Steiner tree in Euclidean space has maximum degree three, that the Steiner points always have degree three, and that each angle spanned by two edges with a common endpoint is at least 120°, and exactly 120° at each Steiner point [1, Section 6.1]. In the plane, there is a ruler-and-compass construction of a minimum Steiner tree once the graph structure (or *topology*) is known. This construction, also known as the Melzak algorithm [7], can be done in linear time [8]. On the other hand, determining the topology of a minimum Steiner tree is hard. There is a super-exponential number of different topologies [9], and it is already NP-hard to decide whether a given set of points in the plane has a Steiner tree of length smaller than a given length [10]. On the other hand, the GeoSteiner package of Warme, Winter and Zachariasen quickly finds minimum Steiner trees on a relatively large number of points in the plane [11].

There are polynomial time approximation schemes to calculate minimum Steiner trees in Euclidean space (Arora [12] and Mitchell [13]; see also [14]). However, for the actual implementation of these schemes, there has been progress so far only for certain planar problems [15]. A major obstacle in the implementation of these schemes for higher-dimensional problems is that their time complexity depends doubly exponentially on the dimension, and there is some evidence that this is unavoidable [16].

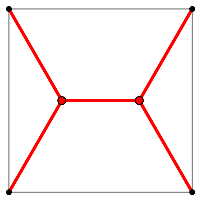


Fig. 1 Minimum Steiner tree (in *red*) of the vertices of a square

In higher dimensions, the Steiner points are not necessarily constructible, and finding the optimal Steiner points results in solving high-degree algebraic equations, or solving a convex optimisation problem numerically [17]. See the papers [9,17–23] for work on finding minimum Steiner trees in Euclidean spaces of dimension at least 3. We mention that Steiner trees in 3-space have been considered in theoretical investigations of multiquarks in particle physics [24] and in higher dimensions have been used to determine phylogenetic trees [25].

One problem arising from a numerical approach is that of estimating how close an approximation is to a locally minimum Steiner tree with a given Steiner topology. Rubinstein et al. [22] studied the relative error in the length of an approximate Steiner tree in terms of how far the angles at Steiner points deviate from 120°. This paper is a further contribution to this topic.

Before we can give an exact definition of the relative error, we introduce our terminology and notation in Sect. 2. Then, in Sect. 3 we define the relative error and formulate the main conjectures from [22]. Our results are stated and summarised in Sect. 4. Sect. 5 is a brief discussion of the monotonicity of the relative error as the number of terminals increases. In Sect. 6, we prove our results for large relative errors. For small relative errors, we subdivide the proofs into a section on upper bounds (Sect. 7) and lower bounds (Sect. 8). We conclude in Sect. 9 with some remarks. There are two tedious induction proofs of results in Sect. 8 which are presented in "Appendix".

2 Terminology

We define a *Steiner topology for n terminals* to be a tree $\mathcal{T}$ with n special vertices $t_1, \ldots, t_n$, called *terminals*, all of degree at most 3, and all other vertices, called *Steiner points*, of degree exactly 3. A Steiner topology is *full* if all terminals have degree 1. Let $N = \{p_1, \ldots, p_n\}$ be a family of n points in $\mathbb{R}^d$ (allowing repeated points). A *Steiner tree* T for N, with topology $\mathcal{T}$, is a representation of $\mathcal{T}$ in $\mathbb{R}^d$, with each t_i represented by p_i, each Steiner point of $\mathcal{T}$ represented by an arbitrary point of $\mathbb{R}^d$, and edges represented by straight-line segments. We say that such a Steiner tree *interconnects* N. A Steiner tree is *full* if its topology is full. We allow Steiner points to coincide with each other and with terminals, hence edges incident to a Steiner point to be of length 0. An edge of length 0 is called *degenerate*, and we say that a Steiner tree that contains a degenerate edge is *degenerate*. We allow edges to intersect each other.

The (convex) angle determined by two edges xy and xz with a common endpoint x is denoted $\sphericalangle yxz$. Its angular measure is also denoted by $\sphericalangle yxz$, and we assume that angular measures are in the interval $[0, \pi]$. We use radians for angular measure throughout the paper, except in a few places, where it will be clear that we use degrees.

We denote the Euclidean length of an edge pq by $|pq|$. The *length* $L(T)$ of a tree T is the sum of the Euclidean lengths of its edges. Among all the trees that interconnect a given set N of terminals there is at least one tree of minimum length, which we call a *minimum Steiner tree* of N. We define a *locally minimum Steiner tree* to be a non-degenerate tree with a Steiner topology and with all angles spanned by the edges at each vertex at least $2\pi/3$. Since each Steiner point in a Steiner topology has degree

3, it easily follows (in any dimension) that each of the three angles at a Steiner point is exactly $2\pi/3$ and that the three edges incident to the Steiner point are coplanar. As mentioned above, any minimum Steiner tree is a locally minimum Steiner tree. A *full minimum Steiner tree* is a minimum Steiner tree that is also full.

We denote the largest integer not greater than x by $\lfloor x \rfloor$.

3 Formulation of the Problem, Conjectures and Previous Results

In [22] the following notions were introduced. Let $\varepsilon \geqslant 0$ be given. An ε- *approximate Steiner tree* is a tree with a Steiner topology, with all the angles spanned by the edges at each Steiner point belonging to the interval $[2\pi/3 - \varepsilon, 2\pi/3 + \varepsilon]$. Note that a 0-approximate Steiner tree is the same as a locally minimum Steiner tree (in [22] the distinction was made between a *pseudo-Steiner point* of an ε-approximate Steiner tree and a *Steiner point* of a locally minimum Steiner tree. For the sake of simplicity we make no such distinction and use the term *Steiner point* for both).

For $d \geqslant 2, n \geqslant 3$ and $\varepsilon \geqslant 0$, let $\mathcal{A}_\varepsilon^d(n)$ denote the set of all full ε-approximate Steiner trees on n terminals in $\mathbb{R}^d$, and let $\overline{\mathcal{A}}_\varepsilon^d(n)$ denote the subset of all $T \in \mathcal{A}_\varepsilon^d(n)$ for which the terminals have a minimum Steiner tree with the same topology as T. In particular, $\mathcal{A}_0^d(n)$ is the set of all full locally minimum Steiner trees on n terminals in $\mathbb{R}^d$, and $\overline{\mathcal{A}}_0^d(n)$ is the set of all full minimum Steiner trees on n terminals in $\mathbb{R}^d$.

Given a tree T in $\mathbb{R}^d$ with Steiner topology $\mathcal{T}$, let $S(T)$ denote the shortest tree in $\mathbb{R}^d$ on the terminals of T with topology $\mathcal{T}$, where we allow degenerate shortest trees. Even though $S(T)$ is not necessarily a Steiner tree (see, for instance, [4, Figure 1.7]), it can be shown that $S(T)$ is always unique [9, Section 4].

Rubinstein, Weng and Wormald [22] defined the following two quantities:

$$F_d(\varepsilon, n) := \sup \left\{ \frac{L(T) - L(S(T))}{(L(S(T))} : T \in \mathcal{A}_\varepsilon^d(n) \right\}$$

and

$$\overline{F}_d(\varepsilon, n) := \sup \left\{ \frac{L(T) - L(S(T))}{(L(S(T))} : T \in \overline{\mathcal{A}}_\varepsilon^d(n) \right\},$$

and made the following conjectures in the case $d \geqslant 3$. Although they did not consider the two-dimensional case, we include it, as it is also still open, and most of our results will be in the plane.

Conjecture 3.1 *For any $d \geqslant 2$ there exist $\varepsilon_0 > 0$ and $C_d > 0$ such that for all $\varepsilon \in]0, \varepsilon_0[$ and $n \in \mathbb{N}$, $F_d(\varepsilon, n) < C_d\varepsilon$.*

Conjecture 3.2 *For any $d \geqslant 2$ there exist $\varepsilon_0 > 0$ and $C_d > 0$ such that for all $\varepsilon \in]0, \varepsilon_0[$ and $n \in \mathbb{N}$, $\overline{F}_d(\varepsilon, n) < C_d\varepsilon$.*

The second conjecture is weaker than the first, but it seems difficult to deduce an upper bound for $\overline{F}_d$ that cannot already be deduced for F_d. Rubinstein, Weng and Wormald [22] showed that for $\varepsilon < 1/n^2$, $F_d(\varepsilon, n) \leqslant C_d(\varepsilon \log n + \varepsilon^2 n^3)$. They also consider larger values of ε.

Table 1 Summary of results

Range	Bound	
$\varepsilon = O(1/n^2)$	$F_2(\varepsilon, n) = O(\varepsilon)$	Theorem 4.1
$\varepsilon < \dfrac{\pi}{n-2}$	$F_2(\varepsilon, n) \leqslant \dfrac{1}{\cos\frac{(n-2)\varepsilon}{2}} - 1 = O(n^2\varepsilon^2)$	Theorem 4.1
$\varepsilon < \dfrac{1}{(\log n)^2}$	$F_2(\varepsilon, n) \geqslant G_2(\varepsilon, n) = \Omega((\log n)^2\varepsilon^2)$	Theorem 4.2
$\varepsilon \leqslant \pi/6$	$F_2(\varepsilon, n) \leqslant 2n - 4$	Proposition 6.1
$0 < \varepsilon < 2\pi/3$	$F_d(\varepsilon, n) = O\left(\left(\dfrac{\cos(\varepsilon/2)}{\sin(\pi/3 - \varepsilon/2)}\right)^n\right)$	Theorem 6.1
$\varepsilon = \pi/3$	$F_2(\varepsilon, n) \geqslant G_2(\varepsilon, n) = \Omega(\log n)$	Theorem 6.2
$\pi/3 < \varepsilon < 2\pi/3$	$F_2(\varepsilon, n) \geqslant G_2(\varepsilon, n) = \Omega(n^{c(\varepsilon)})$ where $0 < c(\varepsilon) \nearrow \infty$ as $\varepsilon \to 2\pi/3$	Theorem 6.2
$0 < \varepsilon < \pi/3$	$F_2(\varepsilon, 3) = G_2(\varepsilon, 3) = \dfrac{1}{\cos(\varepsilon/2)} - 1$	Proposition 7.1
	$F_2(\varepsilon, 4) = G_2(\varepsilon, 4) = \dfrac{1}{\cos\varepsilon} - 1$	Proposition 7.2

4 Overview of New Results

Our results are summarised in Table 1. Our first main result is an upper bound for the relative error in the plane.

Theorem 4.1 *If $n \geqslant 3$ and $0 < \varepsilon < \pi/(n-2)$, then*

$$F_2(\varepsilon, n) \leqslant \frac{1}{\cos\frac{(n-2)\varepsilon}{2}} - 1.$$

The proof is in Sect. 7. As a consequence, Conjecture 3.1 holds in the plane if ε is sufficiently small, depending on n.

Corollary 4.1 *If $0 < \varepsilon < \pi/(n-2)$, then $F_2(\varepsilon, n) = O(n^2\varepsilon^2)$. Consequently, if $\varepsilon = O(1/n^2)$ as $n \to \infty$, then $F_2(\varepsilon, n) = O(\varepsilon)$.*

In [22] an example is given, which shows that Conjecture 3.1 is sharp for each $d \geqslant 3$. Our second main result, Theorem 4.2, is a lower bound for F_2, which shows that Conjecture 3.1 is already sharp in the plane for sufficiently small ε.

Theorem 4.2 *For any $k \geqslant 1$, if $\varepsilon = c/k^2$ with $0 < c < 1$, then*

$$F_2(\varepsilon, 2^k + 1) > \frac{c}{24}\varepsilon.$$

Consequently, if $\varepsilon < (\log_2 n)^{-2}$, then $F_2(\varepsilon, n) = \Omega((\log n)^2\varepsilon^2)$.

The proof is in Sect. 8. In Sect. 6, we show some bounds for larger ε.

In the above definition of F_d, we consider the worst-case relative error between a full ε-approximate Steiner tree T on n terminals and the shortest tree $S(T)$ with the same topology as T, even though $S(T)$ may have a degenerate topology. Instead, we could restrict ourselves to trees T, for which $S(T)$ is non-degenerate. Note that for any $T \in \mathcal{A}^d_\varepsilon(n)$, $S(T)$ is non-degenerate iff $S(T)$ is a locally minimum Steiner tree. We therefore introduce the following variants of the previous two quantities:

$$G_d(\varepsilon, n) := \sup\left\{\frac{L(T) - L(S(T))}{(L(S(T))}: T \in \mathcal{A}^d_\varepsilon(n), S(T) \in \mathcal{A}^d_0(n)\right\}$$

and

$$\overline{G}_d(\varepsilon, n) := \sup\left\{\frac{L(T) - L(S(T))}{(L(S(T))}: T \in \overline{\mathcal{A}}^d_\varepsilon(n), S(T) \in \overline{\mathcal{A}}^d_0(n)\right\}.$$

Clearly, $G_d(\varepsilon, n) \leqslant F_d(\varepsilon, n)$ and $\overline{G}_d(\varepsilon, n) = \overline{F}_d(\varepsilon, n)$. The construction that we make to prove the lower bounds of Theorem 4.2 in fact gives a lower bound for $G_2(\varepsilon, n)$ for certain values of n, as in Theorem 4.3.

Theorem 4.3 *For any $k \geqslant 1$, if $\varepsilon = c/k^2$ with $0 < c < 1$, then*

$$G_2(\varepsilon, 2^k + 1) > \frac{c}{24}\varepsilon.$$

Unfortunately we do not know whether $G_2(\varepsilon, n)$ is monotone in n (see Sect. 5), so we cannot state a lower bound for general n.

5 Monotonicity of F_d and G_d

In many of the examples constructed in this paper, the number of terminals is of a special form such as a power of 2. In order to make general statements for all n, we need to know that F_d and G_d are monotone in n. Monotonicity in ε and in d are straightforward. Indeed, if $0 \leqslant \varepsilon_1 < \varepsilon_2$, then an ε_1-approximate Steiner tree is also an ε_2-approximate Steiner tree, hence $F_d(\varepsilon_1, n) \leqslant F_d(\varepsilon_2, n)$, $G_d(\varepsilon_1, n) \leqslant G_d(\varepsilon_2, n)$ and $\overline{F}_d(\varepsilon_1, n) \leqslant \overline{F}_d(\varepsilon_2, n)$. Clearly F_d, G_d and $\overline{F}_d$ are monotone in d:

$$F_2 \leqslant F_3 \leqslant \cdots, \quad G_2 \leqslant G_3 \leqslant \cdots \quad \text{and} \quad \overline{F}_2 \leqslant \overline{F}_3 \leqslant \cdots.$$

It is still relatively simple to show that F_d is also monotone in n, as we show next.

Proposition 5.1 *For any $d \geqslant 2$, $\varepsilon > 0$ and $n \geqslant 3$, $F_d(\varepsilon, n) \leqslant F_d(\varepsilon, n + 1)$.*

Proof Consider any ε-approximate Steiner tree T with a full Steiner topology on n terminals. Let S be a shortest tree with the same terminals set and with the same (possibly degenerate) topology as T. We show that

$$F_d(\varepsilon, n + 1) \geqslant L(T)/L(S) - 1. \tag{1}$$

Let $\delta > 0$ be arbitrary. Modify T to obtain an ε-approximate Steiner tree T' on $n+1$ terminals as follows. Choose any terminal t of T. It is joined to a Steiner point s of T. Let t_1 and t_2 be two points at distance δ from t such that the three angles at t are equal: $\sphericalangle t_1 t t_2 = \sphericalangle t_1 t s = \sphericalangle t_2 t s$. (Thus, t_1, t_2, t and s have to be coplanar.) If we consider t_1 and t_2 to be two new terminals, and consider t to be a Steiner point, then we obtain an ε-approximate Steiner tree T' on $n+1$ terminals of length $L(T') = L(T) + 2\delta$.

We modify S by adding the edges $t_1 t$ and $t_2 t$ to obtain a tree S' with the same topology as T' (allowing degenerate topologies). Then,

$$L(S(T')) \leqslant L(S') = L(S) + 2\delta,$$

and

$$F_d(\varepsilon, n+1) \geqslant \frac{L(T')}{L(S(T'))} - 1 \geqslant \frac{L(T) + 2\delta}{L(S) + 2\delta} - 1.$$

Since this holds for all $\delta > 0$, (1) follows. Since (1) holds for an arbitrary ε-approximate Steiner tree on n terminals,

$$F_d(\varepsilon, n+1) \geqslant \sup L(T)/L(S) - 1 = F_d(\varepsilon, n).$$

□

The monotonicity of $G_d(\varepsilon, n)$ in n seems to be subtler, and we have only been able to show it for $d \geqslant 3$.

Proposition 5.2 *For any $d \geqslant 3$, $\varepsilon > 0$ and $n \geqslant 3$, $G_d(\varepsilon, n) \leqslant G_d(\varepsilon, n+1)$.*

Proof Let $\delta > 0$ be arbitrary. Let T be a full ε-approximate Steiner tree on n terminals in $\mathbb{R}^d$ such that $S(T)$ is non-degenerate (in particular, $S(T)$ is still full). Choose any terminal t of T. It is joined to a Steiner point s in T and also to a Steiner point s' in $S(T)$. Choose a point t_1 such that $t_1 t$ is perpendicular to ts and to ts', and $|tt_1| = \sqrt{3}\delta$. Let t_2 be the unique point such that t is the midpoint of $t_1 t_2$. Without any loss of generality, $\delta < |ts|, |ts'|$. Then, there exists a unique point s_2 on st such that $\sphericalangle t_1 s_2 t_2 = 2\pi/3$ and a unique point s_2' on $s't$ such that $\sphericalangle t_1 s_2' t_2 = 2\pi/3$. Let T' be the tree obtained from T by removing t and st, and adding the Steiner point s_2, terminals t_1 and t_2, and edges ss_2, $t_1 s_2$ and $t_2 s_2$. Then, T' is an ε-approximate Steiner tree on $n+1$ terminals, and $L(T') = L(T) + 3\delta$. Furthermore, $S(T')$ is the tree obtained from $S(T)$ by removing t and $s't$, and adding the Steiner point s_2', terminals t_1 and t_2, and edges $s's_2'$, $t_1 s_2'$ and $t_2 s_2'$. Then, $L(S(T')) = L(S(T)) + 3\delta$. We conclude that

$$G_d(\varepsilon, n+1) \geqslant \frac{L(T')}{L(S(T'))} - 1 = \frac{L(T) + 3\delta}{L(S(T)) + 3\delta} - 1,$$

and by letting $\delta \to 0$ and taking the sup of the right-hand side, the proof is finished. □

We have not been able to show that $\overline{F}_d(\varepsilon, n) = \overline{G}_d(\varepsilon, n)$ is monotone in n. We are also not sure whether $G_2(\varepsilon, n) \leqslant G_2(\varepsilon, n+1)$ or $\overline{F}_2(\varepsilon, n) \leqslant \overline{F}_2(\varepsilon, n+1)$ always hold.

6 Results for Large ε

This section contains upper and lower bounds for F_d for values of ε that are independent of n. In Proposition 6.1 we obtain the modest upper bound of $2n-4$ for $F_2(\varepsilon, n)$, as long as $\varepsilon \leqslant \pi/6$. We do not know of any better upper bound in the plane for small and fixed ε. In Theorem 6.1 we give an explicit upper bound for $F_d(\varepsilon, n)$ for all values of $\varepsilon < 2\pi/3$. For instance, we obtain $F_d(\varepsilon, n) \leqslant O\left(\left(2/\sqrt{3}+\varepsilon\right)^n\right)$ for small ε.

Theorem 6.2 sharpens Lemma 2.2 of [22] in the range $\varepsilon \in]\pi/3, 2\pi/3[$ by giving a lower bound for F_d for all $d \geqslant 2$ of the form $n^{\alpha(\varepsilon)}$, where $\alpha(\varepsilon)$ is an explicit function of ε. In particular, it will follow that, if $\varepsilon > 105.6\ldots^\circ$, then $\alpha(\varepsilon) > 2$; hence, the lower bound grows super-quadratically. This indicates that Theorem 2.1 of [22] can only hold if ε is sufficiently small. We also obtain a lower bound for $\varepsilon = \pi/3$ of the form $\Omega(\log n)$.

Proposition 6.1 *If $\varepsilon \leqslant \pi/6$ and $n \geqslant 3$, then $F_2(\varepsilon, n) \leqslant 2n-4$.*

Proof Since $2\pi/3 - \varepsilon \geqslant \pi/2$, it follows that each Steiner point of an ε-approximate Steiner tree T is in the convex hull of its neighbours. It easily follows that each Steiner point is in the convex hull K of the terminals. Therefore, each edge of T has length at most diam K. Since T has $2n-3$ edges, and any Steiner tree on the terminals has length at least diam K, it follows that $L(T)/L(S(T)) \leqslant 2n-3$, hence $F_2(\varepsilon, n) \leqslant 2n-4$. □

We will often use the following reverse triangle inequality.

Lemma 6.1 *In $\triangle abc$,*

$$|ab| + |bc| \leqslant \frac{|ac|}{\cos(\theta/2)},$$

where θ is the exterior angle at b.

Proof Let the angular measures of the interior angles of $\triangle abc$ at a, b, c, be α, β, γ, respectively. By the sine rule,

$$\frac{|ab|+|bc|}{|ac|} = \frac{\sin\gamma}{\sin\beta} + \frac{\sin\alpha}{\sin\beta} = \frac{\sin\alpha + \sin\gamma}{\sin\theta} = \frac{2\sin\left(\frac{\alpha+\gamma}{2}\right)\cos\left(\frac{\alpha+\gamma}{2}\right)}{\sin\theta}$$
$$\leqslant \frac{2\sin\left(\frac{\alpha+\gamma}{2}\right)}{\sin\theta} = \frac{2\sin(\theta/2)}{2\sin(\theta/2)\cos(\theta/2)} = \frac{1}{\cos(\theta/2)}.$$

□

We define a *cherry* of a Steiner topology $\mathcal{T}$ to be a subgraph of $\mathcal{T}$, consisting of two terminals with a common Steiner point. It is easy to see that any Steiner topology on at least 3 terminals has at least two cherries. We will later use the fact that for any terminal t there exists a cherry with two terminals not equal to t (to see this, note that in the subtree of $\mathcal{T}$ on the Steiner points, there are at least two leaves, unless $n = 3$).

Lemma 6.2 *Let T be an ε-approximate Steiner tree in $\mathbb{R}^d$, ($0 \leqslant \varepsilon < 2\pi/3$).*

(i) For any cherry with terminals t_1 and t_2 and Steiner point s,

$$|st_1| + |st_2| \leqslant |t_1t_2| / \sin(\pi/3 - \varepsilon/2).$$

(ii) If D is the diameter of the set of terminals, then for any terminal t and Steiner point s,

$$|ts| \leqslant D \cos(\varepsilon/2) / \sin(\pi/3 - \varepsilon/2).$$

Proof For the first statement, we use Lemma 6.1:

$$\frac{|st_1| + |st_2|}{|t_1t_2|} \leqslant \frac{1}{\cos \frac{1}{2}(\pi - \sphericalangle t_1st_2)} = \frac{1}{\sin \frac{1}{2}\sphericalangle t_1st_2} \leqslant \frac{1}{\sin(\pi/3 - \varepsilon/2)}.$$

For the second statement, consider the plane Π through t and the terminals t_1, t_2 of a cherry (if these points are collinear, choose any plane through them). Let o be the midpoint of t_1t_2. Let C_i be the circle with centre t_i and radius D. Denote the half plane bounded by t_1t_2 and containing t by H. Let p be the point where C_1 and C_2 intersect in H. Without any loss of generality, t is inside the angle $\sphericalangle pot_2$.

First, suppose that $\varepsilon \leqslant \pi/6$ (Fig. 2a). Let c be the point on the line op in the half plane H such that $\sphericalangle ct_1t_2 = \pi/6 - \varepsilon$. Let t' be the point where the ray from c through t intersects C_1. Then, $|ct| \leqslant |ct'| \leqslant |cp|$ (Euclid III.7 [26]). Let C be the circle with centre c that passes through t_1 and t_2, and let it intersect the line op in the half plane opposite H in q. Then, for any point $x \in C \cap H$, $\sphericalangle t_1xt_2 = \pi/3 + \varepsilon$ and for any $x \in C \backslash H$, $\sphericalangle t_1xt_2 = 2\pi/3 - \varepsilon$. Since $\pi/3 + \varepsilon < 2\pi/3 - \varepsilon \leqslant \sphericalangle t_1st_2$, s is in the ball B with centre c that passes through t_1 and t_2. In particular, $|cs| \leqslant |cq|$. We conclude that

$$|ts| \leqslant |tc| + |cs| \leqslant |pc| + |cq| = |pq| = D\frac{\sin \sphericalangle pt_1q}{\sin \sphericalangle t_1qp}. \tag{2}$$

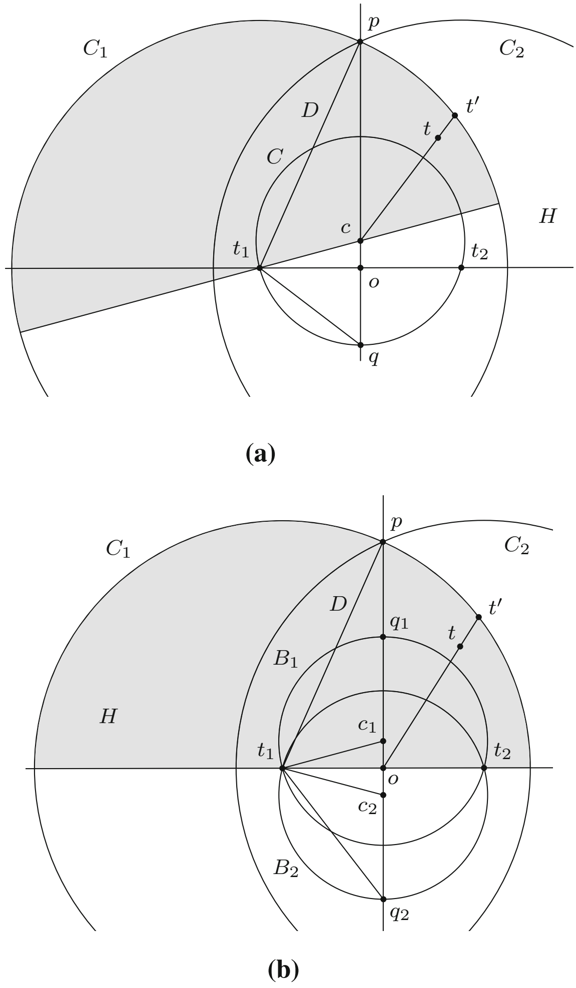
We bound $\sphericalangle pt_1q$ from below as follows. Since $|t_1t_2| \leqslant D = |t_1p|$, $\sphericalangle pt_1t_2 \geqslant \pi/3$. Furthermore, $\sphericalangle qt_1t_2 = \pi/6 + \varepsilon/2$. Therefore, $\sphericalangle pt_1q \geqslant (\pi + \varepsilon)/2$. We substitute this estimate, together with $\sphericalangle t_1qp = \pi/3 - \varepsilon/2$ into (2), to obtain

$$|ts| \leqslant D\frac{\sin(\pi/2 + \varepsilon/2)}{\sin(\pi/3 - \varepsilon/2)} = D\frac{\cos(\varepsilon/2)}{\sin(\pi/3 - \varepsilon/2)}.$$

The case where $\varepsilon > \pi/6$ is similar (Fig. 2b). Let c_1 and c_2 be points on the line op such that $\sphericalangle c_it_1o = \varepsilon - \pi/6$, $i = 1, 2$. Similar to the previous case, $|ot| \leqslant |op|$.

Let B_i be the ball with centre c_i and radius $|c_1t_1| = |c_2t_1|$, $i = 1, 2$. Let q_1 be the point where the line oc_1 intersects B_1 in the half plane H, and q_2 be the point where oc_1 intersects B_2 in the half plane opposite H. Since $B_1 \cup B_2$ is the set of all points x such that $\sphericalangle t_1xt_2 \geqslant 2\pi/3 - \varepsilon$, $s \in B_1 \cup B_2$. If $s \in B_1$, then Euclid III.7 gives that $|os| \leqslant |oq_1| = |oq_2|$. It follows that $|st| = |so| + |ot| \leqslant |q_2o| + |op| = |pq_2|$. Similar to the previous case, we obtain

Fig. 2 Proof of Lemma 6.2



$$|pq_2| = D\frac{\sin \sphericalangle pt_1q_2}{\sin \sphericalangle t_1q_2p} \leqslant D\frac{\sin(\pi/2+\varepsilon/2)}{\sin(\pi/3-\varepsilon/2)} = D\frac{\cos(\varepsilon/2)}{\sin(\pi/3-\varepsilon/2)}.$$

□

Theorem 6.1 *For any* $\varepsilon \in (0, 2\pi/3)$ *and* $d \geqslant 2$*,*

$$F_d(\varepsilon, n) = O\left((\cos(\varepsilon/2)/\sin(\pi/3-\varepsilon/2))^n\right).$$

Proof Let $A = \cos(\varepsilon/2)/\sin(\pi/3-\varepsilon/2)$ and $B = 1/\sin(\pi/3-\varepsilon/2)$. We show by induction on $n \geqslant 2$ that

$$L(T) \leqslant \left(A^{n-2} + \frac{(A^{n-2}-1)B}{A-1}\right) D. \tag{3}$$

If $n = 2$, then $L(T) = D$, which equals the right-hand side. Next let $n > 2$ and assume that (3) holds for ε-approximate Steiner trees on $n-1$ terminals. Consider a cherry of T with Steiner point s and terminals t_1 and t_2. By Lemma 6.2, the distance between s and any terminal of T is at most AD, and $|st_1| + |st_2| \leqslant B|t_1t_2| \leqslant BD$. Remove t_1 and t_2 and the edges st_1 and st_2 from T and change s into a terminal to obtain an ε-approximate Steiner tree T' on $n-1$ terminals. The diameter of this set of terminals is $D' \leqslant AD$. By the induction hypothesis, $L(T') \leqslant \left(A^{n-3} + \frac{(A^{n-3}-1)B}{A-1}\right) D'$. Therefore,

$$\begin{aligned} L(T) &= L(T') + |st_1| + |st_2| \\ &\leqslant \left(A^{n-3} + \frac{(A^{n-3}-1)B}{A-1}\right) AD + BD \\ &= \left(A^{n-2} + \frac{(A^{n-2}-1)B}{A-1}\right) D. \end{aligned}$$

Finally, the length of a Steiner minimal tree joining the terminals of T is at least D, and it follows that

$$\frac{L(T)}{L(S(T))} - 1 \leqslant A^{n-2} + \frac{(A^{n-2}-1)B}{A-1} - 1 = O(A^n).$$

□

The following is a sharper version of Lemma 2.2 in [22]. The proof is along the lines of the proof of Lemma 2.2 in [22], but is done in the plane.

Theorem 6.2 *For each $\varepsilon \in]\pi/3, 2\pi/3[$,*

$$F_2(\varepsilon, n) = \Omega\left(n^{\log_2 C_2(\varepsilon)}\right),$$

where $C_2(\varepsilon) := \left(2\sin\left(\frac{\pi}{3} - \frac{\varepsilon}{2}\right)\right)^{-1}$. Furthermore, $F_2(\pi/3, n) = \Omega(\log n)$.

By making ε large enough, the lower bound in Theorem 6.2 grows faster than any polynomial. In particular, if $\varepsilon > 105.6\ldots^\circ$, then the lower bound is super-quadratic (compare with Theorem 2.1 in [22]). Theorem 6.2 follows from the following lemma (combined with Proposition 5.1).

Lemma 6.3 *Let $k \geqslant 1$ and $\pi/3 < \varepsilon < 2\pi/3$. Then, $F_2(\varepsilon, 2^{k+1}) > \sqrt{3}\frac{C^k-1}{C-1} - 1$, where $C = \left(2\sin\left(\frac{\pi}{3} - \frac{\varepsilon}{2}\right)\right)^{-1}$. Furthermore, for $k \geqslant 1$, $F_2(\pi/3, 2^{k+1}) \geqslant \sqrt{3}k - 1$.*

Proof Let $\pi/3 \leqslant \varepsilon < 2\pi/3$ and $k \geqslant 1$. We construct an ε-approximate Steiner tree with 2^{k+1} terminals. Let $r = \sin\left(\frac{\pi}{3} - \frac{\varepsilon}{2}\right)$. Let $C_0, C_1, \ldots, C_k$ be concentric circles with common centre o and with C_i of radius r^i.

First, we construct "half" the tree with 2^k terminals on C_k and Steiner points on the other circles. Fix any $p_1 \in C_0$. There are two tangent lines from p_1 to C_1. Denote the points where they touch C_1 by p_2 and p_3, chosen such that $\sphericalangle p_2p_1p_3$ is positively oriented. See Fig. 3. Note that $\sphericalangle p_2p_1p_3 = 2\pi/3 - \varepsilon$.

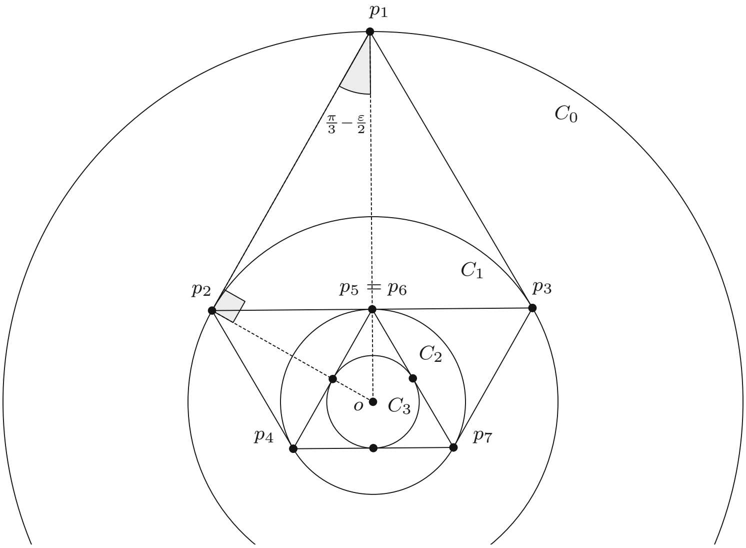


Fig. 3 Constructing a lower bound in the plane

In general, for each $i = 1, \ldots, k$, once $p_{2^{i-1}}, p_{2^{i-1}+1}, \ldots, p_{2^i-1} \in C_{i-1}$ have been determined, for each $p_j \in C_{i-1}$, let p_{2j} and p_{2j+1} be the two points where the tangents from p_j touch C_i, chosen such that $\sphericalangle p_{2j} p_j p_{2j+1}$ is positively oriented. Again, $\sphericalangle p_{2j} p_j p_{2j+1} = 2\pi/3 - \varepsilon$. The points $p_{2^k}, \ldots, p_{2^{k+1}-1} \in C_k$ will be 2^k of the terminals. We join each p_j to p_{2j} and p_{2j+1}, for $j = 1, \ldots, 2^k - 1$.

Next, we "double" the tree, by choosing one of the directions on the tangent line of C_0 at p_1, and moving each p_i in that direction by a distance of δ, where $\delta > 0$ is very small. Denote the moved points by p'_i. We move o in the same direction to obtain o'. The moved points $p'_{2^k}, \ldots, p'_{2^{k+1}-1}$ will give another 2^k terminals. We join p'_j to p'_{2j} and p'_{2j+1}, for $j = 1, \ldots, 2^k - 1$. Finally, we join p_1 and p'_1. All p_j and p'_j with $j < 2^k$ are Steiner points. Each angle at a Steiner point is one of three values $2\pi/3 - \varepsilon$, $5\pi/6 - \varepsilon/2$, and $\pi/6 + \varepsilon/2$. These all belong to the interval $[2\pi/3 - \varepsilon, 2\pi/3 + \varepsilon]$, since $\varepsilon \geqslant \pi/3$. Thus, we obtain a full ε-approximate Steiner tree T on 2^{k+1} terminals, all on the circle C_k of radius r^k. Note that many of the p_j coincide. For instance, it is always the case that $p_5 = p_6$. This is allowed in our definition of an ε-approximate Steiner tree. Alternatively, we could have slightly perturbed the radii of the circles by δ to ensure that all p_j are distinct.

Next, we calculate $L(T)$. An edge from a point of T on C_i to a point on C_{i+1} has length $r^i \cos\left(\frac{\pi}{3} - \frac{\varepsilon}{2}\right)$. Therefore,

$$\begin{aligned} L(T) &= \delta + 2\left(2\cos\left(\tfrac{\pi}{3} - \tfrac{\varepsilon}{2}\right) + 4r\cos\left(\tfrac{\pi}{3} - \tfrac{\varepsilon}{2}\right) + \cdots + 2^k r^{k-1}\cos\left(\tfrac{\pi}{3} - \tfrac{\varepsilon}{2}\right)\right) \\ &= \delta + 4\cos\left(\tfrac{\pi}{3} - \tfrac{\varepsilon}{2}\right)\left(1 + 2r + \cdots + (2r)^{k-1}\right) \\ &= \delta + 4\cos\left(\tfrac{\pi}{3} - \tfrac{\varepsilon}{2}\right)\frac{1 - (2r)^k}{1 - 2r} \end{aligned}$$

if $\varepsilon > \pi/3$, and $L(T) = \delta + 2\sqrt{3}k$ if $\varepsilon = \pi/3$. We form a Steiner tree S with a degeneration of the topology of T by joining each p_i to o, each p_i' to o', and o to o'. Then, $L(S(T)) \leqslant L(S) = \delta + 2(2r)^k$, which equals $\delta + 2$ if $\varepsilon = \pi/3$.

Therefore, if $\varepsilon > \pi/3$, then

$$F_2(\varepsilon, 2^{k+1}) \geqslant \frac{\delta + 4\cos\left(\frac{\pi}{3} - \frac{\varepsilon}{2}\right)\frac{1-(2r)^k}{1-2r}}{\delta + 2(2r)^k} - 1$$

for each $\delta > 0$; hence,

$$\begin{aligned} F_2(\varepsilon, 2^{k+1}) &\geqslant \frac{4\cos\left(\frac{\pi}{3} - \frac{\varepsilon}{2}\right)\left(1-(2r)^k\right)}{2(2r)^k(1-2r)} - 1 \\ &= \frac{2\cos\left(\frac{\pi}{3} - \frac{\varepsilon}{2}\right)}{1-2r}\left(\left(\frac{1}{2r}\right)^k - 1\right) - 1 \\ &= \frac{2\sqrt{1-\frac{1}{4C^2}}}{1-\frac{1}{C}}(C^k - 1) - 1 \\ &= \frac{\sqrt{4C^2-1}}{C-1}(C^k - 1) - 1 > \sqrt{3}\frac{C^k-1}{C-1} - 1, \end{aligned}$$

where $C = 1/(2r) = (2\sin(\frac{\pi}{3} - \frac{\varepsilon}{2}))^{-1}$. Similarly, if $\varepsilon = \pi/3$, then

$$F_2(\varepsilon, 2^{k+1}) \geqslant \frac{\delta + 2\sqrt{3}k}{2+\delta} - 1,$$

and letting $\delta \to 0$, we obtain the required result. □

7 Upper Bounds for Small ε (Proof of Theorem 4.1)

In this section, we prove Theorem 4.1 using an unfolding algorithm described in [18] and [22] based on Melzak's algorithm for finding the shortest Steiner tree for a fixed Steiner topology (if this shortest tree happens to be what we call a locally minimum Steiner tree). This algorithm unfolds an approximate Steiner tree into a broken line segment. First, we describe this unfolding and then use it in the special cases of 3 and 4 terminals in the plane to determine the exact values of $F_2(\varepsilon, 3)$ and $F_2(\varepsilon, 4)$ (Propositions 7.1 and 7.2). Then, the proof of Theorem 4.1 should be clear.

The following inequality and its proof forms the basis for the unfolding algorithm.

Lemma 7.1 *Let $\triangle abc$ be an equilateral triangle in $\mathbb{R}^d$. Then, for any $x \in \mathbb{R}^d$, $|xa| \leqslant |xb| + |xc|$, with equality iff x is on the minor arc $\widehat{bc}$ of the circumcircle of $\triangle abc$.*

Proof The proof is essentially the same as the classical proof that the Fermat point of a triangle with all angles less than $2\pi/3$ minimises the sum of the distances to the vertices. Because there are only 4 points to consider, we may assume without any loss of generality that $x, a, b, c \in \mathbb{R}^3$.

Rotate $\triangle bxc$ by an angle of $\pi/3$ around the axis through b perpendicular to the plane Π through a, b and c such that c is rotated to a. Then, b stays fixed, and x is rotated to x', say. Also, $|xc| = |x'a|$. Let $p\colon \mathbb{R}^3 \to \Pi$ be the orthogonal projection onto Π. Then, $\triangle bp(x)p(x')$ is equilateral. Since xx' is parallel to Π, $|xx'| = |p(x)p(x')| = |bp(x)| \leqslant |bx|$. Therefore, $|xa| \leqslant |xx'| + |x'a| \leqslant |bx| + |xc|$. Equality holds iff x is in the plane Π and a, x', x are collinear, which holds iff $\sphericalangle bx'a = 2\pi/3$, iff $\sphericalangle bxc = 2\pi/3$, iff x is on the minor arc $\widehat{bc}$ of the circumcircle of $\triangle abc$. □

Consider a family of n terminals N_n in $\mathbb{R}^d$ and a full Steiner topology $\mathcal{T}_n$ for those terminals. Choose one of the terminals t_0 as root of $\mathcal{T}_n$. We define a *Melzak sequence* of N_n and $\mathcal{T}_n$ to be two sequences $N_n, N_{n-1}, \ldots, N_2$ and $\mathcal{T}_n, \mathcal{T}_{n-1}, \ldots, \mathcal{T}_2$, where each $\mathcal{T}_i$ is a full Steiner topology on N_i and with root t_0 (thus, $t_0 \in N_i$ for all i). We obtain N_{i-1} and $\mathcal{T}_{i-1}$ from N_i and $\mathcal{T}_i$ as follows. Choose any cherry of $\mathcal{T}_i$ with two terminals $t_1, t_2 \neq t_0$ and Steiner point s with neighbours, say, t_1, t_2 and p. Replace t_1 and t_2 in N_i by any point $t \in \mathbb{R}^d$ such that $\triangle t_1t_2t$ is an equilateral triangle, thus obtaining N_{i-1}. Remove s and its incident edges from $\mathcal{T}_i$ and replace them by the edge pt, to obtain $\mathcal{T}_{i-1}$. If $N_2 = \{t_0, t\}$, say, then we call the line segment t_0t an *unfolding* of N_n with respect to the topology $\mathcal{T}_n$.

It is not difficult to see that, if there is more than one cherry to choose from at a certain stage, it does not matter which we choose first. We may in fact process both cherries in parallel (this is equivalent to saying that in the subtree of $\mathcal{T}_n$ on the Steiner points, it does not matter in which order we remove leaves, and that this may be done in parallel).

Lemma 7.1 and induction immediately give the following, which is Theorem 3.1 of [22] and Theorem 4.2 of [18]:

Lemma 7.2 *The length of any unfolding of a terminal set $N_n \subset \mathbb{R}^d$ with respect to a full Steiner topology $\mathcal{T}_n$ is a lower bound for the shortest tree on N_n, which has $\mathcal{T}_n$ as topology* (*allowing degenerate topologies*).

Next, we describe the plan of the proof of Theorem 4.1. First, we unfold a planar ε-approximate Steiner tree into a polygonal path of the same length, and estimate the *turn* at each internal vertex of the path. By Lemma 7.2, the length between the endpoints of the unfolding is a lower bound on the length of a Steiner minimal tree on the same terminal set. By a result of E. Schmidt [27] (Lemma 7.3 below), this length is minimised among all polygonal paths with the same angles and edges of the same length, by a planar, convex path. Finally, we minimise the length of the endpoint among all polygonal paths of the same total length and the same sum of turns.

Before providing the detail of the general case, we show how to determine exact values for small n.

Proposition 7.1 *For all $\varepsilon \in]0, \pi/3[$, $F_2(\varepsilon, 3) = G_2(\varepsilon, 3) = \frac{1}{\cos \varepsilon/2} - 1$.*

Proof We show that $F_2(\varepsilon, 3) \leqslant (\cos \varepsilon/2)^{-1} - 1$. Consider an ε-approximate Steiner tree T on three terminals t_0, t_1, t_2 in the plane, with Steiner point s and edges $e_i =$

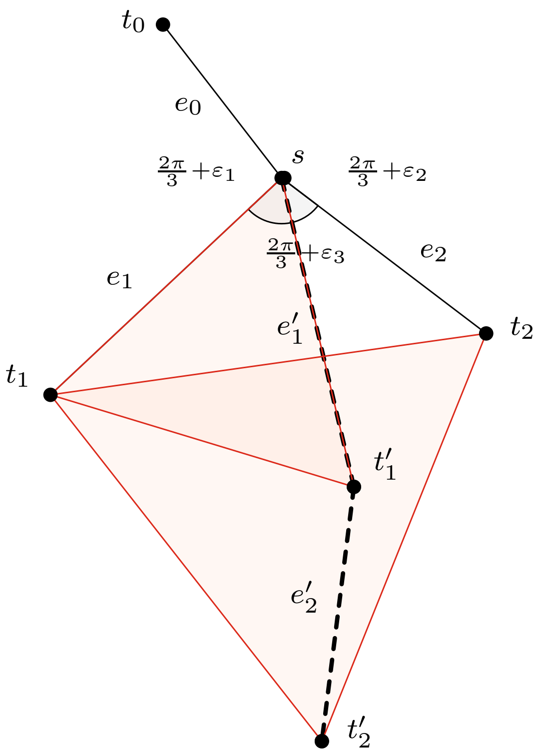


Fig. 4 Unfolding an ε-approximate Steiner tree on 3 terminals

$st_i, i = 0, 1, 2$, numbered in such a way that e_0, e_1, e_2 are in anticlockwise order around s. See Fig. 4. Let $\sphericalangle t_0st_1 = 2\pi/3 + \varepsilon_1$, $\sphericalangle t_0st_2 = 2\pi/3 + \varepsilon_2$ and $\sphericalangle t_1st_2 = 2\pi/3 + \varepsilon_3$, where $|\varepsilon_i| \leqslant \varepsilon, i = 1, 2, 3$. Since $\varepsilon \leqslant \pi/3$, the three angles sum to 2π, and $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 0$.

We unfold the tree into a polygonal line of total length $L(T)$ as follows. We rotate $e_1 = st_1$ by an angle of $\pi/3$ around s to obtain the edge $e'_1 = st'_1$, say. We rotate $e_2 = st_2$ by an angle of $-\pi/3$ around t_1 to obtain the edge $e'_2 = t'_1t'_2$. Then, $t_0st'_1t'_2$ is a polygonal line of length $L(T)$ (see Fig. 4). The turn from edge e_0 to e'_1 equals ε_1, and the turn from e'_1 to e'_2 equals ε_3. Since $|\varepsilon_1 + \varepsilon_3| = |\varepsilon_2| \leqslant \varepsilon < \pi$, the rays $\overrightarrow{t_0s}$ and $\overrightarrow{t'_2t'_1}$ intersect in p, say. Then, $L(T) = |t_0s| + |st'_1| + |t'_1t'_2| \leqslant |t_0p| + |pt'_2|$. By Lemma 7.2, $L(S(T)) \geqslant |t_0t'_2|$. It follows that

$$\frac{L(T)}{L(S(T))} \leqslant \frac{|t_0p| + |pt'_2|}{|t_0t'_2|} \leqslant \frac{1}{\cos\varepsilon/2} \quad \text{by Lemma 6.1,}$$

and

$$F_2(\varepsilon, 3) = \sup \frac{L(T)}{L(S(T))} - 1 \leqslant \frac{1}{\cos\varepsilon/2} - 1.$$

To show that $G_2(\varepsilon, 3) \geqslant (\cos\varepsilon/2)^{-1} - 1$, consider an ε-approximate tree T as above with $\varepsilon_1 = \varepsilon_2 = -\varepsilon/2, \varepsilon_3 = \varepsilon, |t_0s| = \delta$ for arbitrarily small $\delta > 0$, and

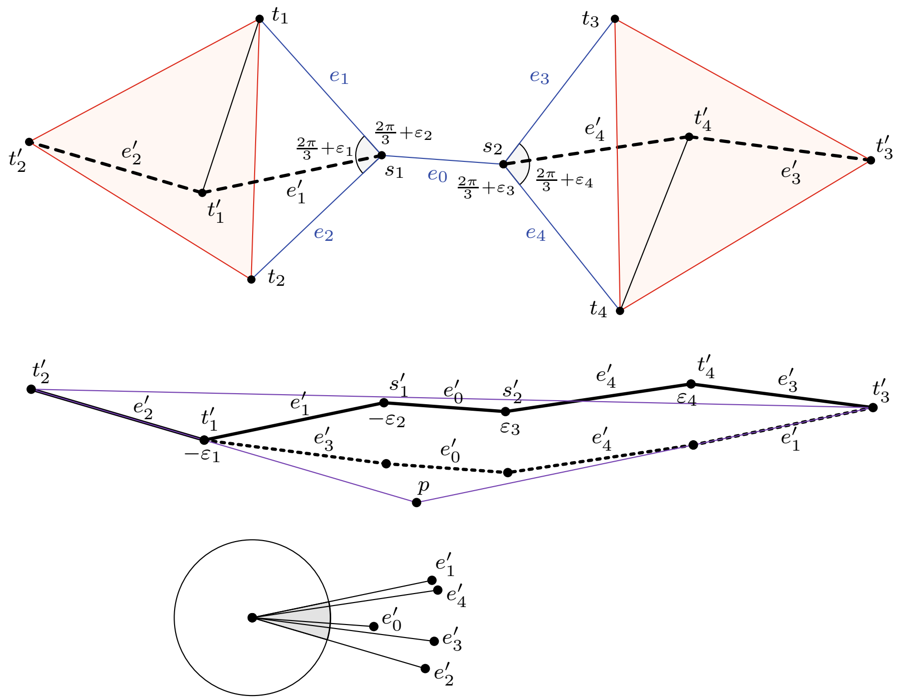


Fig. 5 Unfolding an ε-approximate Steiner tree on 4 terminals

$|t_1 s| = |t_2 s| = 1$. Then, $L(T) = 2 + \delta$ and $L(S(T)) = \delta + 2\cos(\varepsilon/2)$. Since all angles in $\triangle t_0 t_1 t_2$ are less than $2\pi/3$ if δ is small enough, $S(T)$ is not degenerate, hence $G_2(\varepsilon, 3) \geqslant \frac{2+\delta}{\delta+2\cos\varepsilon/2} - 1$ for all $\delta > 0$. It follows that $G_2(\varepsilon, 3) \geqslant (\cos\varepsilon/2)^{-1} - 1$. □

Proposition 7.2 *For all* $\varepsilon \in]0, \pi/3[$, $F_2(\varepsilon, 4) = G_2(\varepsilon, 4) = \frac{1}{\cos\varepsilon} - 1$.

Proof Consider an ε-approximate Steiner tree on four terminals t_1, t_2, t_3, t_4, Steiner points s_1 and s_2, and edges $e_1 = s_1t_1, e_2 = s_1t_2, e_0 = s_1s_2, e_3 = s_2t_3, e_4 = s_2t_4$, labelled in such a way that e_0, e_1, e_2 are in anticlockwise order around s_1, and e_0, e_4, e_3 are in anticlockwise order around s_2. Furthermore, let $\sphericalangle t_1s_1t_2 = 2\pi/3+\varepsilon_1$, $\sphericalangle t_1s_1s_2 = 2\pi/3 + \varepsilon_2$, $\sphericalangle s_1s_2t_4 = 2\pi/3 + \varepsilon_3$ and $\sphericalangle t_3s_2t_4 = 2\pi/3 + \varepsilon_4$, where $|\varepsilon_i| \leqslant \varepsilon, i = 1, 2, 3, 4$, and $|\varepsilon_1 + \varepsilon_2|, |\varepsilon_3 + \varepsilon_4| \leqslant \varepsilon$. See Fig. 5. As in the proof of Proposition 7.1, we unfold the tree into a polygonal line of total length $L(T)$, and with the distance between the endpoints a lower bound to $L(S(T))$. Rotate e_1 by $\pi/3$ around s_1 to obtain $e_1' = s_1t_1'$. Rotate e_2 by $-\pi/3$ around t_1 to obtain $e_2' = t_1't_2'$. Rotate e_3 by $-\pi/3$ around t_4 to obtain $e_3' = t_4't_3'$. Rotate e_4 by $\pi/3$ around s_2 to obtain $e_4' = s_2t_4'$. This gives a polygonal line $P = t_2't_1's_1s_2t_4't_3'$ of length $L(T)$, with turns $-\varepsilon_1$ at t_1', $-\varepsilon_2$ at s_1, ε_3 at s_2, and ε_4 at t_4'. Note that the turn between any two of the five edges of P will be at most 2ε in absolute value. For instance, the absolute turn between e_1' and e_3' equals $|-\varepsilon_2 + \varepsilon_3 + \varepsilon_4| \leqslant |\varepsilon_2| + |\varepsilon_3 + \varepsilon_4| \leqslant 2\varepsilon$. If we reorder the edges of P to make a new, convex polygonal line P' with the same endpoints as P (Fig. 5, middle), then P' will lie inside the triangle $\triangle t_2' t_3' p$ bounded by $t_2' t_3'$ and the lines through the first and last edges of P'. The turn from the first edge to the last edge of P' is exactly the maximum turn between two edges of P, so is at most 2ε. Hence, the angle at the apex of this triangle will be at least $\pi - 2\varepsilon$, and by Lemma 6.1, $L(T)/|t_2' t_3'| \leqslant 1/\cos\varepsilon$. The proof of the upper bound concludes in the same way as that of Proposition 7.1.

To show that $(\cos\varepsilon)^{-1} - 1 \geqslant G_2(\varepsilon, 4)$, fix the above ε-approximate Steiner tree to have $\varepsilon_1 = 0$, $\varepsilon_2 = \varepsilon$, $\varepsilon_3 = -\varepsilon$, $\varepsilon_4 = 0$, $|s_1 s_2| = \delta$ and

$$|s_1 t_1| = |s_1 t_2| = |s_2 t_3| = |s_2 t_4| = 1.$$

It is not difficult to see that the Melzak algorithm obtains a locally minimum Steiner tree $S(T)$ for any $\varepsilon < \pi/3$. □

The following generalises the idea in the above proof of estimating the length of a polygonal path in terms of the distance between its endpoints. We do not know the history of this elementary result, but an extension of this lemma to curves of finite total curvature was proved by Schmidt [27] (see also [28, Theorem 5.8.1] and [29, Proposition 7.1]).

Lemma 7.3 *Consider a polygonal path $p_0 p_1 \dots p_n$ in the plane. For each $i = 1, \dots, n-1$, define the* turn *ε_i at p_i to be the signed angular measure in $[-\pi, \pi]$ by which the ray with source at p_i in the direction opposite to $\overrightarrow{p_i p_{i-1}}$ has to turn to coincide with the ray $\overrightarrow{p_i p_{i+1}}$. Let*

$$\kappa = \max_{1 \leqslant i \leqslant j \leqslant n-1} \left| \sum_{t=i}^{j} \varepsilon_t \right|.$$

If $\kappa < \pi$, then

$$\frac{\sum_{i=0}^{n-1} |p_i p_{i+1}|}{|p_0 p_n|} \leqslant \frac{1}{\cos(\kappa/2)}.$$

Proof The case $n = 2$ is just Lemma 6.1, so assume that $n \geqslant 3$. Since $\kappa < \pi$, the n unit vectors

$$u_i = \|p_{i+1} - p_i\|^{-1} (p_{i+1} - p_i)$$

all lie in an open half circle. The polygonal path $p_0 p_1 \dots p_n$ can be replaced with a convex polygonal path $p_o' p_1' \dots p_n'$ such that $p_0 = p_o'$, $p_n = p_n'$ and each segment of the new path is a translation of a segment of the original path, selected so that the turns all have the same sign. Then, $p_0' p_1' \dots p_n'$ is a convex polygonal path with the same κ and the same endpoints as the original polygonal path. Let the lines $p_0' p_1'$ and $p_{n-1}' p_n'$ intersect in q. Since $\kappa < \pi$, $p_0' p_1' \dots p_n'$ is contained in $\triangle p_o' q p_n'$. By a well-known elementary geometric inequality, $\sum_{i=1}^{n-1} |p_i' p_{i+1}'| \leqslant |p_0' q| + |q p_n'|$. It remains to apply the case $n = 2$ of the lemma to the path $p_0' q p_n'$. □

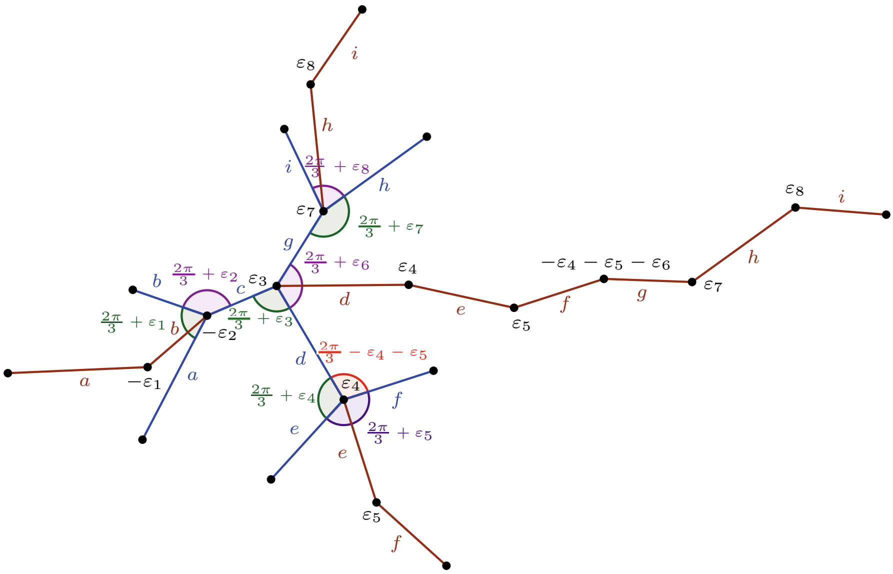


Fig. 6 Proof of Theorem 4.1

Proof of Theorem 4.1 We choose a root edge of an ε-approximate Steiner tree T on n terminals and unfold the two parts of T separated by the root edge to obtain a polygonal path P with $2n-3$ edges, of the same length as T. See Fig. 6, where the blue ε-approximate tree has been unfolded. The turn at each internal vertex of the polygonal path P is indicated. The quantity κ of Lemma 7.3 is the maximum absolute turn between any two edges of P. For example, the total turn between edge a and edge h on P in Fig. 6 equals $-\varepsilon_1-\varepsilon_2+\varepsilon_3+\varepsilon_4+\varepsilon_5-\varepsilon_4-\varepsilon_5-\varepsilon_6=(-\varepsilon_1-\varepsilon_2)-\varepsilon_6$, which is the sum of the errors at the two Steiner points on the path between edges a and h in the tree. Thus, the absolute turn between a and h in P is at most 2ε. In general, since there are at most $n-2$ Steiner points in a full Steiner topology on n terminals, there are at most $n-2$ Steiner points on the path between any two edges in an ε-approximate Steiner tree, each contributing an error of absolute value at most ε. It follows that $\kappa \leqslant (n-2)\varepsilon$. We now apply Lemma 7.3 to obtain that $L(T)/L(S(T)) \leqslant 1/\cos(\frac{1}{2}(n-2)\varepsilon)$. □

8 Construction of an ε-Approximate Full Binary Tree in the Plane

In this section we prove Theorems 4.2 and 4.3 by constructing a sequence of ε-approximate Steiner trees T_k ($k \in \mathbb{N}$), for which it is possible to calculate the ratio between their length and the length of a locally minimum Steiner tree on the same terminals, if $\varepsilon \leqslant 1/k^2$. A somewhat similar construction is made in [30]. The calculation will make essential use of complex numbers. Using complex numbers to solve problems in classical Euclidean geometry is an old trick [31–33], and even in the geometric Steiner tree literature there are papers where complex numbers appear [34,35].

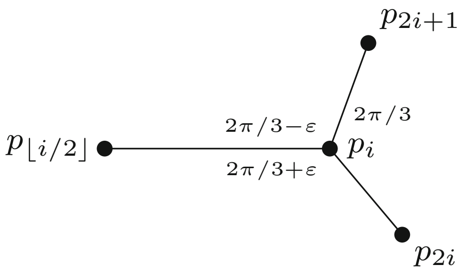


Fig. 7 Angles around a Steiner point in the binary tree construction

Proof of Theorems 4.2 and 4.3 Throughout the proof we denote the largest integer, not greater than x, by $\lfloor x\rfloor$. Fix $k \in \mathbb{N}$. We describe an ε-approximate Steiner tree T_k with 2^k+1 terminals p_i (for $i=0$ and $2^k \leqslant i \leqslant 2^{k+1}-1$), 2^k-1 Steiner points p_i ($1 \leqslant i \leqslant 2^k-1$) and $2^{k+1}-1$ edges $e_i = p_i p_{\lfloor i/2\rfloor}$ ($1 \leqslant i \leqslant 2^{k+1}-1$). Let each e_i have length $2^{-\lfloor \log_2 i\rfloor}$, and let the angles at the edges incident to the Steiner point p_i be

$$\sphericalangle p_{2i} p_i p_{2i+1} = 2\pi/3,\ \sphericalangle p_{2i+1} p_i p_{\lfloor i/2\rfloor} = 2\pi/3-\varepsilon,\ \text{and}\ \sphericalangle p_{\lfloor i/2\rfloor} p_i p_{2i} = 2\pi/3+\varepsilon$$

(Fig. 7). This determines the tree uniquely up to congruence. See Fig. 8 for the case $k=3$. Since there are 2^j edges of length 2^{-j+1} ($j=0,1,\ldots,k$),

$$L(T_k) = k+1. \tag{4}$$

We construct this tree recursively, using complex numbers. Let $p_0 = 0 \in \mathbb{C}$ and $p_1 = 1 \in \mathbb{C}$. Then, $e_1 = p_0 p_1$. Let $\omega = e^{i\pi/3}$ and $z = e^{i\varepsilon}$. □

Once $p_{\lfloor i/2\rfloor}$ and p_i have been defined, define p_{2i} and p_{2i+1} as in Fig. 7. If we walk from $p_{\lfloor i/2\rfloor}$ to p_i and then turn in the direction of p_{2i}, the turn is a right turn by an angle of $\pi/3-\varepsilon$. Furthermore, $|p_i p_{2i}| = \frac{1}{2}|p_{\lfloor i/2\rfloor} p_i|$. Therefore,

$$p_{2i} - p_i = \frac{1}{2}(p_i - p_{\lfloor i/2\rfloor})\omega^{-1} z. \tag{5}$$

Similarly, if we turn instead in the direction of p_{2i+1}, this is a left turn by an angle of $\pi/3+\varepsilon$, which gives

$$p_{2i+1} - p_i = \frac{1}{2}(p_i - p_{\lfloor i/2\rfloor})\omega z. \tag{6}$$

We obtain the following recurrence:

$$\left.\begin{aligned} p_0 &= 0, \quad p_1 = 1, \\ p_{2i} &= p_i + \frac{1}{2}(p_i - p_{\lfloor i/2\rfloor})\omega^{-1} z, \quad i \geqslant 1 \\ p_{2i+1} &= p_i + \frac{1}{2}(p_i - p_{\lfloor i/2\rfloor})\omega z, \quad i \geqslant 1. \end{aligned}\right\} \tag{7}$$

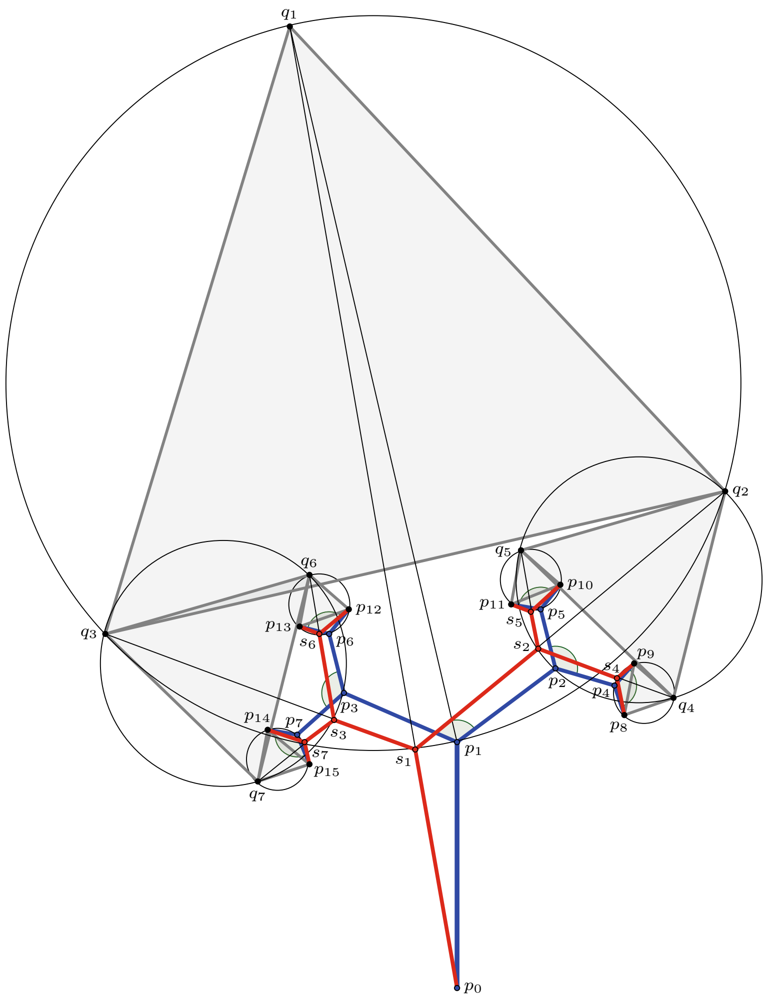


Fig. 8 Construction of T_k, $k = 3$

To describe its solution, we have to consider the sequence of left and right turns as we walk from p_0 to p_i. This can be found from the binary expression of i. Let $h(i) = \lfloor \log_2 i \rfloor$. Let $b_0, b_1, \ldots, b_{h(i)} \in \{0, 1\}$ be the unique values such that

$$i = \sum_{j=0}^{h(i)-1} b_j 2^j + 2^{h(i)}.$$

If we replace 0 by R and 1 by L in the sequence $b_{h(i)-1}, \ldots, b_0$, we obtain the left and right turns in the path from p_0 to p_i. Let $a_j(i)$ be the number of 1s in $b_{h(i)-1}, \ldots, b_{h(i)-j}$ minus the number of 0s in $b_{h(i)-1}, \ldots, b_{h(i)-j}$. In particular, $a_0(i) = 0$.

Lemma 8.1 *For each $i \geqslant 1$,*

$$p_i = \sum_{j=0}^{h(i)} \omega^{a_j(i)} \left(\frac{z}{2}\right)^j . \tag{8}$$

Proof Observe that $h(2i) = h(2i+1) = h(i) + 1$,

$$a_j(2i) = a_j(2i+1) = a_j(i) \text{ for each } j = 0, \ldots, h(i), \text{ and}$$
$$a_{h(i)}(i) = a_{h(2i)}(2i) + 1 = a_{h(2i)-1}(2i) = a_{h(2i+1)}(2i+1) - 1 = a_{h(2i+1)-1}(2i+1). \tag{9}$$

It then follows by induction, using (5) and (6), that

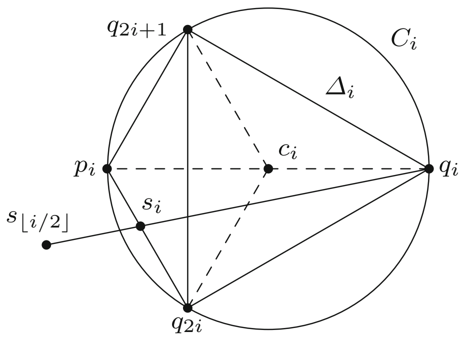
$$p_i - p_{\lfloor i/2 \rfloor} = \omega^{a_{h(i)}(i)} \left(\frac{z}{2}\right)^{h(i)} . \tag{10}$$

Finally, by induction and (7) we obtain (8). □

We remark that each p_i is a polynomial in z of degree $h(i)$ with coefficients in the ring $\mathbb{Z}[1/2, \omega]$. Next, we apply Melzak's Algorithm to the terminals of T_k to obtain the locally minimum Steiner tree $S(T_k)$ with the same topology. Surprisingly, it turns out that the Steiner points of $S(T_k)$ are also polynomials in z with coefficients in $\mathbb{Z}[1/2, \omega]$.

The first step in Melzak's algorithm is to calculate the so-called quasi-terminals q_i ($1 \leqslant i \leqslant 2^{k+1} - 1$) [18]. For each $i = 2^k, \ldots, 2^{k+1} - 1$, let $q_i = p_i$. Then, for each $i = 2^k - 1, \ldots, 1$, once q_{2i} and q_{2i+1} have been defined, let q_i be the unique point such that the triangle $\Delta_i = \triangle q_i q_{2i} q_{2i+1}$ is equilateral, and such that p_i and q_i are on opposite sides of the line $q_{2i}q_{2i+1}$. Let C_i be the circumcircle of Δ_i and c_i its centre (Fig. 9). Since $\sphericalangle p_{2i} p_i p_{2i+1} = 2\pi/3$, $\sphericalangle p_{\lfloor i/2 \rfloor} p_i q_i = \pi - \varepsilon$ and $|p_i p_{2i}| = |p_i p_{2i+1}|$, we obtain by induction that for $i = 1, \ldots, 2^k - 1$, $\sphericalangle q_{2i} p_i q_{2i+1} = 2\pi/3$, $\sphericalangle q_i p_i q_{2i} = \sphericalangle q_i p_i q_{2i+1} = \pi/3$. Hence, p_i is on C_i and the centre c_i of C_i is the midpoint of p_i and q_i. Furthermore, $|p_i q_i| = 2|p_i q_{2i}| = 2|p_i q_{2i+1}|$. Since $c_i q_{2i} p_i q_{2i+1}$ is a parallelogram, we have

Fig. 9 Melzak's algorithm



$$c_i = p_i + (q_{2i} - p_i) + (q_{2i+1} - p_i)$$

and

$$\begin{aligned} q_i &= p_i + 2(c_i - p_i) \\ &= p_i + 2(q_{2i} - p_i) + 2(q_{2i+1} - p_i). \end{aligned} \tag{11}$$

If $2^{k-1} \leqslant i < 2^k$, then we have $q_{2i} = p_{2i}$ and $q_{2i+1} = p_{2i+1}$, hence

$$\begin{aligned} q_i &= p_i + 2(p_{2i} - p_i) + 2(p_{2i+1} - p_i) \\ &= p_i + (p_i - p_{\lfloor i/2 \rfloor})\omega^{-1} z + (p_i - p_{\lfloor i/2 \rfloor})\omega z \quad \text{by (5) and (6)} \\ &= p_i + (p_i - p_{\lfloor i/2 \rfloor}) z \\ &= p_i + \omega^{a_{k-1}(i)} \left(\frac{z}{2}\right)^{k-1} z \quad \text{by (10).} \end{aligned}$$

By induction, we obtain that for each $i < 2^{k-1}$ (use (11), (10), (9); see "Appendix")

$$q_i = p_i + \omega^{a_{h(i)}(i)} \left(\frac{z}{2}\right)^{h(i)} \sum_{j=1}^{k-h(i)} z^j \quad (i \geqslant 1). \tag{12}$$

Therefore, each q_i is a polynomial in z of degree k. In particular,

$$q_1 = \sum_{j=0}^{k} z^j. \tag{13}$$

Furthermore, the centres

$$c_i = \frac{1}{2}(p_i + q_i) = p_i + \frac{1}{2}\omega^{a_{h(i)}(i)} \left(\frac{z}{2}\right)^{h(i)} \sum_{j=1}^{k-h(i)} z^j \tag{14}$$

are polynomials in z of degree k. In particular,

$$c_1 = 1 + \frac{1}{2}\sum_{j=1}^{k} z^j. \tag{15}$$

Finally, we construct the Steiner points s_i, $1 \leqslant i \leqslant 2^k - 1$. Formally, we let $s_0 = p_0 = 0$. Once $s_{\lfloor i/2 \rfloor}$ has been constructed, s_i is the point where the minor arc $\widehat{q_{2i}q_{2i+1}}$ of C_i intersects the segment $s_{\lfloor i/2 \rfloor}q_i$. See Fig. 9. This gives the shortest Steiner tree for this tree topology as long as $\widehat{q_{2i}q_{2i+1}}$ intersects $s_{\lfloor i/2 \rfloor}q_i$. This happens iff $\sphericalangle s_{\lfloor i/2 \rfloor} q_i p_i \leqslant \pi/6$ and $s_{\lfloor i/2 \rfloor}$ is outside C_i. For $i \geqslant 1$, we calculate s_i by solving $|s_i - c_i| = |q_i - c_i|$, where

$$s_i = q_i - \lambda(q_i - s_{\lfloor i/2 \rfloor}), \quad 0 < \lambda < 1. \tag{16}$$

If we square $\left| q_i - \lambda(q_i - s_{\lfloor i/2 \rfloor}) - c_i \right| = |q_i - c_i|$ and use conjugates, we can solve for λ:

$$\lambda = \frac{q_i - c_i}{q_i - s_{\lfloor i/2 \rfloor}} + \frac{\overline{q_i} - \overline{c_i}}{\overline{q_i} - \overline{s_{\lfloor i/2 \rfloor}}},$$

and substitute into (16) to determine s_i:

$$\begin{aligned} s_i &= q_i - \left(\frac{q_i - c_i}{q_i - s_{\lfloor i/2 \rfloor}} + \frac{\overline{q_i} - \overline{c_i}}{\overline{q_i} - \overline{s_{\lfloor i/2 \rfloor}}} \right) (q_i - s_{\lfloor i/2 \rfloor}) \\ &= c_i - \frac{(\overline{q_i} - \overline{c_i})(q_i - s_{\lfloor i/2 \rfloor})}{\overline{q_i} - \overline{s_{\lfloor i/2 \rfloor}}}. \end{aligned} \tag{17}$$

In particular, using (13) and (15), $s_1 = \frac{1}{2} + \frac{1}{2} z^k$. It follows by induction (use (17), (14), (12); see "Appendix") that

$$s_i = p_i + \frac{\omega^{a_{h(i)}(i)}}{2^{h(i)+1}} \left(\sum_{j=0}^{k-h(i)-1} z^j \right) (z^{h(i)+1} - 1) \qquad (i = 1, \ldots, 2^k - 1). \tag{18}$$

Next, we calculate the edge lengths of the Steiner tree.

$$\begin{aligned} s_{2i} - s_i = p_{2i} &+ \frac{\omega^{a_{h(2i)}(2i)}}{2^{h(2i)+1}} \left(\sum_{j=0}^{k-h(2i)-1} z^j \right) (z^{h(2i)+1} - 1) \\ &- p_i - \frac{\omega^{a_{h(i)}(i)}}{2^{h(i)+1}} \left(\sum_{j=0}^{k-h(i)-1} z^j \right) (z^{h(i)+1} - 1) \quad \text{by (18)} \\ &= \frac{\omega^{a_{h(2i)}(2i)}}{2^{h(2i)}} \left[z^{h(2i)} + \frac{1}{2} \left(\sum_{j=0}^{k-h(2i)-1} z^j \right) (z^{h(2i)+1} - 1) \right. \\ &\left. - \omega \left(\sum_{j=0}^{k-h(2i)} z^j \right) (z^{h(2i)} - 1) \right] \quad \text{by (9) and (10).} \end{aligned}$$

Similarly,

$$s_{2i+1} - s_i = \frac{\omega^{a_{h(2i+1)}(2i+1)}}{2^{h(2i+1)}} \left[z^{h(2i+1)} + \frac{1}{2} \left(\sum_{j=0}^{k-h(2i+1)-1} z^j \right) (z^{h(2i+1)+1} - 1) \right.$$
$$\left. - \omega^{-1} \left(\sum_{j=0}^{k-h(2i+1)} z^j \right) (z^{h(2i+1)} - 1) \right].$$

Let $h \in \{1, \ldots, k\}$ and define

$$p_{k,h}(z) := z^h + \frac{1}{2} \left(\sum_{j=0}^{k-h-1} z^j \right) (z^{h+1} - 1) - \omega \left(\sum_{j=0}^{k-h} z^j \right) (z^h - 1)$$

and

$$q_{k,h}(z) := z^h + \frac{1}{2} \left(\sum_{j=0}^{k-h-1} z^j \right) (z^{h+1} - 1) - \omega^{-1} \left(\sum_{j=0}^{k-h} z^j \right) (z^h - 1).$$

It follows that

$$s_{2i} - s_i = 0 \text{ iff } p_{k,h(2i)}(z) = 0,$$

and

$$s_{2i+1} - s_i = 0 \text{ iff } q_{k,h(2i+1)}(z) = 0.$$

Since $p_{k,h}(1) = q_{k,h}(1) = 1$, both $p_{k,h}(z) - 1$ and $q_{k,h}(z) - 1$ have $z - 1$ as a factor. In fact,

$$\begin{aligned} \left| p_{k,h}(z) - 1 \right| &= \left| \sum_{j=0}^{h-1} z^j + \frac{1}{2} \sum_{j=0}^{k-h-1} z^j \sum_{j=0}^{h} z^j - \omega \sum_{j=0}^{k-h} z^j \sum_{j=0}^{h-1} z^j \right| \cdot |z-1| \\ &\leqslant \left(h + \frac{1}{2}(k-h)(h+1) + (k-h+1)h \right) |z-1| \\ &< k^2 \, |z-1| , \end{aligned}$$

and similarly, $\left| q_{k,h}(z) - 1 \right| < k^2 |z-1|$. It follows that, if $|z-1| < 1/k^2$, then $p_{k,h}(z) \neq 0$ and $q_{k,h}(z) \neq 0$. Therefore, the Melzak construction gives a non-degenerate locally minimum Steiner tree for all $\varepsilon \in [0, 1/k^2[$, since $|z-1| \leqslant \varepsilon$.

The length of the Steiner tree is

$$L(S(T_k)) = |p_0 q_1| = \left| \sum_{j=0}^{k} z^j \right|.$$

The modulus of this sum of complex numbers can be interpreted as the distance between the endpoints of a convex polygonal path consisting of $k+1$ segments, of unit length, with a turn of ε between two adjacent segments. This is easily calculated to be $\sin[(k+1)\varepsilon/2]/\sin(\varepsilon/2)$. Thus, the ratio between the length of the approximate tree T_k and the length of the locally minimum Steiner tree $S(T_k)$ is (recall (4))

$$\frac{L(T_k)}{L(S(T_k))} = \frac{(k+1)\sin(\varepsilon/2)}{\sin[(k+1)\varepsilon/2]} \geqslant 1 + \frac{k^2+2k}{24}\varepsilon^2.$$

Therefore, $G_2(\varepsilon, 2^k+1) > (k\varepsilon)^2/24$ if $\varepsilon < 1/k^2$, and Theorems 4.2 and 4.3 follow. □

9 Conclusions

1. In this paper we considered the planar case of the conjectures of Rubinstein, Wormald and Weng [22]. Although we proved one of their conjectures when ε is sufficiently small in terms of the number of terminals (Corollary 4.1), the full conjecture is still open even in the plane, a setting that one would have expected to be simple. It is especially frustrating that for a small constant ϵ (for instance, $\varepsilon = 10^{-3}$), the best upper bound we have is $F_2(\varepsilon, n) = O(n)$ (Proposition 6.1).
2. In the ε-approximate Steiner tree constructed in Sect. 8, the edge lengths are halved at each new level of the tree. If we let the edge lengths decay sufficiently fast, then most likely the topology of the ε-approximate tree will be the same as the topology of a minimum Steiner tree for ε sufficiently small [36]. Thus, the locally minimum tree constructed, using the Melzak algorithm as in Sect. 8, will most likely be a minimum Steiner tree on the terminals. This would then give a (miniscule) lower bound for $\overline{F}_2(\varepsilon, n)$. However, the calculations are much harder when the ratio, at which the edge lengths change, is not exactly $1/2$, and we have not carried these out. For similar ideas, see the papers [36] and [30].
3. In the proof of Theorems 4.2 and 4.3 (Sect. 8) we showed that the polynomials $p_{k,h}$ and $q_{k,h}$ do not have roots at distance smaller than $1/k^2$ from 1. We suspect that these polynomials actually have roots at distance approximately c/k^2 to 1.
4. It is to be expected that the lower bound in Theorem 4.3 should hold for general n, even if it turns out that $G_2(\varepsilon, n)$ is not monotone in n. Most likely the proof can be adapted for values of n other than 2^k+1 by modifying the construction in Sect. 8, but we did not look at this in detail.
5. In the definitions of F_d, $\overline{F}_d$ and G_d in Sects. 3 and 4, we could have included all ε-approximate trees on n points instead of considering only the full ones. However, by decomposing a Steiner tree into full components, it can be shown that the values of F_d, $d \geqslant 2$, and G_d, $d \geqslant 3$, will not change (use the inequality $\frac{a+b}{c+d} \leqslant \max\{\frac{a}{c}, \frac{b}{d}\}$ and Propositions 5.1 and 5.2). We do not know whether the values of $\overline{F}_d$ or G_2 will also be unchanged.

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Appendix: Induction Steps

Here we provide the details of the induction proofs of (12) and (18).

First we assume that (12) holds for q_{2i} and q_{2i+1}:

$$q_{2i} = p_{2i} + \omega^{a_{h(2i)}(2i)} \left(\frac{z}{2}\right)^{h(2i)} (z^1 + z^2 + \cdots + z^{k-h(2i)})$$
$$q_{2i+1} = p_{2i+1} + \omega^{a_{h(2i+1)}(2i+1)} \left(\frac{z}{2}\right)^{h(2i+1)} (z^1 + z^2 + \cdots + z^{k-h(2i+1)}).$$

Then,

$$\begin{aligned} q_i &= p_i + 2(q_{2i} - p_i) + 2(q_{2i+1} - p_i) \quad \text{by (11)} \\ &= p_i + 2\left(p_{2i} - p_i + \omega^{a_{h(i)}(i)-1}\left(\frac{z}{2}\right)^{h(i)+1}\left(z^1 + z^2 + \cdots + z^{k-h(i)-1}\right)\right) \\ &\quad + 2\left(p_{2i+1} - p_i + \omega^{a_{h(i)}(i)+1}\left(\frac{z}{2}\right)^{h(i)+1}\left(z^1 + z^2 + \cdots + z^{k-h(i)-1}\right)\right) \quad \text{by (9)} \\ &= p_i + (p_i - p_{\lfloor i/2 \rfloor})(\omega^{-1} + \omega)z + 2\omega^{a_{h(i)}(i)-1}\left(\frac{z}{2}\right)^{h(i)+1}(z^1 + z^2 + \cdots + z^{k-h(i)-1}) \\ &\quad + 2\omega^{a_{h(i)}(i)+1}\left(\frac{z}{2}\right)^{h(i)+1}(z^1 + z^2 + \cdots + z^{k-h(i)-1}) \quad \text{by (5) and (6)} \\ &= p_i + \omega^{a_{h(i)}(i)}\left(\frac{z}{2}\right)^{h(i)} z + \omega^{a_{h(i)}(i)}\left(\frac{z}{2}\right)^{h(i)}(z^2 + z^3 + \cdots + z^{k-h(i)}) \quad \text{by (10)} \\ &= p_i + \omega^{a_{h(i)}(i)}\left(\frac{z}{2}\right)^{h(i)}(z^1 + z^2 + \cdots + z^{k-h(i)}), \end{aligned}$$

which is (12). □

Next, assume that

$$s_i = p_i + \frac{\omega^{a_{h(i)}(i)}}{2^{h(i)+1}}(1 + z + \cdots + z^{k-h(i)-1})(z^{h(i)+1} - 1).$$

We have to show that

$$s_{2i} = p_{2i} + \frac{\omega^{a_{h(2i)}(2i)}}{2^{h(2i)+1}}(1 + z + \cdots + z^{k-h(2i)-1})(z^{h(2i)+1} - 1) \tag{19}$$

and

$$s_{2i+1} = p_{2i+1} + \frac{\omega^{a_{h(2i+1)}(2i+1)}}{2^{h(2i+1)+1}}(1 + z + \cdots + z^{k-h(2i+1)-1})(z^{h(2i+1)+1} - 1). \quad (20)$$

By (17),

$$s_{2i} = c_{2i} - \frac{(\overline{q_{2i}} - \overline{c_{2i}})(q_{2i} - s_i)}{\overline{q_{2i}} - \overline{s_i}}. \quad (21)$$

By (14),

$$c_{2i} = p_{2i} + \frac{1}{2}\omega^{a_{h(2i)}(2i)}\left(\frac{z}{2}\right)^{h(2i)}\left(z^1 + z^2 + \cdots + z^{k-h(2i)}\right), \quad (22)$$

and by (12),

$$\overline{q_{2i}} - \overline{c_{2i}} = \frac{1}{2}\omega^{-a_{h(2i)}(2i)}(2z)^{-h(2i)}\left(z^{-1} + z^{-2} + \cdots + z^{-k+h(2i)}\right). \quad (23)$$

Next,

$$\begin{aligned}
q_{2i} - s_i &= q_{2i} - p_i - \frac{\omega^{a_{h(i)}(i)}}{2^{h(i)+1}}\left(1 + z + \cdots + z^{k-h(i)-1}\right)(z^{h(i)+1} - 1) \quad \text{by (17)}\\
&= q_{2i} - p_{2i} + p_{2i} - p_i - \frac{\omega^{a_{h(2i)}(2i)+1}}{2^{h(2i)}}\left(1 + z + \cdots + z^{k-h(2i)}\right)(z^{h(2i)} - 1) \quad \text{by (9)}\\
&= \omega^{a_{h(2i)}(2i)}\left(\frac{z}{2}\right)^{h(2i)}\left(z + z^2 + \cdots + z^{k-h(2i)}\right) + \omega^{a_{h(2i)}(2i)}\left(\frac{z}{2}\right)^{h(2i)}\\
&\quad - \frac{\omega^{a_{h(2i)}(2i)+1}}{2^{h(2i)}}\left(1 + z + \cdots + z^{k-h(2i)}\right)(z^{h(2i)} - 1) \quad \text{by (12) and (10)}\\
&= \left(\omega^{a_{h(2i)}(2i)}\left(\frac{z}{2}\right)^{h(2i)}(1 - \omega) + \frac{\omega^{a_{h(2i)}(2i)+1}}{2^{h(2i)}}\right)\left(1 + z + \cdots + z^{k-h(2i)}\right)\\
&= \frac{\omega^{a_{h(2i)}(2i)}}{2^{h(2i)}}\left(\omega^{-1}z^{h(2i)} + \omega\right)\left(1 + z + \cdots + z^{k-h(2i)}\right).
\end{aligned}$$

Hence,

$$\begin{aligned}
\frac{q_{2i} - s_i}{\overline{q_{2i}} - \overline{s_i}} &= \frac{\omega^{a_{h(2i)}(2i)}\left(\omega^{-1}z^{h(2i)} + \omega\right)\left(1 + z + \cdots + z^{k-h(2i)}\right)}{\omega^{-a_{h(2i)}(2i)}\left(\omega z^{-h(2i)} + \omega^{-1}\right)\left(1 + z^{-1} + \cdots + z^{-k+h(2i)}\right)}\\
&= \omega^{2a_{h(2i)}(2i)}z^k. \qquad (24)
\end{aligned}$$

If we substitute (22), (23) and (24) into (21), then we obtain (19). The derivation of (20) is analogous. □

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