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Theorems of the Alternative over Indefinite Inner Product Spaces

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Abstract In this paper, we study the Farkas alternative over indefinite inner product spaces using the recently proposed indefinite matrix product.

Keywords Indefinite matrix products · Indefinite inner product spaces · Farkas alternative

1 Introduction

An indefinite inner product in $\mathbb{C}^n$ is a conjugate symmetric sesquilinear form $[x, y]$ which satisfies the regularity condition: $[x, y] = 0,\ \forall y \in \mathbb{C}^n$ holds only when $x = 0$. Any indefinite inner product is associated with a unique invertible Hermitian matrix P_n with complex entries such that $[x, y] = \langle x, P_n y\rangle$, where $\langle ., .\rangle$ denotes the Euclidean inner product on $\mathbb{C}^n$. The converse is also true. For the sake of ease in algebra, we make an additional assumption on P_n (motivated by the notion of Minkowski space studied by physicists in optics), viz., $P_n = P_n^{-1}$. This is not a restrictive assumption, since the results that we present here can be shown to be true without this assumption on P_n, with appropriate modifications. Finally, we remark that this assumption not only allows us to make a comparison with the Euclidean case but also helps in presenting our results elegantly.

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Investigations of linear maps on indefinite inner product spaces employ the usual multiplication of matrices which is induced by the Euclidean inner product of vectors. See for instance [1, 2] and the references cited therein. This gives rise to a mismatch as there are two different definitions for dot product of vectors. The present authors [3], proposed a new matrix product called indefinite matrix multiplication and studied some of its properties. In particular they established that the Moore-Penrose inverse of any matrix (with real or complex entries) exists over an indefinite inner product space with respect to the indefinite matrix product (Corollary 5, [3]), whereas a similar result does not hold with the conventional product. In this paper, we take this study further and consider linear system of inequalities. Fundamental to solutions of systems of linear equations, namely the alternatives of Fredholm (Theorem 2.4) and Farkas (Theorem 2.2) are generalized in the setting of indefinite inner product spaces. We also demonstrate how Fredholm alternative is useful in proving an identity involving subspaces (Theorem 2.5), well known in the Euclidean case.

One of the well known applications of the Farkas alternative is in providing a proof of the duality theorem in linear programming. It is also used in proving the existence of a stationary probability vector of a Markov matrix (see for instance, [4] for details). Application of Farkas alternative (of this article) in studying the classification problem of duality states of a linear programming problem and its dual (posed in an indefinite inner product space), is currently being undertaken.

2 Farkas Alternative

We first recall the notion of an indefinite multiplication of matrices. We refer the reader to [3] wherein advantages of this product have been discussed in detail.

Definition 2.1 Let A and B be $m \times n$ and $n \times l$ complex matrices, respectively. Let P_n be an arbitrary but fixed $n \times n$ complex matrix such that $P_n = P_n^* = P_n^{-1}$. The indefinite matrix product of A and B (relative to P_n) is defined by $A \circ B = A P_n B$.

Definition 2.2 Let A be an $m \times n$ complex matrix. The adjoint $A^{[*]}$ of A (relative to P_n, P_m) is defined by $A^{[*]} = P_n A^* P_m$.

Definition 2.3 Let A be an $m \times n$ complex matrix. Then, the range space $R(A)$ is defined by $R(A) = \{x \in \mathbb{C}^m : A \circ y = x \text{ for some } y \in \mathbb{C}^n\}$ and the null space $N(A)$ of A is defined by $N(A) = \{x \in \mathbb{C}^n : A \circ x = \mathbf{0}\}$.

Fundamental in studying systems of linear equations in Euclidean spaces are the alternatives of Fredholm and Farkas. These are results that say that either a “primal” system has a solution or a “dual” system has. While Fredholm alternative deals with existence of solutions to systems of equations, Farkas alternative asserts the existence of nonnegative solutions to systems of linear inequalities. In Euclidean space, the well-known Farkas alternative is given as follows: For $x \in \mathbb{R}^n$, we denote $x = (x_i) \geq 0$ to mean that $x_i \geq 0,\ i = 1, 2, \ldots, n$.

Theorem 2.1 *Let A be an $m \times n$ real matrix and $b \in \mathbb{R}^m$. Then, either*

(i) *$Ax = b$ has a solution $x \geq \mathbf{0}$ or (exclusive)*
(ii) *$A^* y \geq \mathbf{0}$, $\langle b, y\rangle < 0$ has a solution y.*

We consider first an analogous form of the Farkas alternative in an indefinite inner product space as follows: For A an $m \times n$ real matrix and $b \in \mathbb{R}^m$, either

(i) $A \circ x = b$ has a solution $x \geq \mathbf{0}$ or (exclusive)
(ii) $A^{[*]} \circ y \geq 0$, $[b, y] < 0$ has a solution y.

The following example shows that Farkas alternative given as above does not hold.

Example 2.1 Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $P_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $b = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Define $A^{[*]} = P_2 A^* P_2$. Then, $A \circ x = b, x \geq 0$ is inconsistent. Set $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$. Then, $A^{[*]} \circ y \geq 0$ implies $y_1 + y_2 = 0$. Thus, $[b, y] = y_1 + y_2 = 0$ for all y. Thus, both the alternatives do not hold.

We next modify the form of the second alternative to get an appropriate extension.

Theorem 2.2 *Let A be an $m \times n$ real matrix and $b \in \mathbb{R}^m$. Then, either*

(i) *$A \circ x = b$ has a solution $x \geq \mathbf{0}$ or (exclusive)*
(ii) *$(I \circ A)^{[*]} \circ y \geq \mathbf{0}$, $[b, y] < \mathbf{0}$ has a solution y.*

Proof It is easy to see that both alternatives do not hold simultaneously. Suppose that (i) does not hold. Then the system $AP_n x = b,\ x \geq 0$ has no solution. By Theorem 2.1 there exists $z \in \mathbb{R}^m$ such that $P_n A^* z \geq 0$ with $\langle b, z\rangle < 0$. Since P_m is invertible there exists $y \in \mathbb{R}^m$ such that $z = P_m y$. Then $P_n A^* P_m y \geq 0$ and $\langle b, P_m y\rangle < 0$. Observe that $[b, y] = \langle b, P_m y\rangle$ and $(I \circ A)^{[*]} \circ y = P_n A^* P_m y$. Thus alternative (ii) holds. □

Example 2.2 Let A, P_2 and b be as given Example 2.1. Choose $y = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$. Then $(I \circ A)^{[*]} \circ y = \mathbf{0}$ and $[b, y] = -2 < 0$. Thus, alternative (ii) holds.

We present next a set-theoretic analogue of the Farkas alternative. We give first a definition and a preliminary result.

Definition 2.4 Let X be a subset of $\mathbb{R}^n$. We define the orthogonal companion of X by

$$X^{\perp} = \left\{ y \in \mathbb{R}^n : [x, y] = 0, \forall x \in X \right\}.$$

The polar of X denoted by X^0 is defined by

$$X^0 = \left\{ y \in \mathbb{R}^n : [x, y] \geq 0, \forall x \in X \right\}.$$

Clearly, $X^{\perp} \subseteq X^0$ and $X^{\perp}$ is a subspace of $\mathbb{R}^n$ for any subset X whereas, X^0 is a subspace if X is a subspace. Also, $X^0 = X^{\perp}$ if X is a subspace of $\mathbb{R}^n$.

In the next result, we determine the polar of $\mathbb{R}^m_+$.

Lemma 2.1 *Let $D = \mathbb{R}^m_+$. Then, $D^0 = P_m \mathbb{R}^m_+$.*

Proof Let $y \in D^0$. Then for all $x \in D, 0 \le [y, x] = \langle P_m y, x\rangle$. Thus $P_m y \ge 0$. Set $z = P_m y$. Then $z \ge 0$ and $y = P_m z \in P_m \mathbb{R}^m_+$. Thus $D^0 \subseteq P_m \mathbb{R}^m_+$. Conversely, if $w \in P_m \mathbb{R}^m_+$, then $w = P_m x$ for $x \in \mathbb{R}^m_+$. Then $P_m w = x \ge 0$ so that for $u \in \mathbb{R}^+_m = D$, $0 \le \langle P_m w, u\rangle = [w, u]$. Thus $w \in D^0$, i.e., $P_m \mathbb{R}^+_m \subseteq D^0$. □

Theorem 2.3 *Let A be an $m \times n$ real matrix, $B = I \circ A$ and $D = \mathbb{R}^m_+$. Then,*

$$R(B) + D = (N(A^{[*]}) \cap D^0)^0.$$

Proof Let $x \in R(B) + D$ and $y \in N(A^{[*]}) \cap D^0$. Then, there exists $u \in \mathbb{R}^n$, $v \in \mathbb{R}^m_+$ such that $x = B \circ u + v$. Thus, $[x, y] = [B \circ u + v, y] = [B \circ u, y] + [v, y]$. We have $[v, y] \ge 0$ as $v \in D$ and $y \in D^0$. Also,

$$[B \circ u, y] = ((I \circ A) \circ u)^{[*]} y = u^{[*]} \circ (A^{[*]} \circ y) = \mathbf{0}$$

as $y \in N(A^{[*]})$. Thus $[x, y] \ge 0$, viz., $x \in (N(A^{[*]}) \cap D^0)^0$. Thus, $R(B) + D \subseteq (N(A^{[*]}) \cap D^0)^0$. On the other hand, let $x \notin R(B) + D$. Then for $u \in \mathbb{R}^n$, $v \in \mathbb{R}^+_m$, the system $B^{[*]} \circ u + v = x$ has no solution. By writing, $u = u^1 - u^2$, with $u^1, u^2 \ge 0$ we infer that the system $((I \circ A), -(I \circ A), P_m) \circ \left(\begin{smallmatrix} u^1 \\ u^2 \\ v \end{smallmatrix}\right) = x$ has no solution $u^1, u^2, v \ge 0$, where we have used the fact that $v = P_m \circ v$. Setting $E = ((I \circ A), -(I \circ A), P_m)$ and $z = \left(\begin{smallmatrix} u^1 \\ u^2 \\ v \end{smallmatrix}\right)$, we infer that the system $E \circ z = x$ has no solution $z \ge 0$. By Theorem 2.2, it then follows that there exists $y \in \mathbb{R}^m$ such that $(I \circ E)^{[*]} \circ y \ge 0$ with $[x, y] < 0$. We note that $I \circ E = (A, -A, I)$ so that y satisfies $A^{[*]} \circ y = 0, I \circ y \ge 0$ with $[x, y] < 0$. Since $I \circ y = P_m y$, by applying Lemma 2.1, we conclude that $y \in D^0$. So $y \in (N(A^{[*]}) \cap D^0)$. Since $[x, y] < 0$, we have $x \notin (N(A^{[*]}) \cap D^0)^0$. Hence, $(N(A^{[*]}) \cap D^0)^0 \subseteq R(B) + D$. □

Corollary 2.1 (See theorem on p. 541, [5]) *For $A \in \mathbb{R}^{m\times n}$, we have*

$$R(A) + \mathbb{R}^m_+ = (N(A^*) \cap \mathbb{R}^m_+)^0.$$

Proof Let $P_m = I$. Then $I \circ A = A$ and $D = \mathbb{R}^m_+ = D^0$, as in this case $[x, y] = \langle y, x\rangle$. Also, $A^{[*]} = A^*$. The proof now follows from Theorem 2.3. □

As an application of the Farkas alternative, we next derive the Fredholm alternative.

Theorem 2.4 *Let A be an $m \times n$ real matrix and $b \in \mathbb{R}^m$. Then, either*

(i) *$A \circ x = b$ has a solution x or (exclusive)*
(ii) *$(I \circ A)^{[*]} \circ y = \mathbf{0}$, $[b, y] \ne 0$ has a solution y.*

Proof Clearly, both alternatives do not hold simultaneously. Suppose that (i) does not hold. Setting $x = x^1 - x^2$ with $x^1, x^2 \ge 0$, we infer that $A \circ x^1 - A \circ x^2 = b$

has no solution $x^1, x^2 \geq 0$. Setting $B = (A, -A)$ and $u = (x^1, x^2)$ this reduces to the statement: $B \circ u = b,\ u \geq 0$ has no solution. By Theorem 2.2 there exists $y \in \mathbb{R}^m$ such that $(I \circ B)^{[*]} \circ y \geq 0$ and $[b, y] < 0$. Finally, $I \circ B = (I \circ A, -(I \circ A))$ so that we have $(I \circ A) \circ y = 0$ and $[b, y] \neq 0$. Thus, alternative (ii) holds. □

The identity $R(A) = N(A^*)^\perp$ is well known for finite matrices A with real or complex entries. We obtain a generalization in an indefinite inner product space. The proof uses the Fredholm alternative.

Theorem 2.5 *For any $m \times n$ real matrix, the following holds*:

$$R(I \circ A) = N(A^{[*]})^\perp.$$

Proof Let $b \in R(I \circ A)$ and $y \in N(A^{[*]})$. If $[b, y] \neq 0$, then $(I \circ B)^{[*]} y = 0$, $[b, y] \neq 0$ has a solution y, where $I \circ B = A$, so that $I \circ A = B$. By Fredholm alternative, $I \circ A \circ x = b$, has no solution, a contradiction. Thus $[b, y] = 0$. Hence $R(I \circ A) \subseteq N(A^{[*]})$. Conversely, suppose that $y \in N(A^{[*]})^\perp$. Then for any u, $A^{[*]} \circ u = 0 \Rightarrow [u, y] = 0$. By setting $B = I \circ A$, we then have $(I \circ B)^{[*]} \circ u = 0 \Rightarrow [u, y] = 0$. Thus (ii) in the Fredholm alternative does not hold. Thus, alternative (i) has a solution. So, there exists x such that $I \circ A \circ x = u$. Thus, $u \in R(I \circ A)$. This completes the proof. □

Remark 2.1 To make a comparison between the conventional matrix product and the indefinite product with regard to Theorem 2.5, we consider the matrix $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $P_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then $R(A) = N(A^{[*]})$ and $R(A)^\perp = R(A)$, with the usual matrix product. Thus, even though the identity $R(A) = N(A^{[*]})^\perp$ holds, the subspaces are not complementary. Moreover, there is no orthogonal complementary subspace for $R(A)$. Interestingly, we note that, the matrix A given here does not have the Moore-Penrose inverse $A^{[\dagger]}$. That this is not a mere coincidence is the essence of the next result.

For the sake of completeness, we recall that the Moore-Penrose inverse of a matrix A is the unique solution (if any) of the four equations: $AXA = A$; $XAX = X$; $(AX)^{[*]} = AX$; $(XA)^{[*]} = XA$. In the Euclidean case, any matrix has a unique Moore-Penrose inverse [6, 7] whereas the same is not true in the indefinite inner product space setting [2].

Theorem 2.6 (See Theorem 5.10, [2]) *Let A be an $m \times n$ complex matrix such that $A^{[\dagger]}$ exists. Then, $R(A)$ and $N(A^{[*]})$ are orthogonal complementary subspaces.*

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