



Mixing Rates of the Geometrical Neutral Lorenz Model

Henk Bruin¹ · Hector Homero Canales Farías¹

Received: 28 May 2023 / Accepted: 18 November 2023 / Published online: 8 December 2023
© The Author(s) 2023

Abstract

The aim of this paper is to obtain polynomial decay of correlations of a Lorenz-like flow where the hyperbolic saddle at the origin is replaced by a neutral saddle. To do that, we take the construction of the geometrical Lorenz flow and proceed by changing the nature of the saddle fixed point at the origin by a neutral fixed point. This modification is accomplished by changing the linearised vector field in a neighbourhood of the origin for a neutral vector field. This change in the nature of the fixed point will produce polynomial tails for the Dulac times, and combined with methods of Araújo and Melbourne (used to prove exponential mixing for the classical Lorenz flow) this will ultimately lead to polynomial upper bounds of the decay of correlations for the modified flow.

Keywords Polynomial decay of correlations · Neutral geometrical Lorenz flow · Mixing · Neutral fixed point

Mathematics Subject Classification Primary 37D25 · Secondary 37C10

Contents

1	Introduction	2
1.1	The Framework	2
1.2	Main Results	4
2	The Poincaré Maps	6
2.1	Neutral Model 1	6
2.2	Neutral Model 2	9
3	The Stable Foliation and the UNI Condition	11
3.1	Existence and Regularity of the Strong Stable Foliation	11
3.2	The UNI Condition	14
4	Decay of Correlations	21
5	Numerical Analysis and Results	23
5.1	Numerics of the 2-Dimensional Case	24
5.2	Numerics of the 3-Dimensional Case	26

Communicated by Peter Balint.

✉ Henk Bruin
henk.bruin@univie.ac.at

¹ Faculty of Mathematics, University of Vienna, Vienna, Austria

5.3 Numerics of the Tails of the Return Map in the 3-Dimensional Case 28
 References 30

1 Introduction

The study of flows on surfaces and higher-dimensional manifolds has caught the interest of many scientists because of its numerous applications such as Hamiltonian flows, geodesic and horocycle flows, billiard flows or flows from meteorological models. These flows are usually equipped with a natural invariant measure μ , for instance the SRB-measure.

The main goal is to have a better understanding of the properties of these flows, such as hyperbolicity, ergodicity, mixing (or at least weak mixing) and, in chaotic settings, rates of mixing; that is, we would like to investigate the asymptotic behaviour of the correlation coefficients

$$\rho_t(v, w) = \left| \int_M v \cdot w \circ f^t d\mu - \int_M v d\mu \int_M w d\mu \right|, \tag{1}$$

where $f^t : M \rightarrow M$ is a flow acting on a manifold M and μ its SRB-measure, and for observables v, w chosen from an appropriate Banach space. Knowing the rates of mixing is very helpful for proving other ergodic and statistical properties since mixing is one of the strongest statistical properties.

Obtaining good mixing rates for flows, even for hyperbolic flows, is far more difficult than for maps. Some seminal ideas were provided by Liverani [29] and Dolgopyat [20, 21], with applications of these methods in e.g., [9, 10, 16]. To obtain sharp estimates in the polynomial setting, the operator renewal theory techniques developed by Sarig [34] and Gou ezel [24] are the only ones available.

The model we would like to study is probably one of the most emblematic ones, the Lorenz flow. In the mid seventies Afraimovi c, Bykov and Shilnikov [1] and independently Guckenheimer and Williams [25] introduced the geometric Lorenz attractor to model the original Lorenz attractor. Our research focuses on a modified version of this geometrical model and study its rate of mixing, based on the precise estimates in [17] of Dulac times associated to a neutral saddle.

Recently, Ara ujo and Melbourne in [3] proved that the geometrical Lorenz flow (and hence the classical Lorenz flow), also enjoys exponential mixing. It is techniques from their papers, specifically $C^{1+\alpha}$ smoothness of the stable foliation, that leads eventually to the claimed mixing rates.

1.1 The Framework

The geometrical Lorenz flow can be seen as the natural extension of a suspension semiflow built over a certain type of one-dimensional expanding map f_{Lor} . We first consider the cross-section Σ transversal to the flow and the Poincar e map $P_{\text{Lor}} : \Sigma \rightarrow \Sigma$, which is decomposed in two parts. The first one is the Dulac map, denoted by P_1 , deals with the local behaviour near the origin and is obtained by considering a linear system in a neighbourhood of the origin; that is, we take the flow X^t obtained from the linear system

$$\begin{aligned} \dot{x} &= \lambda_u x \\ \dot{y} &= -\lambda_s y \\ \dot{z} &= -\lambda_{ss} z \end{aligned} \tag{2}$$

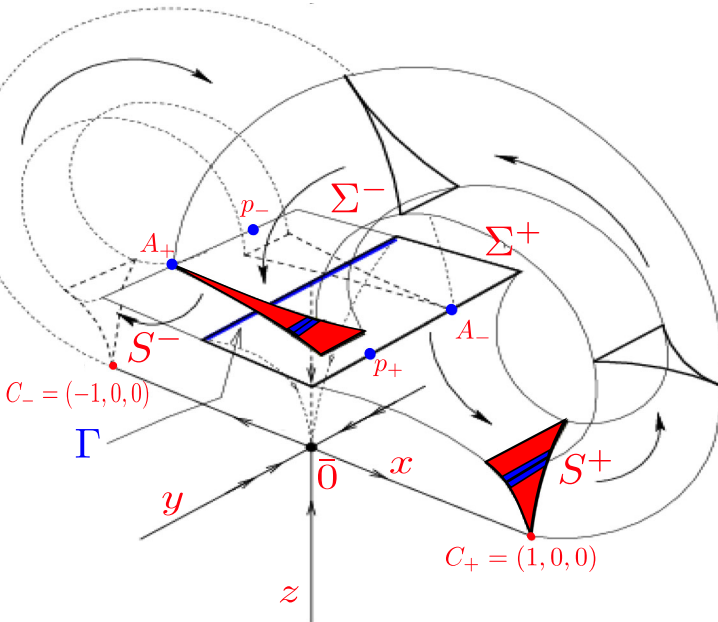


Fig. 1 The Guckenheimer-Williams geometric model of the Lorenz flow (Image taken from [4])

where λ_u, λ_s and λ_{ss} denote the unstable, stable and strong stable eigenvalues of the original Lorenz system, respectively. Then we let points in Σ flow under X^t until flow time $\tau'_{Lor} := \min\{t > 0 : X^t \in S\} = -\lambda_u^{-1} \ln(|x|) + \mathcal{O}(\ln(|x|))$ as $x \rightarrow 0$. Thus we have that $X^{\tau'_{Lor}} = P_1 : \Sigma \rightarrow S$, where S^\pm is the image of Σ^\pm under P_1 and has a cusp-like shape, see Fig. 1.

The second part, denoted by P_2 , consists of the return of S to Σ and simulates the random turns of a regular orbit around the origin and describes a butterfly-like shape. This is done by a composition of a rotation, expansion and translation with hitting time $\tau_2(x) \in C^\epsilon$. Thus, the full return time of the Poincaré map $P_{Lor} = P_2 \circ P_1$ is given by

$$r_{Lor}(x) = \tau'_{Lor}(x) + \tau_2(x). \tag{3}$$

As we will see later, the lines in the y -direction (*i.e.*, parallel to the y axis) in Σ form the stable foliation, invariant under P_{Lor} ; that is, for any leaf γ of this foliation, its image $P_{Lor}(\gamma)$ is contained in a leaf of the same foliation. By quotienting out the stable direction we can rewrite the Poincaré map as a skew-product; that is, $P_{Lor}(x, y) = (f_{Lor}(x), g_{Lor}(x, y))$.

The **geometric Lorenz flow** is the couple (W, X^t_W) , where $W = \{X^t(\bar{x}) \mid \bar{x} \in \Sigma, t \in \mathbb{R}^+\}$. Consider $U = \bigcup_{\bar{x} \in \Sigma} X^{[0, r_{Lor}(x)]}(\bar{x})$, then the **geometric Lorenz attractor** (of the corresponding vector field) is given by $\Lambda_{Lor} = \bigcap_{t>0} X^t_W(U)$.

In [3], exponential mixing for the geometrical Lorenz flow was proven under two conditions: the stable foliation has to be $C^{1+\alpha}$ and a uniform non-integrability (UNI) condition needs to be satisfied.

The modified version is obtained by changing the local behaviour near the origin. We achieve this by replacing the linear system for the following system;

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = Z \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x(a_0x^2 + a_1y^2 + a_2z^2) \\ -\ell y(1 + c_0x^2 + c_2z^2) \\ -z(b_0x^2 + b_1y^2 + b_2z^2) \end{pmatrix} + \mathcal{O}(4), \tag{4}$$

where $a_0, a_1, a_2, b_0, b_1, b_2, c_0, c_2$ and $\ell > 0, a_2b_0 < 9a_0b_2, \Delta := a_2b_0 - a_0b_2 \neq 0$ and $\mathcal{O}(4)$ refers to terms of order four or higher, under the condition that they are of the form $x^2\mathcal{O}(2)$ near the yz -plane and $z^2\mathcal{O}(2)$ near the xy -plane. This system has a polynomial Dulac time (see (8) and Fig. 3) given by;

$$\tau'_{\text{Neu}} := \min\{t > 0 : N^t \in S\} = |x|^{-\frac{1}{\beta_2}} (1 + \mathcal{O}(|x|^{\frac{1}{2\beta_2}})), \tag{5}$$

as $x \rightarrow 0$ and $\beta_2 = \frac{a_2+b_2}{2b_2}$. To obtain the flow time τ'_{Neu} , we make use of the estimates of the Dulac map and the tails of the return map obtained by Bruin and Terhesiu in [17]. This change of flow time, from logarithmic to polynomial, will ultimately allow us to deduce the polynomial decay of correlations.

We denote by N^t the flow obtained from the system given by (4). This change in the local behaviour near the origin leads to a change on the map P_1 ; that is, we have now $N^{\tau'_{\text{Neu}}} = D_1 : \Sigma \rightarrow S$. For the second part, the return of S to Σ , we consider the same diffeomorphism P_2 with same hitting time. In this way, we obtained the modified Poincaré map $P_{\text{Neu}} = P_2 \circ d_1$ with return time given by,

$$r_{\text{Neu}}(x) = \tau'_{\text{Neu}}(x) + \tau_2(x). \tag{6}$$

Similarly, we define the **geometric neutral Lorenz flow** as the couple (W, N^t_W) , where $W = \{N^t(\bar{x}) \mid \bar{x} \in \Sigma, t \in \mathbb{R}^+\}$. We consider again $U = \bigcup_{\bar{x} \in \Sigma} N^{[0, r_{\text{Neu}}(x)]}(\bar{x})$, the **geometric neutral Lorenz attractor** (of the corresponding vector field) is given by $\Lambda_{\text{Neu}} = \bigcap_{t>0} N^t(U)$.

As we will see in more detail in Sect. 2, the geometrical neutral Lorenz flow will be split into three models. **Model 1** is obtained when we take the parameters $c_0 = c_2 = 0$ in (4). **Model 2** when we consider $a_1 = b_1 = 0$. Finally, **Model 3**, the most general, will be given by taking all parameters strictly positive.

1.2 Main Results

Let C^η be the space of functions that are η -Hölder in the space direction, and $C^{m,\eta}$ be the space of functions that are $m + \eta$ -Hölder (i.e., m time differentiable with an η -Hölder m -th derivative) in the flow direction, see Sect. 4 for the precise definitions. The main result in this paper is the following theorem:

Theorem 1.1 *Let $N^t : \Lambda_{\text{Neu}} \rightarrow \Lambda_{\text{Neu}}$ be the geometrical neutral Lorenz flow for Model 1 and Model 2 obtained from the neutral form given by (4), with corresponding parameters. Λ_{Neu} its attractor and its SRB measure μ . Then N^t has polynomial decay of correlations (with exponent $\beta_2 = \frac{a_2+b_2}{2b_2}$); that is, there exist $m \geq 1$ and a constant $C > 0$ such that for observables $v \in C^\eta(M) \cap C^{0,\eta}(M), w \in C^{m,\eta}(M)$, and time $t > 1$ we have*

$$\rho_t(v, w) \leq C(\|v\|_{C^\eta} + \|v\|_{C^{0,\eta}}) \|w\|_{C^{m,\eta}} t^{-\beta_2}.$$

A first question that presents itself is of course if these bounds are sharp, and if current operator renewal theory methods [24, 34] cannot prove that. We say more on this at the end of Sect. 4.

For the proof of Theorem 1.1, we obtain an explicit form of the Poincaré map, since we can solve the differential equation in the y component. Thus we are able to prove polynomial decay of correlations by using the results on non-uniformly hyperbolic flows established by Bálint et al. in [12].

For the third model the situation is more subtle since, to our knowledge, finding the solution of the differential equation in the y component is next to impossible. To overcome this problem we will analyse and compare, with numerical methods, the limit behaviour of the Dulac maps obtained in [15] and [17] and adapted to our framework. More precisely, we will analyse the limit behaviour of the maps $D_1 : \Sigma \rightarrow S$ obtained for each Neutral model. This is sufficient since the Poincaré maps considered in this work are given by $P_{\text{Neu}} = P_2 \circ D_1$, where P_2 is a diffeomorphism and the map D_1 is the Dulac map from the cross-section Σ to the cusps S , which depends on the differential equation being considered. Therefore, the behavioural changes exhibited by the map P_{Neu} are represented by the changes of the map D_1 . Dulac in [22] made a significant contribution to solving Hilbert’s 16th problem by incorporating his map as an element to establish that polynomial vector fields in the plane possess a limited number of limit cycles, demonstrating that they cannot have an infinite number of such cycles.

The numerical analysis on the behaviour of the Dulac maps will give us the plausibility of the following conjecture.

Conjecture 1.2 *Let $N^t : \Lambda_{\text{Neu}} \rightarrow \Lambda_{\text{Neu}}$ be the geometrical neutral Lorenz flow for Model 3 obtained from the neutral form given by (4), with the corresponding parameters. Λ_{Neu} its attractor and its SRB measure μ . Then N^t has polynomial decay of correlations (with exponent $\beta_2 = \frac{a_2+b_2}{2b_2}$).*

The organization of this paper is as follows: In Sect. 2 we will give the construction of the Poincaré maps of the Neutral Model 1 and 2. In Sect. 3 we will be devoted to the proof that the stable foliation for the geometrical neutral models is $C^{1+\alpha}$ and the UNI condition is satisfied by adapting the existing proofs for the geometrical Lorenz model. Section 4 contains the framework of non-uniformly hyperbolic flows and the proof of Theorem 1.1. Finally, in Sect. 5 we will present the numerical analysis and results we obtained for the Dulac map and the tails of the return map.

For the remaining of this paper we will adopt the following notation.

Notation 1.3 *In order to avoid excessive notation of the higher order terms, obtained from the estimates of the Dulac time given in [17], we will write $A_1(x, \beta)$ and $A_2(x, \beta_2)$ to denote $\xi|x|^\beta(1 + \mathcal{O}(|x|^{\frac{1}{2\beta_2}}))$ and $\zeta|x|^{-\frac{1}{\beta_2}}(1 + \mathcal{O}(|x|^{\frac{1}{2\beta_2}}))$, respectively, where $\beta_0 = \frac{a_0+b_0}{2a_0}$, $\beta_2 = \frac{a_2+b_2}{2b_2}$, $\beta = \frac{\beta_0}{\beta_2}$, ξ and ζ are constants given in [17], namely in Theorem 1.1 and the proof of Proposition 2.1. $X \in \mathfrak{X}^r(M)$ will denote the vector space of C^r vector fields in a manifold M with the C^r topology.*

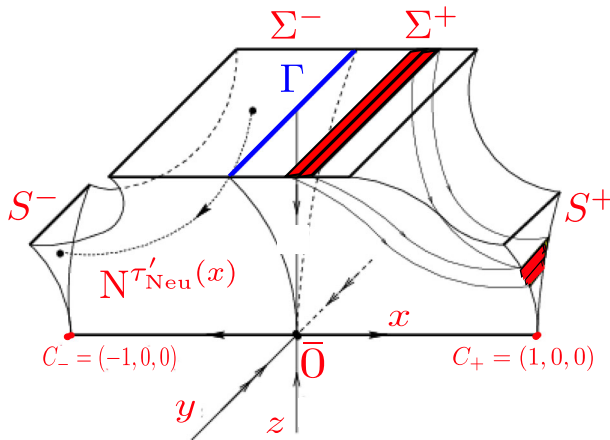


Fig. 2 The map D_1 (Image taken from [4])

2 The Poincaré Maps

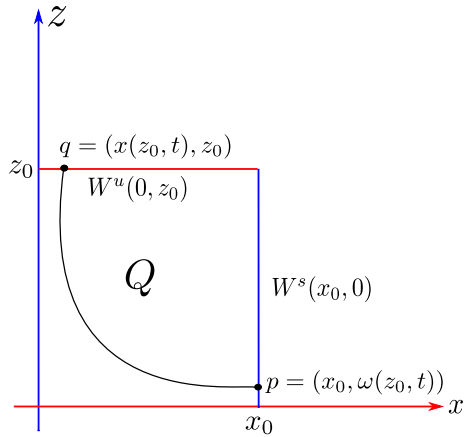
2.1 Neutral Model 1

To create the modified models we will apply local surgery in a neighbourhood of the hyperbolic saddle equilibrium of the geometrical Lorenz model, namely the origin, and transform it into a neutral equilibrium. We do this because we aim to slow down the orbit and thus increase the time that orbits take to flow from the cross-section Σ to the cusps S , see Fig. 1, and see the changes this new motion produces in the decay of correlations. The flow obtained from this modification will be an almost Anosov flow [27, 28]. Existence of a finite or infinite SRB measure for two-dimensional almost Anosov diffeomorphisms was already proven in [27, 28]. Bruin and Terhesiu in [17] proved mixing rates in the infinite SRB measure setting for almost Anosov diffeomorphism and established the required spectral properties for the transfer operator (acting on an appropriate anisotropic Banach space of distributions) of an induced map so as to obtain optimal rates of mixing. Furthermore, they gave more precise tail estimates for the inducing scheme. We will take advantage of these methods and estimates and use them to deduce the rates of mixing of our almost Anosov flow.

We consider now $\Sigma^* = \{(x, y, 1) \in \mathbb{R}^3 \mid |x| \leq 1, |y| \leq 1\}$, $\Sigma^- = \{(x, y, 1) \in \Sigma^* \mid x < 0\}$, $\Sigma^+ = \{(x, y, 1) \in \Sigma^* \mid x > 0\}$, $\Sigma = \Sigma^+ \cup \Sigma^- = \Sigma^* \setminus \tilde{\Gamma}$, where $\tilde{\Gamma} = \{(x, y, 1) \in \Sigma^* \mid x = 0, \}$ and $S = S^+ \cup S^-$ where S^\pm is the image of Σ^\pm under the Dulac map D_1 , see Fig. 2. The section Σ is transversal to the flow and every trajectory eventually crosses Σ in the direction of the negative axis z . Then for each $(x, y, 1) \in \Sigma$, the time τ'_{Neu} such that $N^{\tau'_{\text{Neu}}}(x, y, 1) \in S$ is determined by the estimates of the Dulac map provided in [17], as we will explain now.

Let us start with the Neutral model 1; that is, we consider a neighbourhood U of the fixed point $\bar{0} = (0, 0, 0)$ where the vector field has the form (4) (in local Euclidean coordinates) with $c_0 = c_2 = 0$ and the other parameters satisfying the constraints given before. Note that U is taken much smaller than the scale of Fig. 2. This vector field, denoted by Z , is cubic at $\bar{0}$ in the direction transversal to the stable manifold of $\bar{0}$, but this is the only source of non-hyperbolicity. The y -axis is invariant and all solutions tend to $\bar{0}$. The divergence is given by

Fig. 3 The Dulac map $D : W^u(0, z_0) \rightarrow W^s(x_0, 0)$



$$\text{Div}(Z) = (3a_0 - b_0)x^2 + (a_1 - b_1)y^2 + (a_2 - 3b_2)z^2 - \ell.$$

Since we want the flow to shrink volume exponentially fast, as does the Lorenz flow, we need $\text{Div}(Z) \leq -c < 0$. Therefore, we let ℓ be large enough such that $(3a_0 - b_0)x^2 + (a_1 - b_1)y^2 + (a_2 - 3b_2)z^2 < \ell$ for all $(x, y, z) \in U$. The solution for the y -component is given by $y(t) = y_0 e^{-\ell t}$. Thus we obtain a non-autonomous system of differential equations, since the contribution of the y -component to the x and z -component is exponentially small as time increases, these terms are of smaller order than the higher order terms. Thus we are left with the two-dimensional system studied in [17]:

$$\begin{pmatrix} \dot{x} \\ \dot{z} \end{pmatrix} = Z_{\text{hor}} \begin{pmatrix} x \\ z \end{pmatrix} = \begin{pmatrix} x(a_0x^2 + a_2z^2) \\ -z(b_0x^2 + b_2z^2) \end{pmatrix}. \tag{7}$$

Now let \mathcal{W}^s and \mathcal{W}^u be two mutually transversal foliations of the interior of $U \cap$ [positive quadrant] that is invariant under the flow of (7) and such that

- the leaves of \mathcal{W}^s accumulate in C^1 topology on the stable manifold of $(0, 0)$ and are transversal to the unstable manifold of $(0, 0)$, and
- the leaves of \mathcal{W}^u accumulate in C^1 topology on the unstable manifold of $(0, 0)$ and are transversal to the stable manifold of $(0, 0)$.

One would like to use the stable and unstable foliation of the horizontal flow ϕ_{hor} of (7) for \mathcal{W}^s and \mathcal{W}^u , but as long as we defined the flow only locally, the above properties suffice.

Now fix an unstable leaf $W^u(0, z_0) \in \mathcal{W}^s$ and a stable leaf $W^s(x_0, 0) \in \mathcal{W}^u$, then the Dulac map $D : W^u(0, z_0) \rightarrow W^s(x_0, 0)$, shown in Fig. 3, assigns the first intersection $\phi^T(x, z_0)$ of the integral curve through (x, z_0) with the stable leaf $W^s(x_0, 0)$, where $x \in W^u(0, z_0)$, $\phi^t(x, z)$ is the flow from (7) and T is the exit time. The estimates for the map D and the flow time given in [17] are:

$$\omega = D(x) = c(z_0)x^\beta(1 + \mathcal{O}(x^{2\beta_2})) \tag{8}$$

and

$$\tau'_{\text{Neu}}(x) = A_2(x, \beta_2), \tag{9}$$

where $\beta = \frac{\beta_0}{\beta_2}$ for $\beta_0 = \frac{a_0+b_0}{2a_0}$ and $\beta_2 = \frac{a_2+b_2}{2b_2}$.

More relevant to the proof of the decay of correlations of the neutral geometrical Lorenz flow is the estimate of the tails of the return map which we state in the following theorem.

Theorem 2.1 [17, Theorem 1.1] *Let Q be a region bounded by the stable and unstable manifolds of 0 and a leaf $W^s(x_0, 0) \in \mathcal{W}^s$ and a leaf $W^u(0, z_0) \in \mathcal{W}^u$, see Fig. 3. Let $\beta^* = \frac{1}{2} \min\{1, \frac{a_2}{b_2}, \frac{b_0}{a_0}\}$, then there exists $C_0 > 0$ a constant such that $\mu(\varphi > n) = C_0 n^{-\beta^2} (1 + \mathcal{O}(n^{-\beta^*}))$, where $\varphi = \inf\{t > 0 \mid \phi_{hor}^t(z) \notin Q\}$ and μ is Lebesgue measure¹ on $W^u(0, z_0)$.*

Putting all together we get the following expression for the map D_1

$$D_1(x, y, 1) = N^{\tau'_{\text{Neu}}(x)}(x, y, 1) = \left(1, y e^{-\ell A_2(x, \beta_2)}, A_1(x, \beta)\right), \tag{10}$$

where the functions $A_2(x, \beta_2)$ and $A_1(x, \beta)$ come from Notation 1.3.

We make the following observations:

- Observation 2.2**
- $D_1(\Sigma^\pm)$ has the shape of a cusp at $(\pm 1, 0, 0)$ and (with some abuse of notation) we will denote these images as S^\pm and $S = S^+ \cup S^-$.
 - Denote by $\ell_v(c) = \{(x, y, 1) \in \Sigma \mid x = c\}$, where c is a constant, the line segments in Σ parallel to the y -axis and by $\ell_h(c) = \{(\pm 1, y, z) \in S \mid z = c\}$, the line segments in S parallel to the y -axis. Then $D_1(\ell_v(c_0)) = \ell_h(c_1)$; that is, the map D_1 takes the y -direction lines in Σ to the horizontal line segments in S as illustrated in Fig. 2.

The return of the cusps S to the cross-section Σ is described by the map $P_2 = T \circ E_a \circ R_\theta$, where R_θ is a rotation by an angle of $\theta = \frac{3\pi}{2}$ and the rotation axis are the boundaries of the cross-section Σ parallel to the y -axis, E_a is an expansion by a factor of $a > 1$ in the x -direction and a translation T such that the unstable direction which starts from the origin is sent to the boundary of Σ ; that is, we want to send the cusp points C_\pm to A_\pm , see Fig. 1. Thus, the full Poincaré map $P_{\text{Neu}} = P_2 \circ D_1 : \Sigma \rightarrow \Sigma$, shown in Fig. 4, is given by,

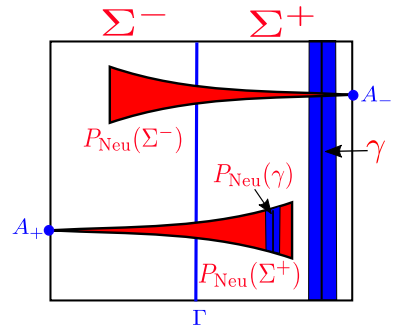
$$P_{\text{Neu}}(x, y) = \begin{cases} \left(aA_1(x, \beta) - 1, y e^{-\ell A_2(x, \beta_2)} - \frac{1}{2}\right), & \text{if } x \in (0, 1]; \\ \left(aA_1(x, \beta) + 1, y e^{-\ell A_2(x, \beta_2)} + \frac{1}{2}\right), & \text{if } x \in [-1, 0). \end{cases} \tag{11}$$

In the positive quadrant the matrix DP_{Neu} has eigenvalues $\lambda_1 = a \frac{\beta_0}{\beta_2} |x|^{\frac{\beta_0}{\beta_2} - 1}$ and $\lambda_2 = e^{-\ell A_2(x, \beta_2)}$. By restricting $\frac{1}{2} < \beta_0 < 2$ we have that $0 < \frac{\beta_0}{\beta_2} < 1$ since $\beta_2 > 2$. Then $\lambda_1 \rightarrow \infty$ as $x \rightarrow 0$. For the other eigenvalue we have that $\lambda_2 < 1$ and $\lambda_2 \rightarrow 0$ as $x \rightarrow 0$ since $A_2(x, \beta_2) \rightarrow \infty$ as $x \rightarrow 0$. Thus the modified Poincaré map P_{Neu} is hyperbolic when x approaches the origin.

The foliation given by the lines $\ell_v(c)$ is invariant under the map P_{Neu} ; that is, given any leaf γ of this foliation its image $P_{\text{Neu}}(\gamma)$ is contained in a leaf of the same foliation (See Fig. 4). Therefore, we can express P_{Neu} as $P_{\text{Neu}}(x, y) = (f_{\text{Neu}}(x), g_{\text{Neu}}(x, y))$, where

¹ In [17], the measure of interest is the SBR measure, but because the SBR measure is flow-invariant (hence, with smooth holonomies in the flow-direction) and conditioned to unstable leaves is absolutely continuous w.r.t. Lebesgue (in fact, equivalent to Lebesgue with a density that is bounded and bounded away from 0, see [17, Proposition 3.1] and [28, Lemma 4.1]) and the separate unstable leaves are C^1 smooth curves, it is justified to work with the Lebesgue measure on $W^u(0, z_0)$.

Fig. 4 The Poincaré map for the neutral geometrical model



$f_{\text{Neu}} : I \setminus \{0\} \rightarrow I$ is a Lorenz-like map with exponent $\beta = \frac{\beta_0}{\beta_2}$ with $\beta_0 = \frac{a_0+b_0}{2a_0}$ and $\beta_2 = \frac{a_2+b_2}{2b_2}$; that is, f_{Neu} is given by,

$$f_{\text{Neu}}(x) = \begin{cases} aA_1(x, \beta) - 1, & \text{if } x \in (0, 1]; \\ 1 - a|A_2(x, \beta)|, & \text{if } x \in [-1, 0), \end{cases} \tag{12}$$

and the function $g_{\text{Neu}} : (I \setminus \{0\}) \times I \rightarrow I$, where $I = [-1, 1]$, satisfies the following:

- g1. The map g_{Neu} is piecewise C^2 and for fixed x_0 , the map $g_{\text{Neu}}(x_0, y)$ is a contraction in the y -direction, i.e.,

$$d(g_{\text{Neu}}(x_0, y_1), g_{\text{Neu}}(x_0, y_2)) \leq cd(y_1, y_2),$$

where d is the Euclidean distance in I and $0 < c < 1$.

- g2. DP_{Neu} has the following bound on its partial derivatives:

- a) For all $(x, y) \in \Sigma$ we have $\partial_y g_{\text{Neu}}(x, y) = e^{-\ell A_2(x, \beta_2)}$. Since $\beta_2 > 2$ and $|x| \leq 1$, there is $0 < \eta < 1$ such that

$$|\partial_y g_{\text{Neu}}(x, y)| < \eta.$$

- b) For $(x, y) \in \Sigma$ with $x \neq 0$ we have $\partial_x g_{\text{Neu}}(x, y) = \frac{\ell}{\beta_2} y (A_2(x, \beta_2))' e^{-\ell A_2(x, \beta_2)}$. Since $1 < 1 + \frac{1}{\beta_2} < \frac{3}{2}$ and $|y|, |x| \leq 1$ we get that $|\partial_x g_{\text{Neu}}(x, y)|$ is bounded. In fact, it tends to zero exponentially fast as x approaches the origin.

- g3. From g2.a) above follows the uniform contraction of the foliation given by the lines $\ell_v(c)$; in other words, there is $C > 0$ such that, for any given leaf γ of the foliation and for $y_1, y_2 \in \gamma$, we have

$$d(P_{\text{Neu}}^n(y_1), P_{\text{Neu}}^n(y_2)) \leq C\eta^n d(y_1, y_2),$$

when $n \rightarrow \infty$.

2.2 Neutral Model 2

Now we will consider the Neutral model 2; that is, we consider the same neighbourhood U of the origin where the flow has the local form given by (4) with $a_1 = b_1 = 0$ and the remaining parameters satisfying the same constraints stated in this framework. Again $\bar{0} = (0, 0, 0)$ is the only neutral periodic orbit and the vector field is cubic in the direction transversal the

stable manifold of $\bar{0}$, but this is the only source of non-hyperbolicity. If $x = 0$ and $z = 0$, then we see that $\dot{x} = 0$, $\dot{y} = -\ell y$ and $\dot{z} = 0$. Hence, the y -axis is invariant and all solutions tend to the origin as in the previous model. Moreover, since \dot{x} and \dot{z} are decoupled from y , we have (7). Thus the asymptotics for the Dulac map and the flow time given in [17] follow. Also $\text{Div}(Z) = (3a_0 - \ell c_1 - b_0)x^2 + (a_2 - \ell c_2 - 3b_2)z^2 - \ell$. Since we want a flow that shrinks volume exponentially fast as before, we take ℓ large enough so that $\frac{(3a_0 - b_0)x^2 + (a_2 - 3b_2)z^2}{1 + c_0x^2 + c_2z^2} < \ell$ for all $(x, y, z) \in U$.

We consider the same cross-section Σ as before and proceed to construct the Poincaré map in the same way. Denote by N^t the flow obtained from (4) with the pertinent constraints in the parameters; that is, $N^t(x, y, z) = (x(t), y(t), z(t))$. By (8) we obtain the following form for the flow,

$$N^{t(x,z)}(x, y, z) = (x_0, y(t(x, z)), \omega(z, t(x, z))). \tag{13}$$

Note that $\dot{y} = y(-\ell(1 + c_0x^2 + c_2z^2))$, applying Grönwall’s Lemma we get,

$$\begin{aligned} y(t) &= y_0 \exp\left(-\ell \int_0^t (1 + c_0x^2 + c_2z^2) ds\right) \\ &= y_0 e^{-\ell t} \exp\left(-\ell \int_0^t (c_0x^2 + c_2z^2) ds\right). \end{aligned} \tag{14}$$

By the estimates of the Dulac map and since $z \in \Sigma$, we obtain that the time $t(x, z)$ becomes a function of the variable x and the integral $\int_0^t (c_0x^2 + c_2z^2) ds$ can be expressed as a function q of the variable x . Observe that $q(x) > 0$ for every x . Therefore we get that,

$$y(t) = y_0 e^{-\ell t} e^{-\ell q(x)}. \tag{15}$$

Hence $y(t)$ decreases exponentially fast as before but with a faster rate. All together, we get that the map $D_1 : \Sigma \rightarrow S$ is given by

$$\begin{aligned} D_1(x, y, 1) &= N^{t_{\text{Neu}}(x)}(x, y, 1) \\ &= \left(1, y e^{-\ell(A_2(x, \beta_2) + q(x))}, A_1(x, \beta)\right), \end{aligned} \tag{16}$$

where $\beta = \frac{\beta_0}{\beta_2}$, compare this with (10). The statements from Observation 2.2 for this new version of the map D_1 will also hold. To finish the construction of the Poincaré map we compose now with the map P_2 . Therefore, the full return map $P_{\text{Neu}} : \Sigma \rightarrow \Sigma$ of Σ is given by

$$P_{\text{Neu}}(x, y) = \begin{cases} \left(aA_1(x, \beta) - 1, y e^{-\ell(A_2(x, \beta_2) + q(x)) - \frac{1}{2}}\right), & \text{if } x \in (0, 1]; \\ \left(1 - a|A_1(x, \beta)|, y e^{-\ell(A_2(x, \beta_2) + q(x)) + \frac{1}{2}}\right), & \text{if } x \in [-1, 0), \end{cases} \tag{17}$$

where $\beta = \frac{\beta_0}{\beta_2} \in (0, 1)$.

The matrix DP_{Neu} has eigenvalues $\lambda_1 = a\beta|x|^{\beta-1}$ and $\lambda_2 = e^{-\ell(A_2(x, \beta_2) + q(x))}$. Since $\beta \in (0, 1)$ we have that $\lambda_1 \rightarrow \infty$ as $x \rightarrow 0$. For the other eigenvalue we have that $\lambda_2 < 1$ and $\lambda_2 \rightarrow 0$ as $x \rightarrow 0$ since $(A_2(x, \beta_2) + q(x)) \rightarrow \infty$ as $x \rightarrow 0$. Thus the modified Poincaré map P_{Neu} is hyperbolic when x approaches the origin.

The properties stated before remain true for this new modified return map P_{Neu} like the invariance of the stable foliation given by the vertical lines $\ell_v(c)$ under the map P_{Neu} . Hence, we can express again P_{Neu} as $P_{\text{Neu}}(x, y) = (f_{\text{Neu}}(x), g_{\text{Neu}}(x, y))$, where $f_{\text{Neu}} : I \setminus \{0\} \rightarrow I$

is again a Lorenz-like map with exponent β (see (12)) and $g_{\text{Neu}} : (I \setminus \{0\}) \times I \rightarrow I$ satisfy the same properties stated in the previous section. The existence of a unique a.c.i.p and statistical properties such as exponential decay of correlations for observables with bounded variation for the Lorenz-like map f_{Neu} are well established, see for example [35].

3 The Stable Foliation and the UNI Condition

3.1 Existence and Regularity of the Strong Stable Foliation

In this subsection we will study the properties of the strong stable foliation \mathcal{F}^{ss} for the neutral geometrical Lorenz model we built in Sect. 2.

For the neutral geometrical Lorenz attractor, denoted by Λ_{Neu} , we consider the Lorenz attractor Λ_{Lor} in an open neighbourhood U of the origin. Instead of considering the linearised vector field we consider the vector field given by (4) with the parameters corresponding for model 1 and 2 described in Sect. 2.1 and 2.2, respectively. More precisely, we take an open neighbourhood U in which the cross-section Σ is contained. Then the Dulac map from Σ to S has the form given by (10) and (16) for the models 1 and 2, respectively.

This modification yields a different flow time from the cross-section Σ to S . In the original Lorenz construction we have a logarithmic Poincaré return time but for these modifications we have a polynomial Poincaré return time given by (5). The rest of the construction, however, remains unchanged; that is, the flow constructed from S to Σ is made by a composition of an expansion, a rotation and a translation. Therefore we have the same hitting time $\tau_2(x)$ and thus the full return time for the modified Poincaré map P_{Neu} is given in (6). The modified Poincaré map $P_{\text{Neu}} : \Sigma \rightarrow \Sigma$ is given by (11) and (17) for the model 1 and 2, respectively. We saw that the lines in the y -direction, denoted by $\ell_v(c)$, in the cross-section Σ form the stable foliation which is preserved by the return map P_{Neu} . Thus by quotienting out the stable direction we can rewrite the Poincaré map as a skew-product; that is, $P_{\text{Neu}}(x, y) = (f_{\text{Neu}}(x), g_{\text{Neu}}(x, y))$, where f_{Neu} is a one-dimensional Lorenz-like map.

Lemma 3.1 *If $a_2b_0 < 9a_0b_2$, then the eigenvalues of $DZ_{(x,y,z)}$ satisfy $0 < -\lambda_s < \lambda_u < -\lambda_{ss}$ for all $(x, y, z) \in U$.*

Proof The derivative matrix $DZ_{(x,y,z)}$ of the vector field Z is

$$\begin{pmatrix} 3a_0x^2 + a_1y^2 + a_0z^2 & 2a_1xy & 2a_2xz \\ -2c_0\ell xy & -\ell(1 + c_0x^2 + c_2y^2) & -2c_2\ell yz \\ -2b_0xz & -2b_1yz & -(b_0x^2 + b_1y^2 + 3b_2z^2) \end{pmatrix}.$$

In finding the eigenvalues, we get $\lambda_{ss} = -\ell$ for Model 1 (i.e., $c_0 = c_2 = 0$) and $\lambda_{ss} = -\ell(1 + c_0x^2 + c_2z^2)$ for Model 2 (i.e., $a_1 = b_1 = 0$). In both cases, the other two eigenvalues are $\frac{1}{2} \left(\text{trace}(A) \pm \sqrt{\text{trace}(A)^2 - 4 \det(A)} \right)$ for the submatrix

$$A = \begin{pmatrix} 3a_0x^2 + a_1y^2 + a_0z^2 & 2a_2xz \\ -2b_0xz & -(b_0x^2 + b_1y^2 + 3b_2z^2) \end{pmatrix}.$$

To ensure that these eigenvalues are real and of opposite sign, it suffices to check that

$$\lambda_u \lambda_s = \det(A) = -(3a_0x^2 + a_1y^2 + a_0z^2)(b_0x^2 + b_1y^2 + 3b_2z^2) + 4a_2b_0x^2z^2 < 0.$$

The worst case is when $y = 0$, so we consider the terms not including y (and divide by 3 for simplicity):

$$(a_2b_0 - 3a_0b_2)x^2z^2 - a_0b_0x^4 - a_2b_2z^2 < 0.$$

Divide by z^4 and introduce the new coordinate $u = x^2/z^2$:

$$-Pu^2 + Qu - R < 0 \quad \text{for} \quad P = a_0b_0, Q = a_2b_0 - 3a_0b_2, R = a_2b_2. \tag{18}$$

The left hand side is indeed negative for $u = 0$, and it is negative for all u if the equation $-Pu^2 + Qu - R = 0$ has no real solution, i.e., if the discriminant is negative:

$$0 > Q^2 - 4PR = (a_2b_0 - 3a_0b_2)^2 - 4a_0b_2a_2b_0 = 9a_0^2b_2^2 + a_2^2b_0^2 - 10a_0b_2a_2b_0.$$

Divide by a_2b_0 and use the new coordinate $\gamma = \frac{a_0b_2}{a_2b_0}$. Then we get the inequality

$$0 > 9\gamma^2 - 10\gamma + 1 = \left(3\gamma - \frac{5}{3}\right)^2 - \frac{16}{9},$$

which is equivalent to $|\gamma - \frac{5}{9}| < \frac{4}{9}$. That is, it fails if $\gamma \leq \frac{1}{9}$ or $\gamma \geq 1$. Now consider equality in (18) and we divide by a_2b_0 , which brings it to the form

$$-\frac{a_0}{a_2}u^2 + (1 - 3\gamma)u - \frac{b_2}{b_0} = 0,$$

with solutions

$$u = \frac{a_2}{2a_0} \left(1 - 3\gamma \mp \sqrt{(1 - 3\gamma)^2 - 4\gamma}\right) = \frac{a_2}{2a_0} \left(1 - 3\gamma \mp \sqrt{9\gamma^2 - 10\gamma + 1}\right).$$

If $\gamma \geq 1$, then these solutions are negative, and since $u = x^2/z^2$, this means that there are no solutions $(x, y, z) \in U$. The remaining case $\gamma \leq \frac{1}{9}$ is exactly the excluded case in the lemma. This concludes the proof. \square

Using our assumption $a_2b_0 < 9a_0b_2$ and Lemma 3.1, we obtain that the origin is the only point where we have $\lambda_{ss} = -\ell$ and $\lambda_s = \lambda_u = 0$. Before continuing, we recall the definitions of a partially hyperbolic set and strongly dissipativity.

Definition 3.2 Let Λ be a compact invariant set of $X \in \mathcal{X}^r(M)$, $c > 0$ and $0 < \lambda < 1$. We say that Λ has a (c, λ) -**dominated splitting** if the tangent bundle $T_\Lambda M$ has a DX^t -invariant splitting of sub-bundles

$$T_\Lambda M = E^1 \oplus E^2,$$

such that for all $t > 0$ and $x \in \Lambda$, we have

$$\left\|DX^t|_{E^1_x}\right\| \cdot \left\|DX^{-t}|_{E^2_{X^t(x)}}\right\| < c \cdot \lambda^t. \tag{19}$$

We say that Λ is **partially hyperbolic** if it has a (c, λ) -dominated splitting such that E^1 is uniformly contracting; that is, for some $c > 0$ and all $t > 0$ and every $x \in \Lambda$ it holds

$$\left\|DX^t|_{E^1_x}\right\| < c \cdot \lambda^t. \tag{20}$$

In this case we will denote E^1 by E^s and call it the contracting direction. Also E^2 will be denoted by E^{cu} and called the center-unstable direction.

Definition 3.3 Let G be a C^∞ vector field on \mathbb{R}^3 with a Lorenz-like equilibrium; that is, the eigenvalues of DG_p are real and satisfy $0 < -\lambda_s < \lambda_u < -\lambda_{ss}$. We say that G is **strongly dissipative** if the divergence of the vector field G is strictly negative; that is, there is $c > 0$ such that $\text{Div}(G)(x) \leq -c$ for every x and the eigenvalues of the singularity at p satisfy the constraint

$$\lambda_u + \lambda_{ss} < \lambda_s. \tag{21}$$

Figure 2 shows how the flows given by the Neutral model 1 and 2 send the lines in the y -direction in Σ to lines in the y -direction in S . Thus, its derivative DN^t preserves the lines in the y -direction. Furthermore, by the way the flow from S to Σ was constructed (Fig. 1) we notice that horizontal lines in S ; that is, parallel to the y -axis, are taken to parallel lines to the same axis in Σ . In other words, the flow from S to Σ preserves parallel lines to the y -axis. Since this flow is a composition of a rotation, an expansion and a translation, the derivative of the flow also preserves planes orthogonal to the y -axis. From this we can deduce that the splitting $\mathbb{R}^3 = E \oplus F$, where $E = \{0\} \times \mathbb{R} \times \{0\}$ and $F = \mathbb{R} \times \{0\} \times \mathbb{R}$, is preserved by the flow; i.e., $DN^t(E) = E$ and $DN^t(F) = F$ for any t , where N^t is the flow obtained from Equation (4) with the corresponding parameters for model 1 and 2. Since we have uniform contraction along E ($\|DN^t|_{E_x}\| \leq e^{\lambda_{ss}t}$ with $\lambda_{ss} = -\ell < 0$ for every $x \in U$) and domination of the splitting ($\|DN^t|_{E_x}\| \cdot \|DN^{-t}|_{F_{N^t(x)}}\| \leq e^{\lambda_{ss}-\lambda_s t}$ with $\lambda_{ss} - \lambda_s < 0$ for every $x \in U$) we can conclude that U is partially hyperbolic. It is worth noticing that the origin is the only point that spoils the singular hyperbolicity condition (a set A is singular hyperbolic if all its singularities are hyperbolic and it has volume expanding central direction) since $J_t^{cu}(\bar{0}) = |\det DN^t|_{F_0}| = e^{(\lambda_u + \lambda_s)t} = 1$ (recall $\lambda_s = 0 = \lambda_u$); that is, there is no area expansion along the subbundle F . Hence, Λ_{Neu} is a partially hyperbolic attractor.

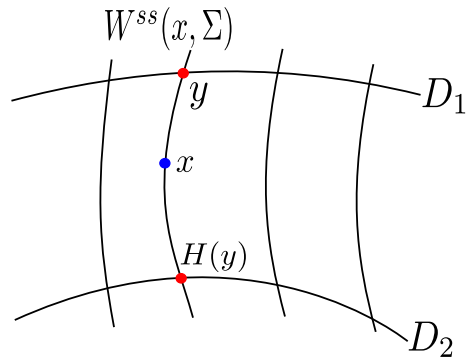
Theorem 6 in [6] provides us with local strong-stable and center-unstable laminations $W_\epsilon^{ss}(x)$ and $W_\epsilon^{cu}(x)$, respectively, through the points $x \in \Lambda_{\text{Neu}} \setminus \{\bar{0}\}$. We note that both, $W_\epsilon^{ss}(x)$ and $W_\epsilon^{cu}(x)$, are embedded discs and hence sub-manifolds of M . Also $W_\epsilon^{ss}(x)$ is uniquely determined since E^s is uniformly contracting. Corollary 6 in [6] shows us that the local strong-stable lamination can be extended to an invariant foliation $\mathcal{F}^{ss}(x)$ of an open neighbourhood of Λ_{Neu} with C^2 leaves and whose foliated charts are C^1 . Moreover, the leaves are uniformly contracted by the action of the flow.

We note that Σ is a C^2 embedded compact disk transversal to the flow N^t . Furthermore, Σ is contained in the open neighbourhood V of Λ_{Neu} . By Theorem 6 and Corollary 6 from [6] we know that local strong-stable lamination $W_\epsilon^{ss}(x)$ extends to an invariant foliation $\mathcal{F}^{ss}(x)$. In this way, for $x \in \Sigma$ we define $W^{ss}(x, \Sigma)$ to be the connected component of $\mathcal{F}^{ss}(x) \cap \Sigma$, where $\mathcal{F}^{ss}(x) = \bigcup_{t \in \mathbb{R}} N^t(\mathcal{F}^{ss}(x))$ is the center-stable leaf. Since the flow $(N^t)_{t \in \mathbb{R}}$ is C^2 , $W^{ss}(x, \Sigma)$ is a C^2 one-dimensional embedded curve for every $x \in \Sigma$ and their leaves form a C^1 foliation \mathcal{F}_Σ^{ss} of Σ .

Given a pair of embedded disks D_1 and D_2 in Σ intersecting transversally a set $\{W^{ss}(x, \Sigma)\}_{x \in \Sigma}$ of stable leaves, the **holonomy map** $H : D_1 \cap W^{ss}(x, \Sigma) \rightarrow D_2 \cap W^{ss}(x, \Sigma)$ assigns to $y \in D_1 \cap W^{ss}(x, \Sigma)$ the unique point in $h(y) \in D_2 \cap W^{ss}(x, \Sigma)$, see Fig. 5.

From the developments in the partially hyperbolicity theory by Brin-Pesin [13] and Pugh-Shub [32] we have that the projection along leaves, also known as holonomies, between pair of transversal surfaces to \mathcal{F}^{ss} have a Hölder continuous Jacobian with respect to Lebesgue measure. This implies a similar conclusion for the holonomies transversal to \mathcal{F}^{cs} . It follows that the holonomy between pairs of transversal curves to \mathcal{F}_Σ^{ss} along the lines of \mathcal{F}_Σ^{ss} can be thought of as interval maps having a Hölder Jacobian. Hence these holonomies are $C^{1+\alpha}$ for

Fig. 5 The holonomy map



some $0 < \alpha < 1$. In this setting, the leaves $W^{ss}(x, \Sigma)$, with $x \in \Sigma$, determine a foliation $\mathcal{F}_{\Sigma}^{ss}$ of Σ with transversal smoothness $C^{1+\alpha}$.

Therefore, we can assume that Σ is the image of the unit square $I \times I$ under the action of a $C^{1+\alpha}$ diffeomorphism h , for some $0 < \alpha < 1$. Furthermore, h sends vertical lines inside the leaves of $\mathcal{F}_{\Sigma}^{ss}$. The next step is to prove that the strong stable foliation $\mathcal{F}^{ss}(x)$ is not only C^1 but also $C^{1+\alpha}$, for some $0 < \alpha < 1$. This was done by Araújo, Melbourne and Varandas in [7], stated as Lemma 2.2. This result is a consequence of domination of the splitting (Equation (19)), uniform contraction along the stable direction (Equation (20)) and strong dissipativity (Definition 3.3). Therefore, we can conclude that the neutral geometrical flow has a strong-stable foliation $\mathcal{F}^{ss}(x)$ which is $C^{1+\alpha}$. Furthermore, the modified return map also has a strong-stable foliation $\mathcal{F}_{\Sigma}^{ss}$, whose transversal smoothness is $C^{1+\alpha}$, with $0 < \alpha < 1$.

A final remark concerning this subsection, Theorem 6 in [6] is stated for a singular hyperbolic attractor. However, the conclusions and arguments still hold true if we consider a compact partially hyperbolic invariant set instead of a singular hyperbolic set. We will also like to mention that the situation for the origin is slightly different, since the splitting of the tangent bundle is given by a one-dimensional strong-stable direction E^s and a two-dimensional center direction E^c . However, we are only concerned with the existence of the strong-stable manifold $W^{ss}(\bar{0})$, which follows from the theory of partial hyperbolicity.

3.2 The UNI Condition

The main goal of this subsection is to show that the stable and unstable manifolds of the modified geometrical model are jointly nonintegrable. The joint nonintegrability of stable and unstable foliations can be interpreted as follows: The stable and unstable foliation of an Anosov flow are always transversal, therefore, if they are jointly integrable, this provides us with a codimension one invariant foliation which is transversal to the flow direction. In contrast, if there exists a codimension one submanifold transversal to the flow direction, then this foliation must be subfoliated by both the stable and unstable foliations. Thus they must be jointly integrable. In this situation it is known [23, Proposition 3.3] that the flow is semiconjugate to a suspension with a locally constant roof-function over a subshift of finite type. Such a flow need not mix! From the work of Araújo, Butterley and Varandas [8], we know that the joint nonintegrability of the stable and unstable manifolds is equivalent to the uniformly nonintegrability (UNI) condition. As we will see later in this section, the UNI condition, stated formally in Definition 3.7, will ensure that the roof function of the suspension flow, which we will use to model the Neutral geometrical Lorenz flow, is not

cohomologous to a constant function; that is, the time it takes for a point in the base to reach the roof of the suspension flow is not the same for every point. This property will guarantee the mixing properties.

In this chapter we will aim to prove the UNI condition. In [5] and [7] Araújo *et al.* prove the UNI condition by exploiting the properties obtained by using hyperbolic times. From now on, when we talk about the neutral geometrical model, we will refer to both models we constructed in Sect. 2.

Let $I = [-1, 1]$ and $f_{\text{Neu}} : I \rightarrow I$ be the one-dimensional Lorenz-like map obtained from the neutral geometrical model. We notice that $\{0\}$ is a nondegenerate² exceptional set for f_{Neu} . From Theorem 4.3 in [5] we know that there are \tilde{X} a neighbourhood of the singularity 0, a countable partition \tilde{Q} of \tilde{X} Lebesgue modulo zero into subintervals, a function $\tau : \tilde{X} \rightarrow \mathbb{Z}^+$ constant on partition elements and the induced map $\tilde{F} = f_{\text{Neu}}^\tau : \tilde{Q} \rightarrow \tilde{X}$ which is a C^2 uniformly expanding diffeomorphism with bounded distortion for any $\tilde{Q} \in \tilde{Q}$.

The motivation behind taking the inducing scheme is that we aim to extend the Gibbs-Markov map \tilde{F} to a two-dimensional Gibbs-Markov map F and build the suspension flow F^t over the map F with roof function R (see Equation (22)). This will allow us to use the results given in [12] to deduce the decay of correlations for the suspension flow F^t and ultimately for the geometrical neutral Lorenz flow N^t . Furthermore, we want to use the properties provided by the hyperbolic times and the bounded distortion of the map \tilde{F} , to obtain the bound stated in Proposition 3.9 for the roof function R , which will help us prove the UNI condition for the geometrical neutral flow. We make now some observations regarding the induced map \tilde{F} .

Observation 3.4 *The map \tilde{F} is obtained by inducing f_{Neu} on the interval \tilde{X} , and the inducing time is given by the sum of a hyperbolic time with a non-negative integer bounded by N , where N is such that $\bigcup_{i=1}^N (f_{\text{Neu}}^i)^{-1}(\{0\})$ is 2δ -dense in \tilde{X} . Furthermore, \tilde{F} is a full branch Markov map onto \tilde{X} since 0 has dense preimages under f_{Neu} . For more details we refer the reader to [5].*

In addition to the bounded distortion and uniform expansion of \tilde{F} , we have the following inequalities for f_{Neu} as a consequence of hyperbolic times. Given $\sigma \in (0, 1)$ and $c > 0$, there is a constant $b > 0$ such that:

1. (Backward contraction) *Let $\tilde{Q}(x)$ denotes the element of the partition \tilde{Q} containing x , for $y \in \tilde{Q}(x)$*

$$|f_{\text{Neu}}^i(y) - f_{\text{Neu}}^i(x)| \leq b\sigma^{\tau(x)-i} |\tilde{F}(y) - \tilde{F}(x)|, \quad i = 0, \dots, \tau(x) - 1.$$

2. (Slow recurrence to the singular point)

$$|f_{\text{Neu}}^i(x)| \geq \sigma^{c(\tau(x)-i)}, \quad i = 0, \dots, \tau(y) - 1.$$

Following [7], our next step is to construct a piecewise uniformly hyperbolic map F with infinitely many branches, which covers \tilde{F} . First, let $W_{P_{\text{Neu}}}^{ss}(x)$ denote the stable leaf under the Poincaré map P_{Neu} containing the point x , $\pi : \Sigma \rightarrow I$ the projection map. We define $X = \bigcup \{W_{P_{\text{Neu}}}^{ss}(x) \mid x \in \tilde{X}\}$ as the union of stable leaves along \tilde{X} . We also extend the induced time τ to a function on X denoted also by τ and given by $\tau(x) = \tau(\pi(x))$. We make the following important observation on the tails of τ .

Observation 3.5 *The tails of the return time τ and its extension also denoted by τ are exponential (see [33]); i.e., there exists a constant $c > 0$ such that $\mu_{\tilde{X}}(\tau > n) = \mathcal{O}(e^{-cn})$ and*

² Nondegenerate here is meant for the derivative of the one-dimensional map f_{Neu} and shouldn't be confused with the degeneracy at the singularity 0 concerning the eigenvalues being zero.

$\mu_X(\tau > n) = \mathcal{O}(e^{-cn})$, where $\mu_{\tilde{X}}$ and μ_X are the SRB-measures of the Gibbs-Markov maps F and \tilde{F} , see below.

Now, we construct $F : X \rightarrow X$ the Poincaré map by setting $F(x) = P_{\text{Neu}}^{\tau(\pi(x))}(x)$ for $x \in X$. Furthermore, let \tilde{Q} be the measurable partition of X by taking $\bigcup\{W_{P_{\text{Neu}}}^{ss}(x) \mid x \in \tilde{Q}\}$ as its elements, with $\tilde{Q} \in \tilde{\mathcal{Q}}$. We will make use of X and F when making the model of the neutral geometrical Lorenz flow by a suspension flow.

It is standard [33] that the map \tilde{F} has a unique a.c.i.p measure $\mu_{\tilde{X}}$ on \tilde{X} . Furthermore, $r_{\text{Neu}} \in L^1(\mu_{\tilde{X}})$ and there exists a unique invariant measure μ_X for F , μ_Σ for P_{Neu} and μ_I for f_{Neu} satisfying $\pi_*(\mu_X) = \mu_{\tilde{X}}$, $\pi_*(\mu_\Sigma) = \mu_I$, and also

$$\begin{aligned} \mu_\Sigma &= \sum_{n \geq 1} \sum_{i=0}^{n-1} (P_{\text{Neu}}^i)_*(\mu_X | \{\tau \circ \pi = n\}), \\ \mu_I &= \sum_{n \geq 1} \sum_{i=0}^{n-1} (f_{\text{Neu}}^i)_*(\mu_{\tilde{X}} | \{\tau \circ \pi = n\}). \end{aligned}$$

Moreover, $\mu_X \ll \mu_\Sigma$ and $\mu_X(X) = 1$, thus $\mu_\Sigma(X) > 0$. We take the induced roof function $R : X \rightarrow \mathbb{R}^+$ given by

$$R(x) = \sum_{k=0}^{\tau(x)-1} r_{\text{Neu}}(P_{\text{Neu}}^k(x)). \tag{22}$$

Notice that R is constant along stable leaves because r_{Neu} is constants along stable leaves. We also call R the quotient induced roof function $R : \tilde{X} \rightarrow \mathbb{R}^+$. With this in mind we can state the definition of the UNI condition. First, we give the definition of suspension flow.

Definition 3.6 Let (Σ, ν) be a probability space and $P : \Sigma \rightarrow \Sigma$ an ergodic measure-preserving transformation. Let $r : \Sigma \rightarrow \mathbb{R}^+$ be a measurable (Hölder continuous) roof function. We define the **suspension space** as $\Sigma^r = \{(x, u) \in \Sigma \times [0, r(x)]\} / \sim$, where $(x, r(x)) \sim (P(x), 0)$. The **suspension flow** $f_t : \Sigma^r \rightarrow \Sigma^r$ is given by $f_t(x, u) = (x, u + t)$ computed modulo identifications and the measure $\mu = \nu \times \lambda$, where λ is the Lebesgue measure, is ergodic and f_t -invariant. In the finite measure case, we normalise by $\bar{r} = \int_\Sigma r d\mu$ so that $\mu = \frac{\nu \times \lambda}{\bar{r}}$ is a probability measure.

Definition 3.7 Let $R : X \rightarrow \mathbb{R}^+$ be a roof function as above, $F^t : X^R \rightarrow X^R$ the suspension flow built over $F : X \rightarrow X$, $R_n(x) = \sum_{i=0}^{n-1} R \circ F^i(x)$. Define $\psi_{h_1, h_2} = R_n \circ h_1 - R_n \circ h_2 : X \rightarrow \mathbb{R}$, for $h_1, h_2 \in \mathcal{H}_n$; that is, inverse branches of F^n . Then the **UNI condition** holds if there exist $D > 0$ and $h_1, h_2 \in \mathcal{H}_{n_0}$ for some sufficiently large $n_0 \geq 1$, such that $|\psi'_{h_1, h_2}| \geq D$.

We saw in the previous subsection that the Poincaré map $P_{\text{Neu}} : \Sigma \rightarrow \Sigma$ has a strong stable foliation. We observe that the leaves of this foliation cross Σ , hence the induced map F has a strong stable manifold $W_F^{ss}(x) = W_{P_{\text{Neu}}}^{ss}(x)$ that crosses X . Araújo et al. in [7, Proposition 2.4] provide us with local unstable manifolds of uniform size for F and defined almost everywhere, and by [7, Proposition 2.4] we obtain a local product structure, see Fig. 6.

To use the arguments given in [7] for the temporal distortion function we need to adapt the proofs of the uniform bound of the derivative of the induced roof function R and a bound on the flow time r_{Neu} . An adjustment is required because we are changing the flow time

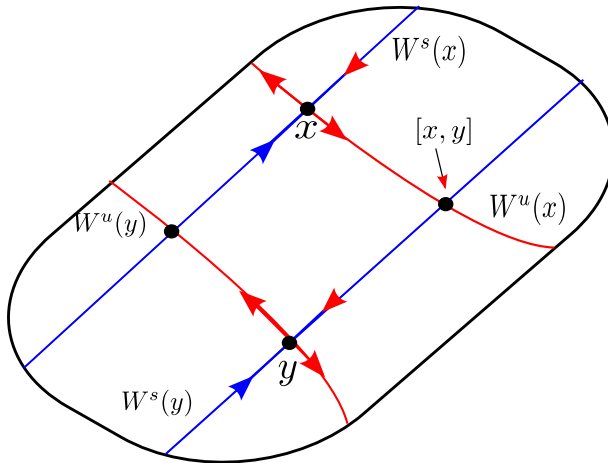


Fig. 6 Local product structure for F

from Σ to S . Recall that the original flow time for the geometrical Lorenz model is given by (3), whereas the modified flow time is given by (6); that is, we change from logarithmic to polynomial. More precisely, we have the following propositions.

Proposition 3.8 *Let $\tilde{Q} \in \tilde{\mathcal{Q}}$ and $\tilde{F} : \tilde{\mathcal{Q}} \rightarrow \tilde{X}$ as above. Denote by \mathcal{H} the set of all inverse branches of \tilde{F} . Then we have*

$$\sup_{h \in \mathcal{H}} \sup_{x \in \tilde{X}} |D(R \circ h)(x)| < \infty.$$

Proof Let $\tilde{Q} \in \tilde{\mathcal{Q}}$ and $h \in \mathcal{H}$; that is, $h : \tilde{X} \rightarrow \tilde{\mathcal{Q}}$ be an inverse branch of \tilde{F} with inducing time $\tau = \tau(\tilde{Q}) \geq 1$ and fix $x \in \tilde{\mathcal{Q}}$. We first observe that

$$\begin{aligned} |D(R \circ h)(x)| &= |DR(h(x))| \cdot |Dh(x)| = \frac{|DR(h(x))|}{|D\tilde{F}(h(x))|} \\ &= \left| \sum_{i=0}^{\tau-1} \frac{(Dr_{\text{Neu}} \circ f_{\text{Neu}}^i) \cdot Df_{\text{Neu}}^i \circ h(x)}{D\tilde{F}} \right|. \end{aligned}$$

From the construction of the inducing partition using hyperbolic times (for more details on hyperbolic times see [2] and [5]), we have that backward contraction and slow recurrence to the singular point, see Observation 3.4; that is, there are constants $\sigma \in (0, 1)$, $b_0 > 0$ and $c \in (0, \frac{1}{2}]$ such that

1. (Backward contraction) For $y \in \tilde{\mathcal{Q}}(x)$

$$|f_{\text{Neu}}^i(y) - f_{\text{Neu}}^i(x)| \leq b_0 \sigma^{\tau-i} |\tilde{F}(y) - \tilde{F}(x)|, \quad i = 0, \dots, \tau - 1.$$

2. (Slow recurrence to the singular point)

$$|f_{\text{Neu}}^i(x)| \geq \sigma^{c(\tau-i)}, \quad i = 0, \dots, \tau - 1.$$

Notice that

$$\left| \frac{Df_{\text{Neu}}^i}{D\tilde{F}} \circ h(x) \right| \leq b_0 \sigma^{\tau-i} \quad i = 0, \dots, \tau - 1, \tag{23}$$

where the inequality follows from the backward contraction.

Moreover, from the slow recurrence to the singularity we also get the following inequality;

$$|(Dr_{\text{Neu}} \circ f_{\text{Neu}}^i) \circ h(x)| \leq b_1 \sigma^{-c(1+\frac{1}{\beta_2})(\tau-i)} \quad i = 0, \dots, \tau - 1, \tag{24}$$

for some constant $b_1 > 0$. Altogether, Equations (23) and (24) imply that

$$|D(R \circ h)(x)| \leq b \sum_{i=0}^{\tau-1} \sigma^{si}, \tag{25}$$

where $s = 1 - c(1 + \frac{1}{\beta_2})$. Since $0 < c \leq \frac{1}{2}$ and $\beta_2 > 2$, we have $s < 1$. Therefore the sum converges, and we have that

$$\sup_{h \in \mathcal{H}} \sup_{x \in \tilde{X}} |D(R \circ h)(x)| < \infty.$$

□

For $x, y \in \tilde{X}$ we define the separation time $s(x, y)$ as the least integer $n \geq 0$ such that $\tilde{F}^n(x)$ and $\tilde{F}^n(y)$ are in different partition elements of \mathcal{Q}_0 . For given $0 < \eta < 1$, the symbolic metric is defined on \tilde{X} as $d_\eta(x, y) = \eta^{s(x,y)}$. Finally, we set $r_{\text{Neu}}^{(k)}(x) = \sum_{i=0}^{k-1} r_{\text{Neu}}(P_{\text{Neu}}^i(x))$.

Proposition 3.9 *There exists $B > 0$ such that for all $x, y \in \tilde{X}$ with $s(x, y) \geq 1$ and $0 \leq k \leq \tau(x) = \tau(y)$ we have $|r_{\text{Neu}}^{(k)}(x) - r_{\text{Neu}}^{(k)}(y)| \leq B|\tilde{F}(x) - \tilde{F}(y)|^\epsilon$. Consequently, there is $\eta \in (0, 1)$ such that $|R|_\eta < \infty$, where $|R|_\eta = \sup_{x \neq y} \frac{|R(x) - R(y)|}{d_\eta(x, y)}$ denotes the Lipschitz constant of the quotient induced roof function $R : \tilde{X} \rightarrow \mathbb{R}^+$ with respect to d_η . Moreover, $|\tilde{F}(x) - \tilde{F}(y)| \leq B d_\eta(x, y)$.*

Proof For convenience, in this proof f will denote f_{Neu} . Let $x, y \in \tilde{X}$ such that $s(x, y) = n \geq 1$ and $0 \leq k \leq \tau(x) = \tau(y)$. Thus $y \in \tilde{Q}^n(x)$, where $\tilde{Q}^n(x) = \bigcap_{i=0}^{n-1} (\tilde{F}^i)^{-1}(\tilde{Q}(x))$

is the n th refinement of $\tilde{Q}(x)$, and so $\tau(\tilde{F}^i(x)) = \tau(\tilde{F}^i(y))$ for $i = 0, \dots, n - 1$. Hence, the choice of the cross-section assures that r_{Neu} is constant along stable leaves and that $r_{\text{Neu}}(x) = |x|^{-\frac{1}{\beta_2}} h(x) + \tau_2(x)$, where $h(x)$ is bounded and bounded away from zero. In fact, $h(x)$ is of the form $h_0 + \mathcal{O}(|x|^\gamma)$ where h_0 is a positive constant and $\gamma > 0$ depending on whether the higher order terms are considered in (7) or not. Also $h(x)$ is differentiable for $x > 0$, because the Dulac map is differentiable. Then we can write

$$\begin{aligned} |R(x) - R(y)| &\leq \sum_{i=0}^{\tau(x)-1} |r_{\text{Neu}}(f^i(x)) - r_{\text{Neu}}(f^i(y))| \\ &\leq \sum_{i=0}^{\tau(x)-1} \left| |f^i(x)|^{-\frac{1}{\beta_2}} h(f^i(x)) - |f^i(y)|^{-\frac{1}{\beta_2}} h(f^i(y)) \right| \\ &\quad + |\tau_2(f^i(x)) - \tau_2(f^i(y))|. \end{aligned}$$

We first notice that,

$$\begin{aligned} \left| \tau_2(f^i(y)) - \tau_2(f^i(x)) \right| &\leq \|\tau_2\|_\epsilon \left| f^i(y) - f^i(x) \right|^\epsilon \\ &\leq \sigma^{\epsilon(\tau(x)-i)} \|\tau_2\|_\epsilon \left| \tilde{F}^i(y) - \tilde{F}^i(x) \right|^\epsilon. \end{aligned} \tag{26}$$

The second inequality in (26) follows from Observation 3.4-1. Now, denote $|f^i(x)|^{-\frac{1}{\beta_2}}$ $h(f^i(x)) - |f^i(y)|^{-\frac{1}{\beta_2}} h(f^i(y))$ by A , then we obtain that,

$$A \leq \left| |f^i(x)|^{-\frac{1}{\beta_2}} - |f^i(y)|^{-\frac{1}{\beta_2}} \right| |h(f^i(x))| + |f^i(y)|^{-\frac{1}{\beta_2}} |h(f^i(x)) - h(f^i(y))|. \tag{27}$$

We notice that $\left| |f^i(x)|^{-\frac{1}{\beta_2}} - |f^i(y)|^{-\frac{1}{\beta_2}} \right|$ is bounded. Indeed we have the following:

$$\begin{aligned} \left| |f^i(x)|^{-\frac{1}{\beta_2}} - |f^i(y)|^{-\frac{1}{\beta_2}} \right| &= |f^i(x)|^{-\frac{1}{\beta_2}} \left| 1 - \left(\frac{|f^i(y)|}{|f^i(x)|} \right)^{-\frac{1}{\beta_2}} \right| \\ &\leq |f^i(x)|^{-\frac{1}{\beta_2}} \left| 1 - \left(1 + \frac{|f^i(y) - f^i(x)|}{|f^i(x)|} \right)^{-\frac{1}{\beta_2}} \right| \\ &\leq |f^i(x)|^{-\frac{1}{\beta_2}} \frac{C_0}{\beta_2} \frac{|f^i(y) - f^i(x)|}{|f^i(x)|} \\ &= \frac{C_0}{\beta_2} |f^i(x)|^{-(1+\frac{1}{\beta_2})} |f^i(y) - f^i(x)| \\ &\leq \frac{C_0}{\beta_2} \sigma^{(1-c(1+\frac{1}{\beta_2}))(\tau(x)-i)} |\tilde{F}(y) - \tilde{F}(x)|, \end{aligned} \tag{28}$$

where $C_0 > 0$ and $0 < c \leq \frac{2}{3}$. The first and second inequalities follow from the Bernoulli inequality and from Observation 3.4-1 and 2, respectively. Since $h(f^i(x))$ is bounded and by (28), we can bound the first term in the sum of (27) by $\frac{C_0}{\beta_2} \sigma^{(1-c(1+\frac{1}{\beta_2}))(\tau(x)-i)} |\tilde{F}(y) - \tilde{F}(x)|$.

To finish the proof it remains to find a bound for

$$|f^i(y)|^{-\frac{1}{\beta_2}} |h(f^i(x)) - h(f^i(y))|. \tag{29}$$

Notice that $|h(f^i(x)) - h(f^i(y))| \approx h'(\xi) |f^i(x) - f^i(y)|$, with $f^i(x) < \xi < f^i(y)$. Hence, by Observations 3.4-1 and 2, we have that (29) is bounded by $C_0 h'(\xi) \sigma^{(1-\frac{c}{\beta_2})(\tau(x)-i)} |\tilde{F}(y) - \tilde{F}(x)|$. Assuming that $h'(\xi)$ is bounded and combining all the previous bounds we have that,

$$\begin{aligned} |R(x) - R(y)| &\leq C \sum_{i=0}^{\tau(x)-1} [(\sigma^{s(\tau(x)-i)} + \sigma^{u(\tau(x)-i)}) |\tilde{F}(y) - \tilde{F}(x)| \\ &\quad + \sigma^{\epsilon(\tau(x)-i)} \|\tau_2\|_\epsilon |\tilde{F}^i(y) - \tilde{F}^i(x)|^\epsilon] \\ &\leq B |\tilde{F}^i(y) - \tilde{F}^i(x)|^\epsilon, \end{aligned}$$

for some constant $B > 0$ where $s = 1 - c(1 + \frac{1}{\beta_2}) > 0$ and $u = 1 - \frac{c}{\beta_2}$. As in the previous proof, the sum converges since $\beta_2 > 2, 0 < c \leq \frac{2}{3}$ and hence $0 < s, u < 1$. This establishes what we were aiming to prove.

One caveat we have to make here is that this argument is only valid if we assume boundedness for h' . This is true if the higher order terms in (7) are not present. If higher order terms

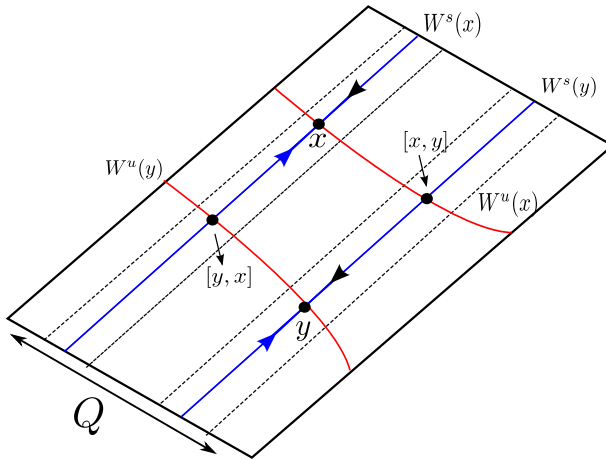


Fig. 7 Local product structure for F

are present, boundedness of h' is plausible, but since the required perturbation argument in [17] is less constructive so as to immediately derive this boundedness, we will try to convince the reader with the numeric analysis performed at the end of this work that this boundedness is indeed true. Thus, rather than a rigorous proof we give a combination of mathematical arguments and numerical verification. \square

Now let $Q \in \mathcal{Q}$ be a partition elements for F . The temporal distortion function $T : Q \times Q \rightarrow \mathbb{R}$ is defined almost everywhere by,

$$\begin{aligned}
 T(x, y) &= \sum_{i=-\infty}^{\infty} [r_{\text{Neu}}(P_{\text{Neu}}^i(x)) - r_{\text{Neu}}(P_{\text{Neu}}^i([x, y])) - r_{\text{Neu}}(P_{\text{Neu}}^i([y, x])) \\
 &\quad + r_{\text{Neu}}(P_{\text{Neu}}^i(y))] \\
 &= \sum_{i=-\infty}^{-1} [r_{\text{Neu}}(P_{\text{Neu}}^i(x)) - r_{\text{Neu}}(P_{\text{Neu}}^i([x, y])) - r_{\text{Neu}}(P_{\text{Neu}}^i([y, x])) \\
 &\quad + r_{\text{Neu}}(P_{\text{Neu}}^i(y))], \tag{30}
 \end{aligned}$$

where $[x, y]$ is the local product of x and y (see Fig. 7). The second inequality follows from the property of r_{Neu} of being constant along stable leaves.

Now, for every $x, y \in X$ in the same unstable manifold for $F : X \rightarrow X$ we define

$$T_0(x, y) = \sum_{i=1}^{\infty} [r_{\text{Neu}}(P_{\text{Neu}}^{-i}(x)) - r_{\text{Neu}}(P_{\text{Neu}}^{-i}(y))]. \tag{31}$$

The continuity and other properties of T_0 are stated in [7, Lemma 3.1]. Furthermore, we can rewrite the temporal distortion function $T(x, y)$ in terms of T_0 ; that is,

$$T(x, y) = T_0(x, [x, y]) + T_0(y, [y, x]).$$

The main result concerning the temporal distortion function establishes the joint nonintegrability of the stable and unstable foliations for the flow by proving that the temporal

distortion function T is not identically zero, that is, there is $Q \in \mathcal{Q}$ and $x, y \in Q$ such that $T(x, y) \neq 0$ and is stated and proven in [7, Theorem 3.4].

We adjusted the proof of the uniform bound of the derivative of the induced roof function for the geometrical neutral Lorenz model since we want to apply [9, Proposition 7.4] in order to use the same arguments given in [3, Corollary 4.3]. Thus, we get the UNI condition for the geometrical neutral Lorenz flow. For completeness we state it in the following theorem.

Theorem 3.10 *The UNI condition holds for the geometrical neutral Lorenz flow.*

For fixed $x \in X$, we define the map $h : W_F^u(x) \rightarrow \mathbb{R}$ given by

$$h(y) = T(x, y) = T_0(x, [x, y]) + T_0(y, [y, x]),$$

the map h is C^1 . Furthermore, there exists a nonempty open set $U \subset W_F^u(x)$ such that $h|_U$ is a C^1 diffeomorphism. For the proofs of the properties of the map h see Proposition 3.6 and Corollary 4.7 in [7].

The next result will be of great help in proving the decay of correlations for the geometrical neutral Lorenz flow. The original statement involves the geometrical Lorenz flow, but the same arguments can be used to prove the same result for our setting. Before stating the result we give a definition.

Definition 3.11 Let X and F be as in the beginning of Section 6.3. A subset $Z_0 \subset X$ is called a **finite subsystem** of X if $Z_0 = \bigcap_{n \geq 0} F^{-n}Z$, where Z is the union of finitely many partition elements of X .

Let Q_1 and $Q_2 \in \mathcal{Q}$ be two partition elements and consider $Q = Q_1 \cup Q_2$. We define the finite subsystem $Q_0 = \bigcap_{n \geq 0} F^{-n}Q$, then we have the following:

Proposition 3.12 [7, Proposition 3.8] *For the finite subsystem Q_0 , the set $T(Q_0 \times Q_0)$ has positive lower box dimension.*

We will like to end this Section with the following remark. To establish their results on decay of correlations, Bálint et al. in [12] and Melbourne in [30] assumed a very important, yet technical property namely, absence of approximate eigenfunctions. They also provide some criteria that guarantees the absence of approximate eigenfunctions. The first one, involves the temporal distortion function providing a nonintegrability condition. This criteria is given by Proposition 3.12; that is, when the temporal distortion function is not identically zero. In other words, when the UNI condition is satisfied. The second one, is a Diophantine condition on the periods of three periodic solutions [21], which is satisfied with probability one. It is important to remark that from these criteria, the UNI condition is robust while the Diophantine condition is not.

4 Decay of Correlations

In this section we prove Theorem 1.1 for the first two neutral models. The third and more general model will be analysed in Sect. 5. We will use the results of Bálint et al. [12] to prove our theorem. In [12] polynomial decay of correlations for non-uniformly hyperbolic flows is proven under absence of approximate eigenfunctions. Let us start by giving the description of a non-uniformly hyperbolic flow described in [12].

First, we observe that the geometrical neutral Lorenz flow $N^t : \Lambda_{\text{Neu}} \rightarrow \Lambda_{\text{Neu}}$, where Λ_{Neu} is the geometrical neutral Lorenz attractor, can be modelled as the suspension flow

$S^t : \Sigma^{\text{Neu}} \rightarrow \Sigma^{\text{Neu}}$ over the Poincaré map P_{Neu} with base the cross-section Σ and roof function r_{Neu} from (6). However, in order to use the results of [12], we take the alternative model $F^t : X^R \rightarrow X^R$, where $X \subset \Sigma$ is a cross-section to the flow with nice hyperbolic structure (local product structure) and with induced roof function $R : X \rightarrow \mathbb{R}^+$ given by $R(x) = \sum_{k=0}^{\tau(x)-1} r_{\text{Neu}}(P_{\text{Neu}}^k(x))$, see Sect. 3.2. Then the suspension flow F^t built over the uniformly hyperbolic map $F : X \rightarrow X$ is identical to the suspension flow S_t , thus F_t is an extension of the underlying flow, namely the neutral geometrical Lorenz flow. Within this framework, N^t is called in [12] a **non-uniformly hyperbolic flow**.

Under suitable conditions it can be shown that the suspension flow F^t described above is a Gibbs-Markov flow [12, Section 6]. Therefore, the mixing rates for non-uniformly hyperbolic flows can be deduced from the corresponding results for Gibbs-Markov flows, see [12, Corollary 8.1].

For observables v and w , let $\rho_t(v, w)$ denote the decay of correlations of the geometrical neutral Lorenz flow; that is,

$$\rho_t(v, w) = \left| \int v \cdot w \circ N^t d\mu - \int v d\mu \int w d\mu \right|, \tag{32}$$

where μ is the SRB measure of N^t . Before giving the proof of Theorem 1.1 for the neutral models 1 and 2, we will give the definitions of the space of observables.

Let (M, d) be a metric space with $\text{diam}(M) \leq 1$ and define a flow $T^t : M \rightarrow M$ on M . We fix $\eta \in (0, 1]$ and for a given observable $v : M \rightarrow \mathbb{R}$ we define

$$|v|_{C^\eta} = \sup_{x \neq y} \frac{|v(x) - v(y)|}{d(x, y)^\eta},$$

and the norm $\|v\|_{C^\eta} = |v|_\infty + |v|_{C^\eta}$. We define the Banach space of Hölder functions on M by $C^\eta(M)$; i.e.,

$$C^\eta(M) = \{v : M \rightarrow \mathbb{R} \mid \|v\|_{C^\eta} < \infty\}.$$

Furthermore, let

$$|v|_{C^{0,\eta}} = \sup_{\substack{x \in M \\ t > 0}} \frac{|v(T^t(x)) - v(x)|}{t^\eta}$$

and define $\|v\|_{C^{0,\eta}} = |v|_\infty + |v|_{C^{0,\eta}}$. We denote the space of Hölder observables in the flow direction by

$$C^{0,\eta}(M) = \{v : M \rightarrow \mathbb{R} \mid \|v\|_{C^{0,\eta}} < \infty\}.$$

We will say that an observable $w : M \rightarrow \mathbb{R}$ is **differentiable in the flow direction** if

$$\partial_t w = \lim_{t \rightarrow 0} \frac{w \circ T^t - w}{t}$$

exists pointwise. Let $\|w\|_{C^{m,\eta}} = \sum_{k=0}^m \|\partial_t^k w\|_{C^\eta}$. We will denote the space of observables that are m -times differentiable in the flow direction by

$$C^{m,\eta}(M) = \{w : M \rightarrow \mathbb{R} \mid \|w\|_{C^{m,\eta}} < \infty\}.$$

For a Borel set $X \subset M$ we define $C^\eta(X)$ as above by using the restriction of the metric d to X .

Proof Theorem 1.1 We first note that the suspension flow F^t projects to a quotient suspension semiflow $\tilde{F}^t : \tilde{X}^R \rightarrow \tilde{X}^R$, where \tilde{X} is the quotient space obtained from X after quotienting out the stable leaves and $\tilde{F} : \tilde{X} \rightarrow \tilde{X}$ is a Gibbs-Markov map. Proposition 3.9 ensures that the following inequality holds.

$$|\varphi(x) - \varphi(y)| \leq C\gamma^{s(x,y)} \inf_{Q_i} \varphi \quad \text{for all } x, y \in \tilde{Q}_i, i \geq 1, \tag{33}$$

where $\{\tilde{Q}_i\}_{i \geq 0}$ is a countable Lebesgue modulo zero partition into subintervals, $0 < \gamma < 1$ and $s(x, y)$ is the separation time.

Therefore, we have that \tilde{F}^t is a Gibbs-Markov semiflow and consequently that F^t is a Gibbs-Markov flow. Then the conclusion follows from [12, Corollary 8.1].

There are still four details concerning the hypothesis in [12, Corollary 8.1] that we have not mentioned yet. The first one is regarding condition (H); for us this condition holds automatically since R is constant along stable leaves.

The second concerns the absence of approximate eigenfunctions for F^t . Melbourne gave in [30, Chapter 5] sufficient conditions for the absence of approximate eigenfunctions, namely the existence of a finite subsystem with positive lower box dimension. Hence, it follows from Proposition 3.12 and Lemma 8.9 in [12] that F^t has absence of approximate eigenfunctions.

The third concerns the tails of R ; that is, we want to estimate $\mu_X(R > t)$. From Observation 3.5 we know that $\mu_X(\tau > n)$ has exponential tails; i.e., there exists a constant $c_0 > 0$ such that $\mu_X(\tau > n) = \mathcal{O}(e^{-c_0n})$. Moreover, by Theorem 2.1 we have that $\mu_X(r_{\text{Neu}} > t)$ has polynomial tails; that is, there is a constant $c_1 > 0$ such that $\mu_X(r_{\text{Neu}} > t) = \mathcal{O}(c_1 t^{-\beta_2})$, where $\beta_2 = \frac{a_2 + b_2}{2b_2}$. Then by [18, Proposition 5.1] we have that $\mu_X(R > t) = \mathcal{O}((\ln t)^{\beta_2} t^{-\beta_2})$.

The fourth and last detail concerns how to improve the estimates for $\mu_X(R > t)$ and remove the logarithmic term. For this, we make use of [11, Lemma 4.1]. There the settings is made for infinite horizon planar periodic Lorentz gases, for that setting the tails of the flow time (in this work denoted by r_{Neu}) is of order $\mathcal{O}(t^{-2})$. By replacing in [11, Lemma 4.1] the order of the tails from $\mathcal{O}(t^{-2})$ to $\mathcal{O}(c_1 t^{-\beta_2})$ we can use the same proof to remove the logarithmic term. Hence, we have that $\mu_X(R > t) = \mathcal{O}(t^{-\beta_2})$. With this we conclude our proof. \square

The natural questions is about the lower bounds, i.e., if the bounds given in this theorem are sharp.

Although we definitely think they are, the currently available literature is insufficient to conclude this, although the margin is fairly narrow.

In [24], the renewal operator methods are developed to get such lower bounds, but his paper is for maps, not flows. Melbourne and Terhesiu come the closest in [31], where they consider suspension semiflows with polynomial roof functions over Gibbs-Markov base maps, and indeed, their results imply polynomial mixing for the flow $F^t : X^R \rightarrow X^R$, for Hölder observables that are constant on the stable fibers. The step from this suspension flow to the actual flow $N^t : \Lambda_{\text{Neu}} \rightarrow \Lambda_{\text{Neu}}$, however, is not trivial at all. (Here the effect of the second inducing step needs to be undone, in a way).

In [18] this step was taken for discrete suspensions (i.e., Young towers), specifically for billiard maps, but not for flows. However, the lower bounds that we obtain for $F^t : X^R \rightarrow X^R$ as a corollary of the results in [31] are sufficient to prove stable laws (with exponent $1/\beta_2 \in (1, 2]$) for the neutral Lorenz flow $N^t : \Lambda_{\text{Neu}} \rightarrow \Lambda_{\text{Neu}}$, cf. [15, 19].

5 Numerical Analysis and Results

In this section we provide the results of the numerical approximation we obtained for the exponent β of the Dulac map (see Equation (8)) and the exponent β_2 of the tails of the

return map (see Theorem 2.1) for the two-dimensional setting (the setting of [17]) and for the 3-dimensional setting concerning this work.

5.1 Numerics of the 2-Dimensional Case

To verify the existing theoretical asymptotics from [17], we will start the numerical analysis in the 2-dimensional case; that is, we consider the framework of [15] and [17]. There, the following neutral form was considered:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} x(a_0x^\kappa + a_2z^\kappa) \\ -y(b_0x^\kappa + b_2y^\kappa) \end{pmatrix} + \mathcal{O}(4), \tag{34}$$

where a_0, a_2, b_0 and $b_2 > 0$ and $\Delta := a_2b_0 - a_0b_2 \neq 0$. For simplicity we let $\kappa = 2$. For the analysis of the Dulac map close to the neutral equilibrium of Equation (34), we take an unstable leaf $W^u(0, y_0)$ and a stable leaf $W^s(x_0, 0)$, then the Dulac map $D : W^u(0, y_0) \rightarrow W^s(x_0, 0)$, shown in Fig. 3, assigns the first intersection of the integral curve through (x, y_0) with the stable leaf $W^s(x_0, 0)$, where $x \in W^u(0, y_0)$ and T is the flow time; that is, the exit time.

For the setting considered in this work, we will perform the numerical experiments with $x_0 = 1$. In order to corroborate the estimates of the Dulac map given by Equation (8), we expect the numerical experiments to show us that

$$\beta \approx \frac{\ln(y)}{\ln(x)}. \tag{35}$$

We actually will see that

$$\beta = \frac{\ln(y)}{\ln(x)} - \frac{\ln(c(y_0))}{\ln(x)} + \mathcal{O}\left(\frac{1}{\ln(x)}\right). \tag{36}$$

From [15] and [17] we know that the constants $c(y_0)$ are given by a specific formula which is not easy to compute. For this reason, we decided to use the least-squares method to calculate these constants.

For the numerical experiments we will take different values of β and 250 points $x \in [1.0 \times 10^{-5}, 1.0 \times 10^{-4}]$ at the unstable leave $W^u(0, y_0)$ with $y_0 = 1.0$. The integration method we will use for the numerical experiments concerning this work is the so-called Radau quadrature method, to deal with the numerical complications of integrating near a neutral stationary point, see [26].

Figure 8 a), b) show us the approximation of β (the red graph), the adjusted approximation of β (green graph), and the theoretical value of β (blue graph) for $\beta = 0.266$ and $\beta = 0.40$, respectively. The approximation of beta is done by taking the last y value of each integral curve and divide it by the x value ranging in $[1.0 \times 10^{-5}, 1.0 \times 10^{-4}]$, the adjusted beta is calculated using Equation (36). The points-axis corresponds to the 250 x values we considered starting from 1.0×10^{-4} and ending in 1.0×10^{-5} ; that is, point 0 corresponds to the value $x = 1.0 \times 10^{-4}$ and point 250 corresponds to the value $x = 1.0 \times 10^{-5}$. The constants $c(y_0) = \ln(0.8)$ and $c(y_0) = 1.2 \ln(1.1)$ for the adjusted approximation correspond to Fig. 8 a) and b), respectively. The approximations show an error that tends to decrease as we get closer to $x = 0$ as depicted in the graphs. The value $\beta = 0.40$ and $\beta = 0.266$ are obtained by taking $a_0 = 15.0, a_2 = 5.0, b_0 = 1.0$ and $b_2 = 3.0$ and $a_0 = 15.0, a_2 = 6.0, b_0 = 1.0$ and $b_2 = 2.0$, respectively, in the vector field from Equation (34).

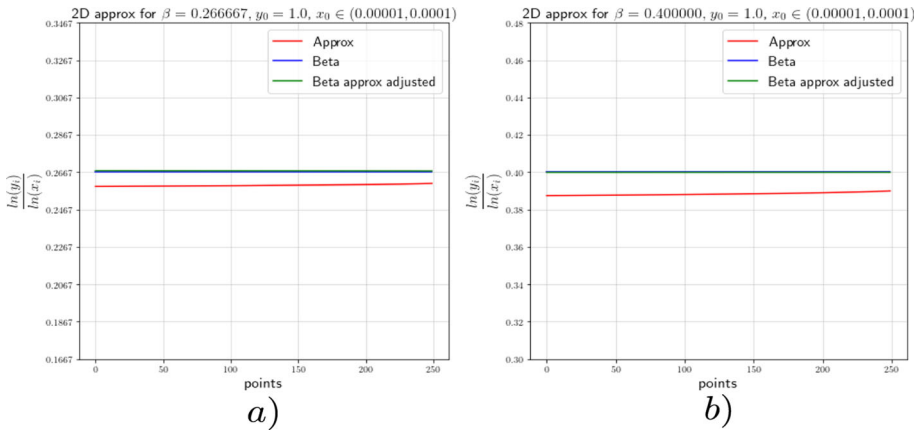


Fig. 8 2-dimensional β approximation

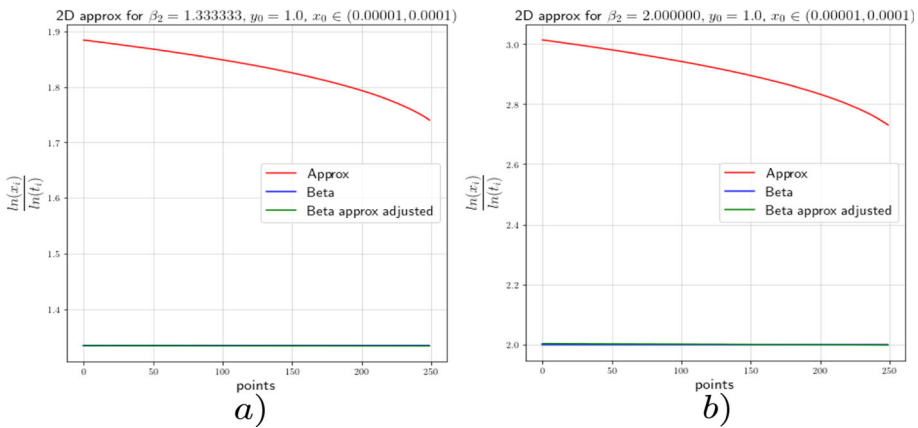


Fig. 9 2-dimensional β_2 approximation

We proceed in the same way to perform the numerical analysis for the exponent β_2 . From the estimates obtained in [17], we can see that,

$$\beta_2 \approx -\frac{\ln(x)}{\ln(t)}, \tag{37}$$

where t is the flow time that it takes a point from the unstable leaf to hit the stable leaf and $x \in [1.0 \times 10^{-5}, 1.0 \times 10^{-4}]$. We will actually show that

$$\beta_2 = \frac{\ln(c(y_0))}{\ln(t)} - \frac{\ln(x)}{\ln(t)} + \mathcal{O}\left(\frac{1}{\ln(t)}\right). \tag{38}$$

Figure 9 shows the approximation of the exponent of the tail of the return map with values $\beta_2 = 0.1333$ and $\beta_2 = 2.0$ which correspond to the values $\beta = 0.400$ and $\beta = 0.266$, respectively. For this case the constants for the adjusted approximations are $c(y_0) = \ln(0.06)$ and $c(y_0) = \ln(0.045)$ for Fig. 9 a) and b), respectively. From the approximations we see, as before, that the error decreases as we approach to $x = 0$.

5.2 Numerics of the 3-Dimensional Case

In this subsection we will perform the numeric experiments for the 3-dimensional models. We will start with Neutral model 1. Recall that the Neutral model 1 was given by Equation (39)

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = N \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x(a_0x^2 + a_1y^2 + a_2z^2) \\ -\ell y \\ -z(b_0x^2 + b_1y^2 + b_2z^2) \end{pmatrix} + \mathcal{O}(4), \tag{39}$$

where $a_0, a_1, a_2, b_0, b_1, b_2$ and $\ell > 0$ and $\Delta := a_2b_0 - a_0b_2 \neq 0$. Precise asymptotics are not available, but since $y(t)$ decreases exponentially fast, the same asymptotics as in (34) are expected, and our numerics indeed confirm this. Note that the strong stable direction of (39) is still purely y -directed.

For the analysis of the Dulac map close to the neutral equilibrium of Equation (39), we will perform the numerical analysis on $N_1 : \Sigma \rightarrow S$. For the purpose of this work, we want to show with the numeric experiment that the x and z components behave like the 2-dimensional model from the previous subsection regardless of the y value. To perform the numerical analysis we will take different unstable leaves $W^u(x, y_0, z_0)$ and a stable leaf $W^s(1, 0, 0)$, where $y_0 = 1.0$ and $z_0 = 1.0$. Hence, like in the 2-dimensional analysis we expect the numerical experiments to show us that,

$$\beta \approx \frac{\ln(z)}{\ln(x)}, \tag{40}$$

where z is the last value of the integral curve with initial condition (x, y_0, z_0) for $x \in [1.0 \times 10^{-5}, 1.0 \times 10^{-4}]$. Again, we will actually show that

$$\beta = \frac{\ln(z)}{\ln(x)} - \frac{\ln(c(z_0))}{\ln(x)} + \mathcal{O}\left(\frac{1}{\ln(x)}\right). \tag{41}$$

As before, we will use the Radau quadrature method and take 250 points for the values of x starting from 1.0×10^{-4} and ending with 1.0×10^{-5} ; that is, point 0 and point 250 correspond to $x = 1.0 \times 10^{-4}$ and $x = 1.0 \times 10^{-5}$, respectively. We will consider the same values of β we considered in the previous subsection. The approximation of β , corresponding to the red line in all figures, is done by taking the last z value of each integral curve with initial condition (x, y_0, z_0) and divide it by the x value ranging in $[1.0 \times 10^{-5}, 1.0 \times 10^{-4}]$, the adjusted β , plotted in green in all figures, is calculated by using Equation (41), and the theoretical value of β , corresponding to the blue graph in all figures, is obtained from the parameters a_0, a_2, b_0 and b_2 as before. The constants $c(z_0)$ were calculated using the least squares method. The constants $c(z_0) = \ln(1.1)$ and $c(z_0) = \ln(1.06)$ correspond to Fig. 10 a) and b), respectively.

Next, we consider the Neutral model 2 given by Equation (42), with the parameters satisfying the usual constraints, and present the numerical results obtained by performing the same experiments we did for the Neutral model 1. We consider this form since it is no longer a skew product like the previous model and poses a new challenge to deduce its asymptotics and u decay of correlations.

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = G \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x(a_0x^2 + a_2z^2) \\ -\ell y(1 + c_0x^2 + c_2z^2) \\ -z(b_0x^2 + b_2z^2) \end{pmatrix} + \mathcal{O}(4). \tag{42}$$

Figure 11 a) and b) show us the the numerical approximations of the Neutral model 2 for $\beta = 0.40$ and $\beta = 0.266$ with constants $c(z_0) = \ln(1.2)$ and $c(z_0) = \ln(1.08)$, respectively.

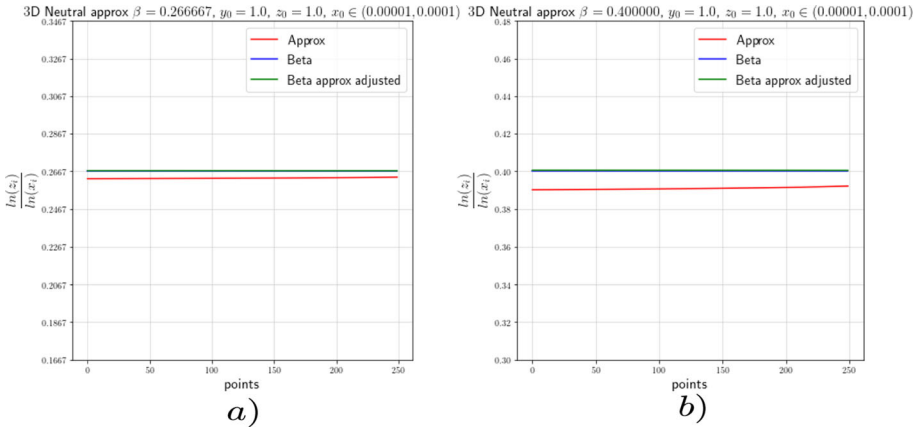


Fig. 10 Neutral model 1 β approximation

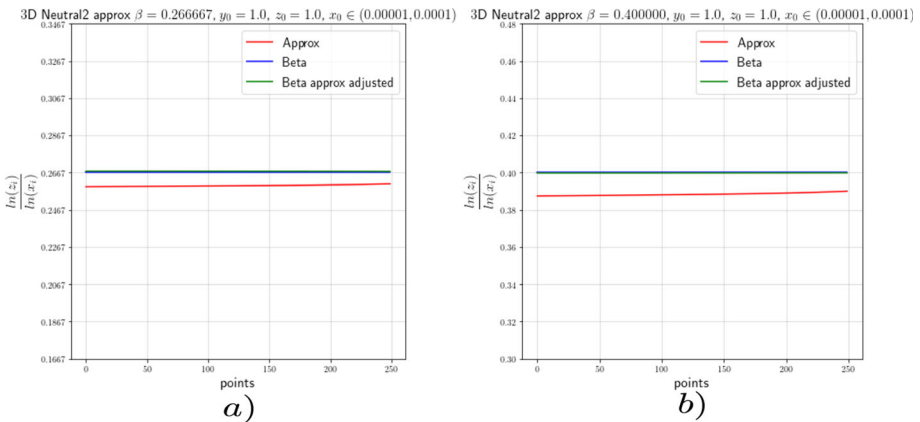


Fig. 11 Neutral model 2 β approximation

We observe that the constants $c(z_0)$ and $c(y_0)$ from Equation (41) and Equation (36), respectively, are almost equal; that is, the x and z components of the 3-dimensional model behaves like the x and y component of the 2-dimensional model.

Until now we have performed the numerical experiments for the Neutral model 1 and 2 corresponding to the normal forms given by Equations (39) and (42), respectively. We saw there that for both models, the asymptotic behaviour of the x and z components are the same as the x and y component of the 2-dimensional model. Recall that for this two models we had explicit formulas for the map $N : \Sigma \rightarrow S$ and hence for the modified Poincaré map $P_{\text{Neu}} : \Sigma \rightarrow \Sigma$. Our goal is to see whether the asymptotic behaviour of the Neutral model 3 given by Equation (4) is similar to the other two models. Therefore, the numerical results obtained for the Neutral models 1 and 2 will be our reference and we will compare them to the numerical results obtained for the third model.

Figure 12 a) and b) show us the the numerical approximations of the Neutral model 2 for $\beta = 0.40$ and $\beta = 0.266$ with constants $c(z_0) = \ln(1.12)$ and $c(z_0) = \ln(1.07)$, respectively. We observe again that the x and z components of the 3-dimensional model behaves like the

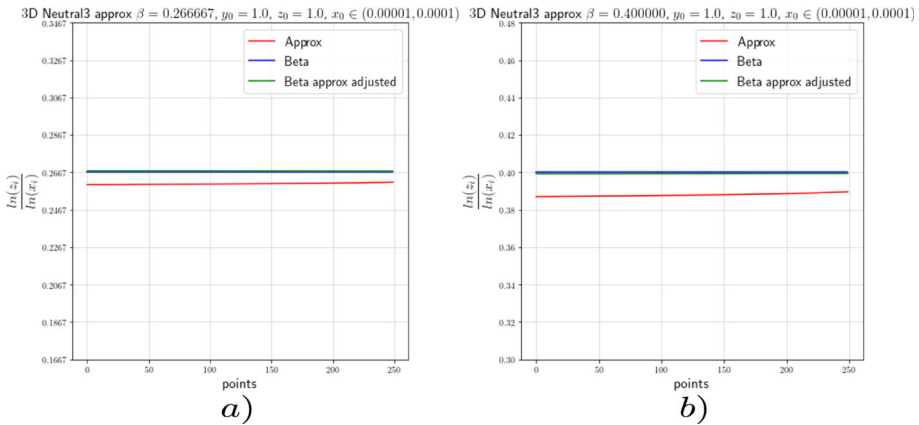


Fig. 12 Neutral model 2 β approximation

x and y component of the 2-dimensional model. From this numerical experiments we can conclude that the behaviour of the map N_1 obtained by considering the neutral model 3 is asymptotically similar to the other two models; that is, the asymptotics of the Dulac map of the three models are similar.

5.3 Numerics of the Tails of the Return Map in the 3-Dimensional Case

Next we will perform the numeric experiments for the 3-dimensional models and see the approximations for the exponent of the decay of correlations; that is, for the exponent β_2 . The general Neutral model or neutral model 3 is given by Equation (4), where $a_0, a_1, a_2, b_0, b_1, b_2, c_0, c_2$ and $\ell > 0$ and $\Delta := a_2b_0 - a_0b_2 \neq 0$. Note that the neutral model 1 and the neutral model 2 are obtained from the general neutral model if we let $c_0, c_2 = 0$ and if we let $a_1, b_1 = 0$, respectively.

In the previous subsection we saw the numerical analysis on $N_1 : \Sigma \rightarrow S$ and showed, with the numeric experimentation, that the x and z components behaves like the 2-dimensional model. For the next numerical analysis, we will take an unstable leaf $W^u(x, y_0, z_0)$ and a stable leaf $W^s(1, 0, 0)$, where $y_0 = 1.0$ and $z_0 = 1.0$. From the estimates obtained in [17] we can see that

$$\beta_2 \approx -\frac{\ln(x)}{\ln(t)}, \tag{43}$$

where t is the flow time that it takes a point from the unstable leaf to hit the stable leaf and $x \in [1.0 \times 10^{-5}, 1.0 \times 10^{-4}]$. We will actually show that

$$\beta_2 = \frac{\ln(c(z_0))}{\ln(t)} - \frac{\ln(x)}{\ln(t)} + \mathcal{O}\left(\frac{1}{\ln(t)}\right). \tag{44}$$

As before, we will use the Radau quadrature method and take 50 points for the values of x starting from 1.0×10^{-4} and ending with 1.0×10^{-5} ; that is, point 0 and point 50 correspond to $x = 1.0 \times 10^{-4}$ and $x = 1.0 \times 10^{-5}$, respectively. We will consider the same values of β we considered in the previous subsection. The approximation of β_2 , corresponding to the red line in all figures, is done by taking the x value, ranging in $[1.0 \times 10^{-5}, 1.0 \times 10^{-4}]$, of each integral curve with initial condition (x, y_0, z_0) and divide it by the flow time t , the adjusted

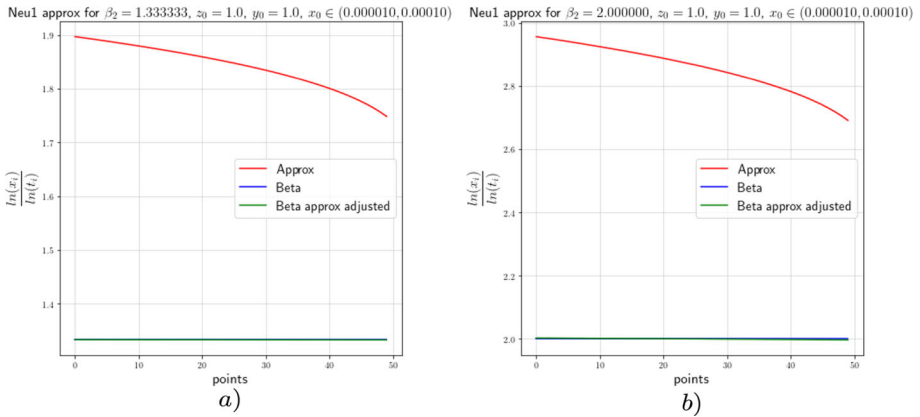


Fig. 13 Neutral model 2 β approximation

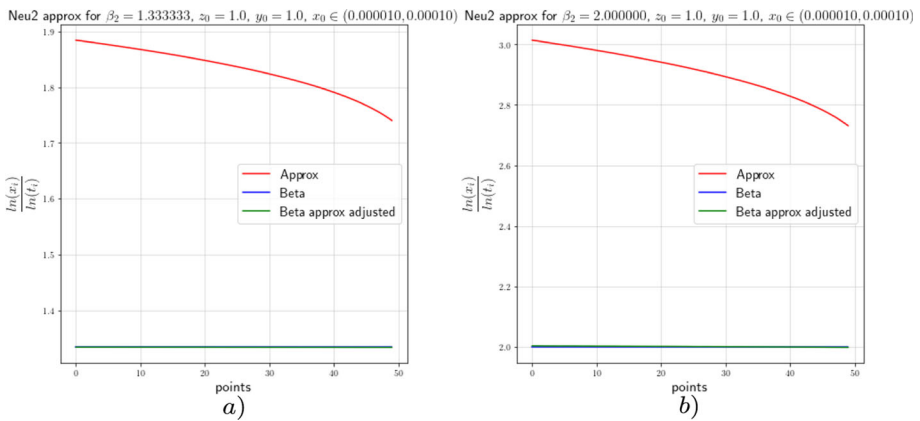


Fig. 14 Neutral model 2 β approximation

approximation of β_2 , shown in green in all figures, is calculated by using Equation (44), and the theoretical value of β_2 , corresponding to the blue graph in all figures, is obtained from the parameters a_2 and b_2 ; that is, $\beta_2 = \frac{a_2 + b_2}{2b_2}$.

We start considering the neutral model 1. Figure 13 a) shows the approximation for $\beta_2 = 1.333$ which corresponds to the case $\beta = 0.40$, for the adjusted approximation the constant is $\ln(c(z_0)) = 0.06$ and b) displays the approximation for $\beta_2 = 2.0$ corresponding to the case $\beta = 0.266$ with adjustment constant $\ln(c(z_0)) = 0.05$.

Next, we consider the neutral model 2. Figure 14 a) and b) show the approximation for $\beta_2 = 1.333$ and $\beta_2 = 2.0$, respectively. Their adjusted approximation the constant are $\ln(c(z_0)) = 0.06$ and $\ln(c(z_0)) = 0.04$, respectively.

Finally, we consider the neutral model 3. Figure 15 a) and b) show the approximation for $\beta_2 = 1.333$ and $\beta_2 = 2.0$, respectively. Their adjusted approximation the constant are $\ln(c(z_0)) = 0.06$ and $\ln(c(z_0)) = 0.04$, respectively.

This numerical experiment has shown us a good approximation of the exponent of the decay of correlations for the 3 neutral models. From this we can deduce the same results concerning the decay of correlations, and obtaining Theorem 1.1 in its full generality.

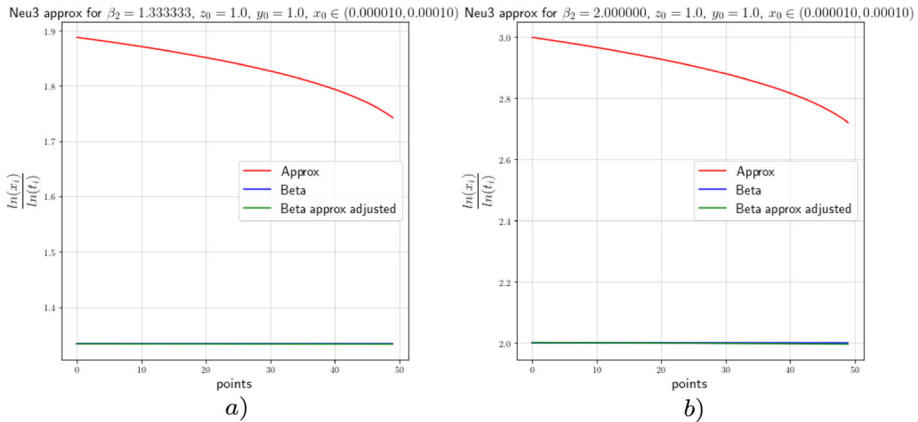


Fig. 15 Neutral model 2 β approximation

Acknowledgements This paper grew out of the thesis work of the second author. We gratefully acknowledge the support of FWF stand-alone grant P31950-N35. We thank the referees for their very useful comments.

Funding Open access funding provided by Austrian Science Fund (FWF).

Data availability The methods of how the numerical graphics were computed are given in the paper. For further details on the hardware and software specifications, as well as the code implemented to produce the graphical results, please see the arXiv version [14].

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

References

1. Afraïmovič, V., Bykov, V., Shilnikov, L.: The origin and structure of the Lorenz attractor. Dokl. Akad. Nauk SSSR **234**(2), 336–339 (1977)
2. Alves, J., Araújo, V.: Hyperbolic times: frequency vs. integrability. Commun. Math. Phys. **311**(1), 215–246 (2012)
3. Araújo, V., Melbourne, I.: Exponential decay of correlations for nonuniformly hyperbolic flows with a $C^{1+\alpha}$ stable foliation. Including the classical Lorenz attractor. Ann. Henri Poincaré **17**(11), 2975–3004 (2016)
4. Araújo, V., Pacifico, M.J.: Three-dimensional flows, vol. 53. Springer Science & Business Media, Germany (2010)
5. Araújo, V., Varandas, P.: Robust exponential decay of correlations for singular-flows. Commun. Math. Phys. **311**(1), 215–246 (2012)
6. Araújo, V., Galatolo, S., Pacifico, M.J.: Decay of correlations for maps with uniformly contracting fibers and logarithm law for singular hyperbolic attractors. Math. Zeitschrift **276**(3–4), 1001–1048 (2014)
7. Araújo, V., Melbourne, I., Varandas, P.: Rapid mixing for the Lorenz attractor and statistical limit laws for their time-1 maps. Commun. Math. Phys. **340**(3), 901–938 (2015)
8. Araújo, V., Butterley, O., Varandas, P.: Open sets of Axiom A flows with exponentially mixing attractors. Proc. Am. Math. Soc. **144**(7), 2971–2984 (2016)

9. Avila, A., Gouëzel, S., Yoccoz, J.-C.: Exponential mixing for the Teichmüller flow. *Publ. Math. Inst.* **104**, 143–211 (2006)
10. Baladi, B., Vallée, B.: Exponential decay of correlations for surface semi-flows without finite Markov partitions. *Proc. Am. Math. Soc.* **133**, 865–874 (2005)
11. Bálint, P., Melbourne, I.: *Decay of correlations for flows with unbounded roof function, including the infinite horizon planar periodic Lorenz gas*, preprint (2010)
12. Bálint, P., Butterley, O., Melbourne, I.: Polynomial decay of correlations for flows, including Lorenz gas examples. *Commun. Math. Phys.* **368**(1), 55–111 (2019)
13. Brin, M.I., Pesin, Y.B.: Partially hyperbolic dynamical systems. *Math. USSR-Izv.* **8**(1), 177 (1974)
14. Bruin, H., Canales Farias H.H.: *Mixing rates of the geometrical neutral Lorenz model*, Preprint (2023) [arXiv:2305.07502](https://arxiv.org/abs/2305.07502)
15. Bruin, H.: *On volume preserving almost Anosov flows*, *Monatsh. Math.* <https://doi.org/10.1007/s00605-022-01807-w> appeared electronically in December (2022)
16. Bruin, H., Terhesiu, D.: The Dolgopyat inequality in BV for non-Markov maps. *Stoch. Dyn.* **18**(02), 1850006 (2018)
17. Bruin, H., Terhesiu, D.: Regular variation and rates of mixing for infinite measure preserving almost Anosov diffeomorphisms. *Ergodic Theo. Dyn. Syst.* **48**(3), 663–698 (2020)
18. Bruin, H., Melbourne, I., Terhesiu, D.: Sharp polynomial bounds on decay of correlations for multidimensional non-uniformly hyperbolic systems and billiards. *Ann. Henri Lebesgue* **4**, 407–451 (2021)
19. Bruin, H., Terhesiu, D., Todd, M.: Pressure function and limit theorems for almost Anosov flows. *Commun. Math. Phys.* **382**, 1–47 (2021)
20. Dolgopyat, D.: On the decay of correlations in Anosov flows. *Annal. Math.* **147**, 357–390 (1998)
21. Dolgopyat, D.: On mixing properties of compact group extensions of hyperbolic systems. *Israel J. Math.* **130**, 157–205 (2002)
22. Dulac, H.: Sur les cycles limites. *Bull. Soc. Math. France* **51**, 45–188 (1923)
23. Field, M., Melbourne, I., Török, A.: Stability of mixing and rapid mixing for hyperbolic flows. *Annal. Math.* 269–291 (2007)
24. Gouëzel, S.: Sharp polynomial estimates for the decay of correlations. *Israel J. Math.* **139**, 29–65 (2004)
25. Guckenheimer, J., Williams, R.: Structural stability of Lorenz attractors. *Inst. Hautes Etudes Sci. Publ. Math.* **50**, 59–72 (1979)
26. Hildebrand, F.B.: Introduction to numerical analysis. Courier Corporation, USA (1987)
27. Hu, H.: Conditions for the existence of SBR measures for “almost Anosov” diffeomorphisms. *Trans. Am. Math. Soc.* **352**(5), 2331–2367 (2000)
28. Hu, H., Young, L.-S.: Nonexistence of SBR measures for some diffeomorphisms that are ‘almost Anosov’. *Ergod. Theo. Dyn. Syst.* **15**(1), 67–76 (1995)
29. Liverani, C.: On contact Anosov flows. *Annal. Math.* **159**(2), 1275–1312 (2004)
30. Melbourne, I.: Superpolynomial and polynomial mixing for semiflows and flows. *Nonlinearity* **31**(10), 268 (2018)
31. Melbourne, I., Terhesiu, D.: Operator renewal theory for continuous time dynamical systems with finite and infinite measure. *Monatsh. Math.* **182**, 377–432 (2017)
32. Pugh, C., Shub, M.: Ergodicity of Anosov actions. *Invent. Math.* **15**(1), 1–23 (1972)
33. Rychlik, M.: Bounded variation and invariant measures. *Stud. Math.* **76**(1), 69–80 (1983)
34. Sarig, O.: Subexponential decay of correlations. *Invent. Math.* **150**, 629–653 (2002)
35. Viana, M.: Stochastic dynamics of deterministic systems, vol. 21. IMPA, Rio de Janeiro (1997)