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Response of a Canonical Ensemble of Quantum Oscillators to a Random Metric

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Abstract
We calculate the susceptibility of a canonical ensemble of quantum oscillators to the singular random metric. If the covariance of the metric is $|\mathbf{x}-\mathbf{x}'|^{-4\alpha}$ $(0<\alpha<\frac{1}{2})$then the expansion of the partition function in powers of the temperature involves non-integer indices.

Keywords Random metric · Random diffusivity · Quantum oscillators

1 Introduction

It is known that the dynamics in an irregular domain can be chaotic and conversely the chaotic motion can have a fractal attractor [1, 2]. Diffusion in fractal domains exhibits their fractal dimension which in general is not a natural number [3–6]. Thermodynamics of a canonical ensemble of particles in an irregular domain depends on the spectral dimension of the domain [7]. In such a case thermodynamic properties (e.g. critical indices) may be functions of a non-integer dimension. For some time fractal geometry has been associated with quantum gravity [5, 8, 9].Quantum gravity can be expressed as a random geometry. It is believed that quantum gravity leads to a fractal geometry by a "foamy" behaviour of the metric at short distances [10]. Such irregular shapes of random figures have been at the basis of the fractal geometry [6].It has been suggested that irregular metric at the Planck scale can modify the short distance behaviour of quantum fields at short distances [5, 10, 11].

In this paper we study quantum oscillators in a random singular metric (or random position dependent mass). We define the susceptibility to the metric which has an expansion in powers of the inverse temperature β if the metric is a regular random field. We show that if the random metric is singular then the susceptibility has an expansion in non-integer powers of β. It is known that an ensemble of oscillators can serve as an approximation to field theory. The quantum statistical mechanics of oscillators will resemble the quantum field theory at finite temperature.

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We consider a Hamiltonian H_γ perturbed around the free theory(harmonic oscillators)

$$H_\gamma = H + \gamma H_1. \tag{1}$$

The statistical expectation value of an observable $\mathcal{A}$ at the temperature $\beta^{-1} = k_B T$ (where k_B is the Boltzman constant) is defined by

$$< \mathcal{A} >_\beta = Z_\gamma^{-1} Tr\Big(\exp(-\beta H_\gamma)\mathcal{A}\Big), \tag{2}$$

where the partition function

$$Z_\gamma = Tr\Big(\exp(-\beta H_\gamma)\Big). \tag{3}$$

For a small γ we have the expansion (till the first order in γ)

$$Z_\gamma = Z_0 - \gamma \int_0^\beta ds Tr\Big(\exp(-\beta H) \exp(sH) H_1 \exp(-sH)\Big).$$

In terms of the partition function we can define other thermodynamic functions as ,e.g.,the internal energy U

$$U_\gamma = -\partial_\beta \ln Z_\gamma. \tag{4}$$

The susceptibility to H_1 can be defined as

$$\chi_\beta = \partial_\gamma (Z_\gamma)_{|\gamma=0}. \tag{5}$$

We can use the expansion

$$\exp(sH)H_1 \exp(-sH) = H_1 + s[H, H_1] + ...$$

in Eq. (5) to see that for a small β

$$\chi_\beta = \partial_\gamma (Z_\gamma)_{|\gamma=0} = -\beta Tr\Big(\exp(-\beta H) H_1\Big). \tag{6}$$

For H_0 of the harmonic oscillator with the frequency ω (the ground state energy subtracted) we have

$$\ln Z_0 = -\ln(1 - \exp(-\beta\omega)) \tag{7}$$

and

$$U = -\partial_\beta \ln Z_0 = \omega(\exp(\beta\omega) - 1)^{-1} \simeq \beta^{-1} \tag{8}$$

for a small β (high temperature).

If the oscillators are the modes of the electromagnetic field in a cavity then $\omega = |\mathbf{k}|c$ where c is the velocity of light and $\mathbf{k}$ is the wave vector in the cavity. In such a case in eqs.(7)–(8) we have a sum over modes. The modified thermodynamics [7] comes from the modified distribution of modes in cavities with fractal geometry.

We consider Hamiltonians H such that its similarity transformation

$$\hat{H} = \Omega^{-1} H \Omega \tag{9}$$

gives a generator of a diffusion process ($\exp(-\beta\hat{H})$ is a Markov semigroup).

According to our assumption

$$(\exp(-\beta\hat{H})\psi)(\xi) = E[\psi(\xi_\beta(\xi))], \tag{10}$$

where ξ are the coordinates of the oscillators, $\xi_\beta(\xi)$ is a Markov process starting from ξ and the expectation value $E[...]$ is over the paths of the process. In our models we consider $\xi = (\mathbf{x}, X) \in R^{n+d}$ and assume that $H_1 = V(P)$ (a function of momentum). Then, according to Eq. (6)

$$\begin{aligned} &Tr\Big(\exp(-\beta\hat{H})V(P)\Big) \\ &\quad = (2\pi)^{-d} E\Big[\delta(\mathbf{x}_\beta(\mathbf{x}) - \mathbf{x})\exp(iP(X_\beta - Y)) < Y|V(P)|X > \Big] d\mathbf{x}dXdYdP, \quad (11) \end{aligned}$$

where

$$< Y|V(P)|X >= (2\pi)^{-d} \int dK \exp(iK(Y - X))V(K). \quad (12)$$

So that

$$Tr(\exp(-\beta\hat{H})V(P)) = (2\pi)^{-d} \int E[\delta(\mathbf{x}_\beta(\mathbf{x}) - \mathbf{x})\exp(iP(X_\beta - X))]V(P)d\mathbf{x}dXdP. \quad (13)$$

2 Statistical Mechanics of Oscillators

Let us consider in R^{n+d} the coordinates ξ^A and a diffusion operator of the form (sum over repeated indices)

$$\hat{H} = -\tfrac{\sigma^2}{2} g^{AB}\partial_A\partial_B + \omega_A\xi^A\partial_A + \tfrac{\omega_D}{2} g^{CD}\partial_C g_{AB}\xi^A\xi^B\partial_D. \quad (14)$$

By means of

$$\Omega = \exp\Big(-\frac{\omega_A}{2\sigma} g_{AB}\xi^A\xi^B\Big)$$

we obtain the Hamiltonian H of Eq. (9)

$$\begin{aligned} H = &-\frac{\sigma^2}{2} g^{AB}\partial_A\partial_B + \frac{\omega_A^2}{2} g_{AB}\xi^A\xi^B \\ &+\sigma\frac{\omega_C}{8} g^{CD}\partial_C g_{RM}\xi^M\xi^R\partial_D g_{AB}\xi^A\xi^B - \frac{1}{2}\sum_A \omega_A. \quad (15) \end{aligned}$$

We divide the coordinates $\xi = (\mathbf{x}, X)$ into two classes $\mathbf{x}$ and X where $\mathbf{x} \in R^n$ and $X \in R^d$. In order to simplify the model we assume that only the X coordinates are coupled to the random metric (or a random mass). So, $(g^{AB}) = (1, g^{\mu\nu})$ and in $g^{\mu\nu}(\mathbf{x})$ the dependence on X is negligible (the coordinates x_j of $\mathbf{x}$ have Latin indices $j = 1, ..., n$ and the coordinates of X the Greek indices $\mu = n + 1, .., n + d$). We can imagine a random mass distribution (producing the metric) which depends only on some coordinates. To make the model simple we assume $\omega_j = \nu$ and $\omega_\mu = \omega$. If ω_A are the modes of a massless field in a cavity which is a rectangular box of sides L_A then $\omega_A = c\frac{2\pi n_A}{L_A}$ where n_A are integers. We could arrange the model so that ω_j are small and the non-linear terms in Eq. (14) with $\omega_j = \nu$ are negligible. Then, $\hat{H}$ is of the form

$$\hat{H} = -\frac{\sigma^2}{2}\nabla_{\mathbf{x}}^2 + \nu\mathbf{x}\nabla_{\mathbf{x}} + \frac{\omega}{2}\nabla_{\mathbf{x}} g_{\mu\nu}X^\mu X^\nu\nabla_{\mathbf{x}} - \frac{\sigma^2}{2} g^{\mu\nu}(\mathbf{x})\partial_\mu\partial_\nu + \omega X^\mu\partial_\mu, \quad (16)$$

where

$$\partial_\mu = \frac{\partial}{\partial X^\mu}.$$

With

$$\Omega = \exp\Big(- \frac{1}{2}\nu\sigma^{-1}\mathbf{x}^2 - \frac{1}{2}\omega\sigma^{-1}g_{\mu\nu}X^\mu X^\nu\Big) \tag{17}$$

the similarity transformation (9) gives

$$\begin{aligned} H = &-\frac{\sigma^2}{2}\nabla_{\mathbf{x}}^2 + \frac{1}{2}\nu^2\mathbf{x}^2 + \sigma\frac{\omega}{8}\nabla_{\mathbf{x}}g_{\mu\nu}X^\mu X^\nu\nabla_{\mathbf{x}}g_{\sigma\rho}X^\sigma X^\rho \\ &-\frac{\sigma^2}{2}g^{\mu\nu}(\mathbf{x})\partial_\mu\partial_\nu + \frac{1}{2}\omega^2 g_{\mu\nu}X^\mu X^\nu - \frac{d}{2}\omega - \frac{n}{2}\nu. \end{aligned} \tag{18}$$

$\exp(-t\hat{H})$ can be expressed (according to Eq. (10)) by the solution of the stochastic equations [12]

$$d\mathbf{x}_t = -\nu\mathbf{x}_t dt - \frac{\omega}{2}\nabla_{\mathbf{x}}g_{\mu\nu}X^\mu X^\nu + \sigma d\mathbf{b}_t, \tag{19}$$

$$dX_t^\mu = -\omega X_t^\mu dt + \sigma e_a^\mu(\mathbf{x}_t)dB_t^a, \tag{20}$$

where we expressed the metric g by vierbeins (tetrads) e

$$g^{\mu\nu} = e_a^\mu e_a^\nu. \tag{21}$$

$(\mathbf{b}_t, B_t)$ is the Brownian motion on R^{n+d} ,i.e., the Gaussian process with mean zero and the covariance

$$E[b_t^j b_s^l] = min(t, s)\delta^{jl}$$

(and similarly for B_t). In order to take the expectation value over the metric we need an explicit solution of Eq. (20). For $\mathbf{x}_t$ we require only some estimates on the behaviour in t . However, for a simplicity of the arguments we neglect the non-linear term in Eq. (19) (we assume that $e_a^\mu = \delta_a^\mu + \kappa\epsilon_a^\mu$, where κ is a small parameter, then $\nabla g \simeq \kappa$). After a negligence of the non-linear term the solution of Eq. (19) with the initial condition $\mathbf{x}$ is

$$\mathbf{x}_t = \exp(-\nu t)\mathbf{x} + \sigma\int_0^t \exp(-\nu(t-s))d\mathbf{b}_s. \tag{22}$$

The solution of Eq. (20) reads

$$X_t^\mu = \exp(-\omega t)X^\mu + \sigma\int_0^t \exp(-\omega(t-s))e_a^\mu(\mathbf{x}_s)dB_s^a. \tag{23}$$

The kernel K of $\exp(-\beta\hat{H})$ can be expressed by means of the Fourier transform

$$\begin{aligned} K_\beta(\mathbf{x}, X; \mathbf{y}, Y) = (2\pi)^{-d}\int dPE\Big[\delta(\mathbf{x}_\beta(\mathbf{x}) - \mathbf{y})\exp\Big(iP(\exp(-\beta\omega)X \\ +\sigma\int_0^\beta \exp(-\omega(\beta-s))e_a(\mathbf{x}_s)dB^a - Y)\Big)\Big]. \end{aligned} \tag{24}$$

3 Random Metric

We assume that e_a^μ are Gaussian variables with the mean δ_a^μ

$$e_a^\mu = \delta_a^\mu + \kappa\epsilon_a^\mu \tag{25}$$

where

$$< \epsilon_a^\mu(\mathbf{x})\epsilon_c^\nu(\mathbf{x}') >= \delta_{ac}^{\mu\nu} G(\mathbf{x} - \mathbf{x}'). \tag{26}$$

We calculate the Gaussian integral in Eqs. (13) and (24) (with $\delta_{ac}^{\mu\nu} = \delta^{\mu\nu}\delta_{ac}$ chosen for simplicity)

$$\begin{aligned}
&< \exp(iP(\exp(-\omega\beta)X + \sigma\int_0^\beta \exp(-\omega(\beta - s))e_a(\mathbf{x}_s)dB^a)) > \\
&\quad = \exp\Big(iP\exp(-\omega\beta)X - \frac{1}{2} < (PQ_\beta)^2 > + i\sigma P_\mu\int_0^\beta \exp(-\omega(\beta - s))dB_s^\mu\Big)) \\
&\quad = \exp\Big(- \frac{1}{2}\kappa^2 P_\mu P_\nu \int_0^\beta\int_0^\beta \exp(-\omega(\beta - s))\exp(-\omega(\beta - s'))G(\mathbf{x}_s - \mathbf{x}_{s'})dB_s^\mu dB_{s'}^\nu \\
&\qquad + iP\exp(-\omega\beta)X + i\sigma P_\mu\int_0^\beta \exp(-\omega(\beta - s))dB_s^\mu\Big).
\end{aligned} \tag{27}$$

In Eq. (27) we used the formula for Gaussian expectation value of ϵ

$$< \exp(iP_\mu Q_\beta^\mu) >= \exp\left(- \frac{1}{2} < (P_\mu Q_\beta^\mu)^2 > \right) \tag{28}$$

with

$$P_\mu Q_t^\mu = P_\mu\sigma\int_0^t \exp(-\omega(t - s))\epsilon_a^\mu(\mathbf{x}_s)dB_s^a \equiv F_t. \tag{29}$$

This expression can be written in a different way using the Ito calculus [12, 14]

$$\int dF_t^2 = 2\int F_t dF_t + \int dF_t dF_t, \tag{30}$$

where we have (here $P^2 = P_\mu P^\mu$)

$$dF_t dF_t = d\kappa^2 G(\mathbf{0})P^2 dt. \tag{31}$$

Next, we consider a singular covariance G. For this purpose at the beginning we treat ϵ_a^μ as a regularized random field. Then, we remove the regularization. In order to make $H\psi$ a well-defined random field we need the normal ordering of H with

$$: g^{\mu\nu} :=: e_a^\mu(\mathbf{x})e_a^\nu(\mathbf{x}) := e_a^\mu(\mathbf{x})e_a^\nu(\mathbf{x}) - \kappa^2 < \epsilon_a^\mu(\mathbf{x})\epsilon_a^\nu(\mathbf{x}) > .$$

The normal ordering in the exponential of Eq. (27) removes the second term $d\kappa^2 G(\mathbf{0})P^2 t$ on the rhs of Eq. (30) whereas the first term there (i.e., $\int_0^\beta F_s dF_s$) becomes a time-ordered integral (renormalization of such expressions appear also in QED [15, 16]). Owing to the normal ordering in Eq. (27)

$$< (PQ_t)^2 > \rightarrow < (PQ_t)^2 > -tP^2\kappa G(0)d \tag{32}$$

and because of the subtraction (32) we can define the action of $\exp(-tH)$ upon a test function ψ (decaying fast in the momentum space) so that $\exp(-Ht)\psi$ is a well-defined random field.

After the averaging over the translation invariant random field $e_a^{\mu}(\mathbf{x})$ and the renormalization (32) we can write $< \exp(-\beta : H :)\psi >$ in terms of Fourier transforms as

$$
\begin{aligned}
&(< \exp(-\beta : H :) > \psi)(\mathbf{x}, X) \\
&= E\Big[\delta(\mathbf{x}_\beta - \mathbf{y}) \exp\Big(- \kappa^2 P_\mu P_\nu \int_0^\beta \exp(-\omega(\beta - s)) dB_s^\mu \\
&\quad \times \int_0^s dB_{s'}^\nu \exp(-\omega(\beta - s')) G(\mathbf{x}_s - \mathbf{x}_{s'}) \\
&\quad + i\sigma P_\mu \int_0^\beta \exp(-\omega(\beta - s)) dB_s^\mu + iP \exp(-\omega\beta) X\Big)\Big]\psi(\mathbf{y}, P) dP d\mathbf{y}. \qquad (33)
\end{aligned}
$$

At $\kappa = 0$ we have

$$
\begin{aligned}
Z_0 &= Tr(\exp(-\beta H_0)) = \int d\mathbf{x} dX < \exp(-\beta H_0) > (\mathbf{x}, X; \mathbf{x}, X) \\
&= (2\pi)^{-d} \int dP d\mathbf{x} dX E\Big[\delta(\mathbf{x}_\beta - \mathbf{x}) \exp(iP((\exp(-\omega\beta) - 1)X \\
&\quad + i\sigma P \int_0^\beta \exp(-\omega(\beta - s)) dB_s))\Big] \\
&= \int K_\beta^{(0)}(\mathbf{x}, X; \mathbf{x}, X) d\mathbf{x} dX, \qquad (34)
\end{aligned}
$$

where $H^{(0)}$ is the Hamiltonian of uncoupled harmonic oscillators and $K^{(0)}$ is the well-known Mehler kernel of the harmonic oscillator. So, at $\kappa = 0$ we obtain the formula (7). We are interested in the κ-term as a perturbation resulting from an interaction with a random metric. In Eq. (33) we apply the identity for B_s and $\mathbf{b}_s$ (in the sense that both sides have the same probability law)

$$B_s = \sqrt{\lambda} B_{\frac{s}{\lambda}}. \qquad (35)$$

After rescaling

$$\mathbf{x}_{s'} = \exp(-\nu s')\mathbf{x} + \sqrt{\lambda} \int_0^{\frac{s'}{\lambda}} \exp\Big(- \lambda\nu(\frac{s'}{\lambda} - \tau)\Big) d\mathbf{b}_\tau. \qquad (36)$$

It is easy to see that for a small t in Eq. (22) (set $\lambda = s'$ in Eq. (36))

$$\mathbf{x}_t = \mathbf{x} + \sqrt{t}\mathbf{q}_t, \qquad (37)$$

where $\mathbf{q}_t \simeq \mathbf{a} + \mathbf{c}\sqrt{t} + ...$ with $|\mathbf{a}| > 0$ for a small t. This scaling behaviour is all what we need to assume about solutions of Eq. (19) with a random metric g of Eq. (26). In Eq. (33) we rescale the Brownian motion $B_{s'}$ and $\mathbf{x}_{s'}$ as in (36) (with $\lambda = s$) and subsequently B_s and $\mathbf{x}_s$ with $\lambda = \beta$. After such a change of variables the integral of the κ-dependent part in the formula (33) reads

$$\exp\Big(- \tfrac{1}{2}\kappa^2 \beta P_\mu P_\nu \textstyle\int_0^1 \exp(-\beta\omega(1 - s)) dB_s^\mu \int_0^s \exp(-\beta\omega(1 - s')) G(\mathbf{x}_s - \mathbf{x}_{s'}) dB_{s'}^\nu\Big), \qquad (38)$$

where $0 \le s' \le s \le 1$, $\mathbf{x}_s = \mathbf{x} + \sqrt{\beta}\mathbf{q}_s$ and $\mathbf{x}_{s'} = \mathbf{x} + \sqrt{\beta}\tilde{\mathbf{q}}_{s'}$ where $\mathbf{q}_s \neq 0$ and $\tilde{\mathbf{q}}_s \neq 0$ at $\beta = 0$. We consider the covariance (the upper bound on $\alpha > 0$ will be discussed at the end of this section)

$$G(\mathbf{x} - \mathbf{x}') = |\mathbf{x} - \mathbf{x}'|^{-2\alpha}. \qquad (39)$$

Then, from Eq. (38) for a small β

$$G(\mathbf{x}_s - \mathbf{x}_{s'}) = \beta^{-\alpha} g(\beta, s, s'), \tag{40}$$

where $s' \leq s \in [0, 1]$ and $g \simeq A + C\beta$ (with $A > 0$) for a small β. Hence, Eq. (33) is of the form

$$E\Big[\exp\Big(iP((\exp(-\omega\beta) - 1)X - \int_0^\beta \exp(-\omega(\beta - s))dB)\Big) \\ \times \exp\Big(-\kappa^2\beta^{1-\alpha}P_\mu P_\nu f^{\mu\nu}(\beta)\Big)\Big], \tag{41}$$

where $|f^{\mu\nu}(\beta)P_\mu P_\nu|$ is bounded from below by a constant.

We can now estimate the behaviour of the partition function in the metric field (26) (note that $: g^{\mu\nu} : - <: g^{\mu\nu} :>$ has the covariance $|\mathbf{x} - \mathbf{x}'|^{-4\alpha}$ if ϵ_a^μ has the covariance (39)). We have

$$g^{\mu\nu} = \delta^{\mu\nu} + 2\kappa\epsilon_\nu^\mu + \kappa^2\epsilon_a^\mu\epsilon_a^\nu. \tag{42}$$

The Hamiltonian (16) is of the form $H = H_0 + \kappa\mathcal{H}_1 + \kappa^2\mathcal{H}_2$.From Eq. (27) we can see that the contribution to the partition function Z is of order κ^2. We consider $H_\gamma = H + \gamma V(P)$ then in the approximation (6)

$$\partial_\gamma Z_\gamma = Tr(\exp(-\beta H)V(P)). \tag{43}$$

The anomalous (fractional) dependence $\beta^{1-\alpha}$ in Eq. (41) of the partition function is a characteristic of the coupling to a singular metric field. We can calculate the κ^2-derivative of the susceptibility (6) to the metric for small β (high temperature)

$$\partial_{\kappa^2}\partial_\gamma Z_{|\kappa=\gamma=0} = \beta^{1-\alpha}E\Big[P_\mu P_\nu f^{\mu\nu}(\beta)\delta(\mathbf{x}_\beta - \mathbf{x}) \\ \exp(iP(\exp(-\omega\beta) - 1)X + i\sigma P\int_0^\beta \exp(-\omega(\beta - s))dB_s))\Big]\beta V(P)d\mathbf{x}dXdP \tag{44}$$

and

$$\partial_{\kappa^2}^2\partial_\gamma Z_{|\kappa=\gamma=0} = \beta^{2-2\alpha}E\Big[\Big(P_\mu P_\nu f^{\mu\nu}(\beta)\Big)^2\delta(\mathbf{x}_\beta - \mathbf{x}) \\ \exp(iP(\exp(-\omega\beta) - 1)X + i\sigma P\int_0^\beta \exp(-\omega(\beta - s))dB_s))\Big]\beta V(P)d\mathbf{x}dXdP. \tag{45}$$

For a small κ and small β the partition function has the expansion in non-integer powers of β

$$\partial_\gamma Z_{\kappa^2} = \partial_\gamma Z_0 + \kappa^2\chi_1\beta\beta^{1-\alpha} + \kappa^4\chi_2\beta\beta^{2-2\alpha} + \dots. \tag{46}$$

with certain constants χ_1 and χ_2.

We still have to estimate the integrals in Eqs. (43)–(46). The integral (43) is expressed by kernels in Eq. (13). The expectation value in Eq. (13) after the renormalization (32) involves the time-ordered stochastic integral in Eq. (33) which fails to be positive definite. Hence, if the expectation value in Eq. (43) is to be finite $V(P)$ must decrease faster than $\exp(-RP^2)$ for any R (the derivatives in Eqs. (44)–(46) impose milder requirements on the decay of $V(P)$ for a large P). There is still the problem of the convergence of the stochastic integrals in the definition of $f_{\mu\nu}P^\mu P^\nu$. This stochastic integral is of the form

$$P_\mu P_\nu \int_0^\beta dB_s^\mu \int_0^s dB_{s'}^\nu G(\mathbf{x}_s - \mathbf{x}_{s'}). \tag{47}$$

The stochastic integrals in eq. (47) can be estimated by ordinary integrals using the formula [13](better estimates on the multiple stochastic integrals (33) and (47) can be obtained using the results of ref.[17])

$$E\Big[\Big(\int FdB_s\Big)^{2k}\Big] \leq C_k E\Big[\int F^{2k}ds\Big] \tag{48}$$

with certain constants C_k. The rhs of Eq. (48) involves the Ornstein-Uhlenbeck process (22) [18] whose transition function is expressed by the Mehler formula. In calculations in Eq. (48) for small β we can approximate the Ornstein-Uhlenbeck process $\mathbf{x}_s$ by the Brownian motion with the transition function $p(s,\mathbf{x}) = (2\pi s)^{-\frac{n}{2}}\exp(-\frac{|\mathbf{x}|^2}{2s})$. Then, the integral in Eq. (47) can be estimated by (use Eq. (48) twice with $k = 1$)

$$\int dsds' \int d\mathbf{x}p(s-s',\mathbf{x})|\mathbf{x}|^{-4\alpha} < \infty \tag{49}$$

if $2\alpha < 1$. The expansion (46) must be terminated at the kth order if $2k\alpha > 1$ because the rhs of the estimate (48) is infinite.

4 The Outlook

Some approximate calculations [8, 9, 11] indicate that the singularity of the quantum gravitational field at small distances can be different than the canonical one which in n dimensions is of the form $|\mathbf{x}-\mathbf{x}'|^{-n+2}$ (where $n = 4$ corresponds to $\alpha = \frac{1}{2}$ in Eq. (39)). The fractal dimensionality of the physical space-time has been discussed in [19, 20] on the basis of the Cosmic Microwave Background measurements. Some limits on the deviation of the observational space-time dimension from the physical four dimensions have been obtained. The effect of quantum gravity could be observed either at small distances or at high energies (which in cosmology are connected with high temperatures). The non-integer indices in the expansion of the partition function in Eqs. (44)–(46) could indicate the relevance of quantum gravity for some extremal processes in astrophysics (which possibly could be tested on the quantum level in gravitational wave interferometers [21]). The model of a random mass distribution which according to Eqs. (14) and (23) is equivalent to a random diffusivity is of interest in condensed matter physics [22, 23]. An anomalous behaviour of the partition function (46) or other thermodynamic functions of complex systems (e.g. molecules or crystals) could be an indication of the random mass or random metric present in these systems.

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