



On a Generalized Central Limit Theorem and Large Deviations for Homogeneous Open Quantum Walks

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Abstract

We consider homogeneous open quantum walks on a lattice with finite dimensional local Hilbert space and we study in particular the position process of the quantum trajectories of the walk. We prove that the properly rescaled position process asymptotically approaches a mixture of Gaussian measures. We can generalize the existing central limit type results and give more explicit expressions for the involved asymptotic quantities, dropping any additional condition on the walk. We use deformation and spectral techniques, together with reducibility properties of the local channel associated with the open quantum walk. Further, we can provide a large deviation principle in the case of a fast recurrent local channel and at least lower and upper bounds in the general case.

Keywords Central limit theorem · Homogeneous open quantum walks · Quantum trajectories · Minimal enclosures · Quantum recurrence and transience

1 Introduction

Quantum walks are interesting mathematical objects, introduced in discrete and continuous time, intensively studied in the last 30 years for their wide range of applications in various fields. Detailed analysis and competent reviews of the subject can be found in [25, 28, 32, 41, 44, 46] and references therein, together with discussions about the possible realizations and about their applications.

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These walks can be thought of as Markov processes on a lattice where the evolution of the walker's position depends on local degrees of freedom. Unitary evolutions are of fundamental importance, but the need for a definition of quantum walks in an open environment naturally arises: for instance, it has been observed that unitary quantum walks are sometimes of difficult physical implementation due to decoherence effects [1, 9, 29].

We study open quantum (random) walks (OQW) as introduced in [3] by Attal et al.. Following some more recent literature about the subject, we drop the adjective "random", but we remark that the mathematical object remains the same. Such processes are a possible noncommutative generalization of classical Markov chains and have potential applications in quantum computing, quantum optics, biology. For a survey illustrating the intensive research activity around the subject, we refer to [41]; some applications can be found e.g. in [4, 34, 38] and [6, 7, 40] show how to obtain open quantum brownian motion as a scaling limit of OQWs.

Quantum trajectories associated with an OQW defined on a lattice V produce a classical V -valued process which is not Markov in general. Nonetheless, some classical notions, commonly associated to Markov chains, can be extended to this class of processes and have been recently studied: reducibility, period, hitting probabilities, the expected number of visits, the expected return time and various types of recurrence [5, 14, 18, 20]. Particular attention has been concentrated from the origin on the case of homogeneous OQWs (HOQWs) on infinite lattices. As for classical homogeneous random walks, also in this case an interesting question is establishing the conditions for different kinds of asymptotic behaviors of the position process, such as the law of large numbers, central limit theorems, and large deviations bounds [2, 15, 30, 31, 33, 37]. We aim at exploring these lines of investigation in particular.

In [2], a central limit theorem (CLT) and a law of large numbers for a special class of HOQWs were proved by the use of Poisson equation and the CLT for martingales. At the same time, the authors of [2] highlighted the difficulty to prove analogous results under weaker assumptions, even if simulations clearly indicated the possibility of appearance of a mixture of Gaussians in general [39]. In [15] a large deviation principle was also proved, and a different approach was used to obtain the CLT, but the assumptions essentially remained the same. Since then, various papers have been devoted to investigating these problems: see for instance [10, 30, 31, 33, 36, 37] (see Remark 4.7 for more details).

Before proceeding, let us now introduce more precisely the mathematical definition of the quantum walks that we treat.

- Let $V \subset \mathbb{R}^d$ denote a locally finite lattice, positively generated by a set $S = \{s_1, \dots, s_n\} \neq \{0\}$. We assume without loss of generality that $0 \in V$. The canonical example is $V = \mathbb{Z}^d$ and $S = \{\pm e_1, \dots, \pm e_d\}$ where (e_1, \dots, e_d) is the canonical basis of \mathbb{R}^d .
- We denote by $\ell^2(V)$ the Hilbert space of square summable sequences indexed in V , describing the position of the particle in the quantum evolution, and we fix $\{|k\rangle\}_{k \in V}$ an orthonormal basis for $\ell^2(V)$. We introduce a finite-dimensional Hilbert space \mathfrak{h} describing the internal degrees of freedom of the particle.
- We consider a quantum system described by the separable complex Hilbert space $\mathcal{H} = \mathfrak{h} \otimes \ell^2(V)$. We denote by $B(\mathcal{H})$ the von Neumann algebra of bounded linear operators on \mathcal{H} and by $L^1(\mathcal{H})$ the Banach space of trace class operators on \mathcal{H} (similarly for $B(\mathfrak{h})$ and $L^1(\mathfrak{h})$). Self-adjoint bounded operators correspond to physical observables, while unit-trace positive operators are called states and represent the noncommutative analog of probability densities.

A HOQW \mathfrak{M} is a particular quantum channel acting on $L^1(\mathcal{H})$ in such a way that, at each time step, the position of the evolution can go only to nearest neighbors and also the change

in the local state only depends on the position shift [3, 15]. More precisely, \mathfrak{M} is defined through its Kraus form as a map acting on trace class operators on \mathcal{H}

$$\mathfrak{M} : L^1(\mathcal{H}) \rightarrow L^1(\mathcal{H})$$

$$\omega \mapsto \sum_{\underline{k} \in V} \sum_{i=1}^v (L_i \otimes |\underline{k} + s_i\rangle\langle \underline{k}|) \omega (L_i^* \otimes |\underline{k}\rangle\langle \underline{k} + s_i|), \tag{1.1}$$

where $\{L_i\}_{i=1}^v$ are operators in $B(\mathfrak{h})$ such that $\sum_{i=1}^v L_i^* L_i = 1_{\mathfrak{h}}$. As already underlined in the original paper [3, Section 10], OQWs are not a generalization of unitary walks, as commonly understood, like Hadamard walks (see e.g. [28]). Anyway we can read some similarities between HOQWs and unitary walks: for both, the position of the particle can only jump to nearest neighbours, but, for HOQWs, the transformation of the local state is described by the operators L_i 's, which somehow replace the role of the coin in unitary walks, allowing to decline it differently according to the chosen shift direction. All these possible local actions L_i 's determine the auxiliary (or local) map, which is the quantum channel \mathfrak{L} , acting on the space $L^1(\mathfrak{h})$ of the trace class operators on the local space \mathfrak{h} , defined by

$$\mathfrak{L} : L^1(\mathfrak{h}) \rightarrow L^1(\mathfrak{h}), \quad \mathfrak{L}(\sigma) = \sum_{i=1}^v L_i \sigma L_i^*.$$

We shall see that this auxiliary map is of primary importance in our study: it contains all essential information and it completely characterizes most properties of \mathfrak{M} .

\mathfrak{M} can be seen as a stochastic process since it represents the analog of the transition operator for classical Markov processes: once the initial condition is selected, the repeated application of the transition operator allows to determine all finite dimensional laws of the process.

Given the open quantum walk \mathfrak{M} , we can then fix an initial state ρ (a positive unit-trace operator in $L^1(\mathcal{H})$), and, following the usual construction for quantum trajectories, we can introduce the stochastic process $(X_n, \rho_n)_{n \geq 0}$, keeping track of the position X_n , valued in V , and of the internal state ρ_n of the particle (a positive unit-trace operator in $L^1(\mathfrak{h})$). See Sect. 2.1 for more precise definitions.

As we already mentioned, the main topic of this paper is about a central limit type result and bounds on the probability of large deviations for the position process $(X_n)_n$. The existing results were established assuming different conditions about the local map \mathfrak{L} . Even if the terminology is not always the same, we can say that precise CLT and large deviation principle have been proved only under the assumption that the local map has a unique invariant state (sometimes also faithful). Some partial results were obtained under particular reducibility conditions on \mathfrak{L} in the case it is fast recurrent (e.g. [2] and [30]). Our aim is to establish the best results in these directions without any restriction on the local map. In order to prove our results, we make use of deformation techniques, spectral theory and Gärtner–Ellis' and Bryc's theorems; they are classical tools for deriving large deviations and central limit results and have already been employed in the study of quantum Markov chains [43], quantum trajectories in continuous time [22], quantum spin chains [35] and quantum trajectories associated to HOQWs [10, 15]. We follow the ideas and lines already used for the irreducible and fast recurrent case in [15]. We shall tackle the difficulties due to the more general context mainly through additional tools based on noncommutative probabilistic features of the local channel \mathfrak{L} : the (in general non-unique) decomposition of the local space in irreducible invariant domains (or enclosures) and the quantum absorption properties of the same domains.

Let us briefly depict the situation for what concerns central limit type results. When \mathfrak{L} has a unique invariant state, previous works (e.g. [2, 11, 15, 31]) show that *there exists a vector m in \mathbb{R}^d and a non negative matrix D such that*

$$\frac{X_n - X_0 - nm}{\sqrt{n}} \xrightarrow{n \rightarrow +\infty} \mathcal{N}(0, D) \quad \text{in law}$$

under the probability measure \mathbb{P}_ρ induced by the initial state ρ of the walk. $\mathcal{N}(0, D)$ denotes the Gaussian law with 0 mean and covariance matrix D .

In case of existence of different invariant states for \mathfrak{L} , simulations clearly show the appearance of different Gaussian-like distributions in the limit behavior of the position process, except for very special cases. This means in particular that there exists in general no vector m such that the sequence $(\frac{X_n - X_0 - nm}{\sqrt{n}})_n$ can converge in law (not to a Gaussian, nor to any other law). So, how can we mathematically describe this situation? Since we have no convergence in law, one natural possibility is identifying a new sequence of laws $(\mu_n)_n$, such that μ_n is a convex combination of Gaussian laws and it approximates the law of $\frac{X_n - X_0}{\sqrt{n}}$. The nature and quality of this approximation procedure will be made precise later through the use of a distance defined on the sets of probability laws on \mathbb{R}^d . As one can imagine, the precise description of this asymptotic behavior requires some efforts, mainly related to technical difficulties, but also involving more complicated structures, sometimes producing heavy notations that we were not able to avoid.

Without giving all the details, we can write the statement of the main theorem of Sect. 4, a “generalized CLT” (see Theorem 4.4 for a more precise statement):

Theorem 1.1 *For any time $n \geq 1$, let $\mathbb{P}_{\rho,n}$ be the law of the random variable $\frac{X_n - X_0}{\sqrt{n}}$ under the measure \mathbb{P}_ρ induced by the initial state ρ of the walk. Then*

$$\lim_{n \rightarrow +\infty} \text{dist}(\mathbb{P}_{\rho,n}, \mu_n) = 0$$

where

$$\mu_n = \sum_{\alpha \in A} a_\alpha(\rho) \mathcal{N}_{n,\alpha}$$

is a convex combination of Gaussian laws $\mathcal{N}_{n,\alpha}$, whose parameters depend on the time n and on a parameter α living in the set A related to the local channel \mathfrak{L} .

All the elements appearing in the previous expression (A , $a_\alpha(\rho)$, the parameters of the Gaussian laws) will be explicitly determined and depend on the initial state of the walk and on the structure of the invariant states of the local channel \mathfrak{L} .

For results on large deviations, the statement is more complicated and we directly refer the reader to Sect. 5. Here, we can simply anticipate that we can prove a large deviation principle in case the local channel is fast recurrent (i.e. there exists at least one faithful invariant state), while, when there is a non trivial transient subspace, we can only find upper and lower bounds through Gärtner–Ellis’ theorem. In both cases we can explicitly write the rate functions using the same ingredients as above.

We are not giving other technical details here, but only a brief description of the contents of the paper.

In Sect. 2 we shall describe the construction of quantum trajectories associated with the HOQW and recall some basic notions about invariant domains and absorption operators related to a quantum channel. Then, at the beginning of Sect. 4, we shall go back to these

topics and add more details about the reducibility properties of a quantum channel and the associated decomposition of the local space \mathfrak{h} in invariant domains.

In Sect. 3, we shall determine a family of probability measures under which the position process verifies a central limit theorem. These probability measures are absolutely continuous with respect to the standard measure \mathbb{P}_ρ , induced by the initial state ρ of the evolution, and are naturally associated with the invariant domains of the local channel. The densities of these measures and the parameters of the limit Gaussian are explicitly written in terms of the initial state and of the particular invariant domain.

Then, in Sect. 4, we shall go to the general case using the decomposition of the space \mathfrak{h} and deducing an expression of \mathbb{P}_ρ as a convex combination of the probability measures described in the previous section. We shall precisely state and demonstrate the previously mentioned main theorem (Theorem 4.4).

Finally, we discuss some examples and numerical simulations in Sect. 6.

2 Preliminaries and Context Description

We recall here some basic definitions, notations, and existing results. In the first subsection, we introduce the precise definition of quantum trajectories and some examples, with simulations suggesting the possible asymptotic behaviors of the position process. In the second one, we recall some general notions about invariant domains (or enclosures) and absorption operators.

A quantum channel is for us a completely positive trace-preserving linear map acting on trace class operators. We already introduced the quantum channel \mathfrak{M} , acting on $L^1(\mathcal{H})$, and defining the HOQW, and the local channel \mathcal{L} acting on trace class operators $L^1(\mathfrak{h})$ on the local Hilbert space.

Notice that the evolution of the system described by a HOQW depends only on the matrix elements of the state which are diagonal with respect to the position observable, hence we can assume that the initial state ρ of the system is of diagonal form

$$\rho = \sum_{\underline{k} \in V} \rho(\underline{k}) \otimes |\underline{k}\rangle\langle \underline{k}| \in L^1(\mathcal{H}), \quad \rho \geq 0, \quad \text{Tr}(\rho) = 1$$

or equivalently

$$\rho(\underline{k}) \in L^1(\mathfrak{h}), \quad \rho(\underline{k}) \geq 0, \quad \sum_{\underline{k} \in V} \text{Tr}(\rho(\underline{k})) = 1.$$

2.1 Quantum Trajectories

The stochastic evolution of the system will depend on the initial state ρ and we shall call \mathbb{P}_ρ the associated probability measure. Let us first define the probability space. We denote by $J = \{1, \dots, v\}$ the set of indices for all possible shifts in $S = \{s_1, \dots, s_v\}$ and we choose the sample set $\Omega = V \times J^{\mathbb{N}}$. On V and J we consider the σ -algebras of the power sets, and on Ω we then consider the σ -algebra \mathcal{F} generated by cylindrical sets.

We define a family of compatible finite dimensional laws which univoquely determines a measure \mathbb{P}_ρ on (Ω, \mathcal{F}) by Kolmogorov extension theorem: for all $\underline{k} \in V, n \geq 1, \underline{j} = (j_1, \dots, j_n) \in J^n$,

$$\begin{aligned} \mathbb{P}_\rho(\{\underline{k}\} \times J^{\mathbb{N}}) &= \text{Tr}(\rho(\underline{k})), \\ \mathbb{P}_\rho(\{\{\underline{k}, \underline{j}\}\} \times J^{\mathbb{N}}) &= \text{Tr}(L_{j_n} \cdots L_{j_1} \rho(\underline{k}) L_{j_1}^* \cdots L_{j_n}^*). \end{aligned}$$

The quantum trajectory is the process $(X_n, \rho_n)_{n \geq 0}$ defined, for $\omega = (\underline{k}, j_1, \dots)$, as

$$X_0(\omega) = \underline{k}, \quad \rho_0(\omega) = \frac{\rho(\underline{k})}{\text{Tr}(\rho(\underline{k}))},$$

$$X_{n+1}(\omega) = X_n(\omega) + s_{j_{n+1}}, \quad \rho_{n+1}(\omega) = \frac{L_{j_{n+1}} \rho_n(\omega) L_{j_{n+1}}^*}{\text{Tr}(L_{j_{n+1}} \rho_n(\omega) L_{j_{n+1}}^*)} \quad \forall n \geq 0.$$

$(X_n, \rho_n)_{n \geq 0}$ is a Markov process on the probability space $(\Omega, \mathcal{F}, \mathbb{P}_\rho)$, with respect to the natural filtration $\{\mathcal{F}_n\}_{n \geq 0}$, with initial law given by

$$\mathbb{P}_\rho \left\{ (X_0, \rho_0) = \left(\underline{k}, \frac{\rho(\underline{k})}{\text{Tr}(\rho(\underline{k}))} \right) \right\} = \text{Tr}(\rho(\underline{k})), \quad \underline{k} \in V$$

and transition probabilities

$$\mathbb{P}_\rho \left(X_{n+1} = X_n + s_j, \rho_{n+1} = \frac{L_j \rho_n L_j^*}{\text{Tr}(L_j \rho_n L_j^*)} \mid X_n, \rho_n \right) = \text{Tr}(L_j \rho_n L_j^*), \quad j \in J, \quad n \geq 1.$$

In order to fix some ideas about the definition of an OQW and on the behavior of the related position process, we introduce two simple examples, both on the lattice $V = \mathbb{Z}$, for which we provide the simulated trajectories of the rescaled position process in the next figures. In this case ($V = \mathbb{Z}$), the HOQW has two possible movements at each time step, i.e. $v = 2$, and the walk is completely determined once we fix the two Kraus operators L_1, L_2 describing the action on the internal state when moving to the right or to the left. For convenience, we shall call them R and L respectively.

Example 2.1 Let us consider a HOQW on $V = \mathbb{Z}$ with local space $\mathfrak{h} = \mathbb{C}^2$ (we denote by $\{e_0, e_1\}$ the canonical basis) and the following local operators:

$$L = \begin{pmatrix} \sqrt{\frac{1}{2}} & 0 \\ -\frac{\sqrt{2}}{3} & \sqrt{\frac{1}{3}} \end{pmatrix}, \quad R = \begin{pmatrix} \sqrt{\frac{1}{6}} & 0 \\ \frac{1}{3} & \sqrt{\frac{2}{3}} \end{pmatrix}$$

corresponding to going to the left and the right respectively. In this case the local map $\mathcal{L}(\cdot) = L \cdot L^* + R \cdot R^*$ admits a unique invariant state $\tau_0 = |e_1\rangle\langle e_1|$. For every initial state ρ , simulations show that, for increasing values of n , the law of $\frac{X_n - X_0}{\sqrt{n}}$ becomes approximately Gaussian, with fixed variance, and mean growing as \sqrt{n} (see Figure 1).

Example 2.2 Consider now always $V = \mathbb{Z}$, but local space $\mathfrak{h} = \mathbb{C}^4$ and local Kraus operators

$$L = \begin{pmatrix} \frac{1}{2\sqrt{2}} & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ -\sqrt{\frac{1}{6}} & 0 & 0 & \frac{\sqrt{2}}{3} \end{pmatrix}, \quad R = \begin{pmatrix} \sqrt{\frac{3}{8}} & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{3}} & 0 & 0 & \frac{1}{\sqrt{3}} \end{pmatrix}.$$

The invariant states of the local map are of the following form: $x\sigma + (1-x)|e_3\rangle\langle e_3|$ where σ is any state supported in $\text{span}\{e_1, e_2\}$ and $x \in [0, 1]$. In this case simulations show that, as n increases, the law of $\frac{X_n - X_0}{\sqrt{n}}$ can approach either a Gaussian or the mixture of two Gaussians, whose parameters will be easy to compute using the results of next sections ($\mathcal{N}(0, 1)$ and $\mathcal{N}(-\sqrt{n}/3, 8/9)$). Figure 2 shows that the weights of such a mixture strongly depend on the initial state.

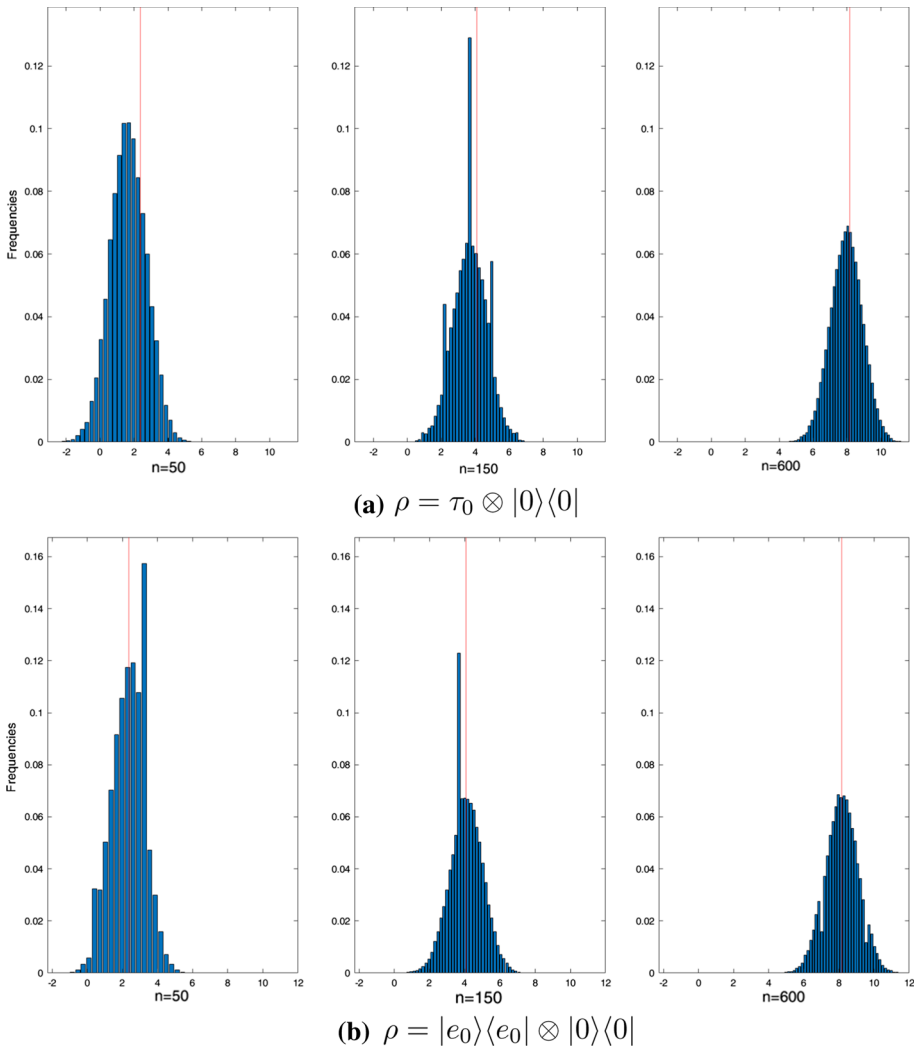


Fig. 1 Panels **a**, **b** show the appearance of the same Gaussian distribution for two different initial states in the model introduced in Example 2.1. We used $N = 5 \times 10^4$ samples of $\frac{X_n}{\sqrt{n}}$ for $n = 50, 150, 600$ in order to draw its profile. The vertical red line corresponds to the mean value of the Gaussian

We shall recover this example in the last section, considering a slightly more general family of Kraus operators.

2.2 Enclosures and Absorption

Let \mathfrak{h} be a finite dimensional Hilbert space and Φ be a quantum channel on $L^1(\mathfrak{h})$, the set of trace-class operators on \mathfrak{h} . Since the topological dual of $L^1(\mathfrak{h})$ is isometrically isomorphic to $B(\mathfrak{h})$, we can define the dual map $\Phi^* : B(\mathfrak{h}) \rightarrow B(\mathfrak{h})$ as the operator verifying

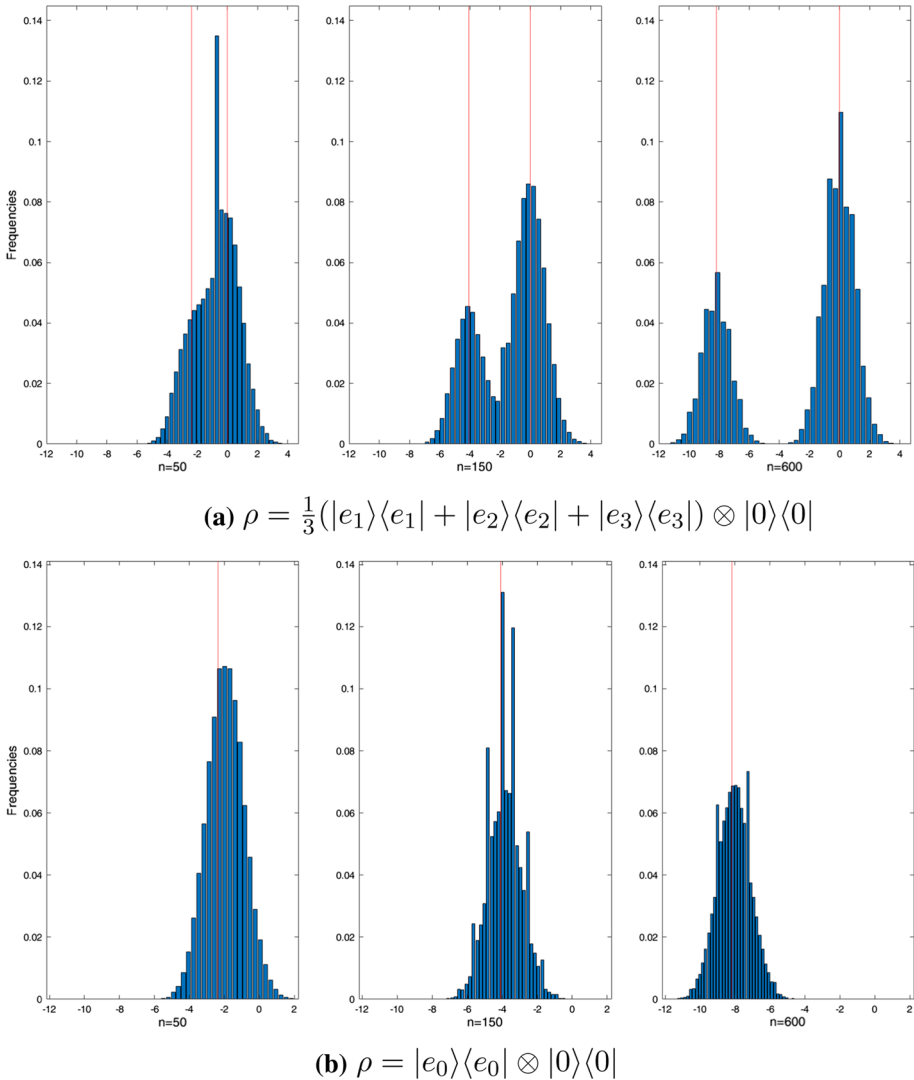


Fig. 2 Panel **a** shows a mixture of two Gaussian distributions for a particular choice of the initial state, while panel **b** shows a single Gaussian for another initial state for the model considered in Example 2.2. We used $N = 5 \times 10^4$ samples of $\frac{X_n}{\sqrt{n}}$ for $n = 50, 150, 600$ in order to draw their profile. The vertical red lines correspond to the mean values of the Gaussians

$$\text{Tr}(\omega\Phi^*(x)) = \text{Tr}(\Phi(\omega)x) \quad x \in B(\mathfrak{h}), \omega \in L^1(\mathfrak{h}).$$

Φ^* is a completely positive unital (i.e. $\Phi^*(1_{\mathfrak{h}}) = 1_{\mathfrak{h}}$) bounded operator. Given any positive operator $x \in B(\mathfrak{h})$, we define its support projection as the orthogonal projection onto $\text{supp}(x) := \ker(x)^\perp$.

Definition 2.3 1. A subspace $\mathcal{V} \subset \mathfrak{h}$ is said to be an enclosure (or invariant domain) for Φ if for every positive $\omega \in L^1(\mathfrak{h})$

$$\text{supp}(\omega) \subset \mathcal{V} \text{ implies } \text{supp}(\Phi(\omega)) \subset \mathcal{V}.$$

An enclosure will be called minimal if it does not contain other non trivial enclosures.

2. A quantum channel is called irreducible if the only enclosures are the trivial ones, i.e. $\{0\}$ and \mathfrak{h} .

A finite dimensional minimal enclosure is always the support of a unique invariant state of the channel Φ , i.e. a positive trace one operator τ such that $\Phi(\tau) = \tau$.

Denoting by $p_{\mathcal{V}}$ the orthogonal projection onto \mathcal{V} , we have the following equivalent characterizations of the notion of enclosure.

Proposition 2.4 (see [16] and references therein) *Let Φ be a quantum channel acting on $L^1(\mathfrak{h})$. The following are equivalent:*

1. \mathcal{V} is an enclosure;
2. $p_{\mathcal{V}}$ is a subharmonic projection, i.e. $\Phi^*(p_{\mathcal{V}}) \geq p_{\mathcal{V}}$;
3. if Φ has a representation with Kraus operators $(V_i)_{i \in I}$, i.e. $\Phi(\cdot) = \sum_{i \in I} V_i(\cdot)V_i^*$, then $V_i p_{\mathcal{V}} = p_{\mathcal{V}} V_i p_{\mathcal{V}}$ for every $i \in I$.

If we consider the restriction of the quantum channel to $p_{\mathcal{V}}L^1(\mathfrak{h})p_{\mathcal{V}} \simeq L^1(\mathcal{V})$, we obtain again a quantum channel, that we shall denote $\Phi|_{\mathcal{V}}$ with an abuse of notation,

$$\begin{aligned} \Phi|_{\mathcal{V}} : p_{\mathcal{V}}L^1(\mathfrak{h})p_{\mathcal{V}} &\rightarrow p_{\mathcal{V}}L^1(\mathfrak{h})p_{\mathcal{V}} \\ p_{\mathcal{V}}\sigma p_{\mathcal{V}} &\mapsto p_{\mathcal{V}}\Phi(p_{\mathcal{V}}\sigma p_{\mathcal{V}})p_{\mathcal{V}} = \Phi(p_{\mathcal{V}}\sigma p_{\mathcal{V}}); \end{aligned}$$

its dual map $\Phi^*|_{\mathcal{V}}$ acts on $p_{\mathcal{V}}B(\mathfrak{h})p_{\mathcal{V}} \simeq B(\mathcal{V})$ and for every $x \in B(\mathfrak{h})$

$$\Phi^*|_{\mathcal{V}}(p_{\mathcal{V}}x p_{\mathcal{V}}) = p_{\mathcal{V}}\Phi^*(p_{\mathcal{V}}x p_{\mathcal{V}})p_{\mathcal{V}} = p_{\mathcal{V}}\Phi^*(x)p_{\mathcal{V}}.$$

The channel $\Phi|_{\mathcal{V}}$ restricted to a minimal enclosure is trivially irreducible by construction. Given an enclosure \mathcal{V} , we can define the associated absorption operator (see [12]) as

$$A(\mathcal{V}) := \lim_{n \rightarrow +\infty} \Phi^{*n}(p_{\mathcal{V}}). \tag{2.2}$$

Absorption operators enjoy remarkable properties that we recall below.

Proposition 2.5 ([12, Proposition 4]) *The following statements hold true:*

1. $0 \leq A(\mathcal{V}) \leq 1_{\mathfrak{h}}$;
2. $A(\mathcal{V})$ is harmonic, that is $\Phi^*(A(\mathcal{V})) = A(\mathcal{V})$;
3. $\ker(A(\mathcal{V}))$ is an enclosure;
4. $A(\mathcal{V}) = p_{\mathcal{V}} + p_{\mathcal{V}^\perp}A(\mathcal{V})p_{\mathcal{V}^\perp}$.

Some additional discussion about the general structure of enclosures for quantum channels and possible decompositions of \mathfrak{h} in orthogonal enclosures will be developed in Sect. 4.

In the same way as for positive matrices, there exists a Perron–Frobenius theorem for quantum channels. We denote the spectral radius of a map Φ as $r(\Phi) := \sup\{|\lambda| : \lambda \in \text{Sp}(\Phi)\}$, where $\text{Sp}(\Phi)$ is the spectrum of Φ .

Theorem 2.6 [45, Theorems 6.4 and 6.5] *Let Φ be a positive map acting on $L^1(\mathfrak{h})$; then $r(\Phi)$ is a eigenvalue and the corresponding eigenvector is positive. If in addition Φ is irreducible, then $r(\Phi)$ is geometrically simple and the corresponding eigenvector is strictly positive.*

In case Φ is a quantum channel $1 = r(\Phi) = \|\Phi\|_\infty$.

3 Selecting a Single Gaussian

In this section, we shall concentrate on a single minimal enclosure \mathcal{V} of the local channel \mathcal{L} and we shall introduce a proper associated probability measure \mathbb{P}'_ρ , which is absolutely continuous with respect to \mathbb{P}_ρ , with a relative density which assigns weights according to the absorption in \mathcal{V} . We shall prove in Theorem 3.6 that the position process $(X_n)_n$ always satisfies a central limit theorem under any of these measures. This will be the main result of this section and already includes previous CLT results proved in [2] and in [15] (see Remark 3.7 below).

According to the notations introduced in the previous section, we shall call $A(\mathcal{V}) = \lim_{n \rightarrow +\infty} \mathfrak{L}^{*n}(p_{\mathcal{V}})$ the absorption operator of the enclosure \mathcal{V} for \mathcal{L} and we shall denote by $\tilde{p}_{\mathcal{V}}$ the support projection of $A(\mathcal{V})$. Due to Proposition 2.5, $\ker(A(\mathcal{V})) = \text{supp}(A(\mathcal{V}))^\perp$ is an enclosure, and by Proposition 2.4 this implies that $(1_{\mathfrak{h}} - \tilde{p}_{\mathcal{V}})$ is a subharmonic projection and its support is preserved by the Kraus operators L_i 's, i.e.

$$\mathfrak{L}^*(1_{\mathfrak{h}} - \tilde{p}_{\mathcal{V}}) \geq 1_{\mathfrak{h}} - \tilde{p}_{\mathcal{V}} \quad \text{and} \quad L_i(1_{\mathfrak{h}} - \tilde{p}_{\mathcal{V}}) = (1_{\mathfrak{h}} - \tilde{p}_{\mathcal{V}})L_i(1_{\mathfrak{h}} - \tilde{p}_{\mathcal{V}}) \quad \text{for } i = 1, \dots, v.$$

Consequently $\tilde{p}_{\mathcal{V}}$ is superharmonic for \mathcal{L} , i.e.

$$\mathfrak{L}^*(\tilde{p}_{\mathcal{V}}) \leq \tilde{p}_{\mathcal{V}} \quad \text{and} \quad \tilde{p}_{\mathcal{V}}L_i = \tilde{p}_{\mathcal{V}}L_i\tilde{p}_{\mathcal{V}} \quad \text{for } i = 1, \dots, v. \tag{3.3}$$

We start introducing the new measure \mathbb{P}'_ρ in the following lemma and then we shall describe more precisely the techniques to prove the first CLT.

Lemma 3.1 *Let $(Y_n)_{n \geq 0}$ be the process defined by $Y_n = \text{Tr}(A(\mathcal{V})\rho_n)$ for any $n \geq 0$.*

1. *Then $(Y_n)_{n \geq 0}$ is a positive and bounded \mathbb{P}_ρ -martingale, converging (almost surely and L^1) to a random variable Y_∞ valued in $[0, 1]$.*
2. *If $\mathbb{E}_\rho[Y_0] = \mathbb{E}_\rho[\text{Tr}(A(\mathcal{V})\rho_0)] > 0$, we can define a new probability measure \mathbb{P}'_ρ such that*

$$\frac{d\mathbb{P}'_\rho}{d\mathbb{P}_\rho} = \frac{Y_\infty}{\mathbb{E}_\rho[Y_0]}, \quad \frac{d\mathbb{P}'_\rho}{d\mathbb{P}_\rho} \Big|_{\mathcal{F}_n} = \frac{Y_n}{\mathbb{E}_\rho[Y_0]}.$$

Moreover the density $\frac{d\mathbb{P}'_\rho}{d\mathbb{P}_\rho}$ is valued in $[0, \mathbb{E}_\rho[Y_0]^{-1}]$ and

$$\left\{ \frac{d\mathbb{P}'_\rho}{d\mathbb{P}_\rho} = \frac{1}{\mathbb{E}_\rho[Y_0]} \right\} = \left\{ \lim_{n \rightarrow +\infty} \|p_{\mathcal{V}}\rho_n p_{\mathcal{V}} - \rho_n\| = 0 \right\},$$

$$\left\{ \frac{d\mathbb{P}'_\rho}{d\mathbb{P}_\rho} = 0 \right\} = \left\{ \lim_{n \rightarrow +\infty} \|\tilde{p}_{\mathcal{V}}^\perp \rho_n \tilde{p}_{\mathcal{V}}^\perp - \rho_n\| = 0 \right\}. \tag{3.4}$$

We remark that, for this lemma, it is not necessary for \mathcal{V} to be minimal. The last sentence of the statement gives a mathematical meaning to the intuition that, given any enclosure \mathcal{V} , the corresponding \mathbb{P}'_ρ encodes the notion of conditioning to the “absorption in \mathcal{V} ”. Nevertheless, Y_∞ is not a Bernoulli random variable in general, hence there does not need to exist any measurable set $B \in \mathcal{F}$ such that $\mathbb{P}'_\rho(\cdot) = \mathbb{P}_\rho(\cdot|B)$, even if it can happen in some cases (see Example 6.2 and in particular the simulations in Figure 5).

Proof Y_n is trivially positive and bounded and

$$\mathbb{E}_\rho[Y_{n+1}|\mathcal{F}_n] = \sum_{i=1}^v \text{Tr}(L_i \rho_n L_i^*) \frac{\text{Tr}(A(\mathcal{V})L_i \rho_n L_i^*)}{\text{Tr}(L_i \rho_n L_i^*)} = \text{Tr}(\mathfrak{L}^*(A(\mathcal{V}))\rho_n) = \text{Tr}(A(\mathcal{V})\rho_n) = Y_n.$$

Since (Y_n) is a positive and bounded martingale, it converges almost surely and in $L^1 := L^1(\Omega, \mathbb{P}_\rho)$ to a positive random variable Y_∞ . When \mathcal{V} and ρ are such that $\mathbb{E}_\rho[\text{Tr}(A(\mathcal{V})\rho_0)] = \sum_{k \in \mathcal{V}} \text{Tr}(A(\mathcal{V})\rho(k)) > 0$, we can introduce the random variables

$$0 \leq Z_n := \frac{Y_n}{\mathbb{E}_\rho[\text{Tr}(A(\mathcal{V})\rho_0)]},$$

and the sequence (Z_n) is a \mathbb{P}_ρ -martingale with expected value equal to 1 and converges almost surely to

$$0 \leq Z_\infty := \frac{Y_\infty}{\mathbb{E}_\rho[\text{Tr}(A(\mathcal{V})\rho_0)]}. \tag{3.5}$$

Note that $Z_\infty \in L^1$. Therefore we can consider the new measure \mathbb{P}'_ρ which has density Z_∞ with respect to \mathbb{P}_ρ , so that

$$\frac{d\mathbb{P}'_\rho}{d\mathbb{P}_\rho} = Z_\infty, \quad \frac{d\mathbb{P}'_\rho}{d\mathbb{P}_\rho} \Big|_{\mathcal{F}_n} = Z_n.$$

The range of $\frac{d\mathbb{P}'_\rho}{d\mathbb{P}_\rho}$ trivially follows from the fact that $0 \leq Y_\infty \leq 1$. We postpone the proof of relations (3.4) to Sect. 4, since we need some notions that we will introduce later on. \square

In order to prove the central limit theorem for the position process, we will apply Bryc’s theorem [11, Proposition 1], that we report below for the reader’s convenience; we refer to [23, Appendix A4] for the multivariate case.

Theorem 3.2 (Bryc) *Let $(T_n)_{n \geq 0}$ be a sequence of random variables defined on the probability spaces $(\Omega_n, \mathcal{B}_n, \mathbb{P}_n)$, $T_n : \Omega_n \rightarrow \mathbb{R}^d$, and suppose there exists $\epsilon > 0$ such that*

$$h(u) = \lim_{n \rightarrow +\infty} \frac{1}{n} \log(\mathbb{E}_n[e^{u \cdot T_n}])$$

exists for every complex u with $|u| < \epsilon$. Then

$$\frac{(T_n - \mathbb{E}_n[T_n])}{\sqrt{n}} \longrightarrow \mathcal{N}(0, D) \quad (\text{in law}),$$

where $\mathcal{N}(0, D)$ denotes a centered Gaussian measure with covariance $D = H(h)(0) \geq 0$ ($H(h)(0)$ is the hessian of h at $u = 0$), and

$$\lim_{n \rightarrow +\infty} \frac{\mathbb{E}_n[T_n]}{n} = \nabla h(0).$$

We shall apply this theorem with $T_n = X_n - X_0$ and all probability spaces $(\Omega_n, \mathcal{B}_n, \mathbb{P}_n)$ coinciding with $(\Omega, \mathcal{F}, \mathbb{P}'_\rho)$, where Ω and \mathcal{F} are the ones defined in Sect. 2.1. The procedure requires some work in order to explicitly compute all the involved quantities. We briefly sum up the main steps of the proof, that will involve all this section.

- First of all, we need to calculate the scaled cumulant generating functions of $X_n - X_0$. We shall show in Lemma 3.3 that they are related to some smooth deformations $\tilde{\mathcal{L}}_u$ of the local map \mathcal{L} restricted to the support of $A(\mathcal{V})$ (defined in (3.6)).
- Once we have calculated the scaled cumulant generating functions, we shall compute the limit function h and this will be done in the proof of Theorem 3.6. It will be shown to coincide with the logarithm of the spectral radius λ_u of the deformations $\tilde{\mathcal{L}}_u$.

- To complete the picture given by Bryc’s Theorem, we need to consider the first and second derivatives of h . Lemmas 3.4 and 3.5 give us the technical instruments to identify these derivatives.

For all $u \in \mathbb{R}^d$, let us define the following operators:

$$L_i^{(u)} = e^{\frac{u \cdot s_i}{2}} L_i, \quad \tilde{L}_i = \tilde{p}_V L_i \tilde{p}_V, \quad \tilde{L}_i^{(u)} = e^{\frac{u \cdot s_i}{2}} \tilde{L}_i, \quad i = 1, \dots, v$$

(recall that \tilde{p}_V is the support projection of $A(V)$) and we call \mathfrak{L}_u and $\tilde{\mathfrak{L}}_u$ the deformations of \mathfrak{L} and $\tilde{\mathfrak{L}} := \tilde{\mathfrak{L}}_0$ respectively, defined as the completely positive operators

$$\mathfrak{L}_u(\sigma) = \sum_{i=1}^v L_i^{(u)} \sigma L_i^{(u)*}, \quad \tilde{\mathfrak{L}}_u(\sigma) = \sum_{i=1}^v \tilde{L}_i^{(u)} \sigma \tilde{L}_i^{(u)*}. \tag{3.6}$$

Notice that \mathfrak{L}_u and $\tilde{\mathfrak{L}}_u$ can be extended for complex values of u and form two analytic families of matrices: $\mathfrak{L}_u(\sigma) = \sum_{i=1}^v e^{u \cdot s_i} L_i \sigma L_i^*$ and $\tilde{\mathfrak{L}}_u(\sigma) = \sum_{i=1}^v e^{u \cdot s_i} \tilde{L}_i \sigma \tilde{L}_i^*$.

Further, all previous mathematical objects depend on the enclosure \mathcal{V} , so it would be more precise to highlight this and denote them $\tilde{L}_i^{(u, \mathcal{V})}$, $\tilde{\mathfrak{L}}_u^{\mathcal{V}}$, ..., $\mathbb{P}_\rho^{\mathcal{V}}$. Since the notations are already quite heavy, we drop the dependence on \mathcal{V} in this section, since we shall use only one enclosure and we shall recover it when necessary, treating the general case.

Lemma 3.3 *Let us denote by \mathbb{E}'_ρ the expected value under the measure \mathbb{P}'_ρ . The scaled cumulant generating function h_n of $X_n - X_0$ under \mathbb{P}'_ρ can be expressed in the following form:*

$$h_n(u) := \frac{1}{n} \log(\mathbb{E}'_\rho[e^{u \cdot (X_n - X_0)}]) = \frac{1}{n} \log \left(\sum_{k \in V} \frac{\text{Tr}(A(V) \tilde{\mathfrak{L}}_u^n(\rho(k)))}{\mathbb{E}_\rho[\text{Tr}(A(V) \rho_0)]} \right), \quad u \in \mathbb{C}^d. \tag{3.7}$$

Proof The proof is just a direct computation. By Lemma 3.1, for $u \in \mathbb{C}^d$

$$\begin{aligned} \mathbb{E}'_\rho[e^{u \cdot (X_n - X_0)}] &= \frac{\mathbb{E}_\rho[\text{Tr}(A(V) \rho_n) e^{u \cdot (X_n - X_0)}]}{\mathbb{E}_\rho[\text{Tr}(A(V) \rho_0)]} \\ &= \frac{1}{\mathbb{E}_\rho[\text{Tr}(A(V) \rho_0)]} \sum_{k \in V} \sum_{s_{j_1}, \dots, s_{j_n}} e^{u \cdot \sum_{k=1}^n s_{j_k}} \text{Tr}(A(V) L_{j_n} \cdots L_{j_1} \rho(k) L_{j_1}^* \cdots L_{j_n}^*) \end{aligned}$$

since we can use the density of \mathbb{P}'_ρ on the algebra \mathcal{F}_n . Now, recalling that \tilde{p}_V is the support projection of $A(V)$ and using relations (3.3), one has

$$A(V) L_{j_n} \cdots L_{j_1} = A(V) \tilde{p}_V L_{j_n} \tilde{p}_V \cdots \tilde{p}_V L_{j_1} \tilde{p}_V = A(V) \tilde{L}_{j_n} \cdots \tilde{L}_{j_1}$$

and replacing in the previous expression we obtain

$$\begin{aligned} \mathbb{E}'_\rho[e^{u \cdot (X_n - X_0)}] &= \frac{1}{\mathbb{E}_\rho[\text{Tr}(A(V) \rho_0)]} \sum_{k \in V} \sum_{s_{j_1}, \dots, s_{j_n}} e^{u \cdot \sum_{k=1}^n s_{j_k}} \text{Tr}(A(V) \tilde{L}_{j_n} \cdots \tilde{L}_{j_1} \rho(k) \tilde{L}_{j_1}^* \cdots \tilde{L}_{j_n}^*) \\ &= \sum_{k \in V} \frac{\text{Tr}(A(V) \tilde{\mathfrak{L}}_u^n(\rho(k)))}{\mathbb{E}_\rho[\text{Tr}(A(V) \rho_0)]}, \end{aligned}$$

where the last equality simply follows from the definition of the operator $\tilde{\mathfrak{L}}_u$ (3.6). □

Equation 3.7 shows the connection between the scaled cumulant generating function of $X_n - X_0$ and the powers of the map $\tilde{\mathfrak{L}}_u$, therefore, in order to derive the limit behavior of $h_n(u)$, a natural thing to do is to study the spectral properties of $\tilde{\mathfrak{L}}_u$. For $u \in \mathbb{R}^d$ we denote

$$\lambda_u = r(\tilde{\mathfrak{L}}_u) \text{ the spectral radius of } \tilde{\mathfrak{L}}_u, \tag{3.8}$$

and Theorem 2.6 ensures that $\lambda_u \in \text{Sp}(\tilde{\mathcal{L}}_u)$ with corresponding positive eigenvector τ_u . Notice that $\lambda_0 = 1$ and τ_0 is the unique minimal invariant state supported on \mathcal{V} . In Lemma 3.4 we shall prove that in a complex neighborhood of the origin the perturbed eigenvalue λ_u and eigenvector τ_u are analytic and that they only depend on the restriction of $\tilde{\mathcal{L}}_u$ (or equivalently \mathcal{L}_u) to the minimal enclosure \mathcal{V} .

We will show that

$$h(u) = \lim_{n \rightarrow +\infty} h_n(u) = \log(\lambda_u)$$

for u in a complex neighborhood of the origin.

Lemma 3.5 will provide an explicit expression for the gradient and the hessian of the limit function h , which identify the asymptotic behaviour of the mean values $\mathbb{E}'_{\rho}[X_n - X_0]$ and the covariance matrix of the limit Gaussian measure.

Lemma 3.4 *Let \mathcal{V} be a minimal enclosure. The operators $\tilde{\mathcal{L}}$ and $\tilde{\mathcal{L}}|_{\mathcal{V}} = \mathcal{L}|_{\mathcal{V}}$ have the same peripheral eigenvalues and eigenvectors with the same multiplicities.*

Moreover in a complex neighborhood of the origin the following hold true:

1. $u \mapsto \lambda_u$ and $u \mapsto \tau_u$ are analytic;
2. $\text{supp}(\tau_u) \subset \mathcal{V}$.

Hence λ_u and τ_u coincide with the analogous quantities for the restricted deformation $\tilde{\mathcal{L}}_{u|_{\mathcal{V}}} = \mathcal{L}_{u|_{\mathcal{V}}}$ (i.e. $\lambda_u = r(\mathcal{L}_{u|_{\mathcal{V}}})$, $\mathcal{L}_{u|_{\mathcal{V}}}(\tau_u) = \lambda_u \tau_u$).

Proof Let $\vartheta \in [0, 2\pi)$ and $\sigma \in L^1(\mathfrak{h})$ such that

$$\tilde{\mathcal{L}}(\sigma) = e^{i\vartheta} \sigma. \tag{3.9}$$

In order to prove that the peripheral eigenvectors and eigenvalues of $\tilde{\mathcal{L}}$ are the same as those of $\tilde{\mathcal{L}}|_{\mathcal{V}}$ we need to prove that $\sigma = p_{\mathcal{V}}\sigma p_{\mathcal{V}}$. Let us consider the orthogonal decomposition $\text{supp}(A(\mathcal{V})) = \mathcal{V} \oplus \mathcal{W}$; by definition $\mathcal{W} = \text{supp}(A(\mathcal{V}) - p_{\mathcal{V}})$ and, since $\dim(\mathfrak{h}) < +\infty$, we know that there exists a constant $\gamma > 0$ such that $p_{\mathcal{W}} \leq \gamma(A(\mathcal{V}) - p_{\mathcal{V}})$, hence by [12, Theorem 14] we have that

$$\tilde{\mathcal{L}}^{*n}(p_{\mathcal{W}}) = \tilde{p}_{\mathcal{V}} \mathcal{L}^{*n}(p_{\mathcal{W}}) \tilde{p}_{\mathcal{V}} \leq \gamma \tilde{p}_{\mathcal{V}} \mathcal{L}^{*n}(A(\mathcal{V}) - p_{\mathcal{V}}) \tilde{p}_{\mathcal{V}} \rightarrow 0.$$

This implies that $\lim_{n \rightarrow +\infty} \|\tilde{\mathcal{L}}^n(\sigma) - p_{\mathcal{V}} \tilde{\mathcal{L}}^n(\sigma) p_{\mathcal{V}}\| = 0$, which, together with Eq. 3.9, implies that $\sigma = p_{\mathcal{V}}\sigma p_{\mathcal{V}}$. If we consider σ as above and ξ is such that $\tilde{\mathcal{L}}(\xi) = e^{i\vartheta} \xi + \sigma$, with the same reasoning as before we can deduce that also $\xi = p_{\mathcal{V}}\xi p_{\mathcal{V}}$ and hence the algebraic multiplicity of $e^{i\vartheta}$ is the same for $\tilde{\mathcal{L}}$ and $\tilde{\mathcal{L}}|_{\mathcal{V}}$.

1. By perturbation theory of linear matrices (see [26]), we only need to show that $\lambda_0 = 1$ is an algebraically simple eigenvalue of $\tilde{\mathcal{L}}$, which, by virtue of what we just showed, is equivalent to prove it for $\tilde{\mathcal{L}}|_{\mathcal{V}} = \mathcal{L}|_{\mathcal{V}}$ and this follows for instance from [45, Proposition 6.2].

2. We recall that $p_{\mathcal{V}}$ is subharmonic for \mathcal{L} , hence $L_i p_{\mathcal{V}} = p_{\mathcal{V}} L_i p_{\mathcal{V}}$ for every $i = 1, \dots, v$; by the definition of $\tilde{\mathcal{L}}_u$, it follows that also $\tilde{\mathcal{L}}_u$ preserves the space $p_{\mathcal{V}} L^1(\mathfrak{h}) p_{\mathcal{V}}$ and eigenvalues and eigenvectors of $\tilde{\mathcal{L}}_{u|_{\mathcal{V}}}$ are also eigenvalues and eigenvectors of $\tilde{\mathcal{L}}_u$. Let $\lambda_u^{\mathcal{V}}$ be the perturbation of 1 for $\tilde{\mathcal{L}}|_{\mathcal{V}}$; by [26, Theorem VII.1.7] and the proof of point 1. in the present Lemma, for small values of u , λ_u is the unique eigenvalue of $\tilde{\mathcal{L}}_u$ in a neighborhood of 1 and it is algebraically simple, however $\lambda_u^{\mathcal{V}}$ is another eigenvalue of $\tilde{\mathcal{L}}_u$ and $\lambda_0^{\mathcal{V}} = 1$ too, hence they must coincide in a neighborhood of the origin (remember that $u \mapsto \lambda_u^{\mathcal{V}}$ is continuous, see [26, Theorem 5.1]). Therefore we have that $\text{supp}(\tau_u) \subset \mathcal{V}$. □

Lemma 3.5 *The function $h : \mathbb{R}^d \ni u \mapsto \log(\lambda_u)$ is infinitely differentiable in 0. For every $u \in \mathbb{R}^d$, we introduce the operators $\mathfrak{L}'_{|\mathcal{V},u}$ and $\mathfrak{L}''_{|\mathcal{V},u}$ by*

$$\begin{aligned} \mathfrak{L}'_{|\mathcal{V},u}, \mathfrak{L}''_{|\mathcal{V},u} : L^1(\mathcal{V}) &\longrightarrow L^1(\mathcal{V}) \\ \mathfrak{L}'_{|\mathcal{V},u}(\sigma) &= \sum_{i=1}^v u \cdot s_i L_i \sigma L_i^*, \quad \mathfrak{L}''_{|\mathcal{V},u}(\sigma) = \sum_{i=1}^v (u \cdot s_i)^2 L_i \sigma L_i^*. \end{aligned}$$

Denoting $\lambda'_u = \frac{d\lambda_u}{dt} \Big|_{t=0}$, $\lambda''_u = \frac{d^2\lambda_u}{dt^2} \Big|_{t=0}$, we have

$$\lambda'_u = \text{Tr}(\mathfrak{L}'_{|\mathcal{V},u}(\tau_0)), \quad \lambda''_u = \text{Tr}(\mathfrak{L}''_{|\mathcal{V},u}(\tau_0)) + 2\text{Tr}(\mathfrak{L}'_{|\mathcal{V},u}(\eta_u))$$

where $\eta_u \in L^1(\mathcal{V})$ is the unique solution with zero trace of the equation

$$(\text{Id} - \mathfrak{L}_{|\mathcal{V}})(\eta_u) = \mathfrak{L}'_{|\mathcal{V},u}(\tau_0) - \text{Tr}(\mathfrak{L}'_{|\mathcal{V},u}(\tau_0))\tau_0.$$

This implies immediately that

$$\nabla h(0) \cdot u = \lambda'_u, \quad \langle u, H(h)(0)u \rangle = \lambda''_u - \lambda'^2_u,$$

where $H(h)(0)$ is the Hessian of h at 0.

Proof Notice that

$$\mathfrak{L}'_{|\mathcal{V},u}(\sigma) = \sum_{i=1}^v u \cdot s_i \{p_{\mathcal{V}} L_i p_{\mathcal{V}}\} \sigma \{p_{\mathcal{V}} L_i^* p_{\mathcal{V}}\},$$

due to the fact that \mathcal{V} is an enclosure (and similarly for $\mathfrak{L}''_{|\mathcal{V},u}(\sigma)$). This fact, together with Lemma 3.4, allows us to reduce the analysis to the irreducible channel $\mathfrak{L}_{|\mathcal{V}}$ and the proof is the same as in [15, Corollary 5.9]. \square

We are now ready to prove the main result of this section.

Theorem 3.6 *Consider a minimal enclosure \mathcal{V} , and τ_0 and λ_u defined as in relation (3.8) before. We introduce the vector*

$$m = \sum_{i=1}^v \text{Tr}(L_i \tau_0 L_i^*) s_i$$

and the matrix D which is the unique matrix satisfying the following formula for every $u \in \mathbb{R}^d$:

$$\langle u, Du \rangle = \lambda''_u - \lambda'^2_u.$$

Then, under \mathbb{P}'_ρ ,

$$\frac{(X_n - X_0) - nm}{\sqrt{n}} \rightarrow \mathcal{N}(0, D) \tag{3.10}$$

where the convergence is in law. Moreover

$$\left| \frac{\mathbb{E}'_\rho[X_n - X_0]}{n} - m \right| = O\left(\frac{1}{n}\right).$$

Remark 3.7 We point out that, when there is a unique minimal enclosure \mathcal{V} , then $A(\mathcal{V}) = 1_{\mathfrak{h}}$, $\mathbb{P}'_{\rho} = \mathbb{P}_{\rho}$, and Theorem 3.6 includes the CLTs for the position process proved in [2] or in [15].

Proof In order to apply Bryc’s theorem (Theorem 3.2), we need to show the existence of

$$\lim_{n \rightarrow +\infty} h_n(u),$$

where $h_n(u)$ is defined in relation (3.7), for u in a complex neighborhood of 0. Let us first consider the case where $\tilde{\mathfrak{L}}|_{\mathcal{V}}$ is aperiodic (since we mimic the proof of [15, Theorem 5.12], we refer to [15] for more information about the notion of period for quantum channels). In this case we have

$$\delta = \sup\{|\lambda| : \lambda \in \text{Sp}(\tilde{\mathfrak{L}}) \setminus \{1\}\} < 1$$

and so, considering the Jordan form of $\tilde{\mathfrak{L}}$, there exists $\epsilon > 0$ such that $\delta + \epsilon < 1$ and for u in a neighbourhood of 0, for $n \in \mathbb{N}$ we have

$$\tilde{\mathfrak{L}}_u^n(\cdot) = \lambda_u^n(\varphi_u(\cdot)\tau_u + O((\delta + \epsilon)^n))$$

where φ_u is a linear form on $L^1(\mathfrak{h})$, analytic in u in the considered neighbourhood of the origin and O is with respect to any norm (remember that in finite dimension all the operator norms are equivalent). Therefore, using equation (3.7), we obtain

$$\begin{aligned} h_n(u) &= \log(\lambda_u) \\ &+ \frac{1}{n} \left[-\log(\mathbb{E}_{\rho}[\text{Tr}(A(\mathcal{V})\rho_0)]) + \log \left(\sum_{k \in \mathcal{V}} \varphi_u(\rho(k))\text{Tr}(A(\mathcal{V})\tau_u) + O((\delta + \epsilon)^n) \right) \right] \\ &\xrightarrow{n \rightarrow +\infty} \log(\lambda_u). \end{aligned}$$

From the proof of Theorem 3.2 we know that all h_n are analytic in a neighborhood of the origin. Further, these functions converge uniformly on compact sets to h and $\sup_{u \in K} |h_n(u) - \log(\lambda_u)| = O(1/n)$ where K is a compact set in the considered neighborhood of the origin. Hence, by Cauchy integral formula we can deduce, since

$$\frac{\mathbb{E}'_{\rho}[X_n - X_0]}{n} = \nabla h_n(0) \text{ and } m = \nabla h(0)$$

that

$$\left| \frac{\mathbb{E}'_{\rho}[X_n - X_0]}{n} - m \right| = O\left(\frac{1}{n}\right)$$

and this allows us to put nm instead of $\mathbb{E}'_{\rho}[X_n - X_0]$ in Eq. (3.10).

On the other hand, if $\mathfrak{L}|_{\mathcal{V}}$ has period $l > 1$ with cyclic resolution p_0, \dots, p_{l-1} , we can write for $n = ql + r$ and $0 \leq r < l$

$$\mathbb{E}'_{\rho}[e^{u \cdot (X_n - X_0)}] = \underbrace{\sum_{j=0}^{l-1} \sum_{z \in \mathcal{V}} \frac{\text{Tr}(A(p_j)\rho(z))}{\mathbb{E}_{\rho}[\text{Tr}(A(\mathcal{V})\rho_0)]}}_{w_j} \underbrace{\sum_{k \in \mathcal{V}} \frac{\text{Tr}(A(p_j)\tilde{\mathfrak{L}}_u^n(\rho(k)))}{\sum_{z \in \mathcal{V}} \text{Tr}(A(p_j)\rho(z))}}_{II}.$$

We can safely define $A(p_j)$ using $\tilde{\mathcal{L}}^l$, for which p_0, \dots, p_{l-1} are minimal enclosures. Furthermore we can express II as

$$II = \sum_{k \in V} \frac{\text{Tr}(A(p_j) \tilde{\mathcal{L}}_u^{lq}(\tilde{\mathcal{L}}_u^r(\rho(\underline{k}))))}{\sum_{z \in V} \text{Tr}(A(p_j) \rho(\underline{z}))}.$$

The support projection of $A(p_j)$, which we call P_j , is superharmonic for $\tilde{\mathcal{L}}^l$, hence, if we consider $\tilde{\mathcal{L}}_{j,u}^l := P_j \tilde{\mathcal{L}}^l(P_j \cdot P_j) P_j$, we can write

$$\text{Tr}(A(p_j) \tilde{\mathcal{L}}_u^{lq}(\tilde{\mathcal{L}}_u^r(\rho(\underline{k})))) = \text{Tr}(A(p_j) \tilde{\mathcal{L}}_{j,u}^{lq}(P_j \tilde{\mathcal{L}}_u^r(\rho(\underline{k})) P_j))$$

and we are back to the aperiodic case. Furthermore the perturbation of 1 for every reduction $\tilde{\mathcal{L}}_{j,u}^l$ is the same as the one of $\tilde{\mathcal{L}}_u^l$ since $P_j \tau_u P_j$ is an eigenvector of $\tilde{\mathcal{L}}_{j,u}^l$ for the eigenvalue λ_u^l :

$$\tilde{\mathcal{L}}_{j,u}^l(P_j \tau_u P_j) = P_j \tilde{\mathcal{L}}_u^l(\tau_u) P_j = \lambda_u^l P_j \tau_u P_j.$$

Therefore we can again prove the statement. □

4 General Case: Mixture of Gaussians

In order to tackle the general case, we now need to consider different enclosures and to handle the simultaneous appearance of different Gaussians. The description of the general context requests the introduction of some additional notions in order to describe an appropriate decomposition of the local Hilbert space \mathfrak{h} . This will induce a decomposition of the measure \mathbb{P}_ρ in terms of measures of the form \mathbb{P}'_ρ as defined in Lemma 3.1.

The solution of the problem is delicate and based on recent results about quantum channels, but some basic ideas can be quite intuitive to grasp. When the local channel \mathcal{L} has a unique invariant state, the system somehow locally converges to it. There is a loss of memory of the position process: the asymptotic behavior do not depend on the initial state and in particular the parameters of the limit Gaussian do not depend on ρ in this case. When we have more invariant states, the system locally converges to some invariant state (in the sense of Frigerio and Verri’s ergodic theorem for instance), but the limit invariant state is a convex combination of extremal invariant states which depends on the initial state ρ . Informally, we can think that the initial state assigns a quantity of mass to the minimal enclosures (i.e. the supports of extremal invariant states), but this mass can be increased by the repeated action of \mathcal{L} because the minimal enclosures absorb mass from the so called transient space. Given the initial state, through absorption operators, we can identify the final quantity of mass flowing to the single minimal enclosures and this will identify the limit behavior.

The first part of this section will recap some definitions about transient and recurrent spaces and about the structure of invariant states and minimal enclosures. In Lemma 4.2, we shall see how this structure is related to the measures of the form \mathbb{P}'_ρ (and to the parameters m and D appearing in the CLT). Then, in Lemma 4.3, we will use the same structure to identify \mathbb{P}_ρ as a convex combination of measures of the form \mathbb{P}'_ρ obtained from different enclosures. Finally, we shall apply these results to deal with the general central limit type asymptotic theorem.

Decomposition of the local Hilbert space and of the recurrent subspace.

We introduce the fast recurrent and the transient space for the local map following [8, 42]; for other notions of recurrence for OQWs we refer to [5, 18, 21] and references therein. We denote by \mathcal{R} the fast recurrent space for the channel \mathfrak{L}

$$\mathcal{R} = \sup\{\text{supp}(\omega) \mid \omega \text{ is an invariant state for } \mathfrak{L}\}. \tag{4.11}$$

\mathcal{R} is an enclosure and, since the space \mathfrak{h} is finite dimensional, any minimal enclosure is included in \mathcal{R} and is the support of a unique extremal invariant state; moreover we have trivial slow recurrent subspace, while \mathcal{R} is always non trivial and ‘‘absorbing’’. Further, the orthogonal complement of \mathcal{R} is the transient space, usually denoted by \mathcal{T} and the absorption in \mathcal{R} is the identity operator (see [8, 15, 42])

$$\mathfrak{h} = \mathcal{R} \oplus \mathcal{T}, \quad A(\mathcal{R}) = 1_{\mathfrak{h}} - \lim_{n \rightarrow +\infty} \mathfrak{L}^{*n}(p_{\mathcal{T}}) = 1_{\mathfrak{h}}.$$

The structure of quantum channels induces a decomposition of the fast recurrent space, also naturally related to the invariant states (see [8] for the finite dimensional case and [16, 24, 42] for infinite dimensional state spaces). This decomposition is the noncommutative counterpart of the decomposition in communication classes for classical Markov chains and plays a fundamental role in different contexts. Here we shall briefly recall the decomposition and the main properties we need.

For a quantum channel acting on $L^1(\mathfrak{h})$, there exists a unique decomposition of \mathcal{R} of the form

$$\mathcal{R} = \bigoplus_{\alpha \in A} \chi_{\alpha},$$

where $(\chi_{\alpha})_{\alpha \in A}$ is a finite set of mutually orthogonal enclosures and every χ_{α} is minimal in the set of enclosures verifying the property:

$$\text{for any minimal enclosure } \mathcal{W} \text{ either } \mathcal{W} \perp \chi_{\alpha} \text{ or } \mathcal{W} \subset \chi_{\alpha}.$$

Every χ_{α} either is a minimal enclosure or can be further decomposed (but not in a unique way!) as the sum of mutually orthogonal isomorphic minimal enclosures, i.e.

$$\chi_{\alpha} = \bigoplus_{\beta \in I_{\alpha}} \mathcal{V}_{\alpha,\beta}, \quad \mathcal{R} = \bigoplus_{\alpha \in A} \chi_{\alpha} = \bigoplus_{\alpha \in A} \bigoplus_{\beta \in I_{\alpha}} \mathcal{V}_{\alpha,\beta}, \tag{4.12}$$

for some finite set $\mathcal{V}_{\alpha,\beta}, \beta \in I_{\alpha}$ of minimal enclosures and, if we fix a particular $\bar{\beta} \in I_{\alpha}$, there exists a unitary transformation U_{α} such that

$$U_{\alpha} : \mathbb{C}^{|I_{\alpha}|} \otimes \mathcal{V}_{\alpha,\bar{\beta}} \rightarrow \chi_{\alpha}. \tag{4.13}$$

Moreover one can define an irreducible quantum channel ψ on $B(\mathcal{V}_{\alpha,\bar{\beta}})$ which completely describes the restriction of the channel to χ_{α}

$$\mathfrak{L}^*_{|\mathcal{R}}(U_{\alpha}(a \otimes b)U_{\alpha}^*) = U_{\alpha}(a \otimes \psi(b))U_{\alpha}^* \quad a \in B(\mathbb{C}^{|I_{\alpha}|}), b \in B(\mathcal{V}_{\alpha,\bar{\beta}}). \tag{4.14}$$

Remark 4.1 χ_{α} is a minimal enclosure if and only if $|I_{\alpha}| = 1$. Otherwise, it is not minimal and it admits infinite possible decompositions in orthogonal minimal enclosures of the form $U_{\alpha}(\mathbb{C}v \otimes \mathcal{V}_{\alpha,\bar{\beta}})$ for $v \in \mathbb{C}^{|I_{\alpha}|}$. In this case, however, a rigid structure of the channel essentially reduces the action on any minimal enclosure inside χ_{α} to be the same up to a unitary transform.

Lemma 4.2 *The parameters $m = m(\mathcal{V})$ and $D = D(\mathcal{V})$ introduced in Theorem 3.6 are independent of the particular minimal enclosure \mathcal{V} in χ_α . Then we define*

$$m_\alpha := \sum_{i=1}^v \text{Tr}(L_i \tau_0^\mathcal{V} L_i^*) s_i, \quad \langle u, D_\alpha u \rangle = \lambda''_u - \lambda'_u{}^2,$$

where λ'_u, λ''_u are defined as in Lemma 3.5 for $\mathfrak{L}|_\mathcal{V}$.

Proof Let us consider two minimal enclosures \mathcal{V} and \mathcal{W} contained in a same χ_α . We just have to prove that the parameters m and D are equal for the two enclosures.

Relations (4.13) and (4.14) imply that there exist two vectors v, w in $\mathbb{C}^{|\mathcal{I}_\alpha|}$ such that

$$\mathcal{V} = U_\alpha((\mathbb{C}v) \otimes \mathcal{V}_{\alpha,\bar{\beta}})U_\alpha^*, \quad \mathcal{W} = U_\alpha((\mathbb{C}w) \otimes \mathcal{V}_{\alpha,\bar{\beta}})U_\alpha^*, \tag{4.15}$$

and we can define a partial isometry $Q = U_\alpha((|w\rangle\langle v|) \otimes 1_{\mathcal{V}_{\alpha,\bar{\beta}}})U_\alpha^*$, from \mathcal{V} to \mathcal{W} , such that

$$Q^*Q = p_\mathcal{V}, \quad QQ^* = p_\mathcal{W} \quad \text{and} \quad \mathfrak{L}_{|\mathcal{V}}^*(x) = Q^* \mathfrak{L}_{|\mathcal{W}}^*(QxQ^*)Q \quad \forall x \in B(\mathcal{V}), \tag{4.16}$$

where $\mathfrak{L}|_\mathcal{V}$ and $\mathfrak{L}|_\mathcal{W}$ are the restrictions of \mathfrak{L} to \mathcal{V} and \mathcal{W} respectively, following the notations introduced before. Due to relation (4.14), Q (and Q^*) is also a fixed point for the dual channel \mathfrak{L}^* , so that it commutes with the Kraus operators L_i, L_i^* for all i (see for instance [13], in particular Proposition 1 applied to the fast recurrent channel \mathfrak{L} restricted to χ_α).

Moreover, since \mathcal{V} and \mathcal{W} are minimal, they are the support of two invariant states, that we can denote by $\tau_0^\mathcal{V}$ and $\tau_0^\mathcal{W}$ and will verify

$$\tau_0^\mathcal{V} = Q^* \tau_0^\mathcal{W} Q.$$

Then we have

$$\text{Tr}(L_i \tau_0^\mathcal{W} L_i^*) = \text{Tr}(L_i Q \tau_0^\mathcal{V} Q^* L_i^*) = \text{Tr}(Q L_i \tau_0^\mathcal{V} L_i^* Q^*) = \text{Tr}(p_\mathcal{V} L_i \tau_0^\mathcal{V} L_i^*) = \text{Tr}(L_i \tau_0^\mathcal{V} L_i^*)$$

so that

$$m(\mathcal{W}) = \sum_i \text{Tr}(L_i \tau_0^\mathcal{W} L_i^*) s_i = \sum_i \text{Tr}(L_i \tau_0^\mathcal{V} L_i^*) s_i = m(\mathcal{V}).$$

Similarly we deduce, for any $u \in \mathbb{R}^d$,

$$\mathfrak{L}'_{|\mathcal{V},u}(Q^* \cdot Q) = Q^* \mathfrak{L}'_{|\mathcal{W},u}(\cdot)Q, \quad \mathfrak{L}''_{|\mathcal{V},u}(Q^* \cdot Q) = Q^* \mathfrak{L}''_{|\mathcal{W},u}(\cdot)Q.$$

Therefore

$$\text{Tr}(\mathfrak{L}'_{|\mathcal{V},u}(\tau_0^\mathcal{V})) = \text{Tr}(\mathfrak{L}'_{|\mathcal{W},u}(\tau_0^\mathcal{W})) \quad \text{and} \quad \text{Tr}(\mathfrak{L}''_{|\mathcal{V},u}(\tau_0^\mathcal{V})) = \text{Tr}(\mathfrak{L}''_{|\mathcal{W},u}(\tau_0^\mathcal{W})).$$

By the same arguments, for all $u \in \mathbb{R}^d$,

$$\eta_u^\mathcal{V} = Q^* \eta_u^\mathcal{W} Q \quad \text{and} \quad \text{Tr}(\mathfrak{L}'_{|\mathcal{V},u}(\eta_u^\mathcal{V})) = \text{Tr}(\mathfrak{L}'_{|\mathcal{W},u}(\eta_u^\mathcal{W}))$$

and we can conclude that $D(\mathcal{V}) = D(\mathcal{W})$. □

Decomposition of the measure \mathbb{P}_ρ .

In Lemma 3.1, we fixed an enclosure \mathcal{V} and we introduced the probability measure denoted by \mathbb{P}'_ρ . Now we need to handle different enclosures, the ones appearing in the decomposition of \mathcal{R} given in relations (4.12). We need to highlight the dependence on the enclosure and we

shall denote from now on by \mathbb{P}^α (resp.ly $\mathbb{P}^{\alpha,\beta}$) the measure \mathbb{P}'_ρ obtained with $\mathcal{V} = \chi_\alpha$ (resp.ly $\mathcal{V} = \mathcal{V}_{\alpha,\beta}$), i.e. with densities satisfying

$$\frac{d\mathbb{P}^\alpha_\rho}{d\mathbb{P}_\rho} \Big|_{\mathcal{F}_n} = \frac{\text{Tr}(A(\chi_\alpha)\rho_n)}{\mathbb{E}_\rho[\text{Tr}(A(\chi_\alpha)\rho_0)]}, \quad \frac{d\mathbb{P}^{\alpha,\beta}_\rho}{d\mathbb{P}_\rho} \Big|_{\mathcal{F}_n} = \frac{\text{Tr}(A(\mathcal{V}_{\alpha,\beta})\rho_n)}{\mathbb{E}_\rho[\text{Tr}(A(\mathcal{V}_{\alpha,\beta})\rho_0)]}. \tag{4.17}$$

We can then decompose \mathbb{P}_ρ into a mixture of \mathbb{P}^α_ρ and $\mathbb{P}^{\alpha,\beta}_\rho$.

Lemma 4.3 *For any $\alpha \in A, \beta \in I_\alpha$ let us define*

$$a_\alpha(\rho) := \mathbb{E}_\rho[Y_0^\alpha] = \mathbb{E}_\rho[\text{Tr}(A(\chi_\alpha)\rho_0)] = \sum_{\underline{k} \in V} \text{Tr}(A(\chi_\alpha)\rho(\underline{k}))$$

$$\text{and } a_{\alpha,\beta}(\rho) := \mathbb{E}_\rho[Y_0^{\alpha,\beta}] = \mathbb{E}_\rho[\text{Tr}(A(\mathcal{V}_{\alpha,\beta})\rho_0)] = \sum_{\underline{k} \in V} \text{Tr}(A(\mathcal{V}_{\alpha,\beta})\rho(\underline{k})).$$

We can write \mathbb{P}_ρ as convex combination

$$\mathbb{P}_\rho = \sum_{\alpha \in A} a_\alpha(\rho)\mathbb{P}^\alpha_\rho = \sum_{\alpha \in A} \sum_{\beta \in I_\alpha} a_{\alpha,\beta}(\rho)\mathbb{P}^{\alpha,\beta}_\rho. \tag{4.18}$$

Proof Indeed, for every $\underline{k} \in V, n \geq 0, \underline{j} \in J^n$

$$\begin{aligned} \mathbb{P}_\rho(\{(\underline{k}, \underline{j})\} \times J^\mathbb{N}) &= \text{Tr}(L_{\underline{j}}\rho(\underline{k})L_{\underline{j}}^*) = \sum_{\alpha \in A} \text{Tr}(A(\chi_\alpha)L_{\underline{j}}\rho(\underline{k})L_{\underline{j}}^*) \\ &= \sum_{\alpha \in A} a_\alpha(\rho) \cdot \text{Tr}(L_{\underline{j}}\rho(\underline{k})L_{\underline{j}}^*) \frac{1}{\mathbb{E}_\rho[\text{Tr}(A(\chi_\alpha)\rho_0)]} \text{Tr} \left(A(\chi_\alpha) \frac{L_{\underline{j}}\rho(\underline{k})L_{\underline{j}}^*}{\text{Tr}(L_{\underline{j}}\rho(\underline{k})L_{\underline{j}}^*)} \right) \\ &= \sum_{\alpha \in A} a_\alpha(\rho)\mathbb{P}^\alpha_\rho(\{(\underline{k}, \underline{j})\} \times J^\mathbb{N}). \end{aligned}$$

where the second equality follows because $\sum_{\alpha \in A} A(\chi_\alpha) = 1_{\mathfrak{h}}$. Similarly one can further decompose the probability measure in $\mathbb{P}^{\alpha,\beta}_\rho$ because for every $\alpha \in A, \sum_{\beta \in B_\alpha} A(\mathcal{V}_{\alpha,\beta}) = A(\chi_\alpha)$. Equation (4.18) is then true because sets of the form $\{(\underline{k}, \underline{j})\} \times J^\mathbb{N}$ generate \mathcal{F} . \square

Before proceeding forward, we can now complete the proof of Lemma 3.1 and deduce relations (3.4).

Proof (of Lemma 3.1—second part).

First notice the following set equivalence:

$$\left\{ \frac{d\mathbb{P}'_\rho}{d\mathbb{P}_\rho} = \frac{1}{\mathbb{E}_\rho[Y_0]} \right\} = \{Y_\infty = 1\}, \quad \left\{ \frac{d\mathbb{P}'_\rho}{d\mathbb{P}_\rho} = 0 \right\} = \{Y_\infty = 0\}.$$

Let us denote by q the orthogonal projection onto the eigenspace corresponding to the eigenvalue 1 of $A(\mathcal{V})$; since $0 \leq A(\mathcal{V}) \leq 1_{\mathfrak{h}}, Y_\infty = 0$ ($Y_\infty = 1$) if and only if $\lim_{n \rightarrow +\infty} \|\tilde{p}_{\mathcal{V}}^{\perp} \rho_n \tilde{p}_{\mathcal{V}}^{\perp} - \rho_n\| = 0$ ($\lim_{n \rightarrow +\infty} q \rho_n q - \rho_n\| = 0$). By [12, Theorem 14], we know that $q - p_{\mathcal{T}} \leq p_{\mathcal{T}}$, hence to conclude we only need to show that $\lim_{n \rightarrow +\infty} \|p_{\mathcal{T}} \rho_n p_{\mathcal{T}}\| = 0$. Since $p_{\mathcal{T}}$ is superharmonic, $T_n := \text{Tr}(p_{\mathcal{T}} \rho_n)$ is a supermartingale:

$$\mathbb{E}_\rho[T_{n+1} | \mathcal{F}_n] = \sum_{i=1}^v \text{Tr}(L_i \rho_n L_i^*) \frac{\text{Tr}(p_{\mathcal{T}} L_i \rho_n L_i^*)}{\text{Tr}(L_i \rho_n L_i^*)} = \text{Tr}(\mathcal{L}^*(p_{\mathcal{T}})\rho_n) \leq \text{Tr}(p_{\mathcal{T}} \rho_n) = T_n.$$

Furthermore $0 \leq T_n \leq 1$, hence T_n converges \mathbb{P}_ρ -a.s. to a certain limit T_∞ . Notice that $\mathbb{E}_\rho[T_\infty] = \lim_{n \rightarrow +\infty} \mathbb{E}_\rho[T_n] = \lim_{n \rightarrow +\infty} \mathcal{L}^{*n}(p_{\mathcal{T}}) = 0$, hence $T_\infty = 0$, which implies that $\lim_{n \rightarrow +\infty} \|p_{\mathcal{T}} \rho_n p_{\mathcal{T}}\| = 0$. \square

Generalized Central Limit Theorem

In the sequel, we aim to provide a precise description of the most general situation where the rescaled position process $(X_n - X_0)/\sqrt{n}$ gets closer to a convex combination of Gaussian laws. In particular, we will fruitfully use the decomposition (4.18) introduced before:

$$\mathbb{P}_\rho = \sum_{\alpha \in A} a_\alpha(\rho) \mathbb{P}_\rho^\alpha.$$

In the case when the above expression becomes a single probability measure \mathbb{P}_ρ^α , Theorem 3.6 together with Lemma 4.2 imply a ‘‘classical’’ Central Limit Theorem for the law of $(X_n - X_0 - \mathbb{E}_\rho[X_n - X_0])/\sqrt{n}$. In the general case, that is, when we allow for two or more probability measures, the situation is completely different. In general, as we mentioned in the Introduction, we cannot expect a convergence in law because there does not need to exist any sequence of vectors $m_n \in \mathbb{R}^d$ such that $(X_n - X_0 - m_n)/\sqrt{n}$ can converge in law (not to a Gaussian, nor to any other law). Nevertheless, we can observe that for increasing values of n the law of $(X_n - X_0)/\sqrt{n}$ under (4.18) gets closer to a mixture of Gaussian measures with means escaping to infinity along different directions. Roughly speaking, with the notations of Theorem 3.6 and Lemma 4.3 before, we expect that, at each step $n \in \mathbb{N}$,

$$\mathbb{P}_{\rho,n} \approx \sum_{\alpha \in A} a_\alpha(\rho) \mathcal{N}(\sqrt{n}m_\alpha, D_\alpha),$$

where $\mathbb{P}_{\rho,n}$ denotes the law of $(X_n - X_0)/\sqrt{n}$ and $\mathcal{N}(\sqrt{n}m_\alpha, D_\alpha)$ denotes the Gaussian measure with mean $\sqrt{n}m_\alpha$ and covariance matrix D_α , respectively. Informally, $\mathbb{P}_{\rho,n}$ can be thought of as a convex combination of Gaussian probabilities up to some error. A possible way to formalize the nature of the above approximation procedure will be through the use of a distance defined on the set of probability laws on \mathbb{R}^d .

The topology of convergence in law is induced by different distances. On this subject, we refer for instance to [19]. Among them, we choose the Fortet–Mourier metric, but the convergence results keep holding true also with a different choice. Let us denote by BL the set of bounded Lipschitz functions on \mathbb{R}^d equipped with the norm

$$\|f\|_{BL} = \sup_{x \in \mathbb{R}^d} |f(x)| + \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|};$$

we introduce the Fortet–Mourier distance between two probability laws P, Q on \mathbb{R}^d ,

$$\text{dist}(P, Q) := \sup \left\{ \left| \int_{\mathbb{R}^d} f dP - \int_{\mathbb{R}^d} f dQ \right| : f \in BL, \|f\|_{BL} \leq 1 \right\}.$$

We recall that [19, Theorem 11.3.3], for P_n, P probability measures on \mathbb{R}^d , the following fact holds

$$P_n \rightarrow P \text{ in law} \quad \text{if and only if} \quad \text{dist}(P_n, P) \rightarrow 0.$$

We are now in a position to state the general central limit type asymptotic theorem.

Theorem 4.4 (Approximation with a mixture of Gaussians)

Take m_α and D_α as in Lemma 4.2 and let $\mathbb{P}_{\rho,n}$ be the law of the random variable $\frac{X_n - X_0}{\sqrt{n}}$ under \mathbb{P}_ρ . Then

$$\lim_{n \rightarrow +\infty} \text{dist} \left(\mathbb{P}_{\rho,n}, \sum_{\alpha \in A} a_\alpha(\rho) \mathcal{N}(\sqrt{n}m_\alpha, D_\alpha) \right) = 0,$$

where $a_\alpha(\rho) = \mathbb{E}_\rho[\text{Tr}(A(\chi_\alpha)\rho_0)]$ and $\mathcal{N}(\sqrt{n}m_\alpha, D_\alpha)$ denotes the Gaussian measure with mean $\sqrt{n}m_\alpha$ and covariance matrix D_α .

Proof By Theorem 3.6, we know that the process $\frac{X_n - X_0 - nm_\alpha}{\sqrt{n}}$ converges in law to a centered normal distribution with covariance matrix D_α under the measure $\mathbb{P}_\rho^{\alpha,\beta}$, so that we can write

$$\lim_{n \rightarrow +\infty} \text{dist} \left(\mathbb{P}_\rho^{\alpha,\beta} \left(\frac{X_n - X_0 - nm_\alpha}{\sqrt{n}} \right), \mathcal{N}(0, D_\alpha) \right) = 0.$$

By definition, the Fortet–Mourier distance is invariant with respect to translations and consequently we deduce

$$\text{dist} \left(\mathbb{P}_\rho^{\alpha,\beta} \left(\frac{X_n - X_0}{\sqrt{n}} \right), \mathcal{N}(\sqrt{n}m_\alpha, D_\alpha) \right) \rightarrow 0, \quad \text{as } n \rightarrow +\infty.$$

Now, since this limit does not depend on β and, by Eq. (4.18) $\mathbb{P}_\rho^\alpha = \sum_{\beta \in I_\alpha} a_{\alpha,\beta}(\rho) \mathbb{P}_\rho^{\alpha,\beta}$ (we denote by \mathcal{N}_α the law $\mathcal{N}(\sqrt{n}m_\alpha, D_\alpha)$ to shorten the expressions in this proof),

$$\begin{aligned} & \text{dist} \left(\mathbb{P}_\rho^\alpha \left(\frac{X_n - X_0}{\sqrt{n}} \right), \mathcal{N}_\alpha \right) = \\ & = \sup \left\{ \left| \int_{\mathbb{R}^d} f \left(\frac{X_n - X_0}{\sqrt{n}} \right) d\mathbb{P}_\rho^\alpha - \int_{\mathbb{R}^d} f d\mathcal{N}_\alpha \right| : f \in BL, \|f\|_{BL} \leq 1 \right\} \\ & \leq \sum_{\beta \in I_\alpha} a_{\alpha,\beta}(\rho) \sup \left\{ \left| \int_{\mathbb{R}^d} f \left(\frac{X_n - X_0}{\sqrt{n}} \right) d\mathbb{P}_\rho^{\alpha,\beta} - \int_{\mathbb{R}^d} f d\mathcal{N}_\alpha \right| : f \in BL, \|f\|_{BL} \leq 1 \right\} \\ & = \sum_{\beta \in I_\alpha} a_{\alpha,\beta}(\rho) \text{dist} \left(\mathbb{P}_\rho^{\alpha,\beta} \left(\frac{X_n - X_0}{\sqrt{n}} \right), \mathcal{N}(\sqrt{n}m_\alpha, D_\alpha) \right) \rightarrow 0, \quad \text{as } n \rightarrow +\infty. \end{aligned}$$

Similarly, always by relation (4.18), $\mathbb{P}_\rho = \sum_{\alpha \in A} a_\alpha(\rho) \mathbb{P}_\rho^\alpha$ and by triangular inequality for any f in BL , we can call $\nu_n = \sum_{\alpha \in A} a_\alpha(\rho) \mathcal{N}_\alpha$ and write

$$\left| \int_{\mathbb{R}^d} f \left(\frac{X_n - X_0}{\sqrt{n}} \right) d\mathbb{P}_\rho - \int_{\mathbb{R}^d} f d\nu_n \right| \leq \sum_{\alpha \in A} a_\alpha(\rho) \left| \int_{\mathbb{R}^d} f \left(\frac{X_n - X_0}{\sqrt{n}} \right) d\mathbb{P}_\rho^\alpha - \int_{\mathbb{R}^d} f d\mathcal{N}_\alpha \right|$$

and we then conclude

$$\text{dist}(\mathbb{P}_{\rho,n}, \nu_n) \leq \sum_{\alpha \in A} a_\alpha(\rho) \text{dist}(\mathbb{P}_{\rho,n}^\alpha, \mathcal{N}_\alpha),$$

which converges to 0 as $n \rightarrow +\infty$. □

Notice that, while the weights $a_\alpha(\rho)$ depend on the initial state and on the transient part of \mathcal{L} , the parameters of the Gaussian measures only depend on the restriction to the fast recurrent part. Theorem 4.4 has the following direct consequence on the convergence of the empirical means.

Corollary 4.5 Let $\hat{\mathbb{P}}_{\rho,n}$ the law of the random variable $\frac{X_n - X_0}{n}$ under \mathbb{P}_ρ , then

$$\lim_{n \rightarrow +\infty} \text{dist} \left(\hat{\mathbb{P}}_{\rho,n}, \sum_{\alpha \in A} a_\alpha(\rho) \delta_{m_\alpha} \right) = 0,$$

where $a_\alpha(\rho)$ are defined as in previous theorem and δ_{m_α} denotes the Dirac measure concentrated in m_α .

Remark 4.6 Possible extensions. As for previous versions of Central Limit Theorems for HOQWs, we can extend our results to more general cases.

1. There is an immediate generalization of HOQWs obtained considering a change in the local state after a shift s_i given by a quantum operation \mathcal{L}_j with more than one Kraus operator, which is the case we considered ($\mathcal{L}_j(\cdot) = L_j \cdot L_j^*$). In this case it suffices to change the notation in the proof of Theorem 4.4 to see that it still holds true.
2. Open quantum walks have been defined also in continuous time [36] and the central limit theorem for the position process has already been proved in [10], under the assumption of irreducibility of \mathcal{R} . Theorem 4.4 can be carried with some technical adaptations to the continuous time case.

Remark 4.7 Comparison with previous results. The first CLT for HOQWs appeared in [2] where the authors proved it by the use of Poisson equation and martingale techniques in the case \mathcal{R} irreducible. Indeed, in [2, Theorem 7.3] they showed the convergence to different Gaussian measures under proper conditional probabilities and under assumptions which can be translated in our language to be

- $\mathcal{T} = \{0\}$,
- χ_α is minimal for every $\alpha \in A$,
- $m_\alpha \neq m_{\alpha'}$ if $\alpha \neq \alpha'$.

These techniques revealed to be successful to treat also other walks and in particular have recently been exploited also in [27] to obtain a CLT for the so-called lazy OQWs. Successively, in [15], an alternative proof of the central limit theorem for an irreducible fast recurrent local channel \mathcal{L} could be deduced from a large deviation principle, proved by deformation techniques. Finally the results in [30, 31, 33] (which are formulated in the setting of homogeneous open quantum walks on crystal lattices) state a kind of convergence to a mixture of Gaussian measures, under some conditions, always essentially implying that the local channel is fast recurrent.

Here, with Theorem 4.4, we can find an improvement of all these previous results since we can drop any condition about recurrence or transience or reducibility of the local channel and we can specify the form of convergence to the mixture of Gaussians introducing a metric on the set of probability measures. Moreover we can specify the weights of the limit mixture in terms of the initial state and of the decomposition of the local space.

We refer the reader to [41] for other hints on the existing literature until 2019 and to [10, 36, 37] for CLT results for different families of open walks.

5 Large Deviations

When the Central Limit Theorem is approached by Bryc’s theorem, it is often treated together with large deviations, and this was indeed the idea in [15], where the proof of the central limit

theorem in the particular case of an irreducible fast recurrent subspace was a byproduct of the large deviation principle. Similarly, it is here natural to wonder whether a large deviation principle can hold in general for the position process of a HOQW, always under the measure \mathbb{P}_ρ induced by the initial state ρ . We shall prove that Gärtner–Ellis’ theorem can be applied and thus large deviations hold when the local map is recurrent. Moreover, the rate function is related to the spectrum of the deformed map \mathcal{L}_u . When instead there is a non-trivial transient subspace for the local channel \mathcal{L} , the limit of the scaled cumulant generating functions is not smooth in general, as [15, Example 7.3] shows, and Gärtner–Ellis’ theorem will simply provide lower and upper bounds.

As for the results in the previous section, only the minimal enclosures in the decomposition of \mathcal{R} that are “reachable” by a initial state ρ will play a role in the large deviations results. For this reason, it is useful to remember the definition of the quantities $a_\alpha(\rho)$, $a_{\alpha,\beta}(\rho)$ (introduced in Lemma 4.3), which are a kind of quantum absorption probabilities of the evolution in the enclosures χ_α , or $\mathcal{V}_{\alpha,\beta}$ respectively, when the initial state is ρ . Differently from the central limit type results, here also the index β , and so the particular enclosures $\mathcal{V}_{\alpha,\beta}$ selected inside χ_α are important, and this is related to the fact that the evolution on the transient subspace affects large deviations results.

Since we need to define restrictions of the channel \mathcal{L} which take into account only proper subspaces reachable by the local initial states $\rho(\underline{k})$, we define the subspace

$$\mathcal{E}(\rho) := \text{span}\{\text{supp}(\mathcal{L}^n(\rho(\underline{k}))), \underline{k} \in V, n \geq 0\} \subset \mathfrak{h}, \tag{5.19}$$

which is an enclosure due to [16, Propositions 4.1 and 4.2].

We recall that by $\hat{\mathbb{P}}_{\rho,n}$ we denote the law of $\frac{X_n - X_0}{n}$ under \mathbb{P}_ρ and, given any enclosure \mathcal{V} , $\tilde{p}_\mathcal{V}$ is the orthogonal projection onto $\text{supp}(A(\mathcal{V}))$.

Theorem 5.1 Large deviation principle. *Suppose that the local map \mathcal{L} is recurrent, i.e. $\mathcal{R} = \mathfrak{h}$. Then $(\hat{\mathbb{P}}_{\rho,n})_{n \geq 1}$ satisfies a large deviation principle with good rate function*

$$\Lambda_\rho(x) = \min_{\alpha: a_\alpha(\rho) \neq 0} \Lambda_\alpha(x),$$

where Λ_α is the Fenchel–Legendre transform of the logarithm of the spectral radius $\lambda_{\alpha,u}$ of $\mathcal{L}|_{\chi_{\alpha,u}}$, i.e.

$$\lambda_{\alpha,u} = r(\mathcal{L}|_{\chi_{\alpha,u}}), \quad \Lambda_\alpha(x) = \sup_{u \in \mathbb{R}^d} \{\langle u, x \rangle - \log(\lambda_{\alpha,u})\} \quad x \in \mathbb{R}^d.$$

Theorem 5.2 Large deviations upper and lower bounds. *For any measurable $B \in \mathcal{B}(\mathbb{R}^d)$*

- $\limsup_{n \rightarrow +\infty} \frac{1}{n} \log(\hat{\mathbb{P}}_{\rho,n}(B)) \leq - \inf_{x \in \bar{B}} \min_{\alpha,\beta: a_{\alpha,\beta}(\rho) \neq 0} \Lambda_{\alpha,\beta}^\rho(x),$
- $\liminf_{n \rightarrow +\infty} \frac{1}{n} \log(\hat{\mathbb{P}}_{\rho,n}(B)) \geq - \min_{\alpha,\beta: a_{\alpha,\beta}(\rho) \neq 0} \inf_{x \in \hat{B} \cap \mathcal{S}_{\alpha,\beta}} \Lambda_{\alpha,\beta}^\rho(x)$

where

- $\lambda_{\alpha,\beta,u}^\rho = r(\mathcal{L}|_{\mathcal{Q}_{\alpha,\beta,u}^\rho})$ for $\mathcal{Q}_{\alpha,\beta}^\rho := \tilde{p}_{\mathcal{V}_{\alpha,\beta}} \mathcal{E}(\rho),$
- $\Lambda_{\alpha,\beta}^\rho(x) = \sup_{u \in \mathbb{R}^d} \{\langle u, x \rangle - \log(\lambda_{\alpha,\beta,u}^\rho)\}$ is the Fenchel–Legendre transform of $\log(\lambda_{\alpha,\beta,u}^\rho),$
- $\mathcal{S}_{\alpha,\beta} = \mathbb{R}^d$ if $\lambda_{\alpha,\beta,u}^\rho$ is smooth, otherwise $\mathcal{S}_{\alpha,\beta}$ is the set of exposed points of $\Lambda_{\alpha,\beta}^\rho$ (see [17, Definition 2.3.3]).

Remark 5.3 We remark that, whenever $a_{\alpha,\beta}(\rho) \neq 0$, $\mathcal{Q}_{\alpha,\beta}^\rho$ is non trivial and

$$\mathcal{Q}_{\alpha,\beta}^\rho = \mathcal{V}_{\alpha,\beta} \oplus (\mathcal{T} \cap \mathcal{Q}_{\alpha,\beta}^\rho) \subset \text{supp}(A(\mathcal{V}_{\alpha,\beta}))$$

(see the first step in the proof of Theorem 5.2).

We shall prove the two theorems in inverse order. The proof will request different steps and we shall proceed similarly as we did for the central limit theorem, first considering the measure \mathbb{P}'_ρ associated with the absorption in a single minimal enclosure (Lemma 3.1), and then generalizing using the expression of \mathbb{P}_ρ as a convex combination given in Lemma 4.3.

Proof Step 1. We fix the initial state ρ and a minimal enclosure \mathcal{V} , whose corresponding absorption operator is denoted as usual by $A(\mathcal{V})$. If $\mathbb{E}_\rho[\text{Tr}(A(\mathcal{V})\rho_0)] > 0$, we introduce the measure \mathbb{P}'_ρ as previously in Lemma 3.1. This first step consists in proving large deviations bounds for the position process under the measure \mathbb{P}'_ρ .

We need to consider a restriction of the channel \mathcal{L} which takes into account only the subspace of $\text{supp}A(\mathcal{V})$ which is somehow reachable by the local initial states $\rho(\underline{k})$. To this aim we use the enclosure $\mathcal{E}(\rho)$ (see equation (5.19)) and define the subspace

$$\Omega = \tilde{p}_\mathcal{V}\mathcal{E}(\rho).$$

1. $\Omega \oplus (\mathcal{E}(\rho)^\perp \cap \text{supp}(A(\mathcal{V}))) = \text{supp}(A(\mathcal{V}))$.

Indeed, $v \in \Omega^\perp \cap \text{supp}(A(\mathcal{V}))$ if and only if

$$v \in \text{supp}(A(\mathcal{V})) \text{ and, } \forall w \in \mathcal{E}(\rho), 0 = \langle v, \tilde{p}_\mathcal{V}(w) \rangle = \langle \tilde{p}_\mathcal{V}(v), w \rangle = \langle v, w \rangle,$$

i.e. $v \in \text{supp}(A(\mathcal{V})) \cap \mathcal{E}(\rho)^\perp$.

2. $\mathbb{E}_\rho[\text{Tr}(A(\mathcal{V})\rho_0)] = 0$ if and only if $\Omega = \{0\}$.

Since $\text{Tr}(A(\mathcal{V})\rho_0)$ is a non negative random variable, it has zero mean if and only if it is almost surely null, that is

$$\Leftrightarrow 0 = \text{Tr}(A(\mathcal{V})\rho(\underline{k})) = \text{Tr}(A(\mathcal{V})\mathcal{L}^n(\rho(\underline{k}))) \quad \forall \underline{k} \in V$$

$$(\text{since } \mathcal{L}^*(A(\mathcal{V})) = A(\mathcal{V})) \Leftrightarrow \text{Tr}(A(\mathcal{V})\mathcal{L}^n(\rho(\underline{k}))) = 0 \quad \forall \underline{k} \in V, n \geq 0$$

$$\Leftrightarrow \tilde{p}_\mathcal{V}(\text{supp}(\mathcal{L}^n(\rho(\underline{k})))) = \{0\} \quad \forall \underline{k}, n$$

3. Otherwise $\mathbb{E}_\rho[\text{Tr}(A(\mathcal{V})\rho_0)] > 0$ and $\mathcal{V} \subset \Omega$.

By using the same ideas as before, if $\mathbb{E}_\rho[\text{Tr}(A(\mathcal{V})\rho_0)] > 0$, Ω is non trivial and there exist some $\underline{k} \in V, n \geq 0$ such that $\text{Tr}(p_\mathcal{V}\mathcal{L}^n(\rho(\underline{k}))) \neq 0$ and this implies

$$\{0\} \neq p_\mathcal{V}(\mathcal{E}(\rho)) = p_\mathcal{V}(p_\mathcal{R}(\mathcal{E}(\rho))) = p_\mathcal{V}(\mathcal{R} \cap \mathcal{E}(\rho))$$

where the last equality follows from [12, Proposition 23]. So $(\mathcal{R} \cap \mathcal{E}(\rho))$ is a non null positive recurrent enclosure (as intersection of enclosures) and it is non orthogonal to \mathcal{V} , hence it contains a minimal enclosure \mathcal{W} which is in the same χ_α as \mathcal{V} and is not orthogonal to \mathcal{V} . Then, by using the representation of \mathcal{V} and \mathcal{W} given by the partial isometry U_α as in relation (4.15), we deduce that

$$\mathcal{V} = p_\mathcal{V}(\mathcal{W}) \subset \tilde{p}_\mathcal{V}(\mathcal{E}(\rho)) = \Omega.$$

We call Φ the restriction of \mathcal{L} to the subspace Ω , i.e. $\Phi(\sigma) := p_\Omega\mathcal{L}(p_\Omega\sigma p_\Omega)p_\Omega$, and Φ_u its deformation. Ω and consequently Φ obviously depend on the enclosure \mathcal{V} and on the initial state ρ , but we do not need to highlight this in the notations.

Lemma 5.4 *Suppose $\mathbb{E}_\rho[\text{Tr}(A(\mathcal{V})\rho_0)] > 0$. For any measurable $B \in \mathcal{B}(\mathbb{R}^d)$*

- $\limsup_{n \rightarrow +\infty} \frac{1}{n} \log(\hat{\mathbb{P}}'_{\rho,n}(B)) \leq -\inf_{x \in \bar{B}} \Lambda(x)$;
- $\liminf_{n \rightarrow +\infty} \frac{1}{n} \log(\hat{\mathbb{P}}'_{\rho,n}(B)) \geq -\inf_{x \in \mathring{B} \cap S} \Lambda(x)$

where

- Λ is the Fenchel–Legendre transform of $\log(\lambda_u^\rho)$,
- λ_u^ρ is the spectral radius of Φ_u ,
- $\mathcal{S} = \mathbb{R}^d$ if λ_u^ρ is smooth, otherwise it corresponds to the set of exposed points of Λ .

Proof In order to apply [17, Theorem 2.3.6], we need to prove that for every $u \in \mathbb{R}^d$ we have

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \log(\mathbb{E}'_\rho[e^{u \cdot (X_n - X_0)}]) = \log(\lambda_u^\rho).$$

Notice that we computed the same limit in the proof of Theorem 3.6, but for u in a complex neighborhood of the origin.

For any $n \in \mathbb{N}$, by construction $\Phi_u^n(\rho(\underline{k})) = \tilde{\mathcal{L}}_u^n(\rho(\underline{k}))$ for all \underline{k} and u , so we can write

$$\begin{aligned} \mathbb{E}_\rho[\text{Tr}(A(\mathcal{V})\rho_0)] \cdot \mathbb{E}'_\rho[e^{u \cdot (X_n - X_0)}] &= \sum_{\underline{k} \in V} \text{Tr}(A(\mathcal{V})\tilde{\mathcal{L}}_u^n(\rho(\underline{k}))) = \sum_{\underline{k} \in V} \text{Tr}(A(\mathcal{V})\Phi_u^n(\rho(\underline{k}))) \\ &= \sum_{\underline{k} \in V} \text{Tr}(\rho(\underline{k})\Phi_u^{*n}(A(\mathcal{V}))) \\ &\leq \left\| \sum_{\underline{k} \in V} \rho(\underline{k}) \right\|_{L^1} \|\Phi_u^{*n}(A(\mathcal{V}))\|_\infty \leq \|\Phi_u^{*n}\|_\infty. \end{aligned}$$

Because of Gelfand formula, we get

$$\limsup_{n \rightarrow +\infty} \frac{1}{n} \log(\mathbb{E}'_\rho[e^{u \cdot (X_n - X_0)}]) \leq \log \left(\lim_{n \rightarrow +\infty} \|\Phi_u^{*n}\|_\infty^{1/n} \right) = \log(\lambda_u^\rho).$$

Now consider $w_u \in B(\mathfrak{h})$ the Perron–Frobenius eigenvector for Φ_u^* , i.e. such that $\Phi_u^*(w_u) = \lambda_u^\rho w_u$. w_u is a non null positive operator supported in Ω , so there exist $N \in \mathbb{N}$ and $\hat{k} \in V$ such that $\text{Tr}(\tilde{\mathcal{L}}^N(\rho(\hat{k}))w_u) \neq 0$. Therefore $\text{Tr}(\Phi_u^N(\rho(\hat{k}))w_u) = \text{Tr}(\tilde{\mathcal{L}}^N(\rho(\hat{k}))w_u) \neq 0$.

Since Ω is finite dimensional, there exists a constant $M > 0$ such that $\rho_\Omega A(\mathcal{V})\rho_\Omega \geq Mw_u$, hence for every $n \geq N$ we have

$$\begin{aligned} \mathbb{E}_\rho[\text{Tr}(A(\mathcal{V})\rho_0)] \cdot \mathbb{E}'_\rho[e^{u \cdot (X_n - X_0)}] &= \sum_{\underline{k} \in V} \text{Tr} \left(A(\mathcal{V})\tilde{\mathcal{L}}_u^n(\rho(\underline{k})) \right) \\ &\geq \text{Tr} \left(A(\mathcal{V})\tilde{\mathcal{L}}_u^n(\rho(\hat{k})) \right) \\ &= \text{Tr} \left(A(\mathcal{V})\Phi_u^n(\rho(\hat{k})) \right) \\ &\geq M \text{Tr} \left(\Phi_u^N(\rho(\hat{k}))\Phi_u^{*(n-N)}(w_u) \right) \\ &= M \text{Tr} \left(\Phi_u^N(\rho(\hat{k}))w_u \right) (\lambda_u^\rho)^{n-N}. \end{aligned}$$

Therefore

$$\liminf_{n \rightarrow +\infty} \frac{1}{n} \log(\mathbb{E}'_\rho[e^{u \cdot (X_n - X_0)}]) \geq \log(\lambda_u^\rho).$$

This allows to compute the desired limit and the statement follows by direct application of the Gärtner–Ellis’ theorem. Notice that we do not have to worry about the domain of $\log(\lambda_u^\rho)$ since it is easy to see that λ_u^ρ is a strictly positive real number for every $u \in \mathbb{R}^d$. \square

Step 2. We complete the proof of the statement of the theorem by using the expression of \mathbb{P}_ρ as convex combinations of the $\mathbb{P}_\rho^{\alpha,\beta}$ deduced in relation (4.18). This implies that a similar decomposition holds for $\hat{\mathbb{P}}_{\rho,n}$ in terms of $(\hat{\mathbb{P}}_{\rho,n}^{\alpha,\beta})_{\alpha,\beta}$, i.e.

$$\hat{\mathbb{P}}_{\rho,n} = \sum_{\alpha \in A} \sum_{\beta \in I_\alpha} a_{\alpha,\beta}(\rho) \hat{\mathbb{P}}_{\rho,n}^{\alpha,\beta} = \sum_{j \in J_\rho} a_j(\rho) \hat{\mathbb{P}}_{\rho,n}^j,$$

where $J_\rho := \{(\alpha, \beta) : \alpha \in A, \beta \in I_\alpha : a_{\alpha,\beta}(\rho) > 0\}$.

Since, for any $j \in J_\rho$ and $B \in \mathcal{B}(\mathbb{R}^d)$, $\hat{\mathbb{P}}_{\rho,n}(B) \geq a_\alpha \hat{\mathbb{P}}_{\rho,n}^j(B)$, we trivially have

$$\liminf_{n \rightarrow +\infty} \frac{1}{n} \log(\hat{\mathbb{P}}_{\rho,n}(B)) \geq \max_{j \in J_\rho} \liminf_{n \rightarrow +\infty} \frac{1}{n} \log(\hat{\mathbb{P}}_{\rho,n}^j(B)),$$

$$\limsup_{n \rightarrow +\infty} \frac{1}{n} \log(\hat{\mathbb{P}}_{\rho,n}(B)) \geq \max_{j \in J_\rho} \limsup_{n \rightarrow +\infty} \frac{1}{n} \log(\hat{\mathbb{P}}_{\rho,n}^j(B)).$$

Then we have

$$\begin{aligned} \limsup_{n \rightarrow +\infty} \frac{1}{n} \log(\hat{\mathbb{P}}_{\rho,n}(B)) &\leq \underbrace{\limsup_{n \rightarrow +\infty} \frac{1}{n} \log(|J_\rho|)}_{=0} + \limsup_{n \rightarrow +\infty} \frac{1}{n} \log \left(\max_{j \in J_\rho} \hat{\mathbb{P}}_{\rho,n}^j(B) \right) \\ &= \max_{j \in J_\rho} \limsup_{n \rightarrow +\infty} \frac{1}{n} \log(\hat{\mathbb{P}}_{\rho,n}^j(B)) \end{aligned}$$

and we are done. □

Proof of Theorem 5.1 Under the hypothesis $\mathfrak{h} = \mathcal{R}$, we have that $A(\mathcal{V}_{\alpha,\beta}) = p_{\mathcal{V}_{\alpha,\beta}}$, which implies $\mathcal{Q}_{\alpha,\beta} = \mathcal{V}_{\alpha,\beta}$ and $\tilde{\mathcal{L}}_{\alpha,\beta,u} = \mathcal{L}_{|\mathcal{V}_{\alpha,\beta},u}$.

Since $\mathcal{L}_{|\mathcal{V}_{\alpha,\beta}}$ is irreducible, $\lambda_{\alpha,\beta,u}$ is an analytic function of $u \in \mathbb{R}^d$ [15, Lemma 5.3] and consequently $\mathcal{S}_{\alpha,\beta} = \mathbb{R}^d$.

Moreover recall (Eq. 4.14) that $\mathcal{L}_{|\chi_\alpha}^*$ is unitarily equivalent to $\text{Id}_{B(\mathbb{C}^{I_\alpha})} \otimes \psi$ where ψ is equal to $\mathcal{L}_{|\mathcal{V}_{\alpha,\beta}}^*$, hence $\mathcal{L}_{|\chi_\alpha}$ and $\mathcal{L}_{|\mathcal{V}_{\alpha,\beta}}$ have the same spectral radius.

Therefore the following equality holds:

$$\min_{(\alpha,\beta) \in J_\rho} \Lambda_{\alpha,\beta} = \min_{\alpha: a_\alpha(\rho) \neq 0} \Lambda_\alpha.$$

Theorem 5.2 ensures that for any measurable $B \in \mathcal{B}(\mathbb{R}^d)$

- $\limsup_{n \rightarrow +\infty} \frac{1}{n} \log(\hat{\mathbb{P}}_{\rho,n}(B)) \leq - \inf_{x \in \bar{B}} \min_{\alpha: a_\alpha(\rho) \neq 0} \Lambda_\alpha(x),$
- $\liminf_{n \rightarrow +\infty} \frac{1}{n} \log(\hat{\mathbb{P}}_{\rho,n}(B)) \geq - \inf_{x \in \mathring{B}} \min_{\alpha: a_\alpha(\rho) \neq 0} \Lambda_\alpha(x),$

which is exactly the definition of large deviation principle with rate function $\Lambda_\rho(x) := \min_{\alpha: a_\alpha(\rho) \neq 0} \Lambda_\alpha(x), x \in \mathbb{R}^d$. Note that Λ_ρ has compact level sets because every Λ_α does (it is a consequence of Gärtner–Ellis’ theorem). □

Consider a minimal enclosure \mathcal{V} such that $\mathbb{E}_\rho[\text{Tr}(A(\mathcal{V})\rho_0)] \neq 0$; taking the notations of the first step in the proof of Theorem 5.2, the following proposition states that λ_u^ρ can be seen as the result of two contributions: one depending on the recurrent dynamic on \mathcal{V} and the other one on the transient dynamic on its orthogonal complement in \mathcal{Q} , which we denote by $\mathcal{W} := \mathcal{Q} \cap \mathcal{J}$.

Proposition 5.5 *Let $\lambda_u^\mathcal{V}$ and $\lambda_u^\mathcal{W}$ be the spectral radii of $\Phi_{|\mathcal{V},u}$ and $\Phi_{|\mathcal{W},u}$ respectively. Then $\lambda_u^\rho = r(\tilde{\mathcal{L}}_u) = \max\{\lambda_u^\mathcal{V}, \lambda_u^\mathcal{W}\}$.*

Proof We only need to prove that if $\lambda_u^\rho > \lambda_u^\mathcal{V}$, then $\lambda_u^\rho = \lambda_u^{\mathcal{W}}$. Theorem 2.6 tells us that there exists a positive $\omega_u \in L^1(\Omega)$ such that $\Phi_u(\omega_u) = \lambda_u^\rho \omega_u$; since $\lambda_u^\rho > \lambda_u^\mathcal{V}$, it must be true that $p_{\mathcal{V}}\omega_u p_{\mathcal{V}} \neq 0$ and we have the following:

$$p_{\mathcal{W}}\Phi_u(p_{\mathcal{W}}\omega_u p_{\mathcal{W}})p_{\mathcal{W}} = p_{\mathcal{W}}\Phi_u(\omega_u)p_{\mathcal{W}} = \lambda_u p_{\mathcal{W}}\omega_u p_{\mathcal{W}}.$$

The first equality follows from the fact that for any $\rho \in L^1(\mathfrak{h})$

$$\Phi_u(p_{\mathcal{V}}\rho) = p_{\Omega}\mathfrak{L}_u(p_{\mathcal{V}}\rho p_{\Omega})p_{\Omega} \stackrel{(\mathcal{V} \text{ is an enclosure})}{=} p_{\mathcal{V}}\mathfrak{L}_u(p_{\mathcal{V}}\rho p_{\Omega})p_{\Omega} = p_{\mathcal{V}}\Phi_u(p_{\mathcal{V}}\rho)$$

and analogously $\Phi_u(\rho p_{\mathcal{V}}) = \Phi(\rho p_{\mathcal{V}})p_{\mathcal{V}}$. □

6 Examples and Numerical Simulations

6.1 Commuting Normal Local Kraus Operators

As a first family of examples, we consider some HOQWs studied in [39]: take $V = \mathbb{Z}^d$ and a local channel with normal commuting Kraus operators $\{L_j\}_{j=1}^{2d}$. In this case, there exists an orthonormal basis $\{\varphi_i\}_{i=1}^h$ that simultaneously diagonalizes the Kraus operators and we can write $L_j = \sum_{i=1}^h \zeta_{i,j}|\varphi_i\rangle\langle\varphi_i|$. The normalization condition for the operators L_j given by Eq. (1.1) implies that $\sum_{j=1}^{2d} |\zeta_{i,j}|^2 = 1$ for any $i = 1, \dots, h$.

It is easy to verify by direct computation that, for every $i = 1, \dots, h$, $\omega_i = |\varphi_i\rangle\langle\varphi_i|$ is a minimal invariant state for \mathfrak{L} , and consequently $\mathcal{V}_i := \text{span}\{\varphi_i\}$ is a minimal recurrent enclosure. Hence \mathfrak{L} is positive recurrent and $\mathfrak{h} = \bigoplus_i \mathcal{V}_i$ is a decomposition of the local space \mathfrak{h} in minimal orthogonal enclosures.

However, for our study, we are interested in a decomposition of the form described in (4.12) and in particular we should identify the enclosures χ_α , which will be given by the direct sum of some of the \mathcal{V}_i 's; indeed, we can see that \mathcal{V}_i and \mathcal{V}_l are in the same χ_α if and only if for every $j = 1, \dots, 2d$, $\zeta_{i,j} = \zeta_{l,j} =: \zeta_{\alpha,j}$. This reflects on the structure of the Kraus operators, that will also be written as $L_j = \sum_{\alpha \in A} \zeta_{\alpha,j} p_{\chi_\alpha}$, $j = 1, \dots, 2d$.

In this simple example, the probability law of the shift $X_n - X_0$ is a convex combination of $|A|$ multinomial distributions with parameters $(|\zeta_{\alpha,1}|^2, \dots, |\zeta_{\alpha,2d}|^2)$: for every $n \geq 1$

$$\mathbb{P}_\rho(X_1 - X_0 = e_{j_1}, \dots, X_n - X_{n-1} = e_{j_n}) = \sum_{\alpha=1}^{|A|} \underbrace{\sum_{k \in \mathbb{Z}^d} \text{Tr}(p_{\chi_\alpha} \rho(k))}_{=: a_\alpha(\rho)} \prod_{k=1}^n |\zeta_{\alpha, j_k}|^2,$$

where e_1, \dots, e_d is the canonical basis of \mathbb{R}^d and $e_{2j} = -e_j$ for $j = 1, \dots, d$. Applying the central limit theorem for the mean of i.i.d. random variables, we see that

$$\lim_{n \rightarrow +\infty} \text{dist} \left(\mathbb{P}_{\rho, n}, \sum_{\alpha=1}^{|A|} a_\alpha(\rho) \mathcal{N}(\sqrt{nm_\alpha}, D_\alpha) \right) = 0 \tag{6.20}$$

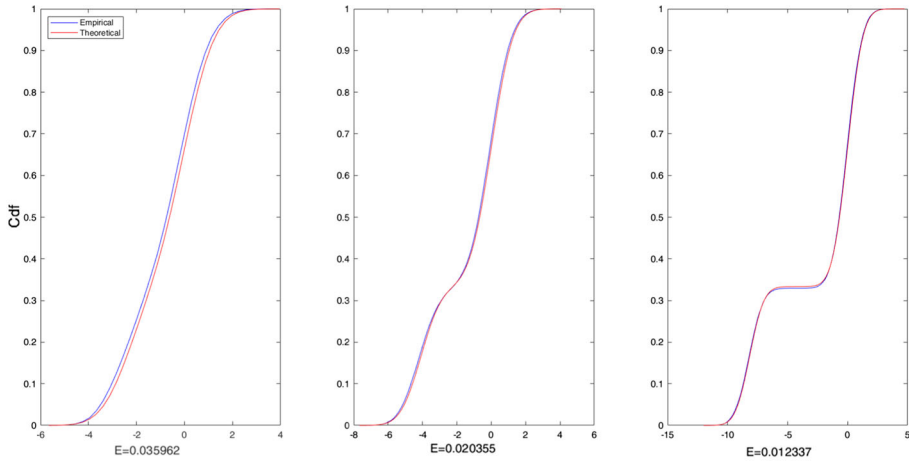
where $m_\alpha = \sum_{j=1}^{2d} |\zeta_{\alpha,j}|^2 e_j$ and $D_\alpha = \sum_{j=1}^d (|\zeta_{\alpha,j}|^2 + |\zeta_{\alpha,2j}|^2) |e_j\rangle\langle e_j| - |m_\alpha\rangle\langle m_\alpha|$.

Similarly, if we apply Theorem 4.4, we find again relation (6.20) (in this case computations for the asymptotic means and covariance matrices are very easy).

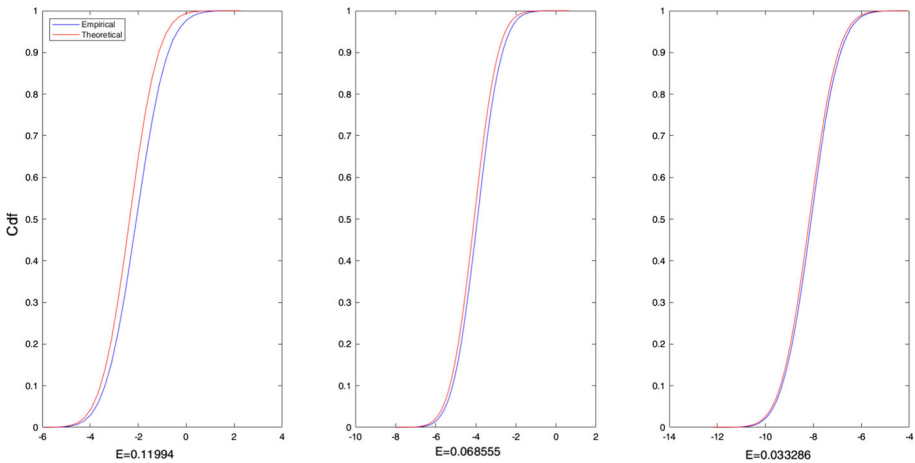
Also, by applying Theorem 5.1, we can state that a large deviations' principle holds for the process $\frac{X_n - X_0}{n}$ and the rate function is given by

$$\Lambda_\rho(x) := \min_{\alpha: a_\alpha(\rho) \neq 0} \Lambda_\alpha(x), \quad x \in \mathbb{R}^d$$

where $\Lambda_\alpha(x) = \sup_{u \in \mathbb{R}^d} \{ \langle u, x \rangle - \log(\lambda_{\alpha, u}) \}$ and $\lambda_{\alpha, u} = \sum_{j=1}^{2d} |\zeta_{\alpha, j}|^2 e^{u \cdot e_j}$.



(a) $\rho = \frac{1}{3}(|e_1\rangle\langle e_1| + |e_2\rangle\langle e_2| + |e_3\rangle\langle e_3|)$ and $p_3 = \frac{1}{2}$.



(b) $\rho = |e_0\rangle\langle e_0|$ and $p_3 = \frac{1}{2}$.

Fig. 3 Example 6.2. Empirical and expected cumulative functions of $\frac{X_n}{\sqrt{n}}$ for $n = 50, 150, 600$. E is the maximum difference in absolute value between the functions

6.2 An Example with Non Trivial Transient Space

We consider a family of HOQWs with local Hilbert space $\mathfrak{h} = \mathbb{C}^4$, including the walk defined in Example 2.2. We introduce the parameters $p_1, p_2, p_3 \geq 0$ such that $\sum_{i=1}^3 p_i = \frac{1}{2}$ and define left and right Kraus operators

$$L = \begin{pmatrix} \frac{1}{2\sqrt{2}} & 0 & 0 & 0 \\ \sqrt{\frac{p_1}{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \sqrt{\frac{p_2}{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ -\sqrt{\frac{p_3}{3}} & 0 & 0 & \frac{2}{\sqrt{3}} \end{pmatrix}, \quad R = \begin{pmatrix} \sqrt{\frac{3}{8}} & 0 & 0 & 0 \\ -\sqrt{\frac{p_1}{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ -\sqrt{\frac{p_2}{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ \sqrt{\frac{2p_3}{3}} & 0 & 0 & \frac{1}{\sqrt{3}} \end{pmatrix}.$$

Notice that Example 2.2 corresponds to the case $p_1 = p_2 = 0, p_3 = 1/2$.

This family of local channels revealed to be very useful since, though with a low dimensional local Hilbert space, it can display already a more sophisticated structure of the decomposition of the local space. Indeed, the transient subspace is non trivial and the recurrent subspace is reducible as a sum of two χ_α , one which is a minimal enclosure and one which is not.

Let $\{e_i\}_{i=0}^3$ be the canonical basis of \mathfrak{h} . It is immediate to see, for instance by computing explicitly the invariant states of the corresponding local channel \mathfrak{L} , that $\mathcal{T} = \text{span}\{e_0\}$, $\mathcal{R} = \text{span}\{e_1, e_2, e_3\}$ and the decomposition of the recurrent space is the following:

$$\mathcal{R} = \underbrace{\text{span}\{e_1, e_2\}}_{\chi_1} \oplus \underbrace{\text{span}\{e_3\}}_{\chi_2}.$$

With simple direct computations one can find the parameters of the limit Gaussians: for the enclosure χ_1 one has mean $m_1 = 0$ and variance $D_1 = 1$, while for χ_2 one has parameters $m_2 = -\frac{1}{3}$ and $D_2 = \frac{8}{9}$.

For this walk, depending on the different choice of the initial state ρ , we can observe either only one of the two Gaussians or various mixtures of the two Gaussians. When the $\rho(k)$'s

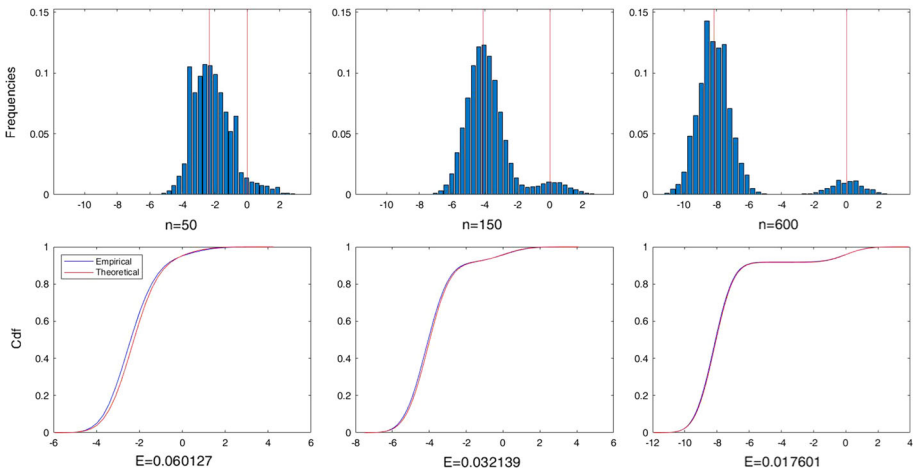


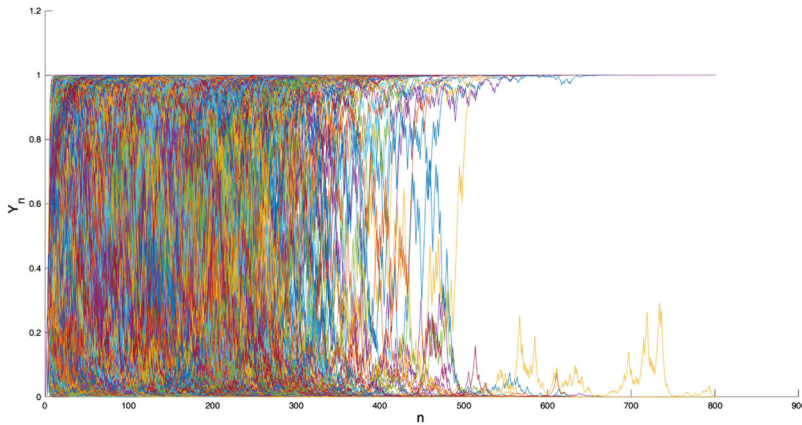
Fig. 4 $\rho_0 = \frac{1}{8}|e_0\rangle\langle e_0| + \frac{7}{8}|e_3\rangle\langle e_3|$ and $p_3 = \frac{1}{6}$

are all contained in a same χ_α , then we shall see only the Gaussian associated with the same $\chi_\alpha, \alpha = 1, 2$.

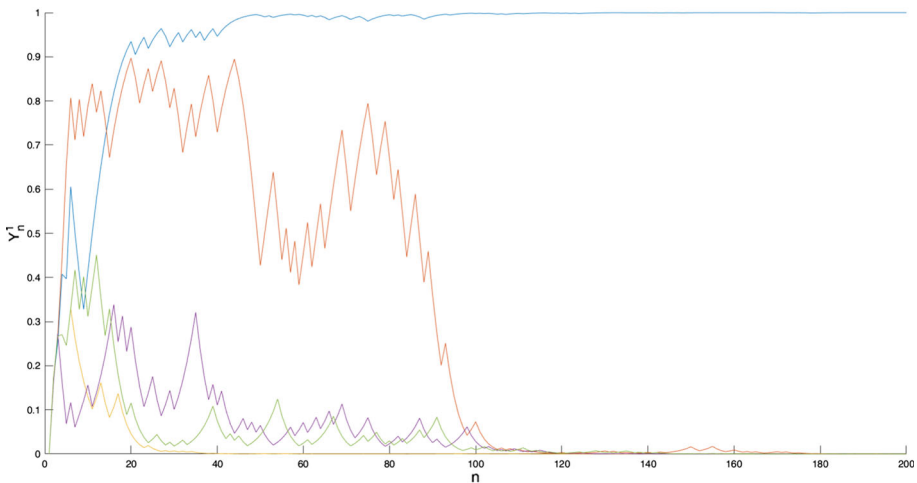
In order to consider the asymptotic behavior, we need the following absorption operators:

$$A(\chi_2) = 2p_3|e_0\rangle\langle e_0| + |e_3\rangle\langle e_3|, \quad A(\chi_1) = I_7 - A(\chi_2).$$

We can take for simplicity $X_0 = 0$ and it will be particularly interesting to consider an initial state ρ supported in the transient subspace, and so of the form $\rho = \rho_0 \otimes |0\rangle\langle 0|$, with $\rho_0 = (\rho_0(i, j))_{i, j=0, \dots, 3}$ a non negative unit-trace matrix in $M_4(\mathbb{C})$. Then we can explicitly compute the weights of the Gaussian mixture appearing in the generalized CLT, which will be given by the quantum absorption probabilities



(a) The graph represents $N = 10^4$ trajectories of Y_n along 800 steps ($\rho_0 = |e_0\rangle\langle e_0|$ and $p_3 = \frac{1}{6}$).



(b) The graph represents $N = 5$ among the previous trajectories.

Fig. 5 The behavior of Y_n

$$a_1(\rho) = 2p_3\rho_0(0, 0) + \rho_0(3, 3),$$

$$a_2(\rho) = 1 - a_1(\rho) = 2(p_1 + p_2)\rho_0(0, 0) + \rho_0(1, 1) + \rho_0(2, 2).$$

We illustrate our result also by numerical simulations. We used $N = 5 \times 10^4$ samples of $\frac{X_n}{\sqrt{n}}$ for $n = 50, 150, 600$ in order to estimate their probability distribution and we compared it with the expected convex combination of Gaussian measures. Figures 1 and 2 show the histograms of $\frac{X_n}{\sqrt{n}}$ at the three different times ($n = 50, 150, 600$) for the choice $p_3 = \frac{1}{2}$ and for two different choices of the local initial state ρ_0 . In Fig. 3 we reported the empirical and the expected cumulative function. The same plots for the choice $p_3 = \frac{1}{6}$ are reported in Fig. 4. Once again we remark that, tuning initial state and absorption rates the Gaussian laws in the mixture do not change, but only their weights.

Finally, numerical simulations can also help us to have a better intuition of the behavior of the processes $(Y_n)_n$ used to introduce the laws of the family \mathbb{P}'_ρ (recall Lemma 3.1). For the enclosure χ_1 , for instance, the corresponding process $Y_n = \text{Tr}(\chi_1 \rho_n)$ should help us to select the trajectories absorbed in some sense in χ_1 . In Fig. 5 we trace the trajectories of $(Y_n)_n$ along 800 steps, which show how Y_∞ is a Bernoulli random variable with parameter $\mathbb{E}_\rho[\text{Tr}(A(\chi_1)\rho_0)]$; hence in this case $\mathbb{P}'_\rho(\cdot)$ (defined as in relation (4.17)) is equal to $\mathbb{P}_\rho(\cdot|B)$ where $B = \{Y_\infty = 1\} = \{\lim_{n \rightarrow +\infty} \|p_{\chi_1} \rho_n p_{\chi_1} - \rho_n\| = 0\}$ and it represents the probability obtained conditioning \mathbb{P}_ρ to the event of “being absorbed in χ_1 ”.

Note that the frequency of trajectories such that $Y_{800} > 0.99$ is equal to 0.3388, and the frequency of trajectories for which $Y_{800} < 0.01$ is 0.6612. This is in agreement with $a_1(\rho) = \mathbb{E}_\rho[\text{Tr}(A(\chi_1)\rho_0)] = \frac{1}{3}$.

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Data Availability Authors can confirm that all relevant data are included in the article.

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