



Vanishing of Drude Weight in Interacting Fermions on \mathbb{Z}^d with Quasi-Periodic Disorder

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Abstract

We consider a fermionic many body system in \mathbb{Z}^d with a short range interaction and quasi-periodic disorder. In the strong disorder regime and assuming a Diophantine condition on the frequencies and on the chemical potential, we prove at $T = 0$ the exponential decay of the correlations and the vanishing of the Drude weight, signaling non-metallic behavior in the ground state. The proof combines Ward Identities, Renormalization Group and KAM Lindstedt series methods.

Keywords Quasi-periodic disorder · Interacting fermions · Localization · Drude weight · Small divisors

1 Introduction

The conductivity properties in fermionic systems, describing electrons in metals, are strongly affected by the presence of disorder, which breaks the perfect periodicity of an ideal lattice and is unavoidable in real systems. Disorder can be represented either by a random variable or by a quasi-periodic potential; the first description is more suitable for impurities in solids while the second appears naturally in quasi-crystals or cold atoms experiments. In absence of many body interaction disorder produces the phenomenon of Anderson localization [1], consisting in an exponential decay of all eigenstates and in an insulating behavior with vanishing conductivity. Such a phenomenon relies on the properties of the single particle Schroedinger equation and it has been the subject of a deep mathematical investigation. With random disorder Anderson localization was established for strong disorder in any dimension [2,3] and in one dimension with any disorder. In the case of quasi-periodic disorder localization in one dimension is present only for large disorder [4,5], while for weak disorder is absent; in higher dimensions localization was proved for strong disorder in $d = 2$ [6,7] and for any d in [8].

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The interplay between disorder and interaction has been deeply analyzed in the physical literature soon after [1]. The presence of many body interaction induces new processes which can indeed destroy localization. At zero temperature $T = 0$ with random disorder qualitative scaling arguments gave evidence of persistence of localization in $d = 3$ [9,10] for short range weak interaction; in $d = 1$ a second order Renormalization Group analysis was shown to produce a complex phase diagram [11]. The case of quasi-random disorder has been less studied, with the exception of [12,13] focusing on the extended weak disorder regime at $T = 0$. In more recent times the properties at $T > 0$ were analyzed in [14], where perturbative arguments for the vanishing of conductivity up to a certain critical T in any dimension were given (many body localized phase). Subsequently numerical simulations found localization in certain systems in all the spectrum and vanishing of conductivity for any T , a phenomenon called *many body localization*, see [15] for random and [16] for quasi-periodic disorder. If all states are localized one expects, in a non-equilibrium setting, that interaction is unable to produce thermalization in an isolated quantum system, a phenomenon that in classical mechanics is due to closeness to an integrable system. Interacting quantum systems with quasi-periodic disorder have been realized in cold atoms experiments [17–19]; quasi-periodic disorder with many body interaction has been extensively numerically analyzed [20–28].

While the above works suggest that localization persists in presence of interaction, results based on numerical or perturbative analysis cannot be conclusive. In particular the presence of *small divisors* has the effect that physical informations are difficult to be extracted by lower order analysis but are typically encoded in convergence or divergence of the whole series. This is a well known phenomenon in classical mechanics; the Birkoff series for prime integrals in Hamiltonian systems are generically diverging while Lindstedt series for Kolomogorov–Arnold–Moser (KAM) tori converge, even if both series are order by order finite and present similar small divisors. Therefore, even if perturbative analysis in [14] or [29] get localization at finite temperature and in any dimension, one cannot exclude that the series are divergent and localization eventually disappear (this would say that thermalization in experiments is eventually reached, even if at long times). A non-perturbative proof of many body localization for all eigenstates has been indeed finally obtained in $d = 1$ with random disorder in [30] but the result is based on a certain unproven assumption. A complete proof have been obtained only with vanishing densities [31,32]. Arguments for breaking of many body localization in $d > 1$ have been indeed presented in [33].

In order to get rigorous results as benchmark for conjectures and approximations, a natural starting point is the zero temperature case in the thermodynamic limit. Our approach is to compute thermodynamical correlations; they not only provide physical observables at equilibrium but give also information on the spectrum (so their computation is of interest even for situation where equilibrium is not reached). In particular at zero temperature they provide information of correlations over the ground state, while the vanishing of conductivity at any temperature is a signal of many body localization in all the spectrum. It has been proven in [34–36] for one dimensional interacting fermions with strong quasi-periodic disorder the $T = 0$ exponential decay of 2-point correlations, indicating persistence of localization in the ground state. Aim of this paper is twofold. The first is to investigate the $d > 1$ case. We consider a disorder of the form $f(\vec{\omega}\vec{x})$ with f periodic, as the one considered in [6] for the single particle Schrodinger equation; more general forms of disorder are however possible, as $f(\vec{\omega}_1\vec{x}, \vec{\omega}_2\vec{x})$ considered in [6]. The second aim is to compute the $T = 0$ conductivity expressed by Kubo formula, whose properties can be analyzed via a combination of information provided by Ward Identities with regularity properties of the current correlations. The thermodynamical quantities are expressed by a series expansion showing a peculiar combinations of properties

appearing in classical and quantum physics; they show a small divisor problem, as in the Lindstedt series for KAM [37], but loop graphs appear in the expansion, a signature of quantum physics totally absent in classical mechanics. In order to achieve convergence and exclude non perturbative effects one has from one side to show that divisors can be controlled by number theoretical conditions on frequencies, and from the other that the huge number of loop graphs is compensated by cancellations from the fermionic anticommutative nature of the problem.

The paper is organized in the following way. In §2 the model is presented and in §3 the main results, together with open problems, are presented. In §4 we discuss the implications of Ward Identities and regularity bounds. In §5 we introduce the Grassmann representation and in §6 we introduce the multiscale analysis. In §7 we prove the convergence of series expansion and in §8 we get the asymptotic decay of correlations.

2 Interacting Fermions with Quasi-Periodic Disorder

We introduce the Fock space $\mathcal{F}_L = \bigoplus_{N \geq 0} \mathfrak{h}_L^{\wedge N}$ where the N particle Hilbert space $\mathfrak{h}_L^{\wedge N}$ is the set of the totally antisymmetric square integrable functions in $\Lambda_L := \{\vec{x} \in \mathbb{Z}^d \mid \vec{x} = n_1 \vec{e}_1 + n_2 \vec{e}_2 + \dots, -L/2 \leq n_i \leq L/2, i = 1, 2, \dots, d\}$ where \vec{e}_i are unit vectors. The $a_{\vec{x}}^{\pm}$ are fermionic creation or annihilation operators sending an element of $\mathfrak{h}_L^{\wedge N}$ in $\mathfrak{h}_L^{\wedge N+1}$ (creation) or $\mathfrak{h}_L^{\wedge N-1}$ (annihilation) and $\{a_{\vec{x}}^+, a_{\vec{y}}^-\} = \delta_{\vec{x}, \vec{y}}, \{a_{\vec{x}}^+, a_{\vec{y}}^+\} = \{a_{\vec{x}}^-, a_{\vec{y}}^-\} = 0$. The Hamiltonian is

$$H = -\frac{\varepsilon}{2} \sum_{\vec{x}} \sum_{i=1}^d (a_{\vec{x}+\vec{e}_i}^+ a_{\vec{x}}^- + a_{\vec{x}}^+ a_{\vec{x}+\vec{e}_i}^-) + u \sum_{\vec{x}} \phi_{\vec{x}} a_{\vec{x}}^+ a_{\vec{x}}^- + \lambda \sum_{\vec{x}} \sum_{i=1}^d a_{\vec{x}}^+ a_{\vec{x}}^- a_{\vec{x}+\vec{e}_i}^+ a_{\vec{x}+\vec{e}_i}^- \tag{1}$$

where $a_{\vec{x}}^{\pm}$ must be interpreted as zero for $\vec{x} \notin \Lambda_L$ and $\phi_{\vec{x}} = \bar{\phi}(\vec{\omega}\vec{x})$ with $\bar{\phi}(t) : \mathbb{T} \rightarrow \mathbb{R}$ periodic of period 1. In order to describe a quasi-periodic disorder we impose that $\vec{\omega}$ is rationally independent and “badly” approximated by rationals (Diophantine condition). The first term in (1) represents the kinetic energy of the fermions hopping on a lattice, the second represents the interaction with a quasi-periodic potential and the last term represents a 2 body interaction.

There are several interesting limits; $\lambda = 0$ is the non interacting limit; $\lambda = u = 0$ is the integrable limit; $\lambda = \varepsilon = 0$ is the anti-integrable limit (the terminology was introduced in [38]). We consider the case in which λ, ε are small with respect to u , and we set $u = 1$ for definiteness; that is we consider a perturbation of the anti-integrable limit.

If $N = \sum_{\vec{x}} a_{\vec{x}}^+ a_{\vec{x}}^-$ we define

$$\langle \cdot \rangle_{\beta, L} = \frac{\text{Tr}_{\mathcal{F}_L} \cdot e^{-\beta(H-\mu N)}}{\mathcal{Z}_{\beta, L}}, \quad \mathcal{Z}_{\beta, L} = \text{Tr}_{\mathcal{F}_L} e^{-\beta(H-\mu N)} \tag{2}$$

where μ is the chemical potential, which is fixed by the density in the Grand-Canonical ensemble, and $\mathcal{Z}_{\beta, L}$ is the partition function. In the limit $\beta \rightarrow \infty$ they provide information on the ground states. We define

$$\langle \cdot \rangle = \lim_{\beta \rightarrow \infty} \lim_{L \rightarrow \infty} \langle \cdot \rangle_{\beta, L} \tag{3}$$

The imaginary-time (or Euclidean) evolution of the fermionic operators is

$$a_{\vec{x}}^{\pm} = e^{x_0(H-\mu N)} a_{\vec{x}}^{\pm} e^{-x_0(H-\mu N)} \tag{4}$$

with $\mathbf{x} = (x_0, \vec{x})$ with $x_0 \in [0, \beta)$, The 2-point function is given by

$$S_{\beta,L}(\mathbf{x}, \mathbf{y}) = \langle T a_{\mathbf{x}}^- a_{\mathbf{y}}^+ \rangle_{\beta,L} \tag{5}$$

and T is the time order product. We also consider the truncated expectations $\langle TA; B \rangle_{\beta,L} = \langle TAB \rangle_{\beta,L} - \langle TA \rangle_{\beta,L} \langle TB \rangle_{\beta,L}$. The density and the current are given by

$$\rho_{\vec{x}} = a_{\vec{x}}^+ a_{\vec{x}}^- \quad j_{\vec{x}}^i = \frac{\varepsilon}{2i} (a_{\vec{x}+\vec{e}_i}^+ a_{\vec{x}}^- - a_{\vec{x}}^+ a_{\vec{x}+\vec{e}_i}^-) \tag{6}$$

The (Euclidean) conductivity density in the zero temperature limit is defined by Kubo formula

$$\sigma_{\vec{y}}^i = \lim_{p_0 \rightarrow 0} \frac{1}{p_0} \lim_{\beta \rightarrow \infty} \lim_{L \rightarrow \infty} [\sum_{\vec{x} \in \Lambda_L} \int_0^\beta dx_0 e^{ip_0 x_0} \langle T j_{\vec{x},x_0}^i; j_{\vec{y},0}^i \rangle_{\beta,L} + \langle \tau_{\vec{y}}^i \rangle_{\beta,L}] \tag{7}$$

where

$$\tau_{\vec{y}}^i = -\frac{\varepsilon}{2} (a_{\vec{y}+\vec{e}_i}^+ a_{\vec{y}}^- + a_{\vec{y}}^+ a_{\vec{y}+\vec{e}_i}^-) \tag{8}$$

The conductivity can be equivalently expressed in terms of the Fourier transform which is, in the $\beta \rightarrow \infty, L \rightarrow \infty$ limit, $i = 1, \dots, d$

$$\widehat{H}_{ii}(\mathbf{p}, \vec{y}) = \sum_{\vec{x} \in \Lambda} \int_{\mathbb{R}} dx_0 e^{i\mathbf{p}\mathbf{x}} \langle T j_{\vec{x},x_0}^i; j_{\vec{y},0}^i \rangle \tag{9}$$

and similarly we define $\widehat{H}_{\mu\nu}(\mathbf{p}, \vec{y})$, with $\mu = 0, 1, \dots, d$ ($\mu = 0$ is the density and $\mu = 1, \dots, d$ the current component). We can rewrite (7) as

$$\sigma_{\vec{y}}^i = \lim_{p_0 \rightarrow 0} \lim_{\vec{p} \rightarrow 0} \frac{1}{p_0} [\widehat{H}_{ii}(\mathbf{p}, \vec{y}) + \langle \tau_{\vec{y}}^i \rangle] \tag{10}$$

Finally the (zero temperature) Drude weight, see e.g., [39,40], is defined as

$$D_{\vec{y}}^i = \lim_{p_0 \rightarrow 0} \lim_{\vec{p} \rightarrow 0} [\widehat{H}_{ii}(\mathbf{p}, \vec{y}) + \langle \tau_{\vec{y}}^i \rangle] \tag{11}$$

In a perfect metal at equilibrium the Drude weight is non-vanishing implying that the conductivity is infinite; a vanishing Drude weight signals a non-metallic behavior.

In the above definitions of conductivity the order in which the limits are taken is essential; already in the integrable limit $u = \lambda = 0$ reversing the order of the limits one obtains a zero result, while the Drude weight is indeed non vanishing as a consequence of the non-continuity of the Fourier transform of the current correlation.

3 Main Result

In the anti-integrable limit $\lambda = \varepsilon = 0$ the eigenvalues of the Hamiltonian are, $\vec{x} \in \Lambda_L$

$$H_0 = \sum_{\vec{x} \in \Lambda_L} \bar{\phi}(\vec{\omega}\vec{x}) n_{\vec{x}} \quad n_{\vec{x}} = 0, 1 \tag{12}$$

and the single particle eigenfunctions have the form of $\delta_{\vec{x},\vec{y}}$. The 2-point function is given by

$$g(\mathbf{x}, \mathbf{y}) = \delta_{\vec{x},\vec{y}} e^{(\phi_{\vec{x}} - \mu)(x_0 - y_0)} \left[\theta(x_0 - y_0) \frac{1}{1 + e^{\beta(\phi_{\vec{x}} - \mu)}} - \theta(y_0 - x_0) \frac{e^{\beta(\phi_{\vec{x}} - \mu)}}{1 + e^{\beta(\phi_{\vec{x}} - \mu)}} \right] \tag{13}$$

which can be equivalently written as

$$g(\mathbf{x}, \mathbf{y}) = \delta_{\vec{x}, \vec{y}} \frac{1}{\beta} \sum_{k_0 = \frac{2\pi}{\beta}(n_0 + \frac{1}{2})} e^{-ik_0(x_0 - y_0)} \widehat{g}(\vec{x}, k_0) = \delta_{\vec{x}, \vec{y}} \bar{g}(\vec{x}; x_0 - y_0) \tag{14}$$

with

$$\widehat{g}(\vec{x}, k_0) = \frac{1}{-ik_0 + \phi_{\vec{x}} - \mu} \tag{15}$$

We define

$$\mu = \bar{\phi}(\alpha) \tag{16}$$

and the occupation number on the ground state is $\theta(\bar{\phi}(\vec{\omega}\vec{x}) - \bar{\phi}(\alpha))$; the choice of μ fixes the averaged density. The conductivity is exactly vanishing as the is proportional to ε . The density correlation is

$$\langle \rho_{\mathbf{x}}; \rho_{\mathbf{y}} \rangle = \delta_{\vec{x}, \vec{y}} \bar{g}(\vec{x}; x_0 - y_0) \bar{g}(\vec{x}; y_0 - x_0) \tag{17}$$

We want to investigate what happens when we consider a non-vanishing hopping $\varepsilon \neq 0$ and interaction $\lambda \neq 0$. As usual in small divisor problems, we need to impose a Diophantine condition on the frequencies $\vec{\omega}$ of the quasi-periodic disorder that is

$$\|(\vec{\omega}\vec{x})\|_{\mathbb{T}} \geq C_0 |\vec{x}|^{-\tau} \quad \vec{x} \in \mathbb{Z}^d / \vec{0} \tag{18}$$

$\|\cdot\|$ being the norm on the one dimensional torus with period 1; we require also a Diophantine condition on the chemical potential, that is

$$\|(\vec{\omega}\vec{x}) \pm 2\alpha\|_{\mathbb{T}} \geq C_0 |\vec{x}|^{-\tau} \quad \vec{x} \in \mathbb{Z}^d / \vec{0} \tag{19}$$

The complementary of the set of numbers ω, α verifying the diophantine conditions for some C_0 has measure $O(C_0)$, see e.g., [41].

In general the value of the chemical potential is modified by the interaction; in order to fix the interacting chemical potential to the value $\bar{\phi}(\alpha)$ we choose the bare one to $\mu = \bar{\phi}(\alpha) + \nu$ with ν chosen properly.

Our main result is the following

Theorem 3.1 *Assume that $\mu = \bar{\phi}(\alpha) + \nu$ and $\phi_{\vec{x}} = \bar{\phi}(\vec{\omega}\vec{x})$ with $\bar{\phi} : \mathbf{T} \rightarrow \mathbb{R}$, even, differentiable and such that $v_0 = \partial\bar{\phi}(\alpha) \neq 0$: in addition $\vec{\omega}$ verifies (18) and α verifies (19). There exists ε_0 and a suitable choice of $\nu = O(\varepsilon_0)$ such that, for $|\lambda| \leq |\varepsilon| \leq \varepsilon_0$ in the zero temperature and infinite volume limit*

1. *The 2-point correlation verifies, for any N*

$$|S(\mathbf{x}, \mathbf{y})| \leq |\log \Delta_{\vec{x}, \vec{y}}| C_N \frac{e^{-\frac{1}{4} |\log |\varepsilon| ||\vec{x} - \vec{y}||}}}{1 + (\Delta_{\vec{x}, \vec{y}} |x_0 - y_0|)^N} \tag{20}$$

with

$$\Delta_{\vec{x}, \vec{y}} = (1 + \min(|\vec{x}|, |\vec{y}|))^{-\tau} \tag{21}$$

2. *The density and current correlations verify*

$$|H_{\mu, \nu}(\mathbf{x}, \mathbf{y})| \leq \Delta_{\vec{x}, \vec{y}}^{-4} C_N \frac{e^{-\frac{1}{4} |\log |\varepsilon| ||\vec{x} - \vec{y}||}}}{1 + (\Delta_{\vec{x}, \vec{y}} |x_0 - y_0|)^N} \tag{22}$$

3. *The Drude weight is vanishing*

$$D_x^j = 0 \tag{23}$$

In the zero temperature limit the correlations (5) reduce to the ground state average, whose exponential decay as in (20) is implied by the localization of the ground state. The result holds with quasi periodic potential of the form $\bar{\phi}(\bar{\omega}\bar{x})$ in any dimension. Moreover the Drude weight at $T = 0$ is vanishing, implying a non-metallic behavior. This result is obtained assuming a Diophantine condition on the frequencies and on the chemical potential (or equivalently on the densities), see (19). As the estimate of the radius of convergence ε_0 is proportional to C_0 to some power, with fixed ε, λ we get a large measure set of densities for which the exponential decay is present (but not on an interval).

Information on the conductivity are obtained by combining the Ward Identities following from the conservation of the current with regularity properties of the Fourier transform of the correlations, which are related to the decay in the coordinate space. In the case of non-interacting fermions, or for $1d$ interacting fermions without disorder, the slow power law decay of correlations implies a non vanishing Drude weight, see [42]. In the present case, the decay in space is exponentially fast but the decay in the imaginary time has rate not uniform in \bar{x}, \bar{y} , due to the lack of translation invariance. As a consequence, we can deduce the vanishing of the Drude weight but not of the conductivity.

The analysis is based on an extension of the Lindstedt series approach to KAM tori with exact Renormalization Group methods for fermions. The correlations are expressed by a series expansion showing a small divisor problem, as in the Lindstedt series for KAM, in graphs with loops, which are a peculiarity of quantum physics. Small divisors are controlled by the Diophantine conditions and the huge number of loop graphs is compensated by cancellations due to anticommutativity.

While we have proved here the vanishing of the Drude weight, it would be interesting to understand if also the conductivity is vanishing; a possibility is that a zero result is found only by a suitable averaging over a phase, that is replacing $\phi(\bar{\omega}\bar{x})$ with $\phi(\bar{\omega}\bar{x} + \alpha)$ and averaging over α .

The effective interaction is irrelevant in the Renormalization Group sense, as consequence of Diophantine conditions and by cancellations due to anticommutativity. The presence of spin [43] and an anisotropic hopping [44] produce extra marginal couplings. They can in principle destroy the convergence result of the present paper, and it is interesting to observe that numerical [45] or cold atoms experiments [19] have found evidence of delocalization in such cases. Another important point would be to extend the analysis to a more general kind of disorder like $f(\bar{\omega}_1\bar{x}, \bar{\omega}_2\bar{x})$. The condition of strong disorder is non technical; in the case of weak quasiperiodic disorder there is no localization; in particular, this is the case of the interacting Aubry-Andre' model [46], of the bidimensional Hofstadter model [47] or of three dimensional Weyl semimetals [48]. Finally, we stress that a rigorous understanding of $T = 0$ properties of interacting fermions with finite density and random disorder is still unknown.

The main open problem is of course to extend the above result on transport coefficients to finite temperature to get information on localization beyond the ground state. In order to do that one needs to perform a Wick rotation at non-zero temperature, but known techniques to do that works only at $T = 0$ [39]. Another difficulty is due to the fact that we do not get ground state localization in an interval of densities, but only in a large measure set. The absence of thermalization in the classical case is considered related to KAM theorem; it is interesting to note that the persistence of localization in a quantum system, which is considered an obstruction to thermalization, is also obtained via the generalization of KAM methods in a quantum context.

4 Vanishing of Drude Weight

We show that the vanishing of Drude weight (23) is consequence of the bound (22) combined with Ward Identities. Note first that the Fourier transform in the infinite volume limit is continuous as

$$\begin{aligned}
 |\widehat{H}_{\mu,v}(\mathbf{p}, \vec{y})| &\leq \sum_{\vec{x}} \int dx_0 |H_{\mu,v}(\mathbf{x}, \mathbf{y})| \leq \sum_{\vec{x}} \int dx_0 \Delta_{\vec{x},\vec{y}}^{-4} C_N \frac{e^{-\frac{1}{4}|\log|\varepsilon||\vec{x}-\vec{y}|}}{1 + (\Delta_{\vec{x},\vec{y}}|x_0|)^N} \\
 &\leq C_1 \sum_{\vec{x}} (|\vec{x} + \vec{y}|^{5\tau} + |\vec{y}|^{5\tau}) e^{-\frac{1}{4}|\log|\varepsilon||\vec{x}|} \\
 &\leq C_2 \sum_{\vec{x}} e^{-\frac{1}{4}|\log|\varepsilon||\vec{x}|} (|\vec{x}|^{5\tau} + 2|\vec{y}|^{5\tau}) \leq C_3 |\vec{y}|^{5\tau} / (|\log|\varepsilon||)^{d+5\tau}
 \end{aligned}
 \tag{24}$$

Ward identities can be deduced from the continuity equation,

$$\partial_0 \rho_{\mathbf{x}} = [H, \rho_{\mathbf{x}}] = -i \sum_i (j_{\mathbf{x}}^i - j_{\mathbf{x}-\mathbf{e}_i}^i)
 \tag{25}$$

we get, setting $\partial_i j_{\mathbf{x}} \equiv j_{\mathbf{x}} - j_{\mathbf{x}-\mathbf{e}_i}$, $i = 1, \dots, d$, $\mathbf{e}_i = (0, \dots, \vec{e}_i)$

$$\begin{aligned}
 \partial_0 \langle T \rho_{\mathbf{x}}; \rho_{\mathbf{y}} \rangle &= -i \sum_i \partial_i \langle T j_{\mathbf{x}}^i; \rho_{\mathbf{y}} \rangle + \delta(x_0 - y_0) \langle [\rho_{\mathbf{x}}, \rho_{\mathbf{y}}] \rangle \\
 \partial_0 \langle T \rho_{\mathbf{x}}; j_{\mathbf{y}}^j \rangle &= -i \sum_i \partial_i \langle T j_{\mathbf{x}}^i; j_{\mathbf{y}}^j \rangle + \delta(x_0 - y_0) \langle [\rho_{\mathbf{x}}, j_{\mathbf{y}}^j] \rangle
 \end{aligned}
 \tag{26}$$

Note that $[\rho_{\vec{x},x_0}, \rho_{\vec{y},x_0}] = 0$ while

$$[\rho_{\vec{x},x_0}, j_{\vec{y},x_0}^j] = -i \delta_{\vec{x},\vec{y}} \tau_{\vec{x}}^j + i \delta_{\vec{x}-\vec{e}_j,\vec{y}} \tau_{\vec{y}}^j
 \tag{27}$$

so that, in the $L, \beta \rightarrow \infty$ limit

$$\begin{aligned}
 \partial_0 \langle T \rho_{\mathbf{x}}; \rho_{\mathbf{y}} \rangle &= -i \sum_i \partial_i \langle T j_{\mathbf{x}}^i; \rho_{\mathbf{y}} \rangle \\
 \partial_0 \langle T \rho_{\mathbf{x}}; j_{\mathbf{y}}^j \rangle &= -i \sum_i \partial_i \langle T j_{\mathbf{x}}^i; j_{\mathbf{y}}^j \rangle \\
 &\quad -i \delta(x_0 - y_0) (-\delta_{\vec{x},\vec{y}} \langle \tau_{\vec{y}}^j \rangle + \delta_{\vec{x}-\vec{e}_j,\vec{y}} \langle \tau_{\vec{y}}^j \rangle)
 \end{aligned}
 \tag{28}$$

Taking the Fourier transform in \mathbf{x} we get, using translation invariance in time and setting $y_0 = 0$

$$\begin{aligned}
 \sum_{\vec{x}} \int dx_0 e^{i\mathbf{p}\mathbf{x}} (\partial_0 \langle T \rho_{\mathbf{x}}; j_{\vec{y}}^j \rangle + i \sum_i \partial_i \langle T j_{\mathbf{x}}^i; j_{\vec{y}}^j \rangle \\
 + i \delta(x_0) (-\delta_{\vec{x},\vec{y}} \langle \tau_{\vec{y}}^j \rangle + \delta_{\vec{x}-\vec{e}_j,\vec{y}} \langle \tau_{\vec{y}}^j \rangle) = 0
 \end{aligned}
 \tag{29}$$

with $p_0 \in \mathbb{R}$ and $\vec{p} \in [-\pi, \pi)^d$ so that

$$-i p_0 \widehat{H}_{0,j}(\mathbf{p}, \vec{y}) + i \sum_i (1 - e^{-ip_1}) (\widehat{H}_{i,j}(\mathbf{p}, \vec{y}) + e^{-i\vec{p}\vec{y}} \langle \tau_{y,0}^j \rangle) = 0
 \tag{30}$$

Setting $j = 1$ for definiteness, we set $\vec{p} = (p_1, 0, 0)$ so that

$$-i p_0 \widehat{H}_{0,1}(\vec{p}, \vec{y}) + i (1 - e^{-ip_1}) (\widehat{H}_{1,1}(\vec{p}, \vec{y}) + e^{-ip_1 y_1} \langle \tau_{y,y_0}^1 \rangle) = 0
 \tag{31}$$

so that

$$\lim_{p_1 \rightarrow 0} (\widehat{H}_{1,1}(0, p_1, \vec{y}) + e^{-ip_1 y_1} \langle \tau_{y, y_0}^1 \rangle) = 0 \tag{32}$$

but $\lim_{p_1 \rightarrow 0} (e^{-ip_1 y_1} - 1) = 0$. In conclusion

$$\lim_{p_1 \rightarrow 0} (\widehat{H}_{1,1}(0, p_1, \vec{y}) + \langle \tau_{y, y_0}^1 \rangle) = 0 \tag{33}$$

Due to (24) $\widehat{H}_{1,1}(\mathbf{p}, \vec{y})$ is continuous in \mathbf{p} so that we can exchange the limits

$$\lim_{p_0 \rightarrow 0} \lim_{\vec{p} \rightarrow 0} (\widehat{H}_{1,1}(\mathbf{p}, \vec{y}) + \langle \tau_{y, y_0}^1 \rangle) = D_x^1 = 0 \tag{34}$$

and this shows that the Drude weight is vanishing. Note the crucial role played by continuity of the Fourier transform, following by the fast decay of the correlations; without quasi-periodic disorder the Fourier transform is not continuous due to its slow decay and the Drude weight is non vanishing.

5 Perturbation Theory and Grassmann Representation

The starting point of the analysis consists in expanding around the anti-integrable limit (12); defining

$$H - \mu N = H_0 + V \tag{35}$$

$$H_0 = \sum_{\vec{x}} (\phi_{\vec{x}} - \bar{\phi}(\alpha)) a_{\vec{x}}^+ a_{\vec{x}}^-$$

$$V = \varepsilon \sum_{\vec{x}, i} (a_{\vec{x}+\vec{e}_i}^+ a_{\vec{x}}^- + a_{\vec{x}}^+ a_{\vec{x}+\vec{e}_i}^-) + \lambda \sum_{\vec{x}, i} a_{\vec{x}}^+ a_{\vec{x}}^- a_{\vec{x}+\vec{e}_i}^+ a_{\vec{x}+\vec{e}_i}^- + \nu \sum_{\vec{x}} a_{\vec{x}}^+ a_{\vec{x}}^- \tag{36}$$

and using the Trotter formula one can write the partition function and the correlations as a power series expansion in λ, ε .

The correlations can be equivalently written in terms of Grassmann integrals. We can write

$$e^{W(\eta, J)} = \int P(d\psi) e^{-\mathcal{V}(\psi) - \mathcal{B}(\psi, J, \eta)} \tag{37}$$

with $\mathbf{e}_i = (0, \vec{e}_i)$

$$\mathcal{V}(\psi) = \varepsilon \sum_i \int d\mathbf{x} (\psi_{\mathbf{x}+\mathbf{e}_i}^+ \psi_{\mathbf{x}}^- + \psi_{\mathbf{x}-\mathbf{e}_i}^+ \psi_{\mathbf{x}}^-) + \lambda \int d\mathbf{x} \sum_i \psi_{\mathbf{x}}^+ \psi_{\mathbf{x}}^- \psi_{\mathbf{x}+\mathbf{e}_i}^+ \psi_{\mathbf{x}+\mathbf{e}_i}^- + \nu \int d\mathbf{x} \psi_{\mathbf{x}}^+ \psi_{\mathbf{x}}^- \tag{38}$$

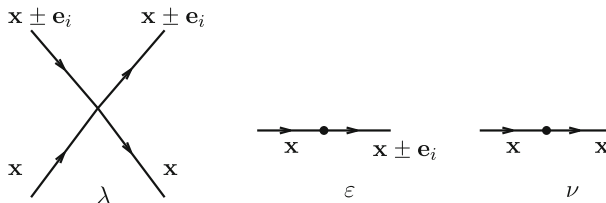


Fig. 1 Graphical representation of the three terms in $\mathcal{V}(\psi)$ Eq.(38)

where $\int d\mathbf{x} = \sum_{x \in \Lambda_L} \int_{-\beta/2}^{\beta/2} dx_0$ and $\psi_{\mathbf{x}}^{\pm}$ is vanishing outside Λ_L ; moreover

$$\mathcal{B}(\psi, J, \eta) = \int d\mathbf{x} [\eta_{\mathbf{x}}^+ \psi_{\mathbf{x}}^- + \psi_{\mathbf{x}}^+ \eta_{\mathbf{x}}^- + \sum_{\mu=0}^d J_{\mu}(\mathbf{x}) j_{\mu}(\mathbf{x})] \tag{39}$$

with

$$j_0(\mathbf{x}) = \psi_{\mathbf{x}}^+ \psi_{\mathbf{x}}^- \quad j_i(\mathbf{x}) = \varepsilon (\psi_{\mathbf{x}+\mathbf{e}_i}^+ \psi_{\mathbf{x}}^- - \psi_{\mathbf{x}}^+ \psi_{\mathbf{x}+\mathbf{e}_i}^-) \tag{40}$$

The 2-point and the current correlations are given by

$$S_2^{L,\beta}(\mathbf{x}, \mathbf{y}) = \frac{\partial^2}{\partial \eta_{\mathbf{x}}^+ \partial \eta_{\mathbf{y}}^-} W(\eta, J)|_{0,0} \quad H_{\mu,\nu}(\mathbf{x}, \mathbf{y}) = \frac{\partial^2}{\partial J_{\mu,\mathbf{x}} \partial J_{\nu,\mathbf{y}}} W(\eta, J)|_{0,0} \tag{41}$$

By expanding in $\lambda, \varepsilon, \nu$ one can write the correlations as a series expansion, which can be expressed in terms of Feynman graphs obtained contracting the half lines of vertices, see Fig. 1, and associating to each line the propagator $g(\mathbf{x}, \mathbf{y})$. There is a basic difference between the perturbative expansion in the non interacting case $\lambda = 0$ and the interacting case $\lambda \neq 0$. In the first case there are only chain graphs, while in the second there are also loops, producing further combinatorial problems. One can verify that the perturbative expansions obtained by Trotter formula for (2) and by the Grassmann generating functions are the same (this is true up to the so called ‘‘tadpoles’’ which can be easily taken into account, see §1 D in [35]). The identity between (2) and (37) is true in a rigorous sense provided that the Grassmann integral representation is analytic in a disk uniformly in L, β , as proven in the following sections. Indeed at finite L, β the partition function in (2) is entire and it coincides order by order with the Grassmann representation, which is analytic in a disk independent on the volume, so they coincide. As the denominator of the correlations is non vanishing in this finite disk and the numerator is entire at finite β, L , also the correlations (2) is analytic and coincide with the Grassmann representation, and the identity holds also in the limit.

6 Multiscale Decomposition and Renormalization

The difficulty in controlling the perturbative expansion is due to a ‘‘small divisor problem’’ related to the size of the propagator; the denominator of $\widehat{g}(\vec{x}, k_0)$ can be arbitrarily small if $\vec{\omega}\vec{x}$ is close to $\pm\alpha$, a fact which can produce in principle $O(n!)$ -terms which could destroy convergence. The starting point of the analysis is to separate the propagator in two terms, one containing the quasi-singularity and a regular part; we write

$$g(\mathbf{x}, \mathbf{y}) = g^{(1)}(\mathbf{x}, \mathbf{y}) + \sum_{\rho=\pm} g_{\rho}^{(\leq 0)}(\mathbf{x}, \mathbf{y}) \tag{42}$$

where

$$g^{(1)}(\mathbf{x}, \mathbf{y}) = \frac{\delta_{\vec{x}, \vec{y}}}{\beta} \sum_{k_0} \chi^{(1)}(\vec{\omega}\vec{x}, k_0) \frac{e^{-ik_0(x_0-y_0)}}{-ik_0 + \overline{\phi}(\vec{\omega}\vec{x}) - \overline{\phi}(\alpha)} = \delta_{\vec{x}, \vec{y}} g^{(1)}(\vec{x}, x_0 - y_0)$$

$$g_{\rho}^{(\leq 0)}(\mathbf{x}, \mathbf{y}) = \frac{\delta_{\vec{x}, \vec{y}}}{\beta} \sum_{k_0} \chi_{\rho}^{(0)}(\vec{\omega}\vec{x}, k_0) \frac{e^{-ik_0(x_0-y_0)}}{-ik_0 + \overline{\phi}(\vec{\omega}\vec{x}) - \overline{\phi}(\alpha)} = \delta_{\vec{x}, \vec{y}} g_{\rho}^{(\leq 0)}(\vec{x}, x_0 - y_0) \tag{43}$$

with $\chi_\rho^{(0)}(\vec{\omega}\vec{x}, k_0) = \tilde{\vartheta}_\rho(\vec{\omega}\vec{x})\bar{\chi}_0(\sqrt{k_0^2 + (\bar{\phi}(\vec{\omega}\vec{x}) - \bar{\phi}(\alpha))^2})$ with $\tilde{\theta}_\rho$ is the periodic theta function ($\tilde{\theta}_\pm = 1$ if $\vec{\omega}\vec{x}$ mod. 1 is positive/negative and zero otherwise) and $\bar{\chi}_0$ such that $C^\infty(\mathbb{R}^+) \rightarrow \mathbb{R}$ such that $\bar{\chi}_0(t) = 1$ with $t \leq 1$ and $\bar{\chi}_0(t) = 0$ for $t \geq \gamma > 1$; moreover $\chi^{(1)} + \sum_{\rho=\pm} \chi_\rho = 1$. The ‘‘infrared’’ propagator $g^{(\leq 0)}(\mathbf{x}, \mathbf{y})$ has denominator arbitrarily small. We can further decompose the infrared propagator as sum of propagators with smaller and smaller denominators

$$g_\rho^{(\leq 0)}(\vec{x}, x_0 - y_0) = \sum_{h=-\infty}^0 g_\rho^{(h)}(\vec{x}, x_0 - y_0) \tag{44}$$

with $g_\rho^{(h)}$ similar $g_\rho^{(\leq 0)}$ with f^h replacing $\bar{\chi}_0$ with

$$f^h = \bar{\chi}_0(\gamma^h \sqrt{k_0^2 + (\bar{\phi}(\vec{\omega}\vec{x}) - \bar{\phi}(\alpha))^2}) - \bar{\chi}_0(\gamma^{h-1} \sqrt{k_0^2 + (\bar{\phi}(\vec{\omega}\vec{x}) - \bar{\phi}(\alpha))^2}) \tag{45}$$

For any integer N one has

$$|g_\rho^{(h)}(\vec{x}, x_0 - y_0)| \leq \frac{C_N}{1 + (\gamma^h |x_0 - y_0|)^N} \tag{46}$$

if C_N is a suitable constant.

The integration of (37) is done iteratively by using two crucial properties of Grassmann integrations. If $P(d\psi^{(1)})$ and $P(d\psi^{(\leq 0)})$ are gaussian Grassmann integrations with propagators $g^{(1)}$ and $g^{(\leq 0)}$, we can write $P(d\psi) = P(d\psi^{(1)})P(d\psi^{(\leq 0)})$ so that

$$\begin{aligned} e^{W(\eta, J)} &= \int P(d\psi^{(1)})P(d\psi^{(\leq 0)})e^{-\mathcal{V}(\psi^{(1)} + \sum_{\rho=\pm} \psi_\rho^{(\leq 0)}) - \mathcal{B}(\psi^{(1)} + \sum_{\rho=\pm} \psi_\rho^{(\leq 0)}, \eta, J)} \\ &= \int P(d\psi^{(\leq 0)})e^{-\mathcal{V}^{(0)}(\psi_\rho^{(\leq 0)}, \eta, J)} \end{aligned} \tag{47}$$

with

$$\mathcal{V}^{(0)}(\psi_\rho^{(\leq 0)}, \eta, J) = \sum_{n=0}^\infty \frac{1}{n!} \mathcal{E}_1^T(\mathcal{V} + \mathcal{B}; n) \tag{48}$$

and \mathcal{E}_1^T are fermionic truncated expectations with propagator $g^{(1)}$. By integrating $\psi^{(0)}, \psi^{(-1)}, \dots, \psi^{(h+1)}$ one obtains a sequence of effective potentials $\mathcal{V}^{(h)}$, $h = 0, -1, -2, \dots$. The way in which we define the integration is dictated by the scaling dimension which is, as we will see below, $D = 1$; that is all terms are relevant in the Renormalization Group sense.

Remark Note that after the integration of ψ^1 one gets a theory defined in terms of two fields ψ_+, ψ_- . This is due to the fact that $\bar{\phi}(t) = \bar{\phi}(\alpha)$ in correspondence of two points $\pm\alpha$. If we consider more general forms of quasi periodic disorder, like $\bar{\phi}(t_1, t_2)$ as the one in [7], then $\bar{\phi}(t_1, t_2) - \mu = 0$ in a set corresponding to a surface. In this case one gets a description in terms of a field ψ_ρ , with ρ a parameter parametrizing this curve, a situation somewhat analogue to what happens in interacting fermions with extended Fermi surface.

The multiscale integration is described iteratively in the following way. Assume that we have already integrated the fields $\psi^{(0)}, \psi^{(-1)}, \dots, \psi^{(h+1)}$ obtaining (we set $\eta = 0$ for the moment)

$$e^{W(0, J)} = \int P(d\psi^{(\leq h)})e^{-\mathcal{V}^{(h)}(\psi^{(\leq h)}, J)} \tag{49}$$

where $P(d\psi^{(\leq h)})$ has propagator

$$g_\rho^{(\leq h)}(\mathbf{x}, \mathbf{y}) = \frac{\delta_{\vec{x}, \vec{y}}}{\beta} \sum_{k_0} \chi_\rho^{(h)}(k_0, \vec{\omega}\vec{x}) \frac{e^{-ik_0(x_0 - y_0)}}{-ik_0 + \bar{\phi}(\vec{\omega}\vec{x}) - \bar{\phi}(\alpha)} = \delta_{\vec{x}, \vec{y}} g_\rho^{(\leq 0)}(\vec{x}, x_0 - y_0) \tag{50}$$

and

$$\mathcal{V}^{(h)}(\psi^{(\leq h)}, J) = \sum_{l \geq 0, m \geq 0} \sum_{\varepsilon, \rho} \int d\mathbf{x}_1 \dots d\mathbf{x}_l d\mathbf{y}_1 \dots d\mathbf{y}_m H_{l,m}^h(\underline{\mathbf{x}}, \underline{\mathbf{y}}) \prod_{i=1}^l \psi_{\rho_i, \mathbf{x}_i}^{\varepsilon_i(\leq h)} \prod_{i=1}^m J_{\mathbf{y}_i} \tag{51}$$

If there is a subset of $\psi_{\rho_i, \mathbf{x}_i}^{\varepsilon_i}$ with the same ε, ρ and \vec{x}_i , by the anticommuting properties of Grassmann variables we can write, if $l > 1$

$$\prod_{i=1}^l \psi_{\vec{x}, x_{0,i}}^\varepsilon = \psi_{\vec{x}, x_{0,1}}^\varepsilon \prod_{i=2}^l D_{\vec{x}, x_{0,i}, x_{0,1}}^\varepsilon \quad D_{\vec{x}, x_{0,i}, x_{0,1}}^\varepsilon = \psi_{\vec{x}, x_{0,i}}^\varepsilon - \psi_{\vec{x}, x_{0,1}}^\varepsilon \tag{52}$$

We can therefore rewrite that effective potential in the following way

$$\mathcal{V}^{(h)}(\psi^{(\leq h)}, J) = \sum_{l \geq 0, m \geq 0} \sum_{\varepsilon, \rho} \int d\mathbf{x}_1 \dots d\mathbf{x}_l d\mathbf{y}_1 \dots d\mathbf{y}_m H_{l,m}^h(\underline{\mathbf{x}}, \underline{\mathbf{y}}) \prod_{i=1}^l d^{\sigma_i} \psi_{\rho_i, \mathbf{x}_i}^{\varepsilon_i} \prod_{i=1}^m J_{\mathbf{y}_i} \tag{53}$$

with $\sigma = 0, 1$ and $d^0\psi = \psi$ and $d^1\psi = D$.

We define *resonant* the terms with fields with the same coordinate \vec{x} , that is $\mathbf{x}_i = (x_{0,i}, \vec{x})$. Note that all the resonant terms with $l \geq 4$ are such that there are at least two D fields; the fields have the same ρ index as have the same $\vec{\omega}\vec{x}$.

We define a *renormalization operation* \mathcal{R} in the following way

1. If $l = 2, m = 0$

$$\mathcal{R} \sum_{\vec{x}} \int dx_{0,1} dx_{0,2} H_{2,0}^{(h)} \psi_{\vec{x}, x_{0,1}, \rho}^{+(\leq h)} \psi_{\vec{x}, x_{0,2}, \rho}^{-(\leq h)} = \sum_{\vec{x}} \int dx_{0,1} dx_{0,2} H_{2,0}^{(h)} \psi_{\vec{x}, x_{0,1}, \rho}^{+(\leq h)} T_{\vec{x}, x_{0,1}, x_{0,2}, \rho}^{-(\leq h)} \tag{54}$$

with

$$T_{\vec{x}, x_{0,1}, x_{0,2}, \rho}^{-(\leq h)} = \psi_{\vec{x}, x_{0,2}, \rho}^{-(\leq h)} - \psi_{\vec{x}, x_{0,1}, \rho}^{-(\leq h)} - (x_{0,1} - x_{0,2}) \partial \psi_{\vec{x}, x_{0,1}, \rho}^{-(\leq h)} \tag{55}$$

2. $\mathcal{R} = 0$ otherwise

We define $\mathcal{R} = 1 - \mathcal{L}$ and by definition $\mathcal{L}\mathcal{V}^{(h)}$ is given by the following expression

$$\mathcal{L}\mathcal{V}^{(h)} = \gamma^h F_v^{(h)} + F_\zeta^{(h)} + F_\alpha^{(h)} \tag{56}$$

where, if $H_{2,0}^{(h)}(\vec{x}, x_0 - y_0) \equiv \bar{H}_{2,0}^{(h)}(\vec{\omega}\vec{x}, x_0 - y_0)$ one has

$$v_h = \int dx_0 \bar{H}_{2,0}^{(h)}(\rho\alpha, x_0) \quad \xi_h(\vec{x}) = \int dx_0 \frac{\bar{H}_{2,0}^{(h)}(\vec{\omega}\vec{x}, x_0) - \bar{H}_{2,0}^{(h)}(\rho\alpha, x_0)}{\vec{\omega}\vec{x} - \rho\alpha} \tag{57}$$

and $\alpha_h(\vec{x}) = \int dx_0 x_0 \bar{H}_{2,0}^{(h)}(\vec{\omega}\vec{x}, x_0)$; moreover

$$F_v^{(h)} = \sum_\rho \sum_{\vec{x}} \int dx_0 v_h \psi_{\mathbf{x}, \rho}^{+(\leq h)} \psi_{\mathbf{x}, \rho}^{-(\leq h)}$$

$$F_\zeta^{(h)} = \sum_\rho \sum_{\vec{x}} \int dx_0 ((\vec{\omega}\vec{x}) - \rho\alpha) \zeta_{h, \rho}(\vec{x}) \psi_{\mathbf{x}, \rho}^{+(\leq h)} \psi_{\mathbf{x}, \rho}^{-(\leq h)}$$

$$F_\alpha^{(h)} = \sum_\rho \sum_{\vec{x}} \int dx_0 \alpha_{h,\rho}(\vec{x}) \psi_{\vec{x},\rho}^{+(\leq h)} \partial_0 \psi_{\vec{x},\rho}^{-(\leq h)} \tag{58}$$

The running coupling constants $\vec{v}_h = (v_h, \alpha_h, \xi_h)$ are independent from ρ , as (37) is invariant under parity $\vec{x} \rightarrow -\vec{x}$. Note also that $(\widehat{g}^{(k)})^*(\vec{x}, k_0) = \widehat{g}^{(k)}(\vec{x}, -k_0)$ so that $(\widehat{H}_{2,\rho}^{(h)}(\vec{x}, k_0))^* = \widehat{H}_{2,\rho}^{(h)}(\vec{x}, -k_0)$, and this implies that v_h is real.

Remark The \mathcal{R} operation is defined in order to act non trivially on the resonant terms with two fields and no J fields; they are the only resonant terms with no D fields. This fact would be not true of there is the spin or an extra degree of freedom, as in the case of lattice Weyl semimetals [48]. In that case the local part of the effective potential would contain also effective interactions.

With the above definitions we can write (49)

$$\begin{aligned} e^{W(0,J)} &= \int P(d\psi^{(\leq h-1)}) \int P(d\psi^{(h)}) e^{-\mathcal{L}\mathcal{V}^{(h)}(\psi^{(\leq h)}, J) - \mathcal{R}\mathcal{V}^{(h)}(\psi^{(\leq h)}, J)} \\ &= \int P(d\psi^{(\leq h-1)}) e^{-\mathcal{L}\mathcal{V}^{(h)}(\psi^{(\leq h-1)}, J)} \end{aligned} \tag{59}$$

and the procedure can be iterated.

7 Convergence of Series Expansion

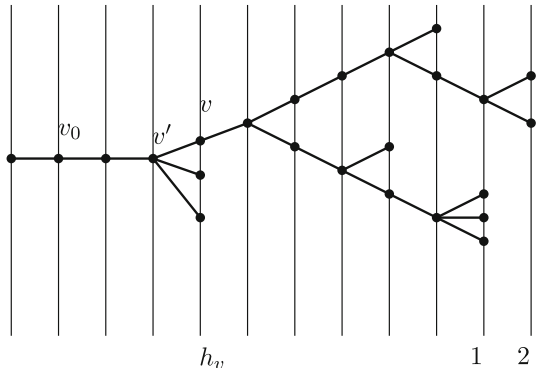
The effective potential can be written as a sum over Gallavotti trees τ , see Fig. 2

$$\mathcal{V}^{(h)}(\psi^{(\leq h)}, J) = \sum_{n=1}^\infty \sum_{\tau \in \mathcal{T}_{h,n}} V^{(h)}(\tau, \psi^{(\leq h)}) \tag{60}$$

where τ are trees constructed adding labels to the unlabeled trees, obtained by joining a point, the *root*, with an ordered set of $n \geq 1$ points, the *endpoints*, so that the root is not a branching point.

The set of labeled trees $\mathcal{T}_{h,n}$ is defined associating a label $h \leq 0$ with the root and introducing a family of vertical lines, labeled by an integer taking values in $[h, 2]$ intersecting all the non-trivial vertices, the endpoints and other points called trivial vertices. To a vertex v is associated h_v and, if v_1 and v_2 are two vertices and $v_1 < v_2$, then $h_{v_1} < h_{v_2}$. Moreover, there is only one vertex immediately following the root, which will be denoted v_0 and can

Fig. 2 A labeled tree



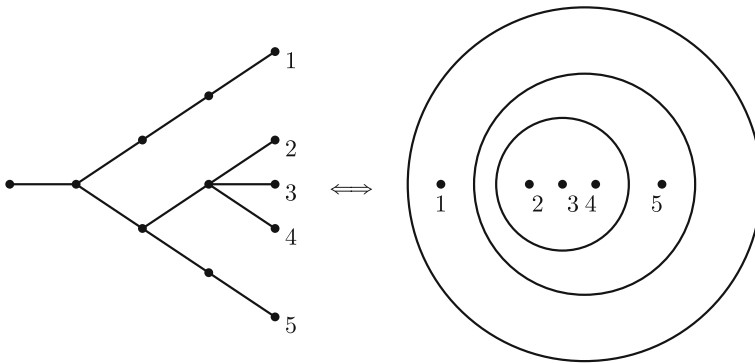


Fig. 3 A tree of order 5 and the corresponding clusters

not be an endpoint; its scale is $h + 1$. To the end-points are associated $\mathcal{V} + \mathcal{B}$, and in such a case the scale is 2; or $\mathcal{L}\mathcal{V}^{h_v-1}(\psi^{(\leq h_v-1)}, J)$ and in this case the scale is $h_v \leq 1$ and there is the constraint that $h_v = h_{\bar{v}} + 1$, if \bar{v} is the first non trivial vertex immediately preceding v . The tree structure induces a jerarchy of end-points which can be represented by clusters, see Fig. 3.

If v_0 is the first vertex of τ and τ_1, \dots, τ_s ($s = s_{v_0}$) are the subtrees of τ with root v_0 , $V^{(h)}(\tau, \psi^{(\leq h)})$ is defined inductively by the relation

$$V^{(h)}(\tau, \psi) = \frac{(-1)^{s+1}}{s!} \mathcal{E}_{h+1}^T [\bar{V}^{(h+1)}(\tau_1, \psi^{(\leq h+1)}); \dots; \bar{V}^{(h+1)}(\tau_s, \psi^{(\leq h+1)})] \quad (61)$$

where $\bar{V}^{(h+1)}(\tau_i, \psi^{(\leq h+1)})$ it is equal to $\mathcal{R}\mathcal{V}^{(h+1)}(\tau_i, \psi^{(\leq h+1)})$ if the subtree τ_i is non trivial; if τ_i is trivial, it is equal to $\mathcal{L}\mathcal{V}^{(h+1)}$. By iterating (61) we get a jerarchy of truncated expectations, with a certain subset of fields contracted in each expectations. We can therefore write $V^{(h)}(\tau, \psi^{(\leq h)})$ as sum over sets defined in the following way. We call I_v the set of ψ associated to the end-points following v and P_v is a subset of I_v denoting the external ψ . We denote by Q_{v_i} the intersection of P_v and P_{v_i} ; they are such that $P_v = \cup_i Q_{v_i}$ and the union \mathcal{I}_v of the subsets $P_{v_i} \setminus Q_{v_i}$ is, by definition, the set of the *internal fields* of v , and is non empty if $S_v > 1$. The effective potential can be therefore written as

$$\mathcal{V}^{(h)}(\tau, \psi^{(\leq h)}) = \sum_{\mathbf{P} \in \mathcal{P}_\tau} \mathcal{V}^{(h)}(\tau, \mathbf{P}) \quad \bar{\mathcal{V}}^{(h)}(\tau, \mathbf{P}) = \int d\mathbf{x}_{v_0} \tilde{\psi}^{(\leq h)}(P_{v_0}) K_{\tau, \mathbf{P}}^{(h+1)}(\mathbf{x}_{v_0}), \quad (62)$$

where $\tilde{\psi}^{(\leq h)}(P) = \prod_{f \in P} \psi_{\mathbf{x}(f)}$. If we expand the truncated expectations by the Wick rule we get a sum of Feynman graphs with an associated cluster structure; an example is in Fig. 4.

The truncated expectations can be written by the Brydges–Battle–Federbush formula

$$\begin{aligned} &\mathcal{E}_{h_v}^T(\tilde{\psi}^{(h_v)}(P_1/Q_1), \dots, \tilde{\psi}^{(h_v)}(P_s/Q_s)) \\ &= \sum_{T_v} \prod_{l \in T_v} [\delta_{\tilde{x}_l, \tilde{y}_l} \bar{g}^{(h_v)}(\tilde{x}_l, x_{0,l} - y_{0,l})] \int dP_T(\mathbf{t}) \det G^{h_v, T}(\mathbf{t}), \end{aligned} \quad (63)$$

where T_v is a set of lines forming an *anchored tree graph* between the clusters of points $\mathbf{x}^{(i)} \cup \mathbf{y}^{(i)}$, that is T_v is a set of lines, which becomes a tree graph if one identifies all the points in the same cluster. Moreover $\mathbf{t} = \{t_{i i'} \in [0, 1], 1 \leq i, i' \leq s\}$, $dP_{T_v}(\mathbf{t})$ is a probability

Fig. 4 An example of graph with λ and ε vertices and the associated cluster structure; the propagator in the cluster, represented as a circle, has scale h smaller than the scales of the propagators external to the cluster

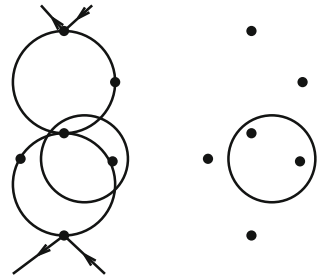
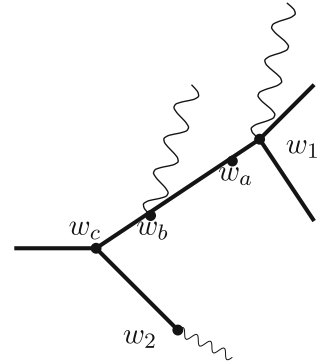


Fig. 5 A tree \tilde{T}_v with attached wiggly lines representing the external lines P_v ; the lines represent propagators with scale $\geq h_v$ connecting w_1, w_a, w_b, w_c, w_2 , representing the end-points following v in τ



measure with support on a set of \mathbf{t} such that $t_{ii'} = \mathbf{u}_i \cdot \mathbf{u}_{i'}$ for some family of vectors $\mathbf{u}_i \in \mathbb{R}^s$ of unit norm.

$$G_{ij,i'j'}^{h,T} = t_{ii'} \delta_{\vec{x}_{ij}, \vec{y}_{i'j'}} \bar{g}^{(h)}(\vec{x}_{ij}, x_{0,ij} - y_{0,i'j'}) , \tag{64}$$

We define $\tilde{T}_v = \bigcup_{w \geq v} T_w$ starting from T_v and attaching to it the trees $T_{v_1}, \dots, T_{v_{S_v}}$ associated to the vertices v_1, \dots, v_{S_v} following v in τ , and repeating this operation until the end-points of τ are reached.

The tree \tilde{T}_v connects the end-points w of the tree τ . To each end-point w we associate a factor $\bar{\delta}_w^i$, and a) $\bar{\delta}_w^i = 0$ if w corresponds to a v_h, α_h, ζ_h end-point; b) $\bar{\delta}_w^i$ one among $\pm \bar{e}_i, i = 1, 2, 3$ if it corresponds to an ε end-point; c) $\bar{\delta}_w^i$ one among $0, \pm \bar{e}_i, i = 1, 2, 3$ if it corresponds to a λ end-point. If \vec{x}_{w_1} and \vec{x}_{w_2} are coordinates of the external fields $\tilde{\psi}(P_v)$ we have, see Fig. 5

$$\vec{x}_{w_1} - \vec{x}_{w_2} = \sum_{w \in c_{w_1, w_2}} \bar{\delta}_w^i \tag{65}$$

where c_{w_1, w_2} is the set of endpoints in the path in \tilde{T} connecting w_1 and w_2 . The above relation implies, in particular, that the coordinates of the external fields $\tilde{\psi}(P_v)$ are determined once that the choice of a single one of them and of τ, \tilde{T}_v and \mathbf{P} is done. We can therefore write the effective potential as sum over trees T , setting the Kronecker deltas in the propagators in $l \in T$ equal to 1

$$\begin{aligned} \mathcal{V}^{(h)}(\tau, \psi^{(\leq h)}) &= \sum_{\mathbf{P} \in \mathcal{P}_\tau} \sum_T \mathcal{V}^{(h)}(\tau, \mathbf{P}, T) \bar{\mathcal{V}}^{(h)}(\tau, \mathbf{P}, T) \\ &= \sum_{\vec{x}} \int dx_{0, v_0} \tilde{\psi}^{(\leq h)}(P_{v_0}) K_{\tau, \mathbf{P}, T}^{(h+1)}(\mathbf{x}_{v_0}) , \end{aligned} \tag{66}$$

where in $K_{\tau, \mathbf{P}, T}^{(h+1)}$ the propagators in T are $g^{(h)}(\vec{x}, x_0 - y_0)$ and the determinants are product of determinants involving propagators with the same \vec{x} . We can bound the propagators in T by

$$\int dx_0 |g^{(h)}(\vec{x}, x_0 - y_0)| \leq C \gamma^{-h} \tag{67}$$

Moreover the determinants in the BFF formula can be bounded by the Gram-Hadamard inequality. We introduce an Hilbert space $\mathcal{H} = \mathbb{R}^s \otimes L^2(\mathbb{R}^1)$ so that

$$\tilde{G}_{ij, i'j'}^{h, T} = \left(\mathbf{u}_i \otimes A(x_{0,ij-}, x_{ij}), \mathbf{u}_{i'} \otimes B(y_{0,i'j'-}, x_{ij}) \right), \tag{68}$$

where $\mathbf{u} \in \mathbb{R}^s$ are unit vectors $(u_i, u_i) = t_{i'}$, and A, B

$$(A, B) = \int dz_0 A(\vec{x}, x_0 - z_0) B^*(\vec{x}, z_0 - y_0) \tag{69}$$

given by

$$A(\vec{x}, x_0 - z_0) = \frac{1}{\beta} \sum_{k_0} e^{-ik_0(x_0-z_0)} \sqrt{f_h} \quad B(\vec{x}, y_0 - z_0) = \frac{1}{\beta} \sum_{k_0} \frac{e^{-ik_0(y_0-z_0)} \sqrt{f_h}}{-ik_0 + \bar{\phi}(\bar{\omega}\vec{x}) - \bar{\phi}(\alpha)}$$

Moreover $\|A_h\|^2 = \int dz_0 |A_h(x', z_0)|^2 \leq C \gamma^h$ and $\|B_h\|^2 \leq C \gamma^{-h}$ so that By Gram-Hadamard inequality we get:

$$|\det \tilde{G}^{h_v, T_v}(\mathbf{t}_v)| \leq C \sum_{i=1}^{S_v} |P_{v_i}| - |P_v| - 2(S_v - 1). \tag{70}$$

One get therefore the bound, for $|\lambda|, |\vec{v}_h| \leq \varepsilon_0$,

$$|K_{\tau, \mathbf{P}, T}^{(h+1)}(\mathbf{x}_{v_0})| \leq C^n \varepsilon_0^n \prod_v \frac{1}{S_v!} \gamma^{-h_v(S_v-1)} \tag{71}$$

which is not suitable for summing over τ and P . In order to improve the above bound we need to implement in the bounds some constraints which have been neglected in the derivation of (71), and to take into account the effect of the presence of the D fields.

We define V_χ the set of non trivial vertices or the trivial ones with non zero internal lines; we define v' the first vertex in V_χ following v . We say that v is a non-resonant vertex if in $\tilde{\psi}(P_v)$ there are at least two different coordinates, and a resonant vertex when all coordinates are equal. We define $S_v = S_v^L + S_v^H$ where S_v^L is the number of non resonant subtrees (including trivial ones) and S_v^H the number of resonant ones (including trivial ones). We also call H the set of $v \in V_\chi$ which are resonant and L the $v \in V_\chi$ which are non resonant. Consider a non resonant vertex v so that there are at least two fields in P_v with different spatial coordinates \vec{x} , say $\vec{x}_{w_1} \neq \vec{x}_{w_2}$. The fields $\tilde{\psi}^{(\leq h_v)}(P_v)$ have scale $\leq \gamma^{h_{v'}}$, $v' \in V_\chi$ the first vertex belonging to V_χ after v so that

$$\|(\bar{\omega}\vec{x}_{w_1}) - \rho_1 \alpha\|_{\mathbb{T}} \leq c v_0^{-1} \gamma^{h_{v'-1}} \quad \|(\bar{\omega}\vec{x}_{w_2}) - \rho_2 \alpha\|_{\mathbb{T}} \leq c v_0^{-1} \gamma^{h_{v'-1}} \tag{72}$$

so that

$$2c v_0^{-1} \gamma^{h_{v'}} \geq \|(\bar{\omega}\vec{x}_{w_1}) - \rho_1 \alpha\|_{\mathbb{T}} + \|(\bar{\omega}\vec{x}_{w_2}) - \rho_2 \alpha\|_{\mathbb{T}} \geq \|(\bar{\omega}(\vec{x}_{w_1} - \vec{x}_{w_2}) - (\rho_1 - \rho_2)\alpha)\|_{\mathbb{T}} \tag{73}$$

and by (65)

$$2c v_0^{-1} \gamma^{h_{v'}} \geq \|(\bar{\omega}(\sum_{w \in c_{w_1, w_2}} \bar{\delta}_w^{i_w}) + (\rho_1 - \rho_2)\alpha)\|_{\mathbb{T}} \geq \frac{C_0}{|\sum_{w \in c_{w_1, w_2}} \bar{\delta}_w^{i_w}|_{\tau}} \tag{74}$$

where the Diophantine conditions have been used. Therefore

$$\sum_{w \in c_{w_1, w_2}} |\vec{\delta}_w^{i_w}| \geq \left| \sum_{w \in c_{w_1, w_2}} \vec{\delta}_w^{i_w} \right| \geq C \gamma^{-h_{v'}/\tau} \tag{75}$$

and, if N_v is the number of end-points following v in τ

$$\sum_{w \in c_{w_1, w_2}} |\vec{\delta}_w^{i_w}| \leq N_v \tag{76}$$

as $|\vec{\delta}_w^{i_w}| = 0, 1$ so that

$$N_v \geq C \gamma^{-h_{v'}/\tau} \tag{77}$$

Note that to each endpoint is associated a small factor ε_0 and the fact that N_v is large by (77) produces a gain for the v with the fields with different \vec{x} . Of course there can be several \vec{T}_v with different v passing through the same end-points. Therefore, given a constant $c < 1$, we can multiply the contribution to each tree τ with n -endpoints by $c^{-n} c^n$ (the factor c^{-n} is of course armless); we can then write

$$c = \prod_{h=-\infty}^0 c^{2^{h-1}} \tag{78}$$

and associate to each v a factor $c^{N_v 2^{h-1}}$. If there are two fields in P_v (that is external to the cluster v) with different \vec{x} we get in the bounds, by assuming $\gamma^{\frac{1}{\tau}}/2 \equiv \gamma^\eta > 1$ than, for any N

$$c^{A \gamma^{\frac{-h}{\tau}} 2^h} = e^{-|\log c| A \gamma^{-\eta h}} \leq \gamma^{N \eta h} \frac{N}{[|\log |c||A|^N e^N]} \tag{79}$$

as $e^{-\alpha x} x^N \leq [\frac{N}{\alpha}]^N e^{-N}$, and we can choose $N = 3/\eta$; therefore given a couple of fields external to a vertex v with different \vec{x} , we can associate a factor $\gamma^{2h_{v'}}$ in the bounds.

On the other hand if there is a D field we get in the bound an extra $\gamma^{h_{v'}-h_v}$ from the expression

$$\bar{g}^{(h_{v'})}(\vec{\omega}\vec{x}, x_{0,1} - z_0) - \bar{g}^{(h_{v'})}(\vec{\omega}\vec{x}, x_{0,2} - z_0) = (x_{0,1} - x_{0,2}) \int_0^1 dt \partial \bar{g}^{(h_{v'})}(\vec{\omega}\vec{x}, \hat{x}_{0,1,2}(t) - z_0) \tag{80}$$

where $\hat{x}_{0,1,2}(t) = x_{0,1} + t(x_{0,2} - x_{0,1})$. In conclusion

1. To each non-resonant v we associate a factor (79) so that we get in the bound an extra factor $\prod_{v \in V_\chi} \gamma^{2h_v S_v^L}$
2. There is a factor $\prod_v^* \gamma^{h_{v'}}$ where v are the endpoints v, α, ξ (it comes from the definition of v and the presence $(x_0 - y_0)$ or $(\vec{\omega}\vec{x} - \rho\alpha)$).
3. In the resonant v with $l \geq 2$ fields there is a factor $\prod_{v \in H} \gamma^{2(h_{v'}-h_v)}$. For $l = 2$ this is due to the \mathcal{R} definition, for $l \geq 4$ by anticommutativity.
4. In the terms with $|P_v| \geq 8$ we can consider the fields ψ_x^ε whose number is maximal; we can group them in couples connected by path in \vec{T} non overlapping, and or have different \vec{x} , hence there is a path in \vec{T} connecting them giving an extra $\gamma^{2h_{v'}}$, or they have the same \vec{x} so that there is an extra $\gamma^{2(h_{v'}-h_v)}$. This produces an extra $\gamma^{-\alpha|P_v|}$, see §F in [36].

We bound first the effective potential ($J = 0$). If $\tau \in \mathcal{T}_{h,n}$, the set of trees with n end-points and defining

$$\|K_{\tau, \mathbf{P}, T}^{(h+1)}\| = \frac{1}{\beta L^d} \sum_{\vec{x}} \int dx_{0, v_0} |K_{\tau, \mathbf{P}, T}^{(h+1)}| \tag{81}$$

we get

$$\|K_{\tau, \mathbf{P}, T}^{(h+1)}\| \leq C^n \varepsilon_0^n \prod_v \frac{1}{S_v!} \gamma^{-h_v(S_v-1)} \prod_{v \in V_\chi} \gamma^{2h_v S_v^L} \prod_v^* \gamma^{h_{v'}} \prod_{v \in H} \gamma^{2(h_{v'}-h_v)} \prod_{v \in V_\chi} \gamma^{-\alpha|P_v|} \tag{82}$$

If the first vertex $v_0 \in V_\chi$ is non resonant we get

$$\prod_{v \in V_\chi} \gamma^{-h_v S_v} \prod_v \gamma^{h_v S_v^L} \prod_v^* \gamma^{h_{v'}} \prod_{v \in H, v \neq v_0} \gamma^{h_{v'}} = 1 \quad \prod_{v \in V_\chi} \gamma^{h_v} \prod_{v \in H, v \neq v_0} \gamma^{-h_v} \leq \gamma^{h_{v_0}} \tag{83}$$

We use that $S_v = S_v^L + S_v^H$, $\prod_v \gamma^{h_v S_v^L} = \prod_{v \in L} \gamma^{h_{v'}} \prod_v^{**} \gamma^{h_v}$, with \prod_v^{**} is over the first vertex $v \in V_\chi$ after the ε, λ endpoints, and that $\prod_{v \in L} \gamma^{h_{v'}} \leq \prod_{v \in L} \gamma^{h_{v'}-h_v}$

$$\|K_{\tau, \mathbf{P}, T}^{(h+1)}\| \leq C^n \varepsilon_0^n \gamma^{h_{v_0}} \prod_v \frac{1}{S_v!} \prod_{v \in V_\chi} \gamma^{(h_{v'}-h_v)} \prod_v^{**} \gamma^{h_v} \prod_{v \in V_\chi} \gamma^{-\alpha|P_v|} \tag{84}$$

where \prod_v^{**} is over the vertices $v \in V_\chi$ following from the end-points associated to ε, λ . Note that $\sum_{\mathbf{P}} [\prod_{v \in V_\chi} \gamma^{-\frac{1}{8}|P_v|}] \leq C^n$; moreover $\sum_{\mathbf{T}} [\prod_v \frac{1}{S_v!}] \leq C^n$. The sum over the trees τ is done performing the sum of unlabeled trees and the sum over scales. The unlabeled trees can be bounded by 4^n by Cayley formula, and the sum over the scales reduces to the sum over h_v , with $v \in V_\chi$, as given a tree with such scales assigned, the others are of course determined.

Let us consider now the case in which the first vertex v_0 is resonant; we can distinguish two cases. If we are considering the contribution to the beta function then there is no \mathcal{R} applied in v_0 so that the same bound as above is found with $h_{v_0} = h + 1$. Instead if \mathcal{R} is applied we get instead of (83), as there is an extra $\gamma^{h_{v_0}'-h_{v_0}}$

$$\prod_{v \in V_\chi} \gamma^{-h_v S_v} \prod_v \gamma^{h_v S_v^L} \prod_v^* \gamma^{h_{v'}} \prod_{v \in H} \gamma^{h_{v'}} = \gamma^{h_{v_0}'} \quad \prod_{v \in V_\chi} \gamma^{h_v} \prod_{v \in H} \gamma^{-h_v} \leq 1 \tag{85}$$

and the same bound is found, as $h_{v_0}' = h + 1$. In conclusion we get

$$\sum_{\tau \in \mathcal{T}_{h,n}} \sum_{\mathbf{P}, T} \|K_{\tau, \mathbf{P}, T}^{(h+1)}\| \leq C^n \varepsilon_0^n \gamma^h \tag{86}$$

The running coupling constant α_h, ξ_h verify

$$\alpha_{h-1} = \alpha_h + O(\varepsilon_0^2 \gamma^{\frac{h}{2}}) \quad \xi_{h-1} = \xi_h + O(\varepsilon_0^2 \gamma^{\frac{h}{2}}) \tag{87}$$

where the factor $\gamma^{\frac{h}{2}}$ is due to the fact that the trees have at least an ε, λ endpoint, from the factor $\prod_v^{**} \gamma^{h_v}$ in (84) (short memory property). The flow of z_h, α_h is therefore summable; in addition one can choose ν so that ν_h is bounded, by proceeding as in Lemma 2.7 of [36].

8 Decay of Correlations

We consider now the current correlations, which can be written as

$$H_{\mu, \nu}(\mathbf{x}, \mathbf{y}) = \sum_{h,n} \sum_{\tau \in \mathcal{T}_{h,n+2}} \sum_{\mathbf{P}, T} G_{\tau, \mathbf{P}, T}(\mathbf{x}, \mathbf{y}) \tag{88}$$

where $\mathcal{T}_{h,n+2}$ is the set of trees with $n + 2$ end-points, two of them associated to the J end-points. In the trees τ we can identify a vertex v_x for the end-point corresponding to $J_{\mathbf{x}}$, and v_y for the end-point corresponding to $J_{\mathbf{y}}$ with $h_{v_x} = h_{v_y} = +2$; we call \widehat{v} , with scale \widehat{h} , the first vertex $v \in V_X$ such that v_x, v_y follows \widehat{v} , and v_0 the first vertex $\in V_X$, with scale h . There are several constraints.

1. By (65) and using that $\vec{x} - \vec{y} = \sum_{w \in C_{v_x, v_y}} \vec{\delta}_w^{i_w}$ we get $n \geq \sum_{w \in C_{v_x, v_y}} |\vec{\delta}_w^{i_w}| \geq |\vec{x} - \vec{y}|$
2. $h \geq \bar{h}(n)$ with, if $|\vec{z}| = 1 + \min(|\vec{x}|, |\vec{y}|)$

$$\gamma^{-\bar{h}} \leq \sup_{\vec{q} = \sum_{i=1}^n \vec{e}_i} \frac{1}{\|\widehat{\omega}(\vec{x} + \vec{q}) - \rho\alpha\|} \leq C(|\vec{z}| + n)^\tau \tag{89}$$

With respect to the bound for the $J = 0$ case there are the following differences. If $T_{\widehat{v}}$ is the tree connecting the 2 J endpoints, we have an extra $\gamma^{\widehat{h}}$ due to the fact that we do not integrate over the coordinates of the J fields, and we can extract from the the propagators in $\prod_{l \in \widehat{T}_{\widehat{v}}} g^{(h_l)}$, $h_l \geq \widehat{h}$ a decay factor

$$\frac{1}{1 + (\gamma^{\widehat{h}}|x_0 - y_0|)^N} \tag{90}$$

Moreover there is no \mathcal{R} in the resonant terms with one or two external J lines. We can multiply and divide by $\gamma^{-4\bar{h}}\gamma^{4\widehat{h}}$: we can select two paths in τ $v_0 < v_1 < ..v_x$ and $v_0 < v'_1 < ..v_y$, writing

$$\gamma^{2\bar{h}} = \gamma^{2(\bar{h}-h_{v_1})} \dots \gamma^{2h_{v'_x}} \quad \gamma^{2\bar{h}} = \gamma^{2(\bar{h}-h_{v'_1})} \dots \gamma^{2h_{v_y}} \tag{91}$$

where v'_x, v'_y are the first vertex $\in V_X$ after v_x, v_y . We get therefore the following bound

$$\begin{aligned} |G_{\tau, \mathbf{P}, T}(\mathbf{x}, \mathbf{y})| &\leq \gamma^{-4\bar{h}} \frac{C^n |\varepsilon|^n \gamma^{\widehat{h}}}{(\gamma^{\widehat{h}}|x_0 - y_0|)^N} \prod_v \frac{1}{S_v!} \gamma^{-h_v(S_v-1)} \prod_{v \in V_X} \gamma^{2h_v S_v^L} \prod_v^* \gamma^{h_v} \\ &\prod_{v \in H} \gamma^{2(h_{v'} - h_v)} \prod_{v \in V_X} \gamma^{-\alpha|P_v|} \end{aligned} \tag{92}$$

where H now includes also resonant terms with one or two J fields. Proceeding as in §7 and for $|x_0 - y_0| > 1$, if \mathcal{T}_n are the trees with n end-points

$$\begin{aligned} \sum_{\tau \in \mathcal{T}_{h,n}} \sum_{\mathbf{P}, T} |G_{\tau, \mathbf{P}, T}(\mathbf{x}, \mathbf{y})| &\leq \gamma^{-3\bar{h}} \frac{C^n |\varepsilon|^n}{1 + (\gamma^{\bar{h}}|x_0 - y_0|)^N} \leq C^n |\varepsilon|^n \frac{|\vec{z}|^{3\tau}}{(|\vec{z}|^{-3\tau}|x_0 - y_0|)^N} \\ &(1 + \frac{n}{|\vec{z}|})^{(N+3)\tau} \end{aligned} \tag{93}$$

The sum over $h \geq \bar{h}$ can be bounded by an an extra $\gamma^{-\bar{h}}$. As $|\vec{z}| \geq 1$ and $n/|\vec{z}| \leq n$; we can sum over n obtaining, remembering the constraint $n \geq |\vec{x} - \vec{y}|$

$$|H_{\mu, \nu}(\mathbf{x}, \mathbf{y})| \leq C \frac{|\vec{z}|^{4\tau}}{(|\vec{z}|^{-3\tau}|x_0 - y_0|)^N} |\varepsilon|^{|\vec{x} - \vec{y}|/4} \tag{94}$$

The analysis of the 2-point function is done in a similar way; there are 2 endpoints associated with the external fields, so with respect to the bound for the effective potential there is an extra factor $\gamma^{-2\bar{h}}$ and an extra $\gamma^{\widehat{h}}$ from the lack of integration; the sum over the scales produces an extra $|\bar{h}|$.

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