Correction To: Clustering in the Three and Four Color Cyclic Particle Systems in One Dimension

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1 Summary

There is an error in the proof of Lemma 4.1. Specifically, there is an ambiguity in the definition of $Z_t(x, y)$, which is such that, depending on how one interprets the ambiguity, the probabilistic interpretation of either $E[\sum_{y \in \mathbb{Z}} Z_t(0, y)]$ or $E[\sum_{y \in \mathbb{Z}} Z_t(y, 0)]$ given afterward is false. Instead of trying to prove the claim of Lemma 4.1—that $p(t) \geq q(t)$ for all $t \geq 0$ —which I suspect may be false, we prove the similar, but asymptotic statement $\limsup_{t \rightarrow \infty} q(t) \leq \inf_{t \geq 0} p(t)$. We then show that this statement can be used in place of $p(t) \geq q(t)$, which is not hard to do. This is done in Section 4 below.

2 Missing Concept: Particle Identity

It appears to me that the reason for which the error was not caught is the absence of a precise notion of particle identity, that is, a criterion that determines whether a particle that exists at time t is the same particle, or a different particle, from one that exists at a time $s \neq t$. As described in Sect. 3 of the paper, edge particles change over time either by movement, or by collisions of various types. It should already be clear how to track a particle over time when it does not move, or when it moves but does not collide, but it is perhaps not immediately clear whether a particle that emerges from a collision is “the same particle” as one of the particles that took part in the collision. For both $\kappa = 3$ and $\kappa = 4$, collisions always involve two particles, and always result in either 0 or 1 particles. Moreover, in every collision there is

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a directed particle (the “attacking particle”) that moves onto a space (the “collision site”) that is already occupied by another particle (the “attacked particle”). Moreover, in cases where a particle remains after the collision, it is always located at the collision site. It seems that the most natural definition is to declare that the remaining particle, when one exists, is the same particle as the attacking particle – of course, in the event of annihilation, both particles are destroyed. In this way, whenever a collision does not result in annihilation of both particles, the attacking particle survives, while the attacked particle is destroyed. This definition ensures that particles can never be created, only destroyed. In addition, the attacking particle may change type. It is also worth noting that since a blockade does not move, it cannot be the attacking particle in a collision, so can never change type – it can either remain where it is, or else be destroyed. We shall represent particle identity by saying that $(e, t) \rightarrow (e', t')$ if there is a particle on edge e at time t that, by either moving or attacking other particles, survives up to time t' and is located at edge e' at time t' .

3 Location and Details of the Error

We now discuss how the lack of a precise notion of particle identity caused the ambiguity that led to a false equality in Lemma 4.1. The lemma is based around the fact that a blockade can only be destroyed by collision from a directed particle, and is built around the function $Z_t(x, y)$, defined as the indicator of the following event: at time t , there is a directed particle at edge $x + 1/2$ and a blockade at edge $y + 1/2$, and the directed particle eventually collides with the blockade. Because the identity of particles after collisions had not been precisely defined, an ambiguity arises as to whether the directed particle is allowed to collide with any other particles before colliding with the blockade. This ambiguity becomes a problem once we examine the following two statements that appear later in the proof:

1. $\sum_{y \in \mathbb{Z}} Z_t(0, y)$ is the indicator that there is a directed particle on the edge $1/2$ at time t that eventually collides with a blockade, and that
2. $\sum_{y \in \mathbb{Z}} Z_t(y, 0)$ is the indicator that there is a blockade on the edge $1/2$ at time t that is eventually removed.

The problem arises from the “reflect” collision, in which a directed particle collides with a blockade, which results in a single directed particle at the collision site. If we assume that the directed particle must not collide with any other particle before colliding with the blockade, then 1 is true, but 2 is false, since a blockade could have been destroyed by a particle that only emerged after time t from a collision. If instead we adopt the above notion of particle identity and allow collisions, then 2 is true but 1 is false, since a single directed particle can potentially remove an arbitrarily large number of blockades.

4 Fixing the Error: Modified Proof

With the error understood, we now give the workaround that fixes it. Recall that $p(t)$ is the density of directed particles, i.e., the (translation-invariant) probability that a given edge is occupied by a directed particle at time t , similarly $q(t)$ is the density of blockades at time t and $r(t) := p(t) + q(t)$ is the total particle density. Lemma 4.1 claimed to prove that $p(t) \geq q(t)$ for all $t \geq 0$. Instead of this, we prove a similar but asymptotic property, namely,

$$\limsup_{t \rightarrow \infty} q(t) \leq \inf_{t \geq 0} p(t), \quad (4.1)$$

then use this together with Lemma 4.2, which says that $t \mapsto r(t)$ is non-increasing and for each $t \geq 0$ there is $s = s(t) \geq t$ such that $r(s) \leq r(t) - p(t)/4$, to prove Theorem 1, namely that $\lim_{t \rightarrow \infty} r(t) = 0$. Note that since there are no blockades when $\kappa = 3$, the error only applies to $\kappa = 4$.

First we use (4.1) and Lemma 4.2 to prove the main result. Let $t_0 = 0$ and recursively let $t_{n+1} = s(t_n)$, where $s(t)$ is obtained from Lemma 4.2 above. Then, as in the previous proof of Theorem 1,

$$r(t_n) \leq r(0) - \frac{1}{4} \sum_{i=0}^{n-1} p(t_i).$$

Let $a_n = \sum_{i=0}^{n-1} p(t_i)$. Then $a_n \leq r(0)$ for each n . Since (a_n) is non-decreasing, it converges. In particular, $p(t_i) \rightarrow 0$ as $i \rightarrow \infty$. Using (4.1), $\lim_{t \rightarrow \infty} q(t) = 0$. In particular, $q(t_i) \rightarrow 0$ as $i \rightarrow \infty$, so $r(t_i) = p(t_i) + q(t_i) \rightarrow 0$ as $i \rightarrow \infty$. Since by Lemma 4.2, $t \mapsto r(t)$ is non-increasing, $\lim_{t \rightarrow \infty} r(t) = 0$.

Now we prove (4.1). Recall from Sect. 2 that if there is a particle at (e', t') and $t < t'$, there is a unique edge e such that $(e, t) \rightarrow (e', t')$. In words, e is the location at time t of the particle that is located at e' at time t' . Following the beginning of the proof of Lemma 1, let $q(t, t')$ be the probability that a given edge is occupied by a blockade on the time interval $[t, t']$. We showed that $\lim_{t' \rightarrow \infty} q(t, t') = 0$ for each $t \geq 0$. Now, let $\tilde{q}(t, t') = q(t') - q(t, t')$, which is the probability that a given edge e' is occupied by a blockade at time t' , and was not occupied by a blockade at some time in the interval $[t, t']$. Based on the definition of particle identity, this means that the particle that is a blockade at (e', t') was not a blockade at some time in the interval $[t, t']$. Since a particle cannot move while it is a blockade, and cannot change type from blockade to a directed particle, this implies that the particle that is a blockade at (e', t') is a directed particle at time t . In other words, $\tilde{q}(t, t')$ is the probability that a given edge e' is occupied by a blockade at time t' , and that the same particle was a directed particle at time t . By appealing either to ergodicity or to a simple version of the mass-transport principle, we can show that $\tilde{q}(t, t') \leq p(t)$ for any $t' > t \geq 0$. We will take the latter approach, since it is more elementary. Once this is shown, the desired result then follows: since $\tilde{q}(t, t') = q(t') - q(t, t')$ and $\lim_{t' \rightarrow \infty} q(t, t') = 0$ for each t ,

$$p(t) \geq \limsup_{t' \rightarrow \infty} \tilde{q}(t, t') = \limsup_{t' \rightarrow \infty} (q(t') - q(t, t')) = \limsup_{t' \rightarrow \infty} q(t').$$

Then, since $t \geq 0$ is arbitrary, $\inf_t p(t) \geq \limsup_{t' \rightarrow \infty} q(t')$ as claimed.

It remains to show that $\tilde{q}(t, t') \leq p(t)$ for $t < t'$. Fixing $t' > t \geq 0$, define $Z(x, y)$ to be the indicator of the event that there is a directed particle at edge $x + 1/2$ at time t , and that the same particle is a blockade at edge $y + 1/2$ at time t' . Since the particle system is translation and reflection invariant in distribution, and using the monotone convergence theorem,

$$\mathbb{E} \left[\sum_y Z(0, y) \right] = \sum_y \mathbb{E}[Z(0, y)] = \sum_y \mathbb{E}[Z(0, -y)] = \sum_y \mathbb{E}[Z(y, 0)] = \mathbb{E} \left[\sum_y Z(y, 0) \right]. \quad (4.2)$$

We now interpret each side of the equation. Note that the interpretation makes implicit use of the fact that, if there is a particle at (e', t') , then there is a unique edge e such that $(e, t) \rightarrow (e', t')$.

1. $\sum_y Z(0, y)$ is the indicator that there is a directed particle at edge $1/2$ at time t that is a blockade (at some edge) at time t' , and

2. $\sum_y Z(y, 0)$ is the indicator that there is a blockade at edge $1/2$ at time t' that was a directed particle at time t .

Using this interpretation, $\mathbb{E}[\sum_y Z(0, y)]$ is the probability of the event in 1, which is at most $p(t)$, and $\mathbb{E}[\sum_y Z(y, 0)]$ is the probability of the event in 2, which is equal to $\tilde{q}(t, t')$. Using this and (4.2),

$$p(t) \geq \tilde{q}(t, t').$$

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