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Renormalizing the Kardar–Parisi–Zhang Equation in $d \geq 3$ in Weak Disorder

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Abstract
We study Kardar–Parisi–Zhang equation in spatial dimension 3 or larger driven by a Gaussian space–time white noise with a small convolution in space. When the noise intensity is small, it is known that the solutions converge to a random limit as the smoothing parameter is turned off. We identify this limit, in the case of general initial conditions ranging from flat to droplet. We provide strong approximations of the solution which obey exactly the limit law. We prove that this limit has sub-Gaussian lower tails, implying existence of all negative (and positive) moments.

Keywords SPDE · Kardar–Parisi–Zhang equation · Stochastic heat equation · Rate of convergence · Edwards–Wilkinson limit · Gaussian free field · Directed polymers · Random environment

Mathematics Subject Classification Primary 60K35 · Secondary 35R60 · 35Q82 · 60H15 · 82D60

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1 Introduction and Main Results

1.1 KPZ Equation and Its Regularization

We consider the *Kardar–Parisi–Zhang* (KPZ) equation written informally as

$$\frac{\partial}{\partial t}h = \frac{1}{2}\Delta h + \Big[\frac{1}{2}|\nabla h|^2 - \infty\Big] + \xi \qquad h\colon \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R} \tag{1.1}$$

and driven by a totally uncorrelated Gaussian space–time white noise ξ. More precisely, ξ on $\mathbb{R}_+ \times \mathbb{R}^d$ is a family $\{\xi(\varphi)\}_{\varphi \in \mathcal{S}(\mathbb{R}_+\times\mathbb{R}^d)}$ of Gaussian random variables

$$\xi(\varphi) = \int_0^\infty \int_{\mathbb{R}^d} \mathrm{d}t\ \mathrm{d}x\ \xi(t,x)\,\varphi(t,x)$$

with mean 0 and covariance

$$\mathbb{E}\big[\xi(\varphi_1)\ \xi(\varphi_2)\big] = \int_0^\infty \int_{\mathbb{R}^d} \varphi_1(t,x)\varphi_2(t,x)\,\mathrm{d}t\,\mathrm{d}x.$$

The Eq. (1.1) describes the evolution of a growing interface in $d+1$ dimension [19,26] and also appears as the scaling limit for $d=1$ of front propagation of the certain exclusion processes ([3,10]) as well as that of the free energy of the discrete directed polymer ([1]). It should be noted that, on a rigorous level, only distribution-valued solutions are expected for (1.1), and thus it is already ill-posed in $d=1$ stemming from the inherent non-linearity of the equation and the fundamental problem of squaring or multiplying random distributions. For $d=1$, studies related to the above equation have enjoyed a huge resurgence of interest in the last decade starting with the important work [16] which gave an intrinsic precise notion of a *solution* to (1.1).

We now fix a spatial dimension $d \geq 3$. As remarked earlier, since (1.1) is a-priori ill-posed, we will study its regularized version

$$\frac{\partial}{\partial t}h_\varepsilon = \frac{1}{2}\Delta h_\varepsilon + \Big[\frac{1}{2}|\nabla h_\varepsilon|^2 - C_\varepsilon\Big] + \beta\varepsilon^{\frac{d-2}{2}}\xi_\varepsilon\ , \qquad h_\varepsilon(0,x) = 0, \tag{1.2}$$

which is driven by the spatially mollified noise

$$\xi_\varepsilon(t,x) = (\xi(t,\cdot)\star\phi_\varepsilon)(x) = \int \phi_\varepsilon(x-y)\xi(t,y)\,\mathrm{d}y,$$

with $\phi_\varepsilon(\cdot) = \varepsilon^{-d}\phi(\cdot/\varepsilon)$ being a suitable approximation of the Dirac measure δ_0 and C_ε being a suitable divergent (renormalization) constant. We will work with a fixed mollifier $\phi : \mathbb{R}^d \to \mathbb{R}_+$ which is smooth and spherically symmetric, with $\mathrm{supp}(\phi) \subset B(0,\frac{1}{2})$ and $\int_{\mathbb{R}^d}\phi(x)\,\mathrm{d}x = 1$. Then, $\{\xi_\varepsilon(t,x)\}$ is a centered Gaussian field with covariance

$$\mathbb{E}[\xi_\varepsilon(t,x)\xi_\varepsilon(s,y)] = \delta(t-s)\,\varepsilon^{-d}V\big((x-y)/\varepsilon\big),$$

where $V = \phi\star\phi$ is a smooth function supported in $B(0,1)$. We also remark that in (1.2), the multiplicative parameter β can be taken to be positive without loss of generality, while by rescaling, no multiplicative parameter is needed in (1.1), see [25]. Also in spatial dimensions $d \geq 3$, the factor $\varepsilon^{\frac{d-2}{2}}$ is the correct scaling—a small enough $\beta > 0$ guarantees a non-trivial random limit of h_ε as $\varepsilon \to 0$, see the discussion in Sect. 1.3.

The goal of the present article is to consider *general solutions* of (1.2), namely the solutions of (1.2) with various initial conditions and prove that as the mollification parameter ε is turned off, the renormalized solution of (1.2) converges to a meaningful random limit as long as β remains small enough. We use Feynman–Kac representation of the solution of stochastic heat equation and results from directed polymers. Not only do we identify the distributional limit of h_ε, but we also provide a sequence (indexed by ε) of functions of the noise such that

- it is a strong approximation of h_ε, i.e. the difference tends to 0 in norm,
- it is a stationary solution of the SPDE in (1.2),
- all terms in the sequence (in ε) have a constant law.

The above functions for the flat initial condition are defined from the martingale limit of a random polymer model taken at some rescaled, shifted and time-reversed version of the noise. The similar approximating functions for other intial conditions can be derived from the martingale limit taken at various version of the noise and the heat kernel. We finally show that it has sub-Gaussian lower tails in this regime, which implies existence of all negative and positive moments of this object. Besides new contributions, we gather and reformulate results which are atomized in the literature, often stated in a primitive form and hidden by necessary technicalities. We end up the introduction with a rather complete account on the state-of-the-art. We now turn to a more precise description of our main results.

1.2 Main Results

In order to state our main results, we will introduce the following notation which will be consistently used throughout the sequel. Recall the definition of the space–time white noise $\xi \in \mathcal{S}'(\mathbb{R} \times \mathbb{R}^d)$ which is a random tempered distribution (defined in all times, including negative ones), and for any $\varphi \in \mathcal{S}(\mathbb{R} \times \mathbb{R}^d)$, $\varepsilon > 0$, $t \in \mathbb{R}$ and $x \in \mathbb{R}^d$,

$$\xi^{(\varepsilon,t,x)}(\varphi) \stackrel{(\text{def})}{=} \varepsilon^{-\frac{d+2}{2}} \int_{\mathbb{R}} \int_{\mathbb{R}^d} \varphi\big(\varepsilon^{-2}(t-s), \varepsilon^{-1}(y-x)\big)\xi(s,y) ds\, dy.$$

Equivalently,

$$\xi^{(\varepsilon,t,x)}(s,y) = \varepsilon^{\frac{d+2}{2}} \xi\left(\varepsilon^2\Big(\frac{t}{\varepsilon^2} - s\Big), \varepsilon\Big(y + \frac{x}{\varepsilon}\Big)\right) \tag{1.3}$$

so that by invariance under space–time diffusive rescaling, time-reversal and spatially translation, $\xi^{(\varepsilon,t,x)}$ is itself a Gaussian white noise and possesses the same law as ξ. This is also the reason why we define the noise above also for negative times. To abbreviate notation, we will also write

$$\xi^{(\varepsilon,t)} = \xi^{(\varepsilon,t,0)}. \tag{1.4}$$

We also need specify the definition(s) of the *critical disorder parameter*. Note that (1.2) is inherently non-linear. The Hopf–Cole transformation suggests that

$$u_\varepsilon = \exp h_\varepsilon \tag{1.5}$$

solves the linear multiplicative noise stochastic heat equation (SHE)

$$\frac{\partial}{\partial t} u_\varepsilon = \frac{1}{2}\Delta u_\varepsilon + \beta \varepsilon^{\frac{d-2}{2}} u_\varepsilon\, \xi_\varepsilon\,, \qquad u_\varepsilon(0,x) = 1, \tag{1.6}$$

provided that the stochastic integral in (1.6) is interpreted in the classical Itô–Skorohod sense and that we choose

$$C_\varepsilon = \beta^2(\phi\star\phi)(0)\varepsilon^{-2}/2 = \beta^2 V(0)\varepsilon^{-2}/2 \tag{1.7}$$

equal to the Itô correction below. Then, the generalized Feynman–Kac formula ([20, Theorem 6.2.5]) provides a solution to (1.6)

$$u_\varepsilon(t,x) = E_x\left[\exp\left\{\beta\varepsilon^{\frac{d-2}{2}} \int_0^t \int_{\mathbb{R}^d} \phi_\varepsilon(W_{t-s}-y)\xi(s,y)\,\mathrm{d}s\,\mathrm{d}y - \frac{\beta^2\, t\, \varepsilon^{-2}}{2}\; V(0)\right\}\right],$$

with E_x denoting expectation with respect to the law P_x of a d-dimensional Brownian path $W=(W_s)_{s\ge 0}$ starting at $x\in\mathbb{R}^d$, which is independent of the noise ξ. Plugging (1.3) in the previous formula, using Brownian scaling and time-reversal, we get the a.s. equality

$$u_\varepsilon(t,x) = \mathscr{Z}_{\frac{t}{\varepsilon^2}}\left(\xi^{(\varepsilon,t)}; \frac{x}{\varepsilon}\right) \tag{1.8}$$

where

$$\mathscr{Z}_T(x) = \mathscr{Z}_T(\xi;x) = E_x\left[\exp\left\{\beta\int_0^T\int_{\mathbb{R}^d}\phi(W_s-y)\xi(s,y)\;\mathrm{d}s\,\mathrm{d}y - \frac{\beta^2\,T}{2}\;V(0)\right\}\right], \tag{1.9}$$

is the *normalized partition function* of the *Brownian directed polymer* in a white noise environment ξ, or equivalently, the total-mass of a *Gaussian multiplicative chaos* in the Wiener space ([24, Sect. 4]).

It follows that there exists $\beta_c\in(0,\infty)$ and a strictly positive non-degenerate random variable $\mathscr{Z}_\infty(x)$ so that, a.s. as $T\to\infty$,

$$\mathscr{Z}_T(x) \to \begin{cases} \mathscr{Z}_\infty(x) & \text{if } \beta\in(0,\beta_c),\\ 0 & \text{if } \beta\in(\beta_c,\infty).\end{cases} \tag{1.10}$$

See [24], or [8] for a general reference. Moreover, $(\mathscr{Z}_T)_{T\ge 0}$ is uniformly integrable for $\beta<\beta_c$, which we will always assume from now on. Now, let $\mathscr{C}^\alpha(\mathbb{R}\times\mathbb{R}^d)$ denote the path-space of the white noise (see Appendix for a precise definition) and

$$\mathfrak{u} = \mathfrak{u}_{\beta,\phi} : \mathscr{C}^\alpha(\mathbb{R}\times\mathbb{R}^d) \to (0,\infty),$$

be any arbitrary representative of the random limit $\mathscr{Z}_\infty = \mathscr{Z}_\infty(0)$; in particular $\mathfrak{u}(\xi)=\mathscr{Z}_\infty$. Then, $\mathbb{E}[\mathfrak{u}]=1$, and throughout the sequel we will write (recall (1.5) and (1.8))

$$\mathfrak{h} = \log\mathfrak{u}\,. \tag{1.11}$$

Since $\mathfrak{u}$ is non constant with $\mathbb{E}\mathfrak{u}=1$, we have $\mathbb{E}\mathfrak{h}<0$.

Finally, we also define another critical disorder parameter:

$$\beta_{L^2} = \sup\left\{\beta>0 : E_0\left[e^{\beta^2\int_0^\infty V(\sqrt{2}W_s)\,\mathrm{d}s}\right]<\infty\right\}$$

which corresponds to the L^2*-region of the polymer model* (see (1.14)). In $d\ge 3$, it is easy to see that for β small enough, $\sup_{x\in\mathbb{R}^d} E_x[\beta\int_0^\infty V(W_s)\,\mathrm{d}s]<1$, so that by Khas'minskii's lemma, $E_0\big[\exp\{\beta\int_0^\infty V(W_s)\,\mathrm{d}s\}\big]<\infty$, so this implies that $\beta_{L^2}>0$. Furthermore, for $\beta<\beta_{L^2}$, convergence (1.10) becomes an L^2-convergence, hence $0<\beta_{L^2}\le\beta_c<\infty$. In fact, it is widely believed that $\beta_{L^2}<\beta_c$, see [8, Remark 5.2 p. 87] for references in the discrete case.

We are now ready to state our main results.

Theorem 1.1 *Assume $d\ge 3$ and recall that $\mathfrak{h}$ is defined in (1.11).*

- *(Flat initial condition). Fix $\beta \in (0, \beta_c)$ and consider the solution h_ε to (1.2) with $h_\varepsilon(0, \cdot) = 0$. Then, for all $t > 0, x \in \mathbb{R}^d$, we have as $\varepsilon \to 0$,*

$$h_\varepsilon(t,x) - \mathfrak{h}\big(\xi^{(\varepsilon,t,x)}\big) \xrightarrow{\mathbb{P}} 0\,.$$

- *(General initial condition). Fix $\beta \in (0, \beta_{L^2})$ and consider the solution h_ε to (1.2) with $h_\varepsilon(0, \cdot) = h_0(\cdot)$ for some $h_0 : \mathbb{R}^d \to \mathbb{R}$ which is continuous and bounded from above. Then, for all $t > 0, x \in \mathbb{R}^d$, we have as $\varepsilon \to 0$,*

$$h_\varepsilon(t,x) - \mathfrak{h}(\xi^{(\varepsilon,t,x)}) - \log \overline{u}(t,x) \xrightarrow{\mathbb{P}} 0\,, \tag{1.12}$$

where

$$\partial_t \overline{u} = \frac{1}{2}\Delta \overline{u}, \qquad \overline{u}(0,x) = \exp h_0(x).$$

- *(Droplet or narrow-wedge initial condition). Fix $\beta \in (0, \beta_{L^2})$ and consider the solution h_ε to (1.2) such that*

$$\lim_{t \searrow 0} \exp h_\varepsilon(t, \cdot) = \delta_{x_0}(\cdot)$$

for some $x_0 \in \mathbb{R}^d$. Then, for all $t > 0, x \in \mathbb{R}^d$, we have as $\varepsilon \to 0$,

$$h_\varepsilon(t,x) - \mathfrak{h}(\xi^{(\varepsilon,t,x)}) - \mathfrak{h}(\xi_{(\varepsilon,x_0)}) - \log \rho(t, x - x_0) \xrightarrow{\mathbb{P}} 0\,, \tag{1.13}$$

where ρ is the d-dimensional Gaussian kernel, and

$$\xi_{(\varepsilon,x_0)}(s,x) = \varepsilon^{\frac{d+2}{2}} \xi(\varepsilon^2 s, x_0 + \varepsilon x)$$

is a space–time Gaussian white noise.

The deterministic terms in (1.12) and (1.13) are logarithms of solutions to heat equation without noise, and one can see that $\mathfrak{h} \to 0$ in L^2 as $\beta \to 0$. This implies that

$$\lim_{\beta \to 0} \lim_{\varepsilon \to 0} h_\varepsilon = \lim_{\varepsilon \to 0} \lim_{\beta \to 0} h_\varepsilon \qquad \text{in distribution.}$$

We obtain an immediate corollary to Theorem 1.1.

Corollary 1.2 *Fix $\beta \in (0, \beta_{L^2})$ and denote by $h_\varepsilon^{(h_0)}$ the solution of (1.2) with initial condition $h_\varepsilon(0, \cdot) = h_0(\cdot)$, where h_0 is continuous and bounded from above. Then for any $x_0 \in \mathbb{R}^d$,*

$$\lim_{e^{h_0} \to \delta_{x_0}} \lim_{\varepsilon \to 0} h_\varepsilon^{(h_0)} \neq \lim_{\varepsilon \to 0} \lim_{e^{h_0} \to \delta_{x_0}} h_\varepsilon^{(h_0)} \qquad \text{in distribution}$$

Our next main result is the following which provides a sub-Gaussian upper tail estimate on the limit $\mathfrak{h}$ defined in (1.11).

Theorem 1.3 *Let $d \geq 3$ and $\beta \in (0, \beta_{L^2})$. Then, there exists a constant $C \in (0, \infty)$ such that for any $\theta > 0$,*

$$\mathbb{P}[\mathfrak{h} \leq -\theta] \leq C e^{-\theta^2/2}.$$

In particular, $\mathfrak{h} \in L^p(\mathbb{P})$ for any $p \in \mathbb{R}$.

From Theorems 1.1 and 1.3, we derive

Corollary 1.4 *In the hypothesis of Corollary 1.2, we have for any $x_0 \in \mathbb{R}^d$,*

$$\lim_{e^{h_0} \to \delta_{x_0}} \lim_{\varepsilon \to 0} \mathbb{E} h_\varepsilon^{(h_0)} - \lim_{\varepsilon \to 0} \lim_{e^{h_0} \to \delta_{x_0}} \mathbb{E} h_\varepsilon^{(h_0)} = -\mathbb{E}\mathfrak{h} > 0\,.$$

1.3 Literature Remarks and Discussion

In the present set up, by finding a non-trivial limit when letting the regularization parameter vanish we have obtained a non-trivial renormalization of KPZ equation (1.1). Let us stress the main specificity of Theorem 1.1. The approximating sequence $(\mathfrak{h}(\xi^{(\varepsilon,t,x)}); \varepsilon > 0)$ combines two interesting properties:

- it is a solution of (1.2) on $\mathbb{R} \times \mathbb{R}^d$ (with null initial condition at time $-\infty$),
- it is constant in law for all (ε, t, x), with law given by the one of $\log \mathscr{Z}_\infty$;
- it approximates $h_\varepsilon(t, x)$ in probability.

(Similar properties hold for the other initial conditions). Since it depends on ε, it is not a (strong) limit, but it can be used similarly. In particular, fluctuations can be studied as shown in [9]. This is quite different from using a deterministic centering, e.g., $\tilde{h}_\varepsilon(t, x) = h_\varepsilon(t, x) - \mathbb{E}h_\varepsilon(t, x)$. As mentioned in [12], $\tilde{h}_\varepsilon$ does not converge to 0 pointwise, but it does as a distribution. Integrating $\tilde{h}_\varepsilon$ in space against test functions cause oscillations to cancel. On the contrary, in our result $h_\varepsilon(t, x) - \mathfrak{h}(\xi^{(\varepsilon,t,x)}) \to 0$ pointwise, and *we do not need any averaging in space.*

The first two claims in Theorem 1.1 can be viewed as counterparts of [13, Theorem 1.1 and Theorem 1.3] dealing with stochastic homogeneization of the stochastic heat equation when the noise is correlated in time. From the last claim in Theorem 1.1—which has no counterpart in [13]—we conclude that the solution with narrow-wedge initial condition performs a “zooming” of the noise around the endpoint (t, x) and the starting point $(0, x_0)$.

We also emphasize that our results concern studying the asymptotic behavior of the solution to the non-linear equation (1.2), and are not restricted to the linear multiplicative noise stochastic heat equation (see (1.6)). Furthermore, the statements of the results concern the solution itself, without need of integrating spatially against test functions. However, note that the limit obtained in Theorem 1.1 *does* depend on the smoothing procedure ϕ as well as on the disorder parameter β and it is not universal (in particular, for $\beta < \beta_{L^2}$, the variance of $\exp(\mathfrak{h})$ can be computed from the RHS of (1.14) for $x = 0$, and it depends on the mollification). Thus the present scenario lies in total contrast with the 1-dimensional spatial case where the limit can be defined by a chaos expansion [1,2] (with the parameter β absorbed by scaling) or via the theory of regularity structures ([17]) which also produces a renormalized limit which does not depend on the mollification scheme.

In [9] we have also investigated the rate of the convergence of $h_\varepsilon \to \mathfrak{h}$ for small enough β. In fact, it is shown there that the finite dimensional distributions of the entire space–time process $\varepsilon^{1-\frac{d}{2}}\big(h_\varepsilon(t, x) - \mathfrak{h}_\varepsilon(t, x)\big)_{t>0, x\in\mathbb{R}^d}$ converge towards that of $\mathscr{H}(t, x) = \gamma(\beta)\int_0^\infty \int_{\mathbb{R}^d} \rho(\sigma + t, x - z)\xi(\sigma, z)\,\mathrm{d}\sigma\,\mathrm{d}z$ where $\rho(t, x)$ is the standard heat kernel and $\gamma(\beta)$ is a positive constant, see below. This limit $\mathscr{H}(t, x)$ is the evolution of a *Gaussian free field* $\mathscr{H}(0, x)$ under the *heat flow*, i.e., the limit $\mathscr{H}$ itself is a pointwise solution of the heat equation $\partial_t \mathscr{H} = \frac{1}{2}\Delta\mathscr{H}$ with a random initial condition $\mathscr{H}(0, x)$ which is a Gaussian free field (GFF). In total constrast to this scenario, for larger β, the so-called *KPZ regime* is expected to take place with different limits, different scaling exponents and non-Gaussian limiting distributions. In particular, the variance in the above Gaussian distribution is given by

$$\gamma^2(\beta) = \beta^2 \int_{\mathbb{R}^d} \mathrm{d}y\; V(y)\; E_y\Big[\mathrm{e}^{\beta^2 \int_0^\infty V(W_{2s})\,\mathrm{d}s}\Big]$$

which already diverges for $\beta > \beta_{L^2}$ indicating that the amplitude of the fluctuations, or at least their distributional nature, changes at this point. However the KPZ regime is not expected before the critical value β_c. Hence this region $\beta \in (\beta_{L^2}, \beta_c)$ remains mysterious.

Finally, we remark on the correlation structure of the limit $\mathfrak{u}$ which was computed in [9]. It was shown that, for β small enough,

$$\mathrm{Cov}\big(\mathscr{Z}_\infty(0), \mathscr{Z}_\infty(x)\big) = \begin{cases} E_{x/\sqrt{2}}\Big[\mathrm{e}^{\beta^2 \int_0^\infty V(\sqrt{2}W_s)ds} - 1\Big] & \forall x \in \mathbb{R}^d, \\ \mathfrak{C}_1\Big(\frac{1}{|x|}\Big)^{d-2} & \forall\, |x| \geq 1, \end{cases} \tag{1.14}$$

with $\mathfrak{C}_1 = E_{\mathbf{e_1}/\sqrt{2}}[\exp\{\beta^2 \int_0^\infty V(\sqrt{2}W_s)ds\} - 1]$. The above correlation structure also underlines that solution $u_\varepsilon(t,x)$ and $u_\varepsilon(t,y)$ become asymptotically independent so that the spatial averages $\int_{\mathbb{R}^d} f(x)\, u_\varepsilon(t,x)\,\mathrm{d}x \to \int f(x)\overline{u}(t,x)\,\mathrm{d}x$ become deterministic and $\overline{u}$ solves the unperturbed heat equation $\partial_t \overline{u} = \frac{1}{2}\Delta\overline{u}$. As remarked earlier, the spatially averaged fluctuations $\varepsilon^{1-\frac{d}{2}} \int_{\mathbb{R}^d} f(x)[u_\varepsilon(t,x) - \overline{u}(t,x)]\,\mathrm{d}x$ were shown to converge ([12,15,21]) to the averages of the heat equation with additive space–time white noise with variance given by (a constant multiple of) $\sigma^2(\beta)$, which also underlines the Edwards–Wilkinson regime in weak disorder. For averaged fluctuations of similar nature in $d = 2$ we refer to [4,7,14].

2 Proof of Theorem 1.1

We now consider the regularized KPZ equation (1.6) as before, but with different initial data and identify the limit of the solution up to leading order. For notational brevity, we will write

$$\Phi_T = \Phi_T(\xi; W) = \exp\left\{\beta \int_0^T \int_{\mathbb{R}^d} \phi(W_s - y)\xi(s,y)\,\mathrm{d}s\,\mathrm{d}y - \frac{\beta^2 T}{2} V(0)\right\} \tag{2.1}$$

where $V = \phi\star\phi$ so that $\mathscr{Z}_T(x) = \mathscr{Z}_T(\xi; x) = E_x[\Phi_T]$ and $\mathbb{E}[\mathscr{Z}_T] = 1$.

We also remind the reader that u_ε solves (1.6) and $h_\varepsilon = \exp[u_\varepsilon]$ solves (1.2) with $C_\varepsilon = \frac{\beta^2\varepsilon^{-2}}{2} V(0)$. Finally, recall that $u_\varepsilon(t,x) = \mathscr{Z}_{\frac{t}{\varepsilon^2}}\big(\xi^{(\varepsilon,t)}; \frac{x}{\varepsilon}\big)$ with $\xi^{(\varepsilon,t)}$ given by (1.3) and (1.4). The proof of the first claim in Theorem 1.1 is plain. Using (1.8) and $\xi^{(\varepsilon,t,x)} = \xi$ in law,

$$\begin{aligned} \|u_\varepsilon(t,x) - \mathfrak{u}(\xi^{(\varepsilon,t,x)})\|_1 &= \|\mathscr{Z}_{\frac{t}{\varepsilon^2}}\big(\xi^{(\varepsilon,t,x)}\big) - \mathfrak{u}(\xi^{(\varepsilon,t,x)})\|_1 \\ &= \|\mathscr{Z}_{\frac{t}{\varepsilon^2}}(\xi) - \mathfrak{u}(\xi)\|_1 \\ &\longrightarrow 0 \end{aligned}$$

as $\varepsilon \to 0$ by (1.10) and uniform integrability if $\beta < \beta_c$. Hence,

$$\begin{aligned} h_\varepsilon(t,x) - \mathfrak{h}\big(\xi^{(\varepsilon,t,x)}\big) &= \log\left(1 + \frac{u_\varepsilon(t,x) - \mathfrak{u}(\xi^{(\varepsilon,t,x)})}{\mathfrak{u}(\xi^{(\varepsilon,t,x)})}\right) \\ &\longrightarrow 0 \end{aligned}$$

in probability as $\varepsilon \to 0$. This yields the first claim.

2.1 General Initial Condition: Proof of (1.12)

Fix continuous functions $u_0 : \mathbb{R}^d \to (0, +\infty)$ and $h_0 : \mathbb{R}^d \to \mathbb{R}$ which are bounded from above, consider the solution of SHE

$$\frac{\partial}{\partial t} u_\varepsilon = \frac{1}{2}\Delta u_\varepsilon + \beta \varepsilon^{\frac{d-2}{2}} u_\varepsilon \, \xi_\varepsilon \ , \qquad u_\varepsilon(0, x) = u_0(x) \ , \tag{2.2}$$

or, equivalently by the relations $u_\varepsilon = \exp h_\varepsilon$ and $u_0 = \exp h_0$, the solution of KPZ

$$\frac{\partial}{\partial t} h_\varepsilon = \frac{1}{2}\Delta h_\varepsilon + \left[\frac{1}{2}|\nabla h_\varepsilon|^2 - C_\varepsilon\right] + \beta \varepsilon^{\frac{d-2}{2}} \xi_\varepsilon \ , \qquad h_\varepsilon(0, x) = h_0(x), \tag{2.3}$$

As before, we have the Feynman–Kac representation

$$u_\varepsilon(t, x) = E_{x/\varepsilon}\left[u_0(\varepsilon W_{\varepsilon^{-2}t})\, \Phi_{\varepsilon^{-2}t}(\xi^{(\varepsilon,t)}; W)\right] \tag{2.4}$$

with $\xi^{(\varepsilon,t)}$ as above.

Lemma 2.1 *For $\beta \in (0, \beta_{L^2})$,*

$$E_{x/\varepsilon}\left[u_0(\varepsilon W_{\varepsilon^{-2}t})\Phi_{\varepsilon^{-2}t}(\xi; W)\right] - \mathfrak{u}(\xi \circ \theta_{x/\varepsilon})\, \overline{u}(t, x) \xrightarrow{L^2} 0,$$

where θ_x denotes the canonical spatial translation in the path space $\mathscr{C}^\alpha$ of the white noise and $\overline{u}$ solves $\partial_t \overline{u} = \frac{1}{2}\Delta \overline{u}$ with $\overline{u}(0, \cdot) = u_0(\cdot)$.

Proof Note that

$$E_0[u_0(x + \varepsilon W_{\varepsilon^{-2}t})] = E_x[u_0(W_t)] = \overline{u}(t, x)$$

Then

$$\mathbb{E}\left[\left(E_{x/\varepsilon}\left[u_0(\varepsilon W_{\varepsilon^{-2}t})\Phi_{\varepsilon^{-2}t}(\xi; W)\right] - \overline{u}(t, x) E_{x/\varepsilon}\left[\Phi_{\varepsilon^{-2}t}(\xi; W)\right]\right)^2\right]$$
$$= E_0^{\otimes 2}\left[e^{\beta^2 \int_0^{\varepsilon^{-2}t} V\left(W_s^{(1)} - W_s^{(2)}\right) \mathrm{d}s}\left(u_0\left(x + \varepsilon W^{(1)}_{\varepsilon^{-2}t}\right) - \overline{u}(t, x)\right)\left(u_0\left(x + \varepsilon W^{(2)}_{\varepsilon^{-2}t}\right) - \overline{u}(t, x)\right)\right]. \tag{2.5}$$

Furthermore,

$$\left(\int_0^{\varepsilon^{-2}t} V(W_s^{(1)} - W_s^{(2)})\, \mathrm{d}s, \varepsilon W^{(1)}_{\varepsilon^{-2}t}, \varepsilon W^{(2)}_{\varepsilon^{-2}t}\right) \xrightarrow{\text{law}} \left(\int_0^\infty V(W_s^{(1)} - W_s^{(2)})\, \mathrm{d}s, Z_t^{(1)}, Z_t^{(2)}\right),$$

where the right hand side is a triplet of three independent random variables, with $Z_t^{(1)}$ and $Z_t^{(2)}$ distributed as W_t. Hence, expectation (2.5) vanishes as $\varepsilon \to 0$, provided that u_0 is bounded and continuous, and because of uniform integrability which is implied by

$$E_0^{\otimes 2}\left[\exp\left\{(1 + \delta)\beta^2 \int_0^\infty V\left(W_s^{(1)} - W_s^{(2)}\right)\, \mathrm{d}s\right\}\right] < \infty, \tag{2.6}$$

for $\beta < \beta_{L^2}$ and $\delta > 0$ small enough. The proof is concluded by the observation that $E_{x/\varepsilon}\left[\Phi_{\varepsilon^{-2}t}(\xi; W)\right] - \mathfrak{u}(\xi \circ \theta_{x/\varepsilon}) \xrightarrow{L^2} 0$ and since the law of the left member does not depend on x, we can set $x = 0$. □

We now end the

Proof of (1.12) For $\beta < \beta_{L^2}$, for all t, x, as $\varepsilon \to 0$, we first show

$$u_\varepsilon(t,x) - \mathfrak{u}\big(\xi^{(\varepsilon,t,x)}\big)\,\overline{u}(t,x) \xrightarrow{L^2} 0\,. \tag{2.7}$$

Note that (2.7) follows directly from Lemma 2.1 and (2.4). Then, since $\mathfrak{u} > 0$, taking logarithm we deduce the convergence in probability (1.12). □

Remark 2.2 Recall that in [24] it was shown that, for any smooth function f with compact support, $\int_{\mathbb{R}^d} u_\varepsilon(t,x) f(x)\, \mathrm{d}x \to \int_{\mathbb{R}^d} \overline{u}(t,x) f(x)\, \mathrm{d}x$. Note that unlike the latter statement, no smoothing in space is needed in the present context. In fact, we can recover the spatially averaged statement from above. Indeed, fast decorrelation in space of $\xi^{(\varepsilon,t,x)}$ as $\varepsilon \to 0$, ergodicity and smoothness justify the equivalence below:

$$\begin{aligned}\int u_\varepsilon(t,x) f(x) dx &\overset{(2.7)}{=} \int \mathfrak{u}(\xi^{(\varepsilon,t,x)})\overline{u}(t,x) f(x) dx + o(1)\\ &\sim \mathbb{E}[\mathfrak{u}(\xi^{(\varepsilon,t,x)})] \int \overline{u}(t,x) f(x)\, \mathrm{d}x\\ &= \int \overline{u}(t,x) f(x)\, \mathrm{d}x.\end{aligned}$$

□

2.2 Narrow-Wedge Initial Condition: Proof of (1.13)

Fix $x_0 \in \mathbb{R}^d$, and consider the solution of SHE

$$\frac{\partial}{\partial t} u_\varepsilon = \frac{1}{2}\Delta u_\varepsilon + \beta \varepsilon^{\frac{d-2}{2}} u_\varepsilon\, \xi_\varepsilon\,, \qquad \lim_{t\searrow 0} u_\varepsilon(t,\cdot) = \delta_{x_0}(\cdot)\,, \tag{2.8}$$

or, equivalently by the relation $u_\varepsilon = \exp h_\varepsilon$, the solution of KPZ

$$\frac{\partial}{\partial t} h_\varepsilon = \frac{1}{2}\Delta h_\varepsilon + \left[\frac{1}{2}|\nabla h_\varepsilon|^2 - C_\varepsilon\right] + \beta \varepsilon^{\frac{d-2}{2}} \xi_\varepsilon\,, \quad \lim_{t\searrow 0} \exp h_\varepsilon(t,\cdot) = \delta_{x_0}(\cdot)\,. \tag{2.9}$$

By Feynman–Kac formula, the solution of SHE now admits a Brownian bridge representation:

$$\begin{aligned} &u_\varepsilon(t,x)\\ &= \rho(t, x-x_0)\, E_{0,\varepsilon^{-1}x_0}^{\varepsilon^{-2}t,\varepsilon^{-1}x}\left[\exp\left\{\beta \int_0^{\varepsilon^{-2}t}\int_{\mathbb{R}^d} \phi(W_s - y)\xi_{(\varepsilon,0)}(s,y)\, \mathrm{d}s\, \mathrm{d}y - \frac{\beta^2\, t}{2\,\varepsilon^2}\; V(0)\right\}\right] \quad (2.10)\\ &= \rho(t,x-x_0) E_{0,0}^{\varepsilon^{-2}t,\varepsilon^{-1}(x-x_0)}\left[\exp\left\{\beta \int_0^{\varepsilon^{-2}t}\int_{\mathbb{R}^d} \phi(W_s - y)\xi_{(\varepsilon,x_0)}(s,y)\, \mathrm{d}s\, \mathrm{d}y - \frac{\beta^2\, t}{2\,\varepsilon^2}\; V(0)\right\}\right] \quad (2.11)\end{aligned}$$

where $E_{0,x}^{t,y}$ denotes expectation with respect to a Brownian bridge starting at x and conditioned to be found at y at time t, ρ is the d-dimensional Gaussian kernel and

$$\xi_{(\varepsilon,x_0)}(S,Y) = \varepsilon^{(d+2)/2}\xi(\varepsilon^2 S, x_0 + \varepsilon Y) \tag{2.12}$$

so that we again have $\xi_{(\varepsilon,x_0)} \overset{\text{law}}{=} \xi$.

The following Lemma follows the approach for proving the local limit theorem as in [27,29].

Lemma 2.3 *For $\beta \in (0, \beta_{L^2})$, for any $A > 0$,*

$$\sup_{|x|\le A} \left\| E_{0,0}^{\varepsilon^{-2}t,\varepsilon^{-1}x}[\Phi_{\varepsilon^{-2}t}(\xi,\cdot)] - \mathfrak{u}(\xi)\, \mathfrak{u}\big(\xi^{(1,\varepsilon^{-2}t,\varepsilon^{-1}x)}\big) \right\|_{L^1(\mathbb{P})} \to 0. \tag{2.13}$$

Proof We will write $X = \varepsilon^{-1}x$, $T = \varepsilon^{-2}t$ and let $m = m_\varepsilon$ be a time parameter, such that $m_\varepsilon \to \infty$ and $m_\varepsilon = o(T)$, as $\varepsilon \to 0$. We use the notation:

$$\Phi_{S,T}(\xi; W) := \exp\left\{\beta \int_S^T \int_{\mathbb{R}^d} \phi(W_s - y)\xi(s, y)\, \mathrm{d}s\, \mathrm{d}y - \frac{\beta^2 (T-S)}{2}\, V(0)\right\}.$$

Step 1 We first want to approximate $E_{0,0}^{T,X}[\Phi_T]$ by $E_{0,0}^{T,X}\left[\Phi_m \Phi_{T-m,T}\right]$ in L^2-norm, so we compute the difference:

$$\mathbb{E}\left[\left(E_{0,0}^{T,X}[\Phi_T - \Phi_m\Phi_{T-m,T}]\right)^2\right]$$
$$= E_{0,0}^{T,0}\left[\mathrm{e}^{\beta^2\int_0^T V(\sqrt{2}W_s)ds} - \mathrm{e}^{\beta^2\int_0^m V(\sqrt{2}W_s)ds}\mathrm{e}^{\beta^2\int_{T-m}^T V(\sqrt{2}W_s)ds}\right].$$

To show that the right hand side goes to 0 as $\varepsilon \to 0$, it suffices to observe that, for all $a > 0$,

$$\lim_{\varepsilon\to 0} E_{0,0}^{T,0}\left[\mathrm{e}^{\beta^2\int_0^T V(\sqrt{2}W_s)\,\mathrm{d}s}\ \mathbf{1}\left\{\int_m^{T-m} V(\sqrt{2}W_s)\ \mathrm{d}s > a\right\}\right] = 0.$$

To prove this, we use Hölder's inequality similarly to (2.6), and use [29, Corollary 3.8] (with a slight modification of the proof to adapt it to the case of a single Brownian bridge, as done in the discrete case in [11, Proposition 3.2]) and transience of Brownian motion for $d \ge 3$, which implies, since $m \to \infty$, that

$$\lim_{\varepsilon\to 0} E_{0,0}^{T,0}\left[\mathbf{1}\left\{\int_m^{T-m} V(\sqrt{2}W_s)\ \mathrm{d}s > a\right\}\right] = 0.$$

Step 2 We wish to use Markov property and symmetry of the Brownian bridge to show that $E_{0,0}^{T,X}\left[\Phi_m\Phi_{T-m,T}\right]$ factorizes asymptotically into the product $E_0\left[\Phi_m\right] E_0[\Phi_m(\xi^{(1,T,X)})]$, which satisfies:

$$\sup_{x\in\mathbb{R}} \left\| E_0\left[\Phi_m\right] E_0[\Phi_m(\xi^{(1,T,X)})] - \mathfrak{u}(\xi)\ \mathfrak{u}(\xi^{(1,\varepsilon^{-2}t,\varepsilon^{-1}x)})\right\|_1 \to 0,$$

as $\varepsilon \to 0$ for $\beta < \beta_{L^2}$, by Cauchy–Schwarz inequality and invariance in law of the white noise with a shift by X. Hence, we compute

$$E_{0,0}^{T,X}\left[\Phi_m\Phi_{T-m,T}\right] = \int_{\mathbb{R}^d} \frac{\rho(T/2, Y)\rho(T/2, X-Y)}{\rho(T,X)} E_{0,0}^{T/2,Y}\left[\Phi_m\right] E_{T/2,Y}^{T,X}\left[\Phi_{T-m,T}\right] \mathrm{d}Y. \tag{2.14}$$

After change of variable by setting $Y = \sqrt{T}y$ in the above integral and since $\rho(Ts, \sqrt{T}z) = T^{-d/2}\rho(s,z)$, observe that by Jensen's inequality and dominated convergence, we can prove that

$$\sup_{|x|\le A} \left\| E_{0,0}^{T,X}\left[\Phi_m\Phi_{T-m,T}\right] - E_0\left[\Phi_m\right] E_0[\Phi_m(\xi^{(1,T,X)})]\right\|_1 \to 0,$$

if we can show that for all fixed $y \in \mathbb{R}^d$,

$$\sup_{|x|\le A} \left\| E_{0,0}^{T/2,\sqrt{T}(y-x)}\left[\Phi_m\right] - E_0\left[\Phi_m\right]\right\|_1 \to 0.$$

To prove this, we use the density of the Brownian bridge, at truncated time horizon, with respect to the Brownian motion ([9, Lemma 4.5]), to get that:

$$E_{0,0}^{T/2,\sqrt{T}(y-x)}[\Phi_m] - E_0[\Phi_m] = E_0\left[\Phi_m\left(\frac{\rho\big(T/2-m,\sqrt{T}(y-x)-W_m\big)}{\rho\big(T/2,\sqrt{T}(y-x)\big)} - 1\right)\right].$$

After rescaling, the difference inside the parenthesis goes almost surely to 0, for y fixed and uniformly in $|x| \leq A$; we conclude the proof of the lemma using Hölder's inequality. □

We can now conclude the

Proof of (1.13) Let h_ε be the narrow-wedge height function solution of (2.9). We need to show that for $\beta < \beta_{L^2}$, for all t, x, as $\varepsilon \to 0$,

$$h_\varepsilon(t,x) - \mathfrak{h}(\xi_{(\varepsilon,x_0)}) - \mathfrak{h}(\xi^{(\varepsilon,t,x)}) - \log\rho(t, x-x_0) \xrightarrow{\mathbb{P}} 0,$$

with $\xi_{(\varepsilon,x_0)}$ in (2.12). We use the representation (2.11) and the property $\xi_{(\varepsilon,x_0)} \overset{\text{law}}{=} \xi$, so that we can exchange ξ with $\xi_{(\varepsilon,x_0)}$, in convergence (2.13) taken with endpoint $\varepsilon^{-1}(x-x_0)$. This leads to the above convergence in probability for the logarithm, proving (1.13). □

3 Proof of Theorem 1.3

We focus on showing that for $\beta < \beta_{L^2}$, $\log \mathscr{Z}_\infty$ admits a sub-Gaussian lower tails estimate, that is, for some $C \in (0,\infty)$ and any $\theta > 0$,

$$\mathbb{P}[\log \mathscr{Z}_\infty \leq -\theta] \leq C e^{-\theta^2/2}. \tag{3.1}$$

We invoke a second moment method combined with the Talagrand's concentration inequality as in [5] (see also [22, Sect. 2.2]). As the Cauchy–Schwarz inequality, which is a central tool in this proof is not directly available in the continuous setting, we choose to introduce a discretization of the white noise to recover it.

Consider $\mathcal{R}_n$ a tiling of $[0,2^n] \times [-2^n,2^n]^d$, composed of cubes of length 2^{-n}, such that every cube of $\mathcal{R}_n$ can be divided in 2^{d+1} cubes of $\mathcal{R}_{n+1}$. We define a discrete version of $\mathscr{Z}_T$ through:

$$\mathscr{Z}_T^{(n)} = E\left[\exp\left\{\beta\int_0^T\int_{\mathbb{R}^d}\phi_W^{(n)}(s,y)\,\xi(s,y)\mathrm{d}s\mathrm{d}y - \frac{\beta^2}{2}\left\|\phi_W^{(n)}\right\|^2_{L^2([0,T]\times\mathbb{R}^d)}\right\}\right],$$

where,

- $\phi_W(s,y) := \phi(W_s - y)$,
- $\phi_W^{(n)}(s,y) = \inf_R \phi_W$ if (s,y) is in a cube R of $\mathcal{R}_n$, and 0 otherwise.

We stress that $\phi_W^{(n)}$ is non-decreasing with n and converges almost surely to ϕ_W.

Using the Gaussian covariance structure, we have that

$$\mathbb{E}\left[\left(\mathscr{Z}_T - \mathscr{Z}_T^{(n)}\right)^2\right] = E^{\otimes 2}\left[e^{\frac{\beta^2}{2}\int_{[0,T]\times\mathbb{R}^d}\phi_{W^{(1)}}\phi_{W^{(2)}}(s,y)\mathrm{d}s\mathrm{d}y}\right]$$
$$- 2E^{\otimes 2}\left[e^{\frac{\beta^2}{2}\int_{[0,T]\times\mathbb{R}^d}\phi^{(n)}_{W^{(1)}}\phi_{W^{(2)}}(s,y)\mathrm{d}s\mathrm{d}y}\right] + E^{\otimes 2}\left[e^{\frac{\beta^2}{2}\int_{[0,T]\times\mathbb{R}^d}\phi^{(n)}_{W^{(1)}}\phi^{(n)}_{W^{(2)}}(s,y)\mathrm{d}s\mathrm{d}y}\right].$$

For $\beta < \beta_{L^2}$ and since $\phi_W^{(n)} \leq \phi_W$, we immediatly obtain from the monotone convergence theorem that the right-hand side goes to zero in the limit $T \to \infty$ followed by $n \to \infty$. By Doob's L^2 inequality applied to the martingale $\mathscr{Z} - \mathscr{Z}^{(n)}$, this implies in particular that

$$\lim_{n\to\infty} \mathbb{E}\left[\left(\mathscr{Z}_\infty - \mathscr{Z}_\infty^{(n)}\right)^2\right] = 0. \tag{3.2}$$

Since $\phi_W^{(n)}$ is set to be 0 outside of $[0, 2^n] \times [-2^n, 2^n]^d$, the set $\mathcal{C}_n$ containing the centers of the cubes of $\mathcal{R}_n$ is finite. Hence, as each cube has volume $2^{-(d+1)n}$, we can write that

$$\int_0^\infty \int_{\mathbb{R}^d} \phi_W^{(n)}(s, y)\, \xi(s, y) \mathrm{d}s \mathrm{d}y = \frac{1}{2^{\frac{(d+1)n}{2}}} \sum_{(i,x)\in\mathcal{C}_n} \phi_W^{(n)}(i, x)\, \xi_n(i, x), \tag{3.3}$$

where the $\xi_n(i, x)$ are independent centered Gaussian random variables of variance 1. Then, we define the polymer measure $\widehat{P}_{\beta,\xi_n}$ of renormalized partition function $\mathscr{Z}_\infty^{(n)}$:

$$\widehat{P}_{\beta,\xi_n}(\mathrm{d}W) = \frac{1}{\mathscr{Z}_\infty^{(n)}}\, \Phi^{(n)}(W) P(\mathrm{d}W),$$

where we have set, using (3.3),

$$\Phi^{(n)}(W) = \exp\left\{ \frac{\beta}{2^{\frac{(d+1)n}{2}}} \sum_{(i,x)\in\mathcal{C}_n} \phi_W^{(n)}(i, x)\, \xi_n(i, x) - \frac{\beta^2}{2} \left\|\phi_W^{(n)}\right\|^2_{L^2([0,\infty]\times\mathbb{R}^d)} \right\}.$$

Finally, we let $\widehat{E}_{\beta,\xi_n}$ denote expectation corresponding to $\widehat{P}_{\beta,\xi_n}$.

Now, we can compare the free energies of two realizations of the noise $\xi_n(i, x)$ and $\xi_n'(i, x)$:

$$\begin{aligned}
&\log \mathscr{Z}_\infty^{(n)}(\xi_n) - \log \mathscr{Z}_\infty^{(n)}(\xi_n') \\
&\quad = \log \widehat{E}_{\beta,\xi_n'}\left[\exp\left\{ \frac{\beta}{2^{\frac{(d+1)n}{2}}} \sum_{(i,x)\in\mathcal{C}_n} \phi_W^{(n)}(i, x)\, \left(\xi_n(i, x) - \xi_n'(i, x)\right)\right\}\right] \\
&\quad \geq \beta \sum_{(i,x)\in\mathcal{C}_n} 2^{-\frac{(d+1)n}{2}}\, \widehat{E}_{\beta,\xi_n'}\left[\phi_W^{(n)}(i, x)\right] \left(\xi_n(i, x) - \xi_n'(i, x)\right) \\
&\quad \geq -\beta \sqrt{\widehat{E}^{\otimes 2}_{\beta,\xi_n'}\left[\int_0^\infty \int_{\mathbb{R}^d} \phi^{(n)}_{W^{(1)}} \phi^{(n)}_{W^{(2)}}(s, y) \mathrm{d}s \mathrm{d}y\right]}\, \mathrm{d}(\xi_n, \xi_n'),
\end{aligned}$$

where $\mathrm{d}(\cdot, \cdot)$ denotes the euclidean distance on $\mathbb{R}^{\mathrm{Card}(\mathcal{C}_n)}$, and where we used Jensen's and Cauchy–Schwarz inequalities for respectively the first and second lower bounds.

Let m and C be two positive constants, and consider the set:

$$\mathcal{A}_n = \left\{ \xi_n : \mathscr{Z}_\infty^{(n)}(\xi_n) \geq m,\ \widehat{E}^{\otimes 2}_{\beta,\xi_n}\left[\int_0^\infty \int_{\mathbb{R}^d} \phi^{(n)}_{W^{(1)}} \phi^{(n)}_{W^{(2)}}(s, y) \mathrm{d}s \mathrm{d}y\right] \leq C^2 \right\}.$$

For $\xi_n' \in \mathcal{A}_n$, the above computation implies that

$$\log \mathscr{Z}_\infty^{(n)}(\xi_n) \geq \log m - \beta C \mathrm{d}\left(\xi_n, \mathcal{A}_n\right), \tag{3.4}$$

therefore, assuming

$$\liminf_{n\to\infty} \mathbb{P}(\mathcal{A}_n) > 0, \tag{3.5}$$

property (3.1) results from (3.4), (3.2) and the following Gaussian concentration inequality (Lemma 4.1 in [5], extracted from [28]):

$$\mathbb{P}\left(\mathrm{d}(\xi_n, \mathcal{A}_n) > u + \sqrt{2\log\frac{1}{\mathbb{P}(\mathcal{A}_n)}}\right) \leq e^{-\frac{u^2}{2}}. \tag{3.6}$$

We now prove (3.5). By convergence (3.2), and since $\mathscr{Z}_\infty > 0$ a.s., we can find $m > 0$, such that for n large enough,

$$\mathbb{P}\left(\mathscr{Z}_\infty^{(n)} > m\right) \geq \frac{1}{2}.$$

Then, for a large enough C,

$$\begin{aligned}\mathbb{P}(\mathcal{A}_n) &\geq \mathbb{P}\left(\mathscr{Z}_\infty^{(n)} \geq m,\ E^{\otimes 2}\left[\int_0^\infty \int_{\mathbb{R}^d} \phi^{(n)}_{W^{(1)}}\phi^{(n)}_{W^{(2)}}(s,y)\mathrm{d}s\mathrm{d}y\, e^{\beta\Phi^{(n)}(W^{(1)})+\beta\Phi^{(n)}(W^{(2)})}\right] \leq mC^2\right)\\ &\geq \mathbb{P}\left(\mathscr{Z}_\infty^{(n)} \geq m\right) - \mathbb{P}\left(E^{\otimes 2}\left[\int_0^\infty \int_{\mathbb{R}^d} \phi^{(n)}_{W^{(1)}}\phi^{(n)}_{W^{(2)}}(s,y)\mathrm{d}s\mathrm{d}y\, e^{\beta\Phi^{(n)}(W^{(1)})+\beta\Phi^{(n)}(W^{(2)})}\right] > mC^2\right) s\\ &\geq \frac{1}{2} - \frac{1}{mC^2}E^{\otimes 2}\left[\int_0^\infty V(W_s^{(1)} - W_s^{(2)})\,\mathrm{d}s\, \mathrm{e}^{\beta^2\int_0^\infty V(W_s^{(1)}-W_s^{(2)})\,\mathrm{d}s}\right] > 0.\end{aligned}$$

In the above display, the first lower bound follows from the definition of $\mathcal{A}$, while we used $\mathbb{P}[A \cap B] \geq \mathbb{P}[A] - \mathbb{P}[B^c]$ in the second lower bound. The third inequality comes from Markov's inequality and the upper-bound $\phi_W^{(n)} \leq \phi_W$. Positivity of the left hand-side of the third line is assured for C large enough, provided that $\beta < \beta_{L^2}$.

This entails that the KPZ limit $|\mathfrak{h}|$ (recall (1.11)) has all positive and negative moments for all $\beta < \beta_{L^2}$. Indeed, letting $\log_- = \log \wedge 0$, the sub-Gaussian decay of the left tail of $\log \mathscr{Z}_\infty$ (3.1) gives $\mathbb{E}[\exp\{\nu \log_- \mathscr{Z}_\infty\}] < \infty$, for all $\nu \in \mathbb{R}$. Moreover, by definition of the L^2 region, we have $\mathbb{E}[\exp\{2\log \mathscr{Z}_\infty\}] < \infty$. Hence, $\log \mathscr{Z}_\infty$ admits all positive and negative moments. □

Remark 3.1 After finishing the writing of the present article, we learned that another proof of the negative moments of $\mathscr{Z}_\infty$ has been recently proposed in [12,18] using a continuous approximation of the white noise. A proof of the corresponding result for the KPZ equation in dimension 2, which relies on the convexity of the free energy and the Malliavin derivative, can be found in [4].

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Compliance with Ethical Standards

Conflict of interest The authors declare that they have no conflict of interest.

Appendix A: The State Space of the Noise

We will include some elementary facts regarding the regularity properties of space–time white noise ξ. For any $z, z' \in \mathbb{R} \times \mathbb{R}^d$, we will denote by $\|\cdot\|$ the *parabolic distance* given by $\|z - z'\| = |t - t'|^{1/2} + \sum_{i=1}^d |x_j - x_j'|$, where $z = (t, x)$ and $z' = (t', x')$. Recall that the Hölder space of positive exponent $\alpha \in (0, 1)$ consists of all functions $u : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ such that for any compact set $K \subset \mathbb{R} \times \mathbb{R}^d$,

$$\sup_{z,z' \in K, z \neq z'} \frac{|u(z) - u(z')|}{\|z - z'\|^\alpha} < \infty.$$

The corresponding Hölder (Besov) space of *negative regularity* is defined as follows. First for any $k \in \mathbb{N}$, let B_k denote the space of all smooth functions $\varphi \colon \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ which are supported on the unit ball in $(\mathbb{R} \times \mathbb{R}^d, \|\cdot\|)$ such that

$$\|\varphi\|_{B_k} \stackrel{\text{(def)}}{=} \sup_{\beta : |\beta| \leq k} \sup_{z \in \mathbb{R} \times \mathbb{R}^d} |D^\beta \varphi(z)| \leq 1.$$

Then for any fixed $\alpha < 0$, we define the space $\mathscr{C}^\alpha$ to be the space of all tempered distributions $\eta \in \mathcal{S}'(\mathbb{R} \times \mathbb{R}^d)$ such that for any compact set $K \subset \mathbb{R} \times \mathbb{R}^d$,

$$\|\eta\|_{\mathscr{C}^\alpha(K)} \stackrel{\text{(def)}}{=} \sup_{z \in K} \sup_{\substack{u \in B_k \\ \lambda \in (0,1]}} \left| \frac{\langle \eta, \Theta_z^\lambda u \rangle}{\lambda^\alpha} \right| < \infty,$$

where $k = \lceil -\alpha \rceil$ and

$$(\Theta_z^\lambda u)(s, y) = \lambda^{-(d+2)} u\big(\lambda^{-2}(t - s), \lambda^{-1}(y - x)\big) \qquad z = (t, x).$$

A crucial estimate on the Besov norm $\|\cdot\|_{\mathscr{C}^\alpha(K)}$ is given by

$$\|\eta\|_{\mathscr{C}^\alpha(K)} \leq C \sup_{n \geq 0} \sup_{z \in (2^{-2n}\mathbb{Z} \times 2^{-n}\mathbb{Z}^d) \cap \widetilde{K}} 2^{-n\alpha} |\langle \eta, \Theta_z^{2^{-n}} u \rangle|, \tag{A.1}$$

where $\widetilde{K}$ is also a compact set slightly larger than K and u is a single, well-chosen test function (see the proof of [23, Lemma 5.2]). Recall that if ξ is space–time white noise (i.e, $\mathbb{E}[\langle \xi, \varphi_1 \rangle \langle \xi, \varphi_2 \rangle] = \int \varphi_1(t, x) \varphi_2(t, x)\, \mathrm{d}t\, \mathrm{d}x$), then with

$$\langle \xi_\lambda, \varphi \rangle = \langle \xi, \Theta_0^\lambda \varphi \rangle \qquad \text{we have} \qquad \mathbb{E}[\langle \xi_\lambda, \varphi \rangle^2] = \lambda^{-(d+2)} \int_{\mathbb{R}^{d+1}} \varphi^2(t, x)\, \mathrm{d}t\, \mathrm{d}x. \tag{A.2}$$

The following result, then implies the desired regularity property of ξ.

Lemma A.1 *Fix $\alpha < 0$ and $p \geq 1$ and let η be a linear map from $\mathcal{S}(\mathbb{R} \times \mathbb{R}^d)$ to the space of random variables. Suppose there exists $C \in (0, \infty)$ such that for all $z \in \mathbb{R} \times \mathbb{R}^d$ and all $u \in \mathcal{S}(\mathbb{R} \times \mathbb{R}^d)$ with compact support in $\mathbb{R} \times \mathbb{R}^d$ with $\sup_w |u(w)| \leq 1$ one has*

$$\mathbb{E}[|\eta(\Theta_z^\lambda u)|^p] \leq C \lambda^{\alpha p} \qquad \forall \lambda \in (0, 1].$$

Then there exists a random distribution $\widetilde{\eta}$ *in* $\mathcal{S}'(\mathbb{R} \times \mathbb{R}^d)$ *such that for all* $\alpha' < \alpha - \frac{(d+2)}{p}$ *and compact set* K*,*

$$\mathbb{E}\big[\|\widetilde{\eta}\|^p_{\mathscr{C}^{\alpha'}(K)}\big] < \infty \qquad \text{and } \eta(u) = \widetilde{\eta}(u) \quad \text{a.s.}$$

Below we will provide a short sketch of the proof of Lemma A.1. First note that since for any $p \geq 1$, $\mathbb{E}[|\langle \xi_\lambda, \varphi \rangle|^p] \leq C_p \mathbb{E}[\langle \xi_\lambda, \varphi\rangle^2]^{p/2}$, then Lemma A.1 and (A.2) imply that ξ has regularity $\mathscr{C}^{-\frac{d}{2}-1-\delta}$ for any $\delta > 0$.

Sketch of proof of Lemma A.1 We follow the proof of Theorem [6, Theorem 2.7] which is based on the arguments of [23, Lemma 5.2] which builds on the classical approach of proving Kolmogorov continuity theorem. Note that if $\mathbb{R} \ni t \mapsto X(t)$ is continuous, then

$$\sup_{\substack{s,t \in [-1,1]\\ s \neq t}} \frac{|X(s) - X(t)|}{|s-t|^{\alpha'}} \leq C \sup_{k \geq 0} \sup_{s \in 2^{-k}\mathbb{Z} \cap [-1,1)} 2^{k\alpha'} |X(s + 2^{-k}) - X(s)|,$$

implying further

$$\begin{aligned}
&\Bigg(\sup_{\substack{s,t \in [-1,1]\\ s \neq t}} \frac{|X(s) - X(t)|}{|s-t|^{\alpha'}} \Bigg)^p \\
&\leq C^p \sup_{k \geq 0} \sup_{s \in 2^{-k}\mathbb{Z} \cap [-1,1)} 2^{kp\alpha'} |X(s + 2^{-k}) - X(s)|^p \\
&\leq C^p \sum_{k \geq 0} \sup_{s \in 2^{-k}\mathbb{Z} \cap [-1,1)} 2^{kp\alpha'} |X(s + 2^{-k}) - X(s)|^p.
\end{aligned}$$

Now taking expectation, we obtain

$$\begin{aligned}
&\mathbb{E}\Bigg[\Bigg(\sup_{\substack{s,t \in [-1,1]\\ s \neq t}} \frac{|X(s) - X(t)|}{|s-t|^{\alpha'}} \Bigg)^p\Bigg] \\
&\leq C^p \sup_{\substack{s,t \in [-1,1]\\ s \neq t}} \bigg(\frac{1}{|t-s|^{\alpha p}} \mathbb{E}\big(|X(s) - X(t)^p\big) \bigg) \sum_{k \geq 0} 2^k 2^{k\alpha' p} 2^{-k\alpha p}.
\end{aligned}$$

Given the crucial estimate (A.1) we can follow the above guiding philosophy of replacing the supremum on the right hand side of (A.1) and summing the geometric series. □

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