Convergence in the p -Contest

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Abstract

We study asymptotic properties of the following Markov system of $N \geq 3$ points in $[0, 1]$. At each time step, the point farthest from the current centre of mass, multiplied by a constant $p > 0$, is removed and replaced by an independent ζ -distributed point; the problem, inspired by variants of the Bak–Sneppen model of evolution and called a p -contest, was posed in Grinfeld et al. (J Stat Phys 146, 378–407, 2012). We obtain various criteria for the convergences of the system, both for $p < 1$ and $p > 1$. In particular, when $p < 1$ and $\zeta \sim U[0, 1]$, we show that the limiting configuration converges to zero. When $p > 1$, we show that the configuration must converge to either zero or one, and we present an example where both outcomes are possible. Finally, when $p > 1$, $N = 3$ and ζ satisfies certain mild conditions (e.g. $\zeta \sim U[0, 1]$), we prove that the configuration converges to one a.s. Our paper substantially extends the results of Grinfeld et al. (Adv Appl Probab 47:57–82, 2015) and Kennerberg and Volkov (Adv Appl Probab 50:414–439, 2018) where it was assumed that $p = 1$. Unlike the previous models, one can no longer use the Lyapunov function based just on the radius of gyration; when $0 < p < 1$ one has to find a more finely tuned function which turns out to be a supermartingale; the proof of this fact constitutes an unwieldy, albeit necessary, part of the paper.

Keywords Keynesian beauty contest · Jante’s law · Rank-driven process

Mathematics Subject Classification Primary 60J05; Secondary 60D05 · 60F15 · 60K35 · 82C22 · 91A15

1 Introduction

This paper extends the results of [3,4] on the so-called *Keynesian beauty contest*, or, as it was called in [4], *Jante’s law process*. Following [3], we recall that in the Keynesian beauty

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contest, we have N players guessing a number, and the person who guesses closest to the mean of all the N guesses wins; see [5, Ch. 12, Sect. V]. The formal version, suggested by Moulin [6, p. 72], assumes that this game is played by choosing numbers on the interval $[0, 1]$, the “ p -beauty contest”, in which the target is the mean value, multiplied by a constant $p > 0$. For the applications of the p -contest in the game theory, we refer the reader to e.g. [7]; see also [1, 3] and references therein for further applications and other relevant papers.

The version of the p -contest with $p \equiv 1$ was studied in [3, 4]. In [3] it was shown that in the model where at each unit of time the point farthest from the center of mass is replaced by a point chosen uniformly on $[0, 1]$, then eventually all (but one) points converge almost surely to some random limit the support of which is the whole interval $[0, 1]$; many of the results were extended for the version of the model on $\mathbb{R}^d$, $d \geq 2$. The results of [3] were further generalized in [4], by removing the assumption that a new point is chosen uniformly on $[0, 1]$, as well as by removing more than one point at once, these points being chosen in such a way that the moment of inertia of the resulting configuration is minimized. However, the case $p \neq 1$ was not addressed in either of these two papers.

Let us now formally define the model; the notation will be similar to those in [3, 4]. Let $\mathcal{X} = \{x_1, x_2, \dots, x_N\} \in \mathbb{R}^N$ be an unordered N -tuple of points in $\mathbb{R}$, and $(x_{(1)}, x_{(2)}, \dots, x_{(N)})$ be these points put in non-decreasing order, that is, $x_{(1)} \leq x_{(2)} \leq \dots \leq x_{(N)}$. As in [3, 4] let us define the barycentre of the configuration as

$$\mu_N(x_1, \dots, x_N) := N^{-1} \sum_{i=1}^N x_i.$$

Fix some $p > 0$ and also define the p -centre of mass as $p\mu_N(x_1, \dots, x_N)$.

The point, farthest from the p -centre of mass, is called the *extreme* point of $\mathcal{X}$, and it can be either $x_{(1)}$ or $x_{(N)}$ (with possibility of a tie), and *the core* of $\mathcal{X}$, denoted by $\mathcal{X}'$, is constructed from $\mathcal{X}$ by removing the extreme point; in case of a tie between the left-most and the right-most point, we choose either of them with equal probability (same as in [3, 4]). Throughout the rest of the paper, $x_{(1)}(t), \dots, x_{(N-1)}(t)$ shall denote the points of the core¹ $\mathcal{X}'(t)$ put into non-decreasing order.

Our process runs as follows. Let $\mathcal{X}(t) = \{X_1(t), \dots, X_N(t)\}$ be an unordered N -tuple of points in $\mathbb{R}$ at time $t = 0, 1, 2, \dots$. Given $\mathcal{X}(t)$, let $\mathcal{X}'(t)$ be the core of $\mathcal{X}(t)$ and replace $\mathcal{X}(t) \setminus \mathcal{X}'(t)$ by a ζ -distributed random variable so that

$$\mathcal{X}(t+1) = \mathcal{X}'(t) \cup \{\zeta_{t+1}\},$$

where ζ_t , $t = 1, 2, \dots$, are i.i.d. random variables with a common distribution ζ .

Finally, to finish the specification of our process, we allow the initial configuration $\mathcal{X}(0)$ to be arbitrary or random, with the only requirement being that all the points of $\mathcal{X}(0)$ must lie in the support of ζ .

Throughout the paper we will use the notation $A \xrightarrow{\text{a.s.}} B$ for two events A and B , whenever $\mathbb{P}(A \cap B^c) = 0$, that is, when $A \subseteq B$ up to a set of measure 0. We will also write, with some abuse of notations, that $\lim_{t \rightarrow \infty} \mathcal{X}'(t) = a \in \mathbb{R}$ or equivalently $\mathcal{X}'(t) \rightarrow a$ as $t \rightarrow \infty$ if $\mathcal{X}'(t) \rightarrow (a, a, \dots, a) \in \mathbb{R}^{N-1}$, i.e. $\lim_{t \rightarrow \infty} x_{(i)}(t) = a$ for all $i = 1, 2, \dots, N-1$. Similarly, for an interval (a, b) we will write $\mathcal{X}'(t) \in (a, b)$ whenever all $x_{(1)}(t), \dots, x_{(N-1)}(t) \in (a, b)$. Finally, we will assume that $\inf \emptyset = +\infty$, and use the notation $y^+ = \max(y, 0)$ for $y \in \mathbb{R}$.

¹ Rather than of $\mathcal{X}(t)$

Also we require that ζ has a *full support* on $[0, 1]$, that is, $\mathbb{P}(\zeta \in (a, b)) > 0$ for all a, b such that $0 \leq a < b \leq 1$.

2 The Case $p < 1$

Throughout this Section we assume that $0 < p < 1$ and that $\text{supp } \zeta = [0, 1]$. Because of the scaling invariance, our results may be trivially extended to the case when $\text{supp } \zeta = [0, A]$, $A \in (0, \infty)$; some of them are even true when $A = \infty$; however, to simplify the presentation from now on we will deal only with the case $A = 1$.

First, we present some general statements; more precise results will follow in case where $\zeta \sim U[0, 1]$.

Proposition 1 *We have*

- (a) $\liminf_{t \rightarrow \infty} x_{(N-1)}(t) = 0$;
- (b) $\mathbb{P}(\exists \lim_{t \rightarrow \infty} \mathcal{X}'(t) \in (0, 1]) = 0$;
- (c) if $p < \frac{1}{2} + \frac{1}{2(N-1)}$ then $\mathbb{P}(\lim_{t \rightarrow \infty} \mathcal{X}'(t) = 0) = 1$;
- (d) if $p < \frac{1}{2} + \frac{1}{N-2}$ then $\{x_{(1)}(t) \rightarrow 0\} \xRightarrow{a.s.} \{\lim_{t \rightarrow \infty} \mathcal{X}'(t) = 0\}$.

Proof (a) Since ζ has full support on $[0, 1]$ it follows that (see [4, Proposition 1]) there exists a function $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that

$$\mathbb{P}(\zeta \in (a, b)) \geq f(b - a) > 0 \quad \text{for all } 0 \leq a < b \leq 1. \quad (2.1)$$

Also, to simplify notations, we write $\mu = \mu_N(\mathcal{X}(t))$ throughout the proof.

Fix a small positive ε such that $p + 2\varepsilon < 1$. Suppose that for some t we have $x_{(N-1)}(t) \leq b \leq 1$. We will show that $x_{(N-1)}(t + N) \leq b(1 - \varepsilon)$ with a strictly positive probability which only depends on p, b, ε and N . Assume that we have $\zeta_{t+1}, \dots, \zeta_{t+N-1} \in (pb, (p + \varepsilon)b) \subset (pb, b)$; this happens with probability no less than $[f(p\varepsilon b)]^{N-1}$. We claim that by the time $t + N$ we have $x_{(N-1)}(t + N - 1) < (p + \varepsilon)b$. Indeed, $p\mu \leq pb$ always lies to the left of the newly sampled points, therefore either there are no more points to the right of $(p + \varepsilon)b$ at some time $s \in [t, t + N - 1]$ (which implies that there will be no points there at time $t + N$ due to the sampling range of the new points), or one of the older points, i.e. present at time t , gets removed (it can be the one to the left of pb). Since we eventually have to replace all the $N - 1$ old points, then $x_{(N-1)}(t + N) \leq b(1 - \varepsilon)$.

Fix a $\delta > 0$ and find M so large that $(1 - \varepsilon)^M < \delta$. Let the event $C(s) = \{x_{(N-1)}(s) < \delta\}$. By iterating the above argument, we get that $\mathbb{P}(C(t + NM) | \mathcal{F}_t) \geq \prod_{i=1}^M [f(p\varepsilon(1 - \varepsilon)^{i-1})]^{N-1}$, since at time t we can set $b = 1$. Therefore, $\sum_m \mathbb{P}(C(NM(m+1)) | \mathcal{F}_{NMm}) = \infty$ and by Lévy's extension of the Borel–Cantelli lemma (see e.g. [8]) infinitely many $C(s)$ occur. Since $\delta > 0$ is arbitrary, we get $\liminf_{t \rightarrow \infty} x_{(N-1)}(t) = 0$.

(b) Let $r = \frac{1+p^{-1}}{2} > 1$. Suppose that the core converges to some point $x \in (0, 1]$; then there exist a rational $q \in (0, 1]$ and a $T > 0$ such that $\mathcal{X}'(t) \in (q, rq)$ for all $t \geq T$, formally

$$\{\exists \lim \mathcal{X}'(t) \in (0, 1]\} \subseteq \bigcup_{q \in \mathbb{Q} \cap (0, 1]} \bigcup_{T > 0} \bigcap_{t \geq T} A_{q,t} \quad (2.2)$$

where $A_{q,t} = \{\mathcal{X}'(t) \in (q, rq)\}$. We will show that

$$\mathbb{P}(A_{q,t+1} | \mathcal{F}_t, A_{q,t}) < 1 - \nu_q \quad \text{for all } t$$

for some $v_q > 0$. This will imply, in turn, that

$$\mathbb{P}\left(\bigcap_{t \geq T} A_{q,t}\right) = 0$$

and hence the RHS (and thus the LHS as well) of (2.2) has the probability 0.

Suppose $A_{q,t}$ has occurred and the newly sampled point $\zeta \in (pq, q)$. Then

$$p\mu_N(\mathcal{X}'(\tau_k) \cup \{\zeta\}) < prq = \frac{pq + q}{2} < \frac{\zeta + x_{(N-1)}}{2}$$

Consequently, $x_{(N-1)}$ lies further from the p -center of mass, and hence it should be removed. The new configuration will, however, contain the point $\zeta \notin (q, rq)$ and hence $A_{q,t+1}$ does not occur. Thus

$$\mathbb{P}(A_{q,t+1} | \mathcal{F}_t, A_{q,t}) \leq 1 - \mathbb{P}(\zeta \in (q, rq)) \leq 1 - f(pq - q) =: 1 - v_q$$

as required.

(c) First, we will show that it is the right-most point of the configuration which should be always removed; note that it suffices to check this only when $x_{(N)} > 0$. Indeed, by the assumption on p we have

$$\mu \leq \frac{(N-1)x_{(1)} + (N-1)x_{(N)}}{N} = \frac{2p(N-1)}{N} \cdot \frac{x_{(1)} + x_{(N)}}{2p} < \frac{x_{(1)} + x_{(N)}}{2p}$$

implying that

$$x_{(N)} - p\mu > p\mu - x_{(1)} \iff x_{(N)} - p\mu > |p\mu - x_{(1)}|$$

Therefore, $x_{(N)}$ is the farthest point from the p -centre of mass. This implies that $x_{(N-1)}(t)$ is non-increasing and therefore result now easily follows from part (a) since $x_{(N-1)}(t)$ is an upper bound for all the core points.

(d) Apply Corollary 3 with $k = 1$; it is possible because of Remark 3. $\square$

We are ready to present the main result of this Section.

Theorem 1 Suppose that $\zeta \sim U[0, 1]$. Then $\mathcal{X}'(t) \rightarrow 0$ a.s.

Proof Proposition 1(c) implies that we now only need to consider the case $p \geq \frac{N}{2(N-1)}$, which we will assume from now on.

Let us introduce a modification of this process on $[0, +\infty)$ which we will call the *borderless p -contest*; it is essentially the same process as the one in [3, Sect. 3.4]. In order to do this, we need the following statement. $\square$

Lemma 1 Suppose that $x_1, \dots, x_{N-1} > 0$. Then there exists an $R = R(x_{(N-1)}) \geq 0$ such that x is the farthest point from $p\mu = \frac{p}{N}(x_1 + \dots + x_{N-1} + x)$ whenever $x > R$.

Proof of Lemma 1 Set $R = 6x_{(N-1)}$. Then $x > x_{(1)}$ is farther from the centre of mass than $x_{(1)}$ if and only if

$$\begin{aligned} x - p\mu > |p\mu - x_{(1)}| &\iff x - p\mu > p\mu - x_{(1)} \\ &\iff x \left(1 - \frac{2p}{N}\right) > 2p \frac{x_1 + \dots + x_{N-1}}{N} - x_{(1)} \end{aligned}$$

This is true, due to the fact that $x > R$ and

$$x \left(1 - \frac{2p}{N}\right) > \frac{x}{3} > 2x_{(N-1)} > 2px_{(N-1)} > 2p \frac{x_1 + \dots + x_{N-1}}{N}$$

since $p < 1$ and $N \geq 3$. $\square$

The borderless process is constructed as follows. Our core configuration starts as before in $[0, 1]$, and we use the same rejection/acceptance criteria for new points. However, we will now allow points to be generated to the right of 1 as well. Let $R_t = R(x_{(N-1)}(t))$ where R is taken from Lemma 1. Then a new point is sampled uniformly and independently of the past on the interval $[0, R_t]$; formally, it is given by $R_t U_t$ where U_t are i.i.d. uniform $[0, 1]$ random variables independent of everything. Observe that if we consider the embedded process only at the times when the core configuration changes, then the exact form of the function $R(\cdot)$ is irrelevant, due to the fact that the uniform distribution conditioned on a subinterval is also uniform on that subinterval.

Next, for $y = \{y_1, \dots, y_{N-1}\}$ define the function

$$h(y) = F(y) + k\mu(y)^2, \quad (2.3)$$

where

$$F(y) = \sum_{i=1}^{N-1} (y_i - \mu(y))^2, \quad \mu(y) = \frac{1}{N-1} \sum_{i=1}^{N-1} y_i, \quad k = \frac{(N-1)^2(1-p)}{N-2}.$$

We continue with the following

Lemma 2 *For the borderless p -contest the sequence of random variables $h(\mathcal{X}'(t)) \geq 0$, $t = 1, 2, \dots$, is a supermartingale.*

Remark 1 Note that the function $F(\cdot)$ defined above is a Lyapunov function for the process in [3]; this is no longer the case as long as $p \neq 1$; that is why we have to use a carefully chosen “correction” factor which involves the barycentre of the configuration.

Proof of Lemma 2 Assume that $x_{(N-1)}(t) > 0$ (otherwise the process has already stopped, and the result is trivial). The inequality, which we want to obtain is

$$\mathbb{E}[h(\mathcal{X}'(t+1)) - h(\mathcal{X}'(t)) | \mathcal{F}_t] \Big|_{x(t)=y} \leq 0$$

for all $y = (y_1, \dots, y_{N-1})$ with $y_i \in [0, 1]$. Note that the function $h(y)$ is homogeneous of degree 2 in y , therefore w.l.o.g. we can assume that $\max y \equiv 1$.

For simplicity let $M = N - 1 \geq 2$, and let

$$z = 6U_t \text{ (the newly sampled point), } a = \min y < 1 \text{ (the leftmost point)}$$

Note also that

$$p \geq \frac{N}{2(N-1)} = \frac{M+1}{2M} = \frac{1}{2} + \frac{1}{2M}. \quad (2.4)$$

Define

$$\begin{aligned} F_{old} &= F(y), & F_{new} &= F((y \cup \{z\})') \\ \mu'_{old} &= \mu(y), & \mu'_{new} &= \mu((y \cup \{z\})'), \\ h_{old} &= F_{old} + k(\mu'_{old})^2, & h_{new} &= F_{new} + k(\mu'_{new})^2 \end{aligned}$$

Thus we need to establish

$$\mathbb{E}[h_{new} - h_{old} | \mathcal{F}_t] \leq 0. \quad (2.5)$$

First of all, observe that if $\tilde{y} = (y \setminus \{y_i\}) \cup \{z\}$, that is, $\tilde{y}$ is obtained from y by replacing y_i with y_0 , then

$$\begin{aligned} F(\tilde{y}) - F(y) &= \frac{z - y_i}{M} [(M - 1)z + (M + 1)y_i - 2M\mu(y)] \\ \mu(\tilde{y})^2 - \mu(y)^2 &= \frac{z - y_i}{M^2} [z - y_i + 2M\mu(y)] \end{aligned}$$

In particular, if we replace point a by the new point z , then

$$\begin{aligned} \Delta_a(z) &:= h_{new} - h_{old} \\ &= \frac{z - a}{M} \left[(M - 1)z + (M + 1)a - 2M\mu(y) + \frac{k}{M}(z - a + 2M\mu(y)) \right] \end{aligned}$$

and if we replace point 1, then

$$\begin{aligned} \Delta_1(z) &:= h_{new} - h_{old} \\ &= \frac{z - 1}{M} \left[(M - 1)z + (M + 1) - 2M\mu(y) + \frac{k}{M}(z - 1 + 2M\mu(y)) \right] \end{aligned}$$

Note that both Δ_a and Δ_1 depend only on four variables (a, z, μ, M) but not the whole configuration. Let us also define

$$m(z) = p \cdot \frac{y_1 + \cdots + y_M + z}{M + 1} = p \cdot \frac{M\mu + z}{M + 1},$$

the p -centre of mass of the old core and the newly sampled point.

There are three different cases that can occur: either (a) the point a is removed, (b) 1, the rightmost point of the previous core, is removed, or (c) the newly sampled point z is removed. In the third case the core remains unchanged, and the change in the value of the function h is trivially zero. The point a can only be removed if $z > a$; the point 1 can only be removed if $z < 1$; the point z can be possibly removed only if $z \in (0, a)$ or $z \in (1, \infty)$. Let us compute the critical values for z , for which there is a tie between the farthest points. $\square$

Which Point to Remove?

(i) Suppose $\boxed{z < a}$. Then there is a tie between z and 1 if and only if $m(z) = \frac{z+1}{2}$, that is if

$$z = t_{z1} := \frac{M(2p\mu - 1) - 1}{M + 1 - 2p} \in \begin{cases} (-\infty, 0) & \text{if } p < p_1 := \frac{M+1}{2M\mu} \\ (0, a) & \text{if } p_1 < p < p_2 := \frac{(M+1)(a+1)}{2M\mu+2a} \\ (a, +\infty) & \text{if } p > p_2. \end{cases}$$

Thus, we have:

- when $p < p_1$, point 1 is removed;
- when $p_1 < p < p_2$, if $z < t_{z1}$ then z is removed; if $z > t_{z1}$ point 1 is removed;
- when $p > p_2$, point z is removed.

(ii) Suppose $a < z < 1$. There is a tie between a and 1 if and only if $m(z) = \frac{a+1}{2}$, that is if

$$z = t_{a1} := \frac{(M+1)(a+1) - 2M\mu p}{2p} \in \begin{cases} (1, +\infty) & \text{if } p < p_3 := \frac{(M+1)(a+1)}{2M\mu+2}, \\ (a, 1) & \text{if } p_3 < p < p_2, \\ (-\infty, a) & \text{if } p > p_2. \end{cases}$$

Thus, we have:

- when $p < p_3$, point 1 is removed;
- when $p_3 < p < p_2$, if $z < t_{a1}$ then 1 is removed; if $z > t_{a1}$ then point a is removed;
- when $p > p_2$, point a is removed.

(iii) Suppose $z > 1$. There is a tie between z and a if and only if $m(z) = \frac{z+a}{2}$, that is if

$$z = t_{za} := \frac{2M\mu p - (M+1)a}{M+1-2p} \in \begin{cases} (-\infty, 1) & \text{if } p < p_3, \\ (1, +\infty) & \text{if } p > p_3. \end{cases}$$

Thus, we have:

- when $p < p_3$, point z is removed;
- when $p > p_3$, if $z < t_{za}$ then a is removed; if $z > t_{za}$ then point z is removed.

We always have $p_1 < p_2$, $p_3 < p_2$ since

$$p_2 - p_1 = \frac{a(M+1)(M\mu-1)}{2M\mu(M\mu+a)} = \frac{a(M+1)(a+(M-2)f)}{2M\mu(M\mu+a)} > 0,$$

$$p_2 - p_3 = \frac{(1-a)^2(M+1)}{2(M\mu+1)(M\mu+a)} > 0,$$

while

$$p_1 < p_3 \iff Ma\mu > 1 \iff f > \frac{1-a-a^2(M-1)}{a(M-2)(1-a)} \text{ (when } M > 2)$$

The final observation is that $t_{za} < 6$, so there is indeed no need to sample the new point outside of the range $(0, 6)$; this holds since $M \geq 2$ and

$$6 - t_{za} = \frac{-2p(M\mu+6) + Ma + 6M + a + 6}{M+1-2p} > \frac{-2M\mu + Ma + 6M + a - 6}{M+1-2p}$$

$$> \frac{-2M\mu + 6M - 6}{M+1-2p} = \frac{2M(1-\mu) + 4M - 6}{M+1-2p} > \frac{2}{M+1-2p} > 0.$$

The Five Cases for the Removal

- $p < \min\{p_1, p_3\}$:
 - when $z < 1$, point 1 is removed
 - when $z > 1$, point z is removed
- $p > p_2$:
 - when $z < a$ or $z > t_{za} \in (1, \infty)$ point z is removed
 - when $a < z < t_{za}$, point a is removed
- $\max\{p_1, p_3\} < p < p_2$

- when $z < t_{z1} \in (0, a)$ or $t > t_{za} \in (1, +\infty)$, point z is removed
- when $t_{z1} < z < t_{a1} \in (a, 1)$, point 1 is removed
- when $t_{a1} < z < t_{za}$, point a is removed
- $p_1 < p < p_3$ ($< p_2$):
 - when $z < t_{z1} \in (0, a)$ or $z > 1$, point z is removed
 - when $t_{z1} < z < 1$, point 1 is removed
- $p_3 < p < p_1$ ($< p_2$):
 - when $z < t_{a1} \in (a, 1)$, point 1 is removed
 - when $t_{a1} < z < t_{za} \in (1, +\infty)$, point a is removed
 - when $z > t_{za}$, point z is removed

Let

$$\begin{aligned} X_1 &= p - p_1 = \frac{M(2\mu p - 1) - 1}{2M\mu}, \\ X_2 &= p - p_2 = \frac{2ap - a - 1 + (2\mu p - a - 1)M}{2(M\mu + a)}, \\ X_3 &= p - p_3 = \frac{2p - a - 1 + (2\mu p - a - 1)M}{2(M\mu + 1)}. \end{aligned}$$

Define

$$\begin{aligned} \tilde{\mathbf{I}}_1 &= \mathbb{E}(h(\mathcal{X}'(t+1)) - h(\mathcal{X}'(t)) | \mathcal{F}_t) \Big|_{x(t)=y} \cdot 1_{X_1 < 0} \cdot 1_{X_3 < 0}, \\ \tilde{\mathbf{I}}_2 &= \mathbb{E}(h(\mathcal{X}'(t+1)) - h(\mathcal{X}'(t)) | \mathcal{F}_t) \Big|_{x(t)=y} \cdot 1_{X_2 > 0}, \\ \tilde{\mathbf{I}}_3 &= \mathbb{E}(h(\mathcal{X}'(t+1)) - h(\mathcal{X}'(t)) | \mathcal{F}_t) \Big|_{x(t)=y} \cdot 1_{X_2 < 0} \cdot 1_{X_1 > 0} \cdot 1_{X_3 > 0}, \\ \tilde{\mathbf{I}}_4 &= \mathbb{E}(h(\mathcal{X}'(t+1)) - h(\mathcal{X}'(t)) | \mathcal{F}_t) \Big|_{x(t)=y} \cdot 1_{X_1 > 0} \cdot 1_{X_3 < 0}, \\ \tilde{\mathbf{I}}_5 &= \mathbb{E}(h(\mathcal{X}'(t+1)) - h(\mathcal{X}'(t)) | \mathcal{F}_t) \Big|_{x(t)=y} \cdot 1_{X_1 < 0} \cdot 1_{X_3 > 0}. \end{aligned}$$

Since $\max y = 1$, because of the comment on the restriction of the uniform distribution on a subinterval, we have $\tilde{\mathbf{I}}_j = c_j \mathbf{I}_j$, $j = 1, 2, 3, 4, 5$, where c_j 's are some positive constants and

$$\begin{aligned} \mathbf{I}_1 &= \mathbf{A}_1 \cdot 1_{X_1 < 0} \cdot 1_{X_3 < 0}, & \mathbf{A}_1 &= \int_0^1 \Delta_1 dz, \\ \mathbf{I}_2 &= \mathbf{A}_2 \cdot 1_{X_2 > 0}, & \mathbf{A}_2 &= \int_a^{t_{za}} \Delta_a dz, \\ \mathbf{I}_3 &= \mathbf{A}_3 \cdot 1_{X_2 < 0} \cdot 1_{X_1 > 0} \cdot 1_{X_3 > 0}, & \mathbf{A}_3 &= \int_{t_{z1}}^{t_{a1}} \Delta_1 dz + \int_{t_{a1}}^{t_{za}} \Delta_a dz, \\ \mathbf{I}_4 &= \mathbf{A}_4 \cdot 1_{X_1 > 0} \cdot 1_{X_3 < 0}, & \mathbf{A}_4 &= \int_{t_{z1}}^1 \Delta_1 dz, \\ \mathbf{I}_5 &= \mathbf{A}_5 \cdot 1_{X_1 < 0} \cdot 1_{X_3 > 0}, & \mathbf{A}_5 &= \int_0^{t_{a1}} \Delta_1 dz + \int_{t_{a1}}^{t_{za}} \Delta_a dz. \end{aligned}$$

Thus to establish (2.5), it suffices to show that $\mathbf{I}_j \leq 0$ for each $j = 1, 2, 3, 4, 5$. This is done by very extensive and tedious calculations, which can be found in the Appendix. $\square$

We now return to our original p -contest process $\mathcal{X}(t)$. For $L \geq 2$ define

$$\tau_L = \inf\{t > 0 : x_{(N-1)}(t) < 1/L\};$$

$$\eta_L = \inf\{t > \tau_L : x_{(N-1)}(t) \geq 1/2\},$$

note that τ_L is a.s. finite for every L by Proposition 1. Let $W(s) = \{w_1(s), \dots, w_N(s)\}$ be a borderless p -contest with $W(0) = \mathcal{X}(\tau_L)$; let $W'(s)$ be its core. By Lemma 2 the quantity $\xi_t = h(W'(t \wedge \eta_L))$ is a supermartingale, that converges to some ξ_∞ . Since ξ_t is bounded,

$$\begin{aligned} \mathbb{E}\xi_0 &\geq \mathbb{E}\xi_\infty = \mathbb{E}[\xi_\infty \cdot 1_{\eta_L < \infty}] + \mathbb{E}[\xi_\infty \cdot 1_{\eta_L = \infty}] \geq \mathbb{E}[\xi_\infty \cdot 1_{\eta_L < \infty}] \\ &\geq \frac{k}{(2(N-1))^2} \mathbb{P}(\eta_L < \infty) \end{aligned}$$

since on $\{\eta_L < \infty\}$ we have $\xi_\infty = W'(\eta_L)$ and the largest coordinate of $W'(\eta_L)$ is larger than $1/2$, implying that $\mu(W'(\eta_L)) \geq \frac{1}{2(N-1)}$ and thus $h(W'(\eta_L)) = F(W'(\eta_L)) + k\mu(W'(\eta_L))^2 \geq \frac{k}{(2(N-1))^2}$. We also have

$$\begin{aligned} \xi_0 &= h(\mathcal{X}'(\tau_L)) = F(\mathcal{X}'(\tau_L)) + k\mu(\mathcal{X}'(\tau_L))^2 \leq \frac{N-1}{L^2} + \frac{k}{L^2} \\ &\implies \mathbb{E}\xi_0 \leq \frac{N+k-1}{L^2} \end{aligned}$$

since $\mathcal{X}'(\tau_L) \subset [0, 1/L]$ and so $\mu(\mathcal{X}'(\tau_L)) \in [0, 1/L]$.

Combining the above inequalities, we conclude that $\mathbb{P}(\eta_L < \infty) \rightarrow 0$ as $L \rightarrow \infty$. However, on $\eta_L = \infty$ the core of the regular p -contest process can be trivially coupled with the core of the borderless process $W'(s)$ which converges to zero, so $\mathcal{X}'(t) \rightarrow 0$ as well. Since $\mathbb{P}(\eta_L = \infty)$ can be made arbitrarily close to 1 by choosing a large L , we conclude that $\mathcal{X}'(t) \rightarrow 0$ a.s. $\square$

3 The Case $p > 1$

Throughout this section we suppose that ζ has a full support on $[0, 1]$, and, unless explicitly stated otherwise, that $p > 1$.

Theorem 2 (a) $\mathbb{P}(\{\mathcal{X}'(t) \rightarrow 0\} \cup \{\mathcal{X}'(t) \rightarrow 1\}) = 1$;

(b) if $x_{(1)}(0) \geq 1/p$ then $\mathbb{P}(\mathcal{X}'(t) \rightarrow 1) = 1$;

(c) if $x_{(k)}(0) > 0$, where k satisfies

$$\{2p(N-k) > N-2p\} \iff \left\{k < N - \frac{N}{2p} + 1\right\}, \quad (3.1)$$

then $\mathbb{P}(\mathcal{X}'(t) \rightarrow 1) > 0$.

Remark 2 In general, both convergences can have a positive probability. Let $N = 3$, $p \in (1, 3/2)$, and

$$\zeta = \begin{cases} U, & \text{with probability } 1/3; \\ 0, & \text{with probability } 1/3; \\ 1, & \text{with probability } 1/3, \end{cases}$$

where $U \in U[0, 1]$ (so ζ has full support). Suppose we sample the points of $\mathcal{X}(0)$ from ζ . If they all start off in 0, then $p\mu \leq p/3 < 1/2$, so they cannot escape from 0. On the other hand, there is a positive probability they all start in $(1/p, 1]$, and then Theorem 2(b) says that they converge to 1.

The key idea behind the proof of Theorem 2 is that one can actually find the “ruling” order statistic of the core; namely, there exists some non-random $k = k(N, p) \in \{1, 2, \dots, N-1\}$ such that $x_{(k)}(t) \rightarrow 0$ implies $\mathcal{X}'(t) \xrightarrow{\text{a.s.}} 0$, while $x_{(k)}(t) \not\rightarrow 0$ implies that $\mathcal{X}'(t) \xrightarrow{\text{a.s.}} 1$.

We start with the following two results, which tells us that there is an *absorbing area* $[\frac{1}{p}, 1]$ for the process, such that, once the core enters this area, it will never leave it, and moreover the core will keep moving to the right.

Claim 1 Suppose that $x_1 \leq x_2 \leq x_3 \leq \dots \leq x_N \leq 1$ and $x_2 \geq p^{-1}$. Then $\{x_1, \dots, x_N\}' = \{x_2, \dots, x_N\}$

Proof Let $\mu = \frac{x_1 + \dots + x_N}{N}$. If $p\mu \geq x_N$ then the claim follows immediately; assume instead that $p\mu < x_N$. We need to check if $p\mu - x_1 > x_N - p\mu$, that is, if

$$2p(x_2 + \dots + x_{N-1}) > (N-2p)(x_1 + x_N) \quad (3.2)$$

However, since $x_i \geq x_2$ for $i = 3, \dots, N-1$ we have

$$2p(x_2 + \dots + x_{N-1}) \geq 2px_2(N-2) \geq 2(N-2)$$

while $(N-2p)(x_1 + x_N) \leq 2(N-2p) < 2(N-2)$. Hence (3.2) follows. $\square$

Lemma 3 If $x_{(1)}(t_0) \geq 1/p$ for some t_0 , then $\mathcal{X}'(t) \rightarrow 1$ a.s.

Proof If $x_{(1)}(t_0) \geq 1/p$, then any point that lands in $[0, 1/p]$ is extreme, so $x_{(2)}(t) \geq 1/p$ for all $t \geq t_0$. Choose any positive $\varepsilon < 1 - \frac{1}{p}$, and let $A_t = \{\zeta_{t+1}, \dots, \zeta_{t+N-1} \in (1-\varepsilon, 1]\}$. Then if A_t happens for $s > t_0$, any point in $[0, 1-\varepsilon]$ is removed in preference to any of the new points coming in, so $x_{(2)}(s+N-1) > 1-\varepsilon$. As a result, by Claim 1 we get that $\mathcal{X}'(t) \in [0, 1-\varepsilon]$ for all $t \geq s$.

On the other hand, $\mathbb{P}(A_t) \geq [f(\varepsilon)]^{N-1} > 0$ (see (2.1)) for any t , and the events $A_t, A_{t+N}, A_{t+2N}, \dots$ are independent. Hence, eventually with probability 1, one of the A_t 's must happen for some $t > t_0$, so a.s. $\mathcal{X}'(t) \in [0, 1-\varepsilon]$ for all large t . Since ε can be chosen arbitrary small, we get the result. $\square$

The next two results show that if there is some $\varepsilon > 0$ such that infinitely often the core does not have any points in $[0, \varepsilon]$, then it must, in fact, converge to 1.

Lemma 4 If $x_{(1)}(t_0) \geq \varepsilon$ for some t_0 and $\varepsilon > 0$, then $\mathbb{P}(x_{(1)}(t_0 + \ell) \geq p^{-1} | \mathcal{F}_t) \geq \delta$ for some $\ell = \ell(\varepsilon)$ and $\delta = \delta(\varepsilon) > 0$.

Proof Suppose that for some t we have $x_{(1)}(t) \geq \varepsilon$. We claim that it is possible to move $x_{(1)}$ to the right of $\frac{1+p}{2}\varepsilon$ in at most $N-1$ steps with positive probability, depending only on p and ε . Indeed, if $x_{(1)}(t) > \frac{1+p}{2}\varepsilon$ then we are already done. Otherwise, if the new point ζ_{t+1} is sampled in $\left(\frac{1+p}{2}\varepsilon, p\varepsilon\right] \subset [0, 1]$ it cannot be rejected. If at this stage $x_{(1)}(t+1) > \frac{1+p}{2}\varepsilon$, then we are done. If not, we proceed again by sampling $\zeta_{t+2} \in \left(\frac{1+p}{2}\varepsilon, p\varepsilon\right]$, etc. After at most $N-1$ steps of sampling new points in $\left(\frac{1+p}{2}\varepsilon, p\varepsilon\right]$, the leftmost point $x_{(1)}$ will have moved to the right of $\frac{1+p}{2}\varepsilon$.

Thus, in no more than $N-1$ steps, with probability no less than $\left[f\left(\frac{p-1}{2}\varepsilon\right)\right]^{N-1} > 0$, $x_{(1)}$ is to the right of $\frac{1+p}{2}\varepsilon$. By iterating this argument at most m times, where $m \in \mathbb{N}$ is chosen such that $\left[\frac{1+p}{2}\right]^m \varepsilon > 1/p$, we achieve that $x_{(1)}$ is to the right of $1/p$ (for definiteness, one can choose $\ell = (N-1)m$ and $\delta = \left[f\left(\frac{p-1}{2}\varepsilon\right)\right]^{(N-1)m}$.) $\square$

Lemma 5 Let $\varepsilon \in (0, 1)$, and define $B(\varepsilon) := \{x_{(1)}(t) \geq \varepsilon \text{ i.o.}\}$. Then $B(\varepsilon) \xRightarrow{\text{a.s.}} \{\mathcal{X}'(t) \rightarrow 1\}$.

Corollary 1 We have $\{\liminf_{t \rightarrow \infty} x_{(1)}(t) > 0\} \xRightarrow{\text{a.s.}} \{\mathcal{X}'(t) \rightarrow 1\}$.

Proof of Lemma 5 Assume that $\varepsilon < \frac{1}{p}$ (otherwise the result immediately follows from Lemma 3). Also suppose that $\mathbb{P}(B(\varepsilon)) > 0$, since otherwise the result is trivial. Let ℓ and δ be the quantities from Lemma 4.

Define

$$\begin{aligned}\tau_0 &= \inf\{t > 0 : x_{(1)}(t) > \varepsilon\}, \\ \tau_k &= \inf\{t > \tau_{k-1} + \ell : x_{(1)}(t) > \varepsilon\}, \quad k \geq 1,\end{aligned}$$

with the convention that if $\tau_k = \infty$ then $\tau_m = \infty$ for all $m > k$. Notice that $B(\varepsilon) = \bigcap_{k=0}^{\infty} \{\tau_k < \infty\}$. On $B(\varepsilon)$ we can also define $D_{\tau_k} = \{x_{(1)}(\tau_k + \ell) \geq 1/p\}$. Since $\tau_k - \tau_{k-1} > \ell$ whenever both are finite, we have from Lemma 4 we have $\mathbb{P}(D_{\tau_{k+1}} | \mathcal{F}_{\tau_k}) \geq \delta$. Therefore,

$$B(\varepsilon) \xRightarrow{\text{a.s.}} \left\{ \sum_{k \geq 0} \mathbb{P}(D_{\tau_{k+1}} | \mathcal{F}_{\tau_k}) = \infty \right\}$$

hence by Lévy's extension of the Borel–Cantelli lemma it follows that a.s. on $B(\varepsilon)$ infinitely many (and hence at least one) of D_{τ_k} occur, that is, $x_{(1)}(\tau_k + \ell) \geq 1/p$. Now the result follows from Lemma 3. $\square$

Assume for now that $p < \frac{N}{2}$; in this case $N - \frac{N}{2p} + 1 < N$ (see (3.1)). The case $p \geq \frac{N}{2}$ will be dealt with separately.

The following statement shows that if all the points to the right of $x_{(k)}$ lie very near each other, while the left-most one lies near zero, then it is to be removed.

Claim 2 Let $a \in (0, 1]$ and suppose that $k \in \{2, \dots, N-1\}$ satisfies (3.1). Then there exist small $\delta, \Delta > 0$, depending on N, k, p, a such that if

$$\begin{aligned}0 &\leq x_1 \leq \delta; \\ x_1 &\leq x_i \leq x_N \quad \text{for } i = 2, \dots, N-1; \\ x_k, x_{k+1}, \dots, x_N &\in [a(1-\Delta), a]\end{aligned}$$

then $\{x_1, \dots, x_N\}' = \{x_2, \dots, x_N\}$.

Proof The condition to remove the leftmost point is $p\mu - \frac{x_1 + x_N}{2} > 0$ where $\mu = (x_1 + \dots + x_N)/N$. However,

$$\begin{aligned}2N \left(p\mu - \frac{x_1 + x_N}{2} \right) &= 2p(x_2 + \dots + x_{N-1}) - (N-2p)x_1 - (N-2p)x_N \\ &\geq 2p(x_k + \dots + x_{N-1}) - (N-2p)\delta - (N-2p)a \\ &\geq 2p(N-k)a(1-\Delta) - (N-2p)\delta - (N-2p)a \\ &= a[2p(N-k)(1-\Delta) - (N-2p)] - (N-2p)\delta\end{aligned}$$

The RHS is linear in δ and Δ , and when $\delta = \Delta = 0$ it is strictly positive by the assumption on k ; hence it can also be made positive, by allowing $\delta > 0$ and $\Delta > 0$ to be sufficiently small. $\square$

Corollary 2 Suppose that $\mathcal{X}(t) = \{x_1, \dots, x_N\}$ satisfies the conditions of Claim 2 for some a and k . Let δ be the quantity from this claim. Then

$$\mathbb{P}(x_{(1)}(t+j) > \delta \text{ for some } 1 \leq j \leq k | \mathcal{F}_t) \geq c = c_{a\Delta} > 0.$$

Proof The probability to sample a new point $\zeta \in (a(1-\Delta), a]$ is bounded below by $f(a\Delta)$ where f is the same function as in (2.1). On the other hand, if the new point is sampled in $(a(1-\Delta), a]$ then $\mathcal{X}(t+1)$ continues to satisfy the conditions of Claim 2 as long as the leftmost point is in $[0, \delta]$. By repeating this argument at most k times and using the induction, we get the result with $c = [f(a\Delta)]^k > 0$. $\square$

Lemma 6 Let $k \in \mathbb{N}$ satisfy (3.1). Then

$$\{x_{(k)}(t) \not\rightarrow 0\} \xRightarrow{\text{a.s.}} \{\mathcal{X}'(t) \rightarrow 1\}.$$

Proof Note that by Lemma 5, it suffices to show that $\{x_{(k)}(t) \not\rightarrow 0\} \xRightarrow{\text{a.s.}} \{x_{(1)}(t) \not\rightarrow 0\}$.

If $x_{(k)}(t) \not\rightarrow 0$, there exists an $a > 0$ such that $x_{(k)}(t) \geq a$ for infinitely many t 's. Let s be such a time. Now suppose that $\zeta_{s+i} \in I := (a(1-\Delta), a]$ for $i = 0, 1, \dots, N-1$ where Δ is defined in Claim 2; the probability of this event is strictly positive and depends only on a and δ (see (2.1)). As long as there are points of $\mathcal{X}(s+i)$ on both sides of the interval I , none of the points inside I can be removed; hence, for some $u \in \{s, s+1, \dots, s+N-1\}$ we have that either $\min \mathcal{X}(u) > a(1-\Delta)$ or $\max \mathcal{X}(u) \leq a$. In the first case, $x_{(1)}(u) > a(1-\Delta)$.

In the latter case, both $x_{(N)}(u) \in I$ and $x_{(k)}(u) \in I$, since every time we replaced a point, the number of points to the left of I did not increase (and there were initially at most $k-1$ of them). As a result

$$a(1-\Delta) \leq x_{(k)}(u) \leq x_{(k+1)}(u) \leq \dots \leq x_{(N)}(u) \leq a.$$

Together with Corollary 2, this yields

$$\{x_{(k)}(t) \geq a \text{ i.o.}\} \xRightarrow{\text{a.s.}} \{x_{(1)}(t) \geq \min\{a(1-\Delta), \delta\} \text{ i.o.}\} \xRightarrow{\text{a.s.}} \{x_{(1)}(t) \not\rightarrow 0\}$$

which proves Lemma 6. $\square$

Claim 3 Let $A_i := \{x_{(i)}(t) \rightarrow 0\}$ and suppose that for some $1 \leq k \leq N-2$ we have

$$\{2p(N-k-1) < N\} \iff \left\{k > N - \frac{N}{2p} - 1\right\}. \quad (3.3)$$

Then $A_k \subseteq \{\exists \lim_{t \rightarrow \infty} x_{(k+1)}(t)\}$.

Proof Fix any $a > 0$. Let $\delta > 0$ be so small that

$$2pN\delta < [N - 2p(N-k-1)]a. \quad (3.4)$$

In the event A_k there exists a finite $\tau = \tau_\delta(\omega)$ such that

$$\left\{\sup_{t \geq \tau} x_{(k)}(t) \leq \delta\right\} \iff \{\text{card}(\mathcal{X}'(t) \cap [0, \delta]) \geq k \text{ for all } t \geq \tau.\}$$

From now on assume that $t \geq \tau$. We will show below that $x_{(k+1)}(t+1) \leq \max\{x_{(k+1)}(t), a\}$.

To begin, let us prove that $x_{(k+1)}(t+1) \leq x_{(k+1)}(t)$ as long as $x_{(k+1)}(t) > \delta$. Indeed, if the new point ζ is sampled to the left of $x_{(k+1)}(t)$, then regardless of which point is to be removed, $x_{(k+1)}(t+1) \leq x_{(k+1)}(t)$. If the new point ζ is sampled to the right, then the farthest point from the p -centre of mass must be the rightmost one (and hence $x_{(k+1)}(t+1) = x_{(k+1)}(t)$).

since there are exactly k points in $[0, \delta]$ and none of these can be removed by the definition of τ .

On the other hand, if $x_{(k+1)}(t) \leq \delta$ then either $x_{(k+2)}(t) \leq a$ or $x_{(k+2)}(t) > a$. In the first case, $x_{(k+1)}(t+1) \leq x_{(k+2)}(t) \leq a$ even if $x_{(1)}$ is removed. In the other case, when $x_{(k+2)}(t) > a$, we have $x_{(N-1)} > a$ as well, and

$$\begin{aligned} p\mu(\mathcal{X}(t+1)) &\leq p \frac{(k+1)\delta + (N-k-1)x_{(N)}}{N} \\ &< \frac{2pN\delta - [N - 2p(N-k-1)]x_{(N)} + Nx_N}{2N} \\ &\leq \frac{Nx_N - \{[N - 2p(N-k-1)]a - 2pN\delta\}}{2N} < \frac{x_{(N)}}{2} \end{aligned}$$

by (3.4), so $x_{(N)} = x_{(N)}(t)$ must be removed and thus $x_{(k+1)}(t+1) \leq x_{(k+1)}(t)$.

Consequently, we obtained

$$\begin{aligned} A_k &\subseteq \bigcap_{t \geq \tau} \{x_{(k+1)}(t+1) \leq \max\{x_{(k+1)}(t), a\}\} \\ &\subseteq \left(\bigcup_{t \geq 0} \{x_{(k+1)}(s) \leq a \text{ for all } s \geq t\} \right) \\ &\quad \cup \left(\bigcup_{t \geq 0} \{x_{(k+1)}(s) \leq x_{(k+1)}(s+1) \text{ for all } s \geq t\} \right) \\ &\subseteq \left\{ \limsup_{t \rightarrow \infty} x_{(k+1)}(t) \leq a \right\} \cup \left\{ \exists \lim_{t \rightarrow \infty} x_{(k+1)}(t) \right\} \end{aligned}$$

since $a > 0$ is arbitrary, we get

$$A_k \subseteq \left\{ \limsup_{t \rightarrow \infty} x_{(k+1)}(t) \leq 0 \right\} \cup \left\{ \exists \lim_{t \rightarrow \infty} x_{(k+1)}(t) \right\} = \left\{ \exists \lim_{t \rightarrow \infty} x_{(k+1)}(t) \geq 0 \right\}$$

□

Lemma 7 Suppose that (3.3) holds for some $1 \leq k \leq N-2$. Then $A_k \xrightarrow[a.s.]{} A_{k+1}$.

Proof Let $\tilde{A}_{k+1}^{\geq a} := \{\lim_{t \rightarrow \infty} x_{(k+1)}(t) \geq a\}$ (the existence of this limit on A_k follows from Claim 3). It suffices to show that $\mathbb{P}(A_k \cap \tilde{A}_{k+1}^{\geq a}) = 0$ for all $a > 0$; then from the continuity of probability we get that $\mathbb{P}(A_k \cap \{\lim_{t \rightarrow \infty} x_{(k+1)}(t) > 0\}) = 0$ and hence $A_k \xrightarrow[a.s.]{} A_{k+1}$.

Fix an $a > 0$. Let

$$C_t = \left\{ x_{(k)}(t) < \frac{a}{3} \text{ and } x_{(k+1)}(t) > \frac{2a}{3} \right\}, \quad \bar{C}_T = \bigcap_{t \geq T} C_t,$$

then

$$A_k \cap \tilde{A}_{k+1}^{\geq a} \subseteq \bigcup_{T \geq 0} \bar{C}_T = \left\{ \exists T > 0 : x_{(k)}(t) < \frac{a}{3} \text{ and } x_{(k+1)}(t) > \frac{2a}{3} \text{ for all } t \geq T \right\}.$$

If the probability of the LHS is positive, then, using the continuity of probability and the fact that $\bar{C}_T$ is an increasing sequence of events, we obtain that $\lim_{T \rightarrow \infty} \mathbb{P}(\bar{C}_T) > 0$. Consequently, there exists a non-random T_0 such that $\mathbb{P}(\bar{C}_{T_0}) > 0$.

This is, however, impossible, as at each time point t , with probability at least $f(a/3)$ (see (2.1)) the new point ζ_t is sampled in $B := (\frac{a}{3}, \frac{2a}{3})$ and then either $x_{(k)}(t+1) \in B$ or $x_{(k+1)}(t+1) \in B$. Formally, this means that

$$\mathbb{P}(C_{t+1}|C_t, \mathcal{F}_t) \leq 1 - f(a/3) \quad \text{for all } t \geq 0.$$

By induction, for all $k \geq 1$,

$$\mathbb{P}(\bar{C}_{T_0}|\mathcal{F}_{T_0}) \leq \mathbb{P}\left(\bigcap_{T=T_0}^{T_0+k} C_T|\mathcal{F}_{T_0}\right) \leq [1 - f(a/3)]^k.$$

Since k is arbitrary, and $f(a/3) > 0$, by taking the expectation, we conclude that $\mathbb{P}(\bar{C}_{T_0}) = 0$ yielding a contradiction.

Hence the probability of the event $A_k \cap \tilde{A}_{k+1}^{\geq a}$ is zero. $\square$

Corollary 3 Suppose that (3.3) holds for some $1 \leq k \leq N-2$. Then

$$\{x_{(k)}(t) \rightarrow 0\} \xRightarrow{\text{a.s.}} \{\mathcal{X}'(t) \rightarrow 0\}.$$

Proof Observe that if k satisfies (3.3) then $k+1$ satisfies (3.3) as well. Thus by iterating Lemma 7 we obtain that $A_k \xRightarrow{\text{a.s.}} A_{k+1} \xRightarrow{\text{a.s.}} A_{k+2} \xRightarrow{\text{a.s.}} \dots \xRightarrow{\text{a.s.}} A_{N-1}$, i.e. $x_{(N-1)}(t) \rightarrow 0$, which is equivalent to the statement of Corollary. $\square$

Remark 3 Note that the condition (3.3) does not assume $p > 1$; hence the conclusion of Corollary 3 holds for the case $0 < p \leq 1$ as well.

For the case $p \geq \frac{N}{2}$ we have

Lemma 8 If $p \geq \frac{N}{2}$ then $\mathcal{X}'(t) \rightarrow 1$ a.s.

Proof The case $p > \frac{N}{2}$ is easy: with a positive probability the newly sampled point $\zeta > 0$ and then

$$p \frac{x_{(1)} + \dots + x_{(N-1)} + \zeta}{N} > \frac{x_{(1)} + \dots + x_{(N-1)} + \zeta}{2} \geq \frac{x_{(1)} + \zeta}{2}$$

hence it is the left-most point which is always removed, implying $\liminf_{t \rightarrow \infty} x_{(1)}(t) > 0$. Hence by Corollary 1, $\mathcal{X}'(t) \rightarrow 1$ a.s.

For the case $p = \frac{N}{2}$ we notice that at each moment of time we either have a tie (between the left-most and right-most point) or remove the left-most point. However, we can only have a tie if $x_{(1)}(t) = \dots = x_{(N-1)}(t) = 0$; in this case, eventually the right-most point will be kept and the left-most removed. After this moment of time, there will be more ties, and the left-most point will always be removed, leading to the same conclusion as in the case $p > N/2$. $\square$

Proof of Theorem 2 Part (b) follows from Lemma 3.

To prove part (c), note that unless $x_{(1)}(0) > 0$ already, by repeating the arguments from the beginning of the proof of Lemma 6, with a positive probability we can “drag” the whole configuration in at most $N-1$ steps to the right of zero, that is, there is $0 \leq t_0 \leq N-1$ such that $\mathbb{P}(\min \mathcal{X}'(t_0) > 0) > 0$. Now we can apply Lemma 4 and then Lemma 3.

Let us now prove part (a). First, assume $p < \frac{N}{2}$. It is always possible to find an integer k which satisfies both (3.1) and (3.3), so let k be such that

$$N - \frac{N}{2p} - 1 < k < N - \frac{N}{2p} + 1$$

(if $N/(2p) \in \mathbb{N}$ this k will be unique). Now the statement of the theorem follows from Corollary 3 and Lemma 6.

Finally, in case $p \geq \frac{N}{2}$ the theorem follows from Lemma 8. $\square$

4 Non-convergence to Zero for $p > 1$ and $N = 3$

In this section we prove the following

Theorem 3 *Suppose that $N = 3$, $p > 1$ and ζ , restricted to some neighbourhood of zero, is a continuous random variable with a non-decreasing density (e.g. uniformly distributed). Then $\mathcal{X}'(t) \rightarrow 1$ as $t \rightarrow \infty$ a.s.*

Remark 4 • In case $p \geq 3/2$ we already know that $\mathcal{X}'(t) \rightarrow 1$ for any initial configuration and any distribution (see Lemma 8), so we have to prove the theorem only for $p \in (1, 3/2)$.

- Simulations suggest that the statement of Theorem 3 holds, in fact, for a much more general class of distributions ζ .

Let $\varepsilon \in (0, 1/2)$ be such that ζ conditioned² on $\{\zeta \leq 2\varepsilon\}$ has a non-decreasing density; according to the statement of the Theorem 3 such an ε must exist. Let us fix this ε from now on.

The idea of the proof will be based on finding a non-negative function $h : (0, 1]^2 \rightarrow \mathbb{R}_+$ which has the following three properties:

- (i) $h(\cdot, \cdot)$ is non-increasing in each of its arguments;
- (ii) $h(x_{(1)}(t), x_{(2)}(t))$ is a supermartingale as long as $x_{(2)}(t) \leq \varepsilon$;
- (iii) $h(\cdot, \cdot)$ goes to infinity when the first coordinate goes to zero.

From the supermartingale convergence theorem it would then follow that

$$\mathbb{P}\left(\liminf_{t \rightarrow \infty} x_{(1)}(t) > 0 \text{ or } \limsup_{t \rightarrow \infty} x_{(2)}(t) \geq \varepsilon\right) = 1.$$

Let us formally prepare for the proof of Theorem 3. As before, denote by $x_1, \dots, x_N$ N distinct points on $[0, 1]$, and let $x_{(1)}, \dots, x_{(N)}$ be this unordered N -tuple sorted in the increasing order. Let

$$\{y_1, \dots, y_{N-1}\} = \{x_1, \dots, x_N\}'_p$$

be the unordered N -tuple $\{x_1, \dots, x_N\}$ with the farthest point from p -centre of mass removed; w.l.o.g. assume that y_i are already in the increasing order.

Lemma 9 *The operation $\{\dots\}'_p$ is monotone in p , that is, if $\hat{p} \geq \tilde{p}$ and*

$$\{\hat{y}_1, \dots, \hat{y}_{N-1}\} = \{x_1, \dots, x_N\}'_{\hat{p}},$$

$$\{\tilde{y}_1, \dots, \tilde{y}_{N-1}\} = \{x_1, \dots, x_N\}'_{\tilde{p}}$$

then $\hat{y}_i \geq \tilde{y}_i$, $i = 1, \dots, N - 1$.

² Note that the full support assumption ensures that the probability of this event is positive.

Proof Assume w.l.o.g. $x_1 \leq \dots \leq x_N$, and let $\mu = \mu(\{x_1, \dots, x_N\})$. Notice that, regardless of the value of p , the only points which can possibly be removed are x_1 or x_N (since they are the two extreme points). Therefore, it suffices to show that $\{x_1, \dots, x_N\}'_p = \{x_2, \dots, x_N\}$ implies $\{x_1, \dots, x_N\}'_{\hat{p}} = \{x_2, \dots, x_N\}$. Note also that $|x_1 - p\mu| = p\mu - x_1$ for all $p \geq 1$.

If $\tilde{p}\mu - x_1 > |\tilde{p}\mu - x_N|$ and $\tilde{p}\mu - x_N > 0$, that is, the p -centre of mass lies to the right of x_N , then $\hat{p}\mu > \tilde{p}\mu > x_N$ as well, and hence x_1 is discarded.

On the other hand, if $\tilde{p}\mu - x_1 > |\tilde{p}\mu - x_N|$ and $\tilde{p}\mu < x_N$ then either $\hat{p}\mu < x_N$, or $\hat{p}\mu \geq x_N$. In the first case,

$$\hat{p}\mu - x_1 > \tilde{p}\mu - x_1 > |\tilde{p}\mu - x_N| = x_N - \tilde{p}\mu > x_N - \hat{p}\mu = |x_N - \hat{p}\mu|$$

so x_1 is discarded. In the second case, p -centre of mass lies to the right of x_N and so x_1 is also discarded. $\square$

Lemma 10 Let h be a real-valued function on the sets of N real numbers. Suppose that h is non-increasing in each of its arguments, namely

$$h(x_1, x_2, \dots, x_{i-1}, x'_i, x_{i+1}, \dots, x_N) \leq h(x_1, x_2, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_N)$$

whenever $x'_i \geq x_i$. Let $\mathcal{E}_t$ be some $\mathcal{F}_t$ -measurable event, and suppose that

$$\mathbb{E}(h(\mathcal{X}'(t+1))|\mathcal{F}_t) \leq h(\mathcal{X}'(t)) \text{ on } \mathcal{E}_t \quad (4.1)$$

for $p = 1$. Then (4.1) holds for $p > 1$ as well.

Proof Let

$$G_p(\mathcal{X}'(t), \zeta_{t+1}) = \{x_{(1)}(t), x_{(2)}(t), \dots, x_{(N-1)}(t), \zeta_{t+1}\}'_p$$

be the new core after the new point ζ_{t+1} is sampled and the farthest point from the p -centre of mass is removed; note that $\mathcal{X}'(t+1) = G_p(\mathcal{X}'(t), \zeta_{t+1})$. Then on $\mathcal{E}_t$

$$\mathbb{E}(h(\mathcal{X}'(t+1))|\mathcal{F}_t) = \mathbb{E}(h(G_p(\mathcal{X}'(t), \zeta_{t+1}))|\mathcal{F}_t) \leq \mathbb{E}(h(G_1(\mathcal{X}'(t), \zeta_{t+1}))|\mathcal{F}_t) \leq h(\mathcal{X}'(t))$$

since the operation $\{\dots\}'_p$ is monotone in p by Lemma 9 and h is decreasing in each argument. $\square$

From now on assume $N = 3$ and $p = 1$. Denote $x_{(1)}(t) = a$, $x_{(2)}(t) = b$ and consider the events

$$\begin{aligned} L_b &= \{\zeta_{t+1} \in ((2a-b)^+, a)\}, & R_a &= \{\zeta_{t+1} \in (b, 2b-a)\}, \\ B_b &= \{\zeta_{t+1} \in (a, \frac{a+b}{2})\}, & B_a &= \{\zeta_{t+1} \in (\frac{a+b}{2}, b)\} \end{aligned}$$

(we assume that b is smaller than $1/2$, yielding $2b-a < 1$.) If $x_{(2)}(t) \leq \varepsilon$ then $\mathcal{X}'(t+1) \neq \mathcal{X}'(t)$ implies that one of the events L_b, B_b, B_a or R_a occurs (i.e. all points sampled outside of $((2a-b)^+, 2b-a)$ are rejected at time $t+1$). Let us study the core $\mathcal{X}'(t+1) = \{\zeta_{t+1}, a, b\}'$ on these events: on L_b and B_b we have $\mathcal{X}'(t+1) = \{x, a\}$, while on B_a and R_a we have $\mathcal{X}'(t+1) = \{x, b\}$.

We have, assuming $x_{(1)}(t) = a$ and $x_{(2)}(t) = b$,

$$\mathbb{E}(h(\mathcal{X}'(t+1)) - h(\mathcal{X}'(t))|\mathcal{F}_t) = \mathbb{E}(h(\{\zeta, a, b\}') - h(a, b)).$$

When $0 \leq a \leq b \leq \varepsilon$ we have $2b - a \leq 2\varepsilon$. Define

$$g(x) = h(\{x, a, b\}') - h(a, b) = \begin{cases} h(x, a) - h(a, b), & \text{if } x \in ((2a - b)^+, a); \\ h(a, x) - h(a, b), & \text{if } x \in (a, (a + b)/2); \\ h(x, b) - h(a, b), & \text{if } x \in ((a + b)/2, b); \\ h(b, x) - h(a, b), & \text{if } x \in (b, 2b - a) \\ 0, & \text{otherwise,} \end{cases}$$

which is positive in the first two cases, and negative in the next two. Let $\varphi(x)$ be the density of ζ conditioned on $\{\zeta \in [0, 2\varepsilon]\}$. By the monotonicity of φ and the positivity (negativity resp.) of g on the first (second resp.) interval,

$$\begin{aligned} \Delta(a, b) &:= \mathbb{E}[g(\zeta)1_{\zeta \in [0, 2\varepsilon]}] = \int_{(2a-b)^+}^{\frac{a+b}{2}} g(x)\varphi(x)dx + \int_{\frac{a+b}{2}}^{2b-a} g(x)\varphi(x)dx \\ &\leq \varphi\left(\frac{a+b}{2}\right) \int_{(2a-b)^+}^{\frac{a+b}{2}} g(x)dx \\ &\quad + \varphi\left(\frac{a+b}{2}\right) \int_{\frac{a+b}{2}}^{2b-a} g(x)dx = \varphi\left(\frac{a+b}{2}\right) \cdot \Lambda, \end{aligned}$$

where

$$\begin{aligned} \Lambda &= \Lambda(a, b) = \int_{(2a-b)^+}^a (h(x, a) - h(a, b))dx + \int_a^{\frac{a+b}{2}} (h(a, x) - h(a, b))dx \\ &\quad + \int_{\frac{a+b}{2}}^b (h(x, b) - h(a, b))dx + \int_b^{2b-a} (h(b, x) - h(a, b))dx. \end{aligned}$$

So if we can establish that $\Lambda \leq 0$ for a suitable function h , then indeed $\Delta(a, b) \leq 0$, and the supermartingale property follows.

Remark 5 Notice that the method of proof, presented here, could possibly work for $N > 3$ as well; that is, if one can find a function $h(x_1, \dots, x_{N-1})$, which is positive and decreasing in each of its arguments, and $h(\mathcal{X}'(t))$ is a supermartingale provided $\max \mathcal{X}'(t) < \varepsilon$ for some $\varepsilon > 0$. Unfortunately, however, we were not able to find such a function.

Set

$$h(x, y) = -2 \log \left(\max \left\{ x, \frac{y}{2} \right\} \right) \geq 0; \quad (4.2)$$

it is easy to check h is indeed monotone in each of its arguments as long as $x, y \in (0, 1]$. Let us now compute the integrals in the expression for Λ . We have

$$\Lambda = \begin{cases} 3(a-b) \ln 2 - 3a + 2b, & \text{if } a \leq \frac{b}{3}; \\ (a+b) \ln(a+b) - (a+b) \ln a + (a-5b) \ln 2 + b, & \text{if } \frac{b}{3} < a \leq \frac{b}{2}; \\ (a+b) \ln(a+b) + (2a-4b) \ln b + 3(b-a) \ln a + (b-5a) \ln 2 + b, & \text{if } \frac{b}{2} < a \leq \frac{2b}{3}; \\ (a+b) \ln(a+b) + (2a-4b) \ln b + (5b-7a) \ln a - (a+b) \ln 2 \\ \quad + 3(b-a) + (4a-2b) \ln(2a-b), & \text{if } \frac{2b}{3} < a \leq b. \end{cases}$$

It turns out that $h(\mathcal{X}'(t))$ indeed has a non-positive drift, provided $0 < a \leq b \leq \varepsilon$, as is shown by the following

Lemma 11 $\Lambda \leq 0$ for $a, b \in (0, 1/2]$.

Proof Substitute $a = bv$ in the expression for Λ . Then for $v \leq 1/3$ we easily obtain $\Lambda = -b[3v(1 - \ln 2) + \ln 8 - 2] \leq 0$.

For $1/3 < v \leq 1/2$ we have $2\Lambda = -bC_1(v) \leq 0$ where

$$C_1(v) = (1+v) \ln \frac{v}{1+v} + (5-v) \ln 2 - 1 > 0,$$

since $\frac{\partial^2 C_1(v)}{\partial^2 v} = -\frac{1}{v^2(1+v)} < 0$ and hence $\min_{1/3 \leq v \leq 1/2} C_1(v)$ is achieved at one of the endpoints $v = 1/3$ or $v = 1/2$; the values there are $C_1(1/3) = \ln(4) - 1 > 0$ and $C_1(1/2) = \frac{1}{2} \ln\left(\frac{512}{27}\right) - 1 > 0$ respectively.

For $1/2 < v \leq 2/3$ we have $\Lambda = -bC_2(v) \leq 0$ where

$$C_2(v) = -(1+v) \ln(1+v) + (3v-3) \ln v - 1 + (5v-1) \ln 2 > 0,$$

since $\frac{\partial^2 C_2(v)}{\partial^2 v} = \frac{2v^2+6v+3}{v^2(1+v)} > 0$ and $\left.\frac{\partial C_2(v)}{\partial v}\right|_{v=2/3} = \ln\left(\frac{256}{45}\right) - \frac{5}{2} < 0$ implies that $\frac{\partial C_2(v)}{\partial v} < 0$ for all $v \in [1/2, 2/3]$ and hence $\min_{1/2 \leq v \leq 2/3} C_2(v) = C_2(2/3) = \frac{1}{3} \ln\left(\frac{104976}{3125}\right) - 1 > 0$.

Finally, for $2/3 < v \leq 1$ we have $\Lambda = -bC_3(v) \leq 0$, where

$$C_3(v) = v \log \frac{2v^7}{(2v-1)^4(v+1)} + \log \frac{2(2v-1)^2}{v^5(v+1)} + 3(v-1) > 0$$

since

$$\frac{d^2 C_3(v)}{dv^2} = \frac{(2v+5)(2v^2-1)}{(2v-1)v^2(v+1)}$$

changes its sign from $-$ to $+$ at $1/\sqrt{2} \in (2/3, 1)$ and therefore $\frac{\partial C_3(v)}{\partial v}$ achieves its maximum at the endpoints of the interval; thus

$$\max_{2/3 \leq v \leq 1} \frac{\partial C_3(v)}{\partial v} = \max_{v=2/3, 1} \frac{\partial C_3(v)}{\partial v} = \max \left\{ -\frac{5}{2} + \ln\left(\frac{256}{45}\right), 0 \right\} = 0$$

Therefore, $C_3(v)$ is decreasing and hence $\min_{2/3 \leq v \leq 1} C_3(v) = C_3(1) = 0$. $\square$

Proof of Theorem 3 We will show that $\mathbb{P}(\mathcal{X}'(t) \rightarrow 0) = 0$, which will imply by Theorem 2(a) that $\mathbb{P}(\mathcal{X}'(t) \rightarrow 1) = 1$; we shall do it by showing that

$$0 \leq \mathbb{P}(\mathcal{X}'(t) \rightarrow 0) \leq \mathbb{P} \left(\left\{ \liminf_{t \rightarrow \infty} x_{(1)}(t) = 0 \right\} \cap \left\{ \limsup_{t \rightarrow \infty} x_{(2)}(t) < \varepsilon \right\} \right) = 0. \quad (4.3)$$

Indeed, fix some $\varepsilon \in (0, 1/2)$ and let $\tau_0 = 0$. For $\ell = 1, 2, \dots$, define the sequence of stopping times

$$\begin{aligned} \eta_\ell &= \inf\{t > \tau_{\ell-1} : x_{(2)}(t) \leq \varepsilon\}, \\ \tau_\ell &= \inf\{t > \eta_\ell : x_{(2)}(t) > \varepsilon\}, \end{aligned}$$

so that $\tau_0 < \eta_1 < \tau_1 < \eta_2 < \tau_2 < \dots$ with the conventions that if one of the stopping times is infinite, so is the rest of them. Define also $\ell^* = \inf\{\ell \geq 1 : \tau_\ell = +\infty\}$.

If $\ell^* = \infty$, that is, $\tau_\ell < \infty$ for all ℓ , we immediately have $\limsup_{t \rightarrow \infty} x_{(2)}(t) \geq \varepsilon$ and we are done; so assume that for some $\ell^* \geq 1$ we have $\tau_{\ell^*-1} < \infty = \tau_{\ell^*}$. If $\eta_{\ell^*} = \infty$, then $x_{(2)}(t) > \varepsilon$ for all $t \geq \tau_{\ell^*-1}$ and thus again $\limsup_{t \rightarrow \infty} x_{(2)}(t) \geq \varepsilon$; hence $\ell^* < \infty$ and $\eta_{\ell^*} < \infty$ on the event $\{\limsup_{t \rightarrow \infty} x_{(2)}(t) < \varepsilon\}$.

On the other hand, as long as $\eta_\ell < \infty$, we can define

$$\xi_s^{(\ell)} = h(\mathcal{X}'(\eta_\ell + s)), \quad s \geq 0 \quad \text{and} \quad \tilde{\xi}_s^{(\ell)} = \xi_{\min\{s, \tau_\ell - \eta_\ell\}}^{(\ell)}.$$

where h is given by (4.2).

By Lemmas 10 and 11 we have

$$\mathbb{E}([h(\mathcal{X}'(t+1)) - h(\mathcal{X}'(t))] 1_{x_{(2)}(t) \leq \varepsilon} | \mathcal{F}_t) \leq 0,$$

hence $\tilde{\xi}_s^{(\ell)}$, the process $\xi^{(\ell)}$ stopped at the time when $x_{(2)}$ exceeds ε , is a non-negative supermartingale, hence it must converge to a finite value. In case $\tau_\ell = +\infty$ this means that $\liminf_{t \rightarrow \infty} x_{(1)}(t) > 0$ since the function $h(a, b)$ goes to infinity when $a \downarrow 0$. Thus we have established (4.3). $\square$

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5 Appendix: The Calculations for the Proof of Lemma 2.

Observe that all expressions for $\mathbf{A}_j$ are fractions of the polynomials in (a, f, p, M) ; moreover, their denominators

$$\begin{aligned} & 3M(M-1) && (\text{for } \mathbf{A}_1), \\ & 3M(M-1)(M+1-2p)^3 && (\text{for } \mathbf{A}_2 \text{ and } \mathbf{A}_4), \\ & 12M(M-1)(M+1-2p)^3 p^3 && (\text{for } \mathbf{A}_3 \text{ and } \mathbf{A}_5) \end{aligned}$$

are always positive. Throughout the rest of the proof let $n(w)$ denote the numerator of such a fraction w .

Case 1: $\mathbf{I}_1 \leq 0$

Observe that

$$n(\mathbf{A}_1) = -2M^2 - 3M\mu + 2M + 1 + [3M\mu - 1]Mp$$

and the term in the square brackets is positive as $M\mu \geq 1$, so the maximum of $n(\mathbf{A}_1)$ is achieved at the highest possible value of p . However, in this case we have $p \leq p_1$, hence

$$n(\mathbf{A}_1) 1_{X_1 \leq 0} \leq n(\mathbf{A}_1)|_{p=p_1} = -\frac{s_1}{2\mu}$$

where

$$s_1 = (M^2 - 2)\mu + (1 - 6\mu)(1 - \mu)M + 1$$

$$= \begin{cases} 3(2\mu - 1)^2, & \text{if } M = 2; \\ 4\mu^2 + 1/2 + 14(\mu - 1/2)^2, & \text{if } M = 3; \\ (M - 3)[(M - 4)\mu + 6\mu^2 + 1] + s_1|_{M=3}, & \text{if } M \geq 4 \end{cases}$$

Hence $s_1 \geq 0$ for $M = 2, 3, \dots$ and thus $\mathbf{I}_1 \leq 0$.

Case 2: $\mathbf{I}_2 \leq 0$

Here

$$n(\mathbf{A}_2) = -4[a(M - p + 1) - M\mu p]^2 s_2$$

where

$$\begin{aligned} s_2 = & M^3\mu p - 4M^2\mu p^2 - M^3a + 2M^2ap + 5M^2\mu p \\ & + 2Map^2 - 3M^2\mu - 6Ma * p + 4M\mu p \\ & + 3Ma - 3M\mu - 2ap + 2a, \end{aligned}$$

and we need to show that $s_2 \geq 0$.

Assume first $M = 2$. Then (using the fact that $\mu = (1 + a)/2$)

$$X_2 \geq 0 \iff p \geq \frac{3a + 3}{4a + 2} \geq 1$$

which is impossible; so from now on $M \geq 3$.

To establish $\mathbf{I}_2 \leq 0$, it will suffice to demonstrate that

$$s_3 := 2Ms_2 - 2M^3(M\mu + a)X_2 \geq 0$$

as $\mathbf{I}_2$ has a factor $1_{X_2 \geq 0}$, and $s_2 1_{X_2 \geq 0} \geq \frac{s_3}{2M} 1_{X_2 \geq 0}$. Substituting

$$p = \left[\frac{1}{2} + \frac{1}{2M} \right] + \left[\frac{1}{2} - \frac{1}{2M} \right] w$$

where $w \in [0, 1)$ corresponding to the condition (2.4), we get

$$\begin{aligned} s_3 = & M(-2M^2\mu w^2 + M^2\mu w + 4M\mu w^2 + M^3 \\ & - 3M^2\mu - M\mu w - 2\mu w^2 + M^2 - M\mu + 2\mu) \\ & - a \cdot (M - 1)[M((M - 1)^2 - (w - 2)^2) + (1 - w)(M^2 - w - 1)] \end{aligned}$$

The expression in the square brackets is non-negative for $M \geq 3$, so the minimum of s_3 is achieved when $a = 1$; i.e.

$$\begin{aligned} s_3 \geq s_3|_{a=1} = & -2M^3\mu w^2 + M^3\mu w + 4M^2\mu w^2 \\ & - 3M^3\mu + M^3w - M^2\mu w + M^2w^2 \\ & - 2M\mu w^2 + 3M^3 - M^2\mu - 5M^2w - 2Mw^2 \\ & + 2M^2 + 2M\mu + 4Mw + w^2 - 2M - 1 =: s_4 \end{aligned}$$

But

$$\frac{\partial s_4}{\partial \mu} = -M((3 - w)M^2 + (1 + w)M - 2 + 2(M - 1)^2w^2) < 0$$

so

$$s_4 \geq s_4|_{\mu=1} = (1-w)(M-1)(wM(2M-3) + M + w + 1) \geq 0.$$

Case 3: $l_3 \leq 0$

Here

$$n(\mathbf{A}_3) = -(M+1)(1-a)s_5$$

and it suffices to show that $s_5 \geq 0$. If $M = 2$, then $\mu = (a+1)/2$ and $p \geq 3/4$, so

$$s_5 = 3(3-2p) \left[(1-a)^2(8p-5) + (32(1-a)^2 + 144a)(1-p)^4 + 12(1-p)^2(4p + a(4ap + 10p - 3)) \right] \geq 0.$$

For $M \geq 3$, let $M = 3 + \delta$, $\delta = 0, 1, \dots$. Then $s_5 = \sum_{i=0}^5 e_{i+1} \delta^i$ where we will show that all $e_i \geq 0$. Indeed, we have

$$\begin{aligned} e_1 &= -432a\mu p^5 - 1296\mu^2 p^5 + 288a^2 p^4 + 2736a\mu p^4 \\ &\quad - 144ap^5 + 432p^4 \mu^2 - 432\mu p^5 - 1632p^3 a^2 \\ &\quad - 2880a\mu p^3 + 1200ap^4 + 4320\mu^2 p^3 + 2736\mu p^4 \\ &\quad + 2624p^2 a^2 - 1152a\mu p^2 - 3744ap^3 - 1728\mu^2 p^2 \\ &\quad - 2880\mu p^3 + 288p^4 - 1536pa^2 + 4928p^2 a - 1152p^2 \mu \\ &\quad - 1632p^3 + 768a^2 - 3072ap + 2624p^2 \\ &\quad + 1536a - 1536p + 768 \\ e_2 &= -288a\mu p^5 - 1296\mu^2 p^5 + 168a^2 p^4 + 2208a\mu p^4 - 48ap^5 \\ &\quad - 360p^4 \mu^2 - 288\mu p^5 - 1160p^3 a^2 \\ &\quad - 1392a\mu p^3 + 600ap^4 + 6984\mu^2 p^3 + 2208\mu p^4 \\ &\quad + 1760p^2 a^2 - 3840a\mu p^2 - 2264ap^3 - 2016\mu^2 p^2 \\ &\quad - 1392\mu p^3 + 168p^4 - 576pa^2 + 2720p^2 a - 3840p^2 \mu \\ &\quad - 1160p^3 + 768a^2 - 1152ap + 1760p^2 \\ &\quad + 1536a - 576p + 768 \\ e_3 &= -48a\mu p^5 - 432\mu^2 p^5 + 24a^2 p^4 + 576a\mu p^4 - 600p^4 \mu^2 \\ &\quad - 48\mu p^5 - 268p^3 a^2 + 216a\mu p^3 + 72ap^4 \\ &\quad + 4404\mu^2 p^3 + 576\mu p^4 + 324p^2 a^2 - 3240a\mu p^2 \\ &\quad - 412ap^3 - 876\mu^2 p^2 + 216\mu p^3 + 24p^4 + 336pa^2 \\ &\quad + 180p^2 a - 3240p^2 \mu - 268p^3 + 288a^2 + 672ap \\ &\quad + 324p^2 + 576a + 336p + 288 \\ e_4 &= -48\mu^2 p^5 + 48a\mu p^4 - 216p^4 \mu^2 - 20p^3 a^2 + 192a\mu p^3 \\ &\quad + 1356\mu^2 p^3 + 48\mu p^4 - 4p^2 a^2 - 1164a\mu p^2 \\ &\quad - 20ap^3 - 168\mu^2 p^2 + 192\mu p^3 + 228pa^2 - 112p^2 a \end{aligned}$$

$$\begin{aligned}
& -1164p^2\mu - 20p^3 + 48a^2 + 456ap - 4p^2 \\
& + 96a + 228p + 48 \\
e_5 = & -24p^4\mu^2 + 24a\mu p^3 + 204\mu^2 p^3 - 4p^2 a^2 - 192a\mu p^2 \\
& - 12\mu^2 p^2 + 24\mu p^3 + 45pa^2 - 16p^2 a \\
& - 192p^2\mu + 3a^2 + 90ap - 4p^2 + 6a + 45p + 3 \\
e_6 = & 360p(a + 1 - 2\mu p)^2 \geq 0.
\end{aligned}$$

The fact that $e_6 \geq 0$ is trivial; we will prove separately that $e_1, \dots, e_5 \geq 0$ below. In what follows, we substitute $p = \frac{1+v}{2}$, where $v \in (0, 1)$.

Proof that $e_1 \geq 0$

We have

$$\frac{\partial^2 e_1}{\partial a^2} = 4[9v^4 - 66v^3 + 76v^2 + 2v + 235] > 0,$$

hence e_1 achieves its minimum at

$$a_{cr} = \frac{9v^5 - 105v^4 + 426v^3 - 46\mu v^2 + 397v - 1669 + 9\mu(1+v)^2(3v^3 - 29v^2 + 13v + 109)}{8[9v^4 - 66v^3 + 76v^2 + 2v + 235]}$$

which solves $\frac{\partial e_1}{\partial a} = 0$. Note that it is possible that $a_{cr} \notin [0, 1]$. However, in any case,

$$e_1 \geq e_1|_{a=a_{cr}} = \frac{1}{32} \cdot \frac{3(1+v)^2 c_1}{9v^4 - 66v^3 + 76v^2 + 2v + 235}$$

so it will suffice to show that

$$\begin{aligned}
c_1 = & 16(1-v)^2 c_{1a} + 3(1-\mu)c_{1b}, \quad \text{where} \\
c_{1a} = & -27v^6 + 144v^5 - 102v^4 + 1620v^3 - 9883v^2 + 12484v + 1732 \\
c_{1b} = & 81\mu v^8 - 108\mu v^7 + 135v^8 - 1260\mu v^6 \\
& - 828v^7 - 12276\mu v^5 + 276v^6 + 84774\mu v^4 \\
& - 4404v^5 - 157140\mu v^3 + 69170v^4 + 152628\mu v^2 \\
& - 198372v^3 - 156108\mu v + 182084v^2 \\
& + 27969\mu - 60588v + 73967
\end{aligned}$$

is positive. We have

$$c_{1a} = 3v^3(540 - 9v^3 + 48v^2 - 34v) + v(12484 - 9883v) + 1732 > 0.$$

Similarly,

$$c_{1b} = 61440(1-\mu) + (1-v)[c_{1b1} + c_{1b2}\mu]$$

where

$$\begin{aligned}
c_{1b1} = & (-135v^7 + 693v^6 + 417v^5 + 4821v^4) - 64349v^3 + 134023v^2 - 48061v + 12527 \\
& \geq -64349v^3 + 134023v^2 - 48061v + 12527 \geq 1000(-67v^3 + 134v^2 - 67v + 12)
\end{aligned}$$

$$= \frac{1000}{27} [56 + 67(4 - 3v)(1 - 3v)^2] > 0$$

and

$$\begin{aligned} c_{1b2} &= (-81v^7 + 27v^6 + 1287v^5 + 13563v^4) - 71211v^3 + 85929v^2 - 66699v + 894 \\ &\geq -71211v^3 + 85929v^2 - 66699v + 89409 > 80000(-v^3 + v^2 - v + 1) \geq 0. \end{aligned}$$

So, $c_{1b1}, c_{1b2} > 0 \implies c_{1b} > 0$ and since $c_{1a} > 0$ we have $c_1 \geq 0$ and thus $e_1 \geq 0$.

Proof that $e_2 \geq 0$

We have

$$\frac{\partial^2 e_2}{\partial a^2} = 21v^4 - 206v^3 + 136v^2 + 398v + 1571 > 0$$

so, similarly to the previous case,

$$e_2 \geq e_2|_{a=a_{cr}} = \frac{3(1+v^2)[582912(1-\mu)^2 + (1-v)c_2]}{8[21v^4 - 206v^3 + 136v^2 + 398v + 1571]}$$

where

$$a_{cr} = \frac{3v^5 - 60v^4 + 296v^3 - 82v^2 - 155v - 2786 + (18v^3 - 222v^2 - 150v + 2010)(1+v)^2\mu}{2[21v^4 - 206v^3 + 136v^2 + 398v + 1571]}$$

solves $\frac{\partial e_2}{\partial a} = 0$ and

$$\begin{aligned} c_2 &= 3v^7 - 123v^6 + 1330v^5 - 1918v^4 - 28897v^3 + 65177v^2 + 93100v + 120544 \\ &\quad + (36v^7 - 624v^6 + 348v^5 + 25616v^4 - 7332v^3 - 272368v^2 - 134556v + 688784) \\ &\quad + (108v^7 - 72v^6 - 4848v^5 - 35916v^4 + 247548v^3 - 252720v^2 + 144456v - 647676)\mu^2 \end{aligned}$$

Now,

$$\begin{aligned} \frac{\partial^2 c_2}{\partial \mu^2} &= v^4(216v^3 - 144v^2 - 9696v - 71832) \\ &\quad + v^3(495096v - 505440) + (288912v - 1295352) < 0 \end{aligned}$$

hence the minimum of c_2 w.r.t. $\mu \in [0, 1]$ can be achieved either at $\mu = 0$ or at $\mu = 1$. At the same time

$$\begin{aligned} c_2|_{\mu=0} &= 3v^7 + 1330v^5 + 65177v^2 + 93100v \\ &\quad + (120544 - 123v^6 - 1918v^4 - 28897v^3) > 0, \\ c_2|_{\mu=1} &= (1-v)(161652 - 147v^6 + 672v^5 + 3842v^4 \\ &\quad + 16060v^3 + (264652 - 195259v)v) \geq 0, \end{aligned}$$

so $c_2 \geq 0$ and hence $e_2 \geq 0$.

Proof that $e_3 \geq 0$

We have

$$\frac{\partial^2 e_3}{\partial a^2} = 3v^4 - 55v^3 - 21v^2 + 471v + 1010 > 0$$

so, similarly to the previous case,

$$e_3 \geq e_3|_{a=a_{cr}} = \frac{3(1+v)^2[(1-v)^2 c_{3a} + (1-\mu)c_{3b}]}{8(3v^4 - 55v^3 - 21v^2 + 471v + 1010)}$$

where

$$a_{cr} = \frac{-9v^4 + 67v^3 + 165v^2 - 579v - 1820 + 3(1+v)^2\mu(v^3 - 21v^2 - 63v + 499)}{2(3v^4 - 55v^3 - 21v^2 + 471v + 1010)}$$

solves $\frac{\partial e_3}{\partial a} = 0$ and

$$\begin{aligned} c_{3a} &= -3v^6 + 12v^5 + 632v^4 + 1794v^3 - 37624v^2 + 65244v + 64877 > 0 \\ c_{3b} &= 2(1-v)(-3v^7 + 12v^6 + 652v^5 + 2417v^4 \\ &\quad - 42561v^3 + 73864v^2 + 41336v + 91323) \\ &\quad + (1-\mu)(3v^6(220 - v^2 + 4v) \\ &\quad + 2v(1490v^4 - 22993v^3 + 39898v^2 + 890v + 109262) + 8477) \geq 0 \end{aligned}$$

Hence $e_3 \geq 0$.

Proof that $e_4 \geq 0$

We have

$$\frac{\partial^2 e_4}{\partial a^2} = 209v + 317 - 5v^3 - 17v^2 > 0$$

so, similarly to the previous case,

$$e_4 \geq e_4|_{a=a_{cr}} = \frac{3(1+v)^2[(1-v)^2 c_{4a} + 4(1-\mu)c_{4b}]}{8(209v + 317 - 5v^3 - 17v^2)}$$

where

$$a_{cr} = \frac{5v^3 + 71v^2 - 329v - 587 + 6\mu(1+v)^2(88 - 10v - v^2)}{2(209v + 317 - 5v^3 - 17v^2)}$$

solves $\frac{\partial e_4}{\partial a} = 0$ and

$$\begin{aligned} c_{4a} &= 8v^4 + 40v^3 - 1395v^2 + 4354v + 4757 > 0 \\ c_{4b} &= 4(1-v)(4v^5 + 21v^4 - 712v^3 + 2011v^2 + 3102v + 3050) \\ &\quad + (1-\mu)(2v^6 + 11v^5 - 360v^4 + 912v^3 + 1705v^2 + 3655v + 543) \geq 0. \end{aligned}$$

Hence $e_4 \geq 0$.

Proof that $e_5 \geq 0$

We have

$$\frac{\partial^2 e_5}{\partial a^2} = 49 + 41v - 2v^2 > 0$$

so, similarly to the previous case,

$$e_5 \geq e_5|_{a=a_{cr}} = \frac{3(1+v)^2[(1-v)^2 c_{5a} + (1-\mu)c_{5b}]}{2(49 + 41v - 2v^2)}$$

where

$$a_{cr} = \frac{4v^2 - 37v - 47 + 3\mu(15-v)(1+v)^2}{49 + 41v - 2v^2}$$

solves $\frac{\partial e_5}{\partial a} = 0$ and

$$\begin{aligned} c_{5a} &= 15 - v^2 + 15v > 0 \\ c_{5b} &= 2(1-v)(14v^2 + 28v + 19 - v^3) \\ &\quad + (1-\mu)(13v^3 + 40v^2 + 49v + 11 - v^4) \geq 0. \end{aligned}$$

Hence $e_5 \geq 0$. As a result, $s_5 \geq 0$ and thus $\mathbf{I}_3 \leq 0$.

Case 4: $\mathbf{I}_4 \leq 0$

Here

$$\begin{aligned} n(\mathbf{A}_4) &= -4(M\mu p - M + p - 1)^2 s_6, \\ s_6 &= 2p - 2 + (3\mu + 6p - 3 - 4\mu p - 2p^2)M \\ &\quad + (4\mu p^2 - 5\mu p + 3\mu - 2p)M^2 + (1 - \mu p)M^3 \end{aligned}$$

Then, substituting $M = 2 + \delta$,

$$\begin{aligned} \frac{\partial s_6}{\partial \delta} &= [5(1-\mu) + 2(1-p)(p+2+10\mu-8\mu p)] \\ &\quad + [8(1-\mu) + 2(1-p)(2+7\mu-4\mu p)]\delta \geq 0 \end{aligned}$$

and as a result for $\delta \geq 0$ we have

$$s_6 \geq s_6|_{\delta=0} = 2(3-2p)[p(1-\mu) + \mu(1-3p)] \geq 0.$$

Case 5: $\mathbf{I}_5 \leq 0$

Here

$$n(\mathbf{A}_5) = -s_7.$$

We need to show that $s_7 \geq 0$ when $X_1 \leq 0$ and $X_3 \geq 0$.

Since $X_1 \leq 0$, we have $2Mp\mu \leq M+1$. Together with $X_3 \geq 0$ this implies

$$0 \leq n(X_3) = 2Mp\mu - (M+1) - a(M+1) + 2p \leq -a(M+1) + 2p$$

whence

$$a \leq \frac{2p}{M+1}.$$

Let us show that for this a we have $s_7 \geq 0$; substitute $a = b \cdot \frac{2p}{M+1}$, where $b \in [0, 1]$.

First, let $\boxed{M=2}$, then $\mu = \frac{1+a}{2}$, $p \in [3/4, 1)$, and $s_7 = \frac{3-2p}{27}s_8$ where

$$\begin{aligned} s_8 = & 512b^3p^8 - 2688b^3p^7 + 5760b^3p^6 + 3456b^2p^7 \\ & - 6912b^3p^5 - 12672b^2p^6 + 5184b^3p^4 \\ & + 16416b^2p^5 + 5184bp^6 - 1944b^3p^3 - 11664b^2p^4 \\ & - 10368bp^5 + 7776b^2p^3 + 1728p^5 - 2916b^2p^2 \\ & + 11664bp^3 - 11664bp^2 - 7776p^3 + 4374bp + 17496p^2 - 17496p + 6561 \end{aligned}$$

Note that we can write $s_8 = e_1 + e_2(1-p) + e_3(1-p)^2$, where

$$\begin{aligned} 128e_1 &= (9-v^2-6v)(81-v^3-9v^2-63v)(v^3+15v^2+81-9v) > 0 \\ 128e_2 &= 3(9-v^2)(v^6+21v^5+168v^4+666v^3+81v^2+81v+486) > 0 \\ 64(e_1+e_3) &= [2v^8+33v^7+234v^6+783v^5] \\ &\quad + [-648v^4-6561v^3+30618v^2-28431v+13122] \\ &\geq -648v^4-6561v^3+30618v^2-28431v+13122 \\ &\geq -1000v^4-7000v^3+24000v^2-29000v+13000 \\ &= 1000(1-v)(5+8(1-v)^2+v^3) \geq 0. \end{aligned}$$

with $p = \frac{3+v}{4}$, $v \in [0, 1]$. Consequently, since $(1-p)^2 < 1$ and $e_1 > 0$,

$$s_8 = e_1 + e_2(1-p) + e_3(1-p)^2 \geq e_2(1-p) + (e_1+e_3)(1-p)^2 \geq 0$$

and thus $s_7 \geq 0$ as required.

For $\boxed{M \geq 3}$, set $M = 3 + \delta$, $\delta \geq 0$. Then

$$s_7 = \sum_{i=0}^9 e_{i+1} \delta^i$$

where

$$\begin{aligned} e_1 &= 196608 + (98304b - 393216)p \\ &\quad + (-49152b^2 - 442368\mu^2 - 196608b - 737280\mu + 589824)p^2 \\ &\quad + (-24576b^3 + 221184b\mu^2 + 98304b^2 \\ &\quad + 1990656\mu^2 + 294912b + 663552\mu - 540672)p^3 \\ &\quad + (49152b^3 + 73728b^2\mu - 552960b\mu^2 - 331776\mu^3 \\ &\quad - 147456b^2 - 2322432\mu^2 - 270336b \\ &\quad + 110592\mu + 233472)p^4 \\ &\quad + (-83968b^3 + 184320b^2\mu - 55296b\mu^2 + 774144\mu^3 \end{aligned}$$

$$\begin{aligned}
& + 135168b^2 + 1050624\mu^2 + 116736b \\
& - 239616\mu - 36864)p^5 \\
& + (52224b^3 - 175104b^2\mu + 165888b\mu^2 - 331776\mu^3 \\
& - 58368b^2 - 165888\mu^2 - 18432b + 55296\mu)p^6 \\
& + (-9216b^3 + 27648b^2\mu + 9216b^2)p^7 \\
e_2 = & 393216 + (172032b - 540672)p \\
& + (-73728b^2 - 958464\mu^2 - 221184b - 2088960\mu + 835584)p^2 \\
& + (-30720b^3 + 423936b\mu^2 + 86016b^2 + 4589568\mu^2 \\
& + 344064b + 1898496\mu - 823296)p^3 \\
& + (30720b^3 + 282624b^2\mu - 1308672b\mu^2 - 663552\mu^3 \\
& - 135168b^2 - 5031936\mu^2 - 344064b \\
& - 18432\mu + 344064)p^4 \\
& + (-77312b^3 + 181248b^2\mu + 4608b\mu^2 + 1658880\mu^3 \\
& + 138240b^2 + 2068992\mu^2 + 142848b \\
& - 360960\mu - 49152)p^5 \\
& + (50176b^3 - 228864b^2\mu + 290304b\mu^2 - 691200\mu^3 \\
& - 56832b^2 - 290304\mu^2 - 19968b + 78336\mu)p^6 \\
& + (-7680b^3 + 32256b^2\mu + 7680b^2)p^7 \\
e_3 = & 344064 + (129024b - 208896)p + (-46080b^2 - 906240\mu^2 \\
& - 49152b - 2558976\mu + 430080)p^2 \\
& + (-15360b^3 + 347136b\mu^2 + 3072b^2 + 4718592\mu^2 \\
& + 129024b + 2217984\mu - 522240)p^3 \\
& + (-6144b^3 + 334848b^2\mu - 1337856b\mu^2 - 566784\mu^3 \\
& - 30720b^2 - 4778496\mu^2 - 175104b \\
& - 198144\mu + 210432)p^4 \\
& + (-24448b^3 + 42240b^2\mu + 100992b\mu^2 + 1543680\mu^3 \\
& + 52992b^2 + 1744512\mu^2 + 69504b \\
& - 223872\mu - 26112)p^5 \\
& + (17856b^3 - 118464b^2\mu + 210816b\mu^2 - 615168\mu^3 \\
& - 20544b^2 - 210816\mu^2 - 8064b + 44160\mu)p^6 \\
& + (-2112b^3 + 14016b^2\mu + 2112b^2)p^7 \\
e_4 = & 172032 + (53760b + 64512)p \\
& + (-15360b^2 - 488448\mu^2 + 44544b - 1790976\mu + 64512)p^2 \\
& + (-3840b^3 + 157440b\mu^2 - 23040b^2 \\
& + 2843904\mu^2 + 1416960\mu - 176640)p^3
\end{aligned}$$

$$\begin{aligned}
& + (-9984b^3 + 193536b^2\mu - 772608b\mu^2 \\
& - 268032\mu^3 + 7680b^2 - 2598912\mu^2 - 44544b \\
& - 170496\mu + 68352)p^4 \\
& + (93024b\mu^2 - 13632b^2\mu + 32\mu(25448\mu^2 + 25515\mu - 2283) \\
& - 32b(77b^2 - 282b - 525) - 6912)p^5 \\
& + (2784b^3 - 30336b^2\mu + 81312b\mu^2 - 303168\mu^3 \\
& - 3264b^2 - 81312\mu^2 - 1440b + 12384\mu)p^6 \\
& + (-192b^3 + 2688b^2\mu + 192b^2)p^7 \\
e_5 = & 53760 + (13440b + 91392)p + (-2880b^2 \\
& - 164160\mu^2 + 34560b - 792768\mu - 26880)p^2 \\
& + (-480b^3 + 42720b\mu^2 - 11520b^2 \\
& + 1109088\mu^2 - 13440b + 548640\mu - 33600)p^3 \\
& + (62496b^2\mu - 275952b\mu^2 - 885744\mu^2 - 67536\mu \\
& - 75792\mu^3 - 96b(34b^2 - 50b + 59) + 12432)p^4 \\
& + (160b^3 - 8544b^2\mu + 38928b\mu^2 + 266192\mu^3 + 576b^2 \\
& + 229104\mu^2 + 2016b - 13200\mu - 912)p^5 \\
& + (160b^3 - 3840b^2\mu + 17568b\mu^2 - 89344\mu^3 - 192b^2 \\
& - 17568\mu^2 - 96b + 1728\mu)p^6 + 192b^2\mu p^7 \\
e_6 = & 10752 + (2016b + 38976)p \\
& + (-288b^2 - 35232\mu^2 + 10848b - 230880\mu - 14784)p^2 \\
& + (-24b^3 + 6936b\mu^2 - 2544b^2 + 290664\mu^2 \\
& - 4032b + 132888\mu - 3408)p^3 \\
& + (11568b^2\mu - 456b^3 - 62472b\mu^2 - 12816\mu^3 + 816b^2 \\
& - 193752\mu^2 - 288b - 14328\mu + 1200)p^4 \\
& + (32b^3 - 1536b^2\mu + 8688b\mu^2 + 55168\mu^3 \\
& + 38544\mu^2 + 96b - 1248\mu - 48)p^5 \\
& + (-192b^2\mu + 2016b\mu^2 - 15744\mu^3 - 2016\mu^2 + 96\mu)p^6 \\
e_7 = & 1344 + (168b + 9072)p + (-12b^2 - 4716\mu^2 \\
& + 1824b - 44340\mu - 3024)p^2 \\
& + (624b\mu^2 - 276b^2 + 51252\mu^2 - 504b + 19764\mu - 144)p^3 \\
& + (-24b^3 + 1152b^2\mu - 8760b\mu^2 - 1200\mu^3 \\
& + 48b^2 - 26568\mu^2 - 1584\mu + 48)p^4 \\
& + (-96b^2\mu + 1008b\mu^2 + 7072\mu^3 + 3600\mu^2 - 48\mu)p^5 \\
& + (96b\mu^2 - 1536\mu^3 - 96\mu^2)p^6 \\
e_8 = & 96 + (6b + 1236)p + (-360\mu^2 + 162b - 5424\mu - 300)p^2
\end{aligned}$$

$$\begin{aligned}
& + (24b\mu^2 - 12b^2 + 5868\mu^2 - 24b + 1656\mu)p^3 \\
& + (48b^2\mu - 696b\mu^2 - 48\mu^3 - 2088\mu^2 - 72\mu)p^4 \\
& + (48b\mu^2 + 512\mu^3 + 144\mu^2)p^5 - 64\mu^3p^6
\end{aligned}$$

and the expressions for e_9 and e_{10} are given a little bit further.

First, we will show that $e_i \geq 0$, $i = 1, \dots, 8$.

Proof that $e_1, \dots, e_8 > 0$

It turns out that it is easiest is to use a computer-assisted proof in this case; to this end we developed the method which we call a *Box method*; it may have been described by other authors, but since we do not have the reference to the right source, we give its description below.

First of all, we substitute

$$p = \frac{1+x_1}{2}, \quad b = x_2, \quad \mu = x_3; \quad x_i \in [0, 1], \quad i = 1, 2, 3.$$

Let $m = \min_{a_i \leq x_i \leq b_i, i=1,2,3} f(x_1, x_2, x_3)$ where

$$f(x_1, x_2, x_3) = f_+(x_1, x_2, x_3) - f_-(x_1, x_2, x_3)$$

and f_+ and f_- are polynomials with non-negative coefficients. We want to show that $m > 0$.

Let

$$G_{f;M} = \min_{i_1, i_2, i_3=0, \dots, M-1} \left[f_+ \left(\frac{i_1}{M}, \frac{i_2}{M}, \frac{i_3}{M} \right) - f_- \left(\frac{i_1+1}{M}, \frac{i_2+1}{M}, \frac{i_3+1}{M} \right) \right].$$

Since

$$m \geq G_{f;M} \rightarrow m$$

as $M \rightarrow \infty$, we conclude that $m > 0$ if and only if $G_{f;M} \geq 0$ for some $M \geq 1$. Checking that $G_{f;M} \geq 0$ can be quite tedious and time-consuming for large M , however, this could be easily accomplished with the help of a computer; please note, that the results are still *completely rigorous*, unlike e.g. simulations.

The results of application of this method to $e_1, \dots, e_8$ are presented in the following table:

$$\begin{aligned}
G_{e_1,2000} &> 825, & G_{e_2,500} &> 25, & G_{e_3,400} &> 1860, & G_{e_4,300} &> 2397, \\
G_{e_5,200} &> 672, & G_{e_6,200} &> 148, & G_{e_7,200} &> 5, & G_{e_8,400} &> 3.
\end{aligned}$$

Consequently, $e_j > 0$ for all $j = 1, \dots, 8$.

Proof that $e_9 \geq 0$ and $e_{10} \geq 0$

The Box method of the previous section would not work for e_9 and e_{10} , since these functions do touch zero in the required area, and hence the minimum is, in fact, 0. Therefore, we have to handle these two cases analytically.

We have

$$\begin{aligned}
e_9 &= 4p^2\mu(4\mu^2p^3 - 18\mu p^2 + 99\mu p - 3 + 15p - 96) \\
&\quad - 12p^2 + 93p + 3 + [6p^2(1 - 2\mu p)(2\mu p + 1)]b,
\end{aligned}$$

hence, the minimum is achieved either at $b = 0$ or $b = 1$.

For $\mu < 1/(2p)$ we have $e_9 \geq e_{9a}$, where

$$\begin{aligned} e_{9a} &= e_9|_{b=0} = 2s^3p^2 - 18p^2s^2 + 30sp^2 + 99s^2p - 12p^2 - 192ps - 3s^2 + 93p + 3 \\ &= 2p^2 + (1-s)[6(1-p) + (1-s)(99p + 2p^2s - 14p^2 - 3)] \geq 0 \end{aligned}$$

where $s = 2p\mu \in [0, 1]$.

In case $\mu \geq 1/(2p)$ we have $e_9 \geq e_{9b}$, where

$$\begin{aligned} e_{9b} &= e_9|_{b=1} = 16p^5s^3 - 24p^4s^3 - 72p^4s^2 + 12p^3s^3 \\ &\quad + 468p^3s^2 - 2s^3p^2 - 24p^3s - 426p^2s^2 \\ &\quad + 24sp^2 + 111s^2p + 2p^2 - 18ps - 3s^2 + 6s \end{aligned}$$

where $\mu = \frac{1}{2p} + s \left(1 - \frac{1}{2p}\right)$, $s \in [0, 1]$. Now,

$$\frac{\partial^2}{\partial s^2} e_{9b} = 6(2p-1)^2(14 + (2p-1)(2p^2s - 3p + 15)) \geq 0$$

so the minimum of e_{9b} w.r.t. s is achieved where $\frac{\partial}{\partial s} e_{9b} = 0$, i.e.

$$s_{cr} = \frac{6p^2 - 33p + 1 + R}{2p^2(2p-1)}, \quad \text{where } R = \sqrt{44p^4 - 400p^3 + 1105p^2 - 66p + 1}$$

and equals

$$\begin{aligned} &\frac{3996p^5 - 284p^6 - 19956p^4 + 37329p^3 - 3291p^2 + 99p - 1 + (400p^3 - 44p^4 - 1105p^2 + 66p - 1)R}{2p^4} \\ &\geq 22120.5 - 1576\sqrt{197} = 0.285896 > 0 \end{aligned}$$

for $p \geq 1/2$.

Finally, trivially, we have $e_{10} = 3p(2\mu p - 1)^2 \geq 0$. Consequently, $s_7 \geq 0$ and $\mathbf{I}_5 \leq 0$.

Combining this with the previously established inequalities $\mathbf{I}_j \leq 0$, $j = 1, 2, 3, 4$, we complete the proof Lemma 2.

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