J Sched (2012) 15:63–75
DOI 10.1007/s10951-009-0110-3

Absolute approximation ratios for packing rectangles into bins

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Received: 27 February 2009 / Accepted: 9 March 2009 / Published online: 25 April 2009

Abstract We consider the problem of packing rectangles into bins that are unit squares, where the goal is to minimize the number of bins used. All rectangles have to be packed non-overlapping and orthogonal, i.e., axis-parallel. We present an algorithm with an absolute worst-case ratio of 2 for the case where the rectangles can be rotated by 90 degrees. This is optimal provided $\mathcal{P} \neq \mathcal{NP}$. For the case where rotation is not allowed, we prove an approximation ratio of 3 for the algorithm HYBRID FIRST FIT which was introduced by Chung et al. (SIAM J. Algebr. Discrete Methods 3(1):66–76, 1982) and whose running time is slightly better than the running time of Zhang's previously known 3-approximation algorithm (Zhang in Oper. Res. Lett. 33(2):121–126, 2005).

Keywords Bin packing · Rectangle packing · Approximation algorithm · Absolute worst-case ratio

1 Introduction

In the rectangle packing problem, a list $I = \{r_1, \ldots, r_n\}$ of rectangles of width $w_i \leq 1$ and height $h_i \leq 1$ is given. An unlimited supply of unit sized bins is available to pack all items from I such that no two items overlap and all items are packed axis-parallel into the bins. The goal is to minimize the number of bins used. The problem is also known as two-dimensional orthogonal bin packing and has many applications, for instance in stock-cutting or scheduling on partitionable resources. In many applications, rotations are not allowed because of the pattern of the cloth or the grain of the wood. However, in other applications, it might be possible to rotate the items.

Most of the previous work on rectangle packing has focused on the *asymptotic* approximation ratio, i.e., the long term behavior of the algorithm, and on packing *without rotations*. Chung et al. (1982) proposed the algorithm HYBRID FIRST FIT and proved that its asymptotic approximation ratio is at most 2.125. Caprara was the first to present an algorithm with an asymptotic approximation ratio less than 2 for rectangle packing without rotations. Indeed, he considered 2-stage packing, in which the items must first be packed into shelves that are then packed into bins, and showed that the asymptotic worst case ratio between rectangle packing and 2-stage packing is $T_\infty = 1.691\ldots$. Therefore, the asymptotic $\mathcal{FPTAS}$ for 2-stage packing from Caprara et al. (2005) achieves an approximation guarantee arbitrary close to T_∞.

Recently, Bansal et al. (2006a) presented a general framework to improve subset oblivious algorithms and obtained asymptotic approximation guarantees arbitrarily close to $1.525\ldots$ for packing with or without rotations. These are the currently best-known approximation ratios for these problems. For packing squares into square bins, Bansal et al. (2006b) gave an asymptotic $\mathcal{PTAS}$. On the other hand, the same paper showed the $\mathcal{APX}$-hardness of rectangle packing without rotations, thus no asymptotic $\mathcal{PTAS}$ exists unless $\mathcal{P} = \mathcal{NP}$. Chlebík and Chlebíková (2006) were the first

A preliminary version was published in the Proceedings of the 11th Scandinavian Workshop on Algorithm Theory (SWAT): LNCS 5124, pp. 306–318, Springer-Verlag Berlin Heidelberg 2008.
Research supported by German Research Foundation (DFG). Parts of this work were done while this author was at the University of Karlsruhe.

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to give explicit lower bounds of $1 + 1/3792$ and $1 + 1/2196$ on the asymptotic approximability of rectangle packing with and without rotations, respectively.

In the current paper, we consider the absolute worst-case ratio. Attaining a good absolute worst-case ratio is more difficult than attaining a good asymptotic worst-case ratio, because in the second case an algorithm is allowed to waste a constant number of bins, which allows, e.g., the classification of items followed by a packing where each class is packed separately. Zhang (2005) presented an approximation algorithm with an absolute approximation ratio of 3 for the problem without rotations. For the special case of packing squares, van Stee (2004) showed that an absolute 2-approximation is possible.

A related two-dimensional packing problem is the strip packing problem, where the items have to be packed into a strip of unit basis and unlimited height such that the height is minimized. Steinberg (1997) and Schiermeyer (1994) presented absolute 2-approximation algorithms for strip packing without rotations. Kenyon and Rémila (2000) and Jansen and van Stee (2005) gave asymptotic $\mathcal{FPTAS}$s for the problem without rotations and with rotations, respectively. The additive constant of these algorithms was recently improved from $\mathcal{O}(1/\varepsilon^2)$ to 1 by Jansen and Solis-Oba (2007). Thus, most versions of the strip packing problem are now closed.

Our contribution We present an approximation algorithm for rectangle packing with rotations with an absolute approximation ratio of 2. As Leung et al. (1990) showed that it is strongly $\mathcal{NP}$-complete to decide whether a set of *squares* can be packed into a given square, this is best possible unless $\mathcal{P} = \mathcal{NP}$. The algorithm is based on a separation of large and small items according to their area. It is very time-efficient for inputs consisting of small items but uses a less efficient subroutine to deal with large items. Our main lemma on the packability of certain sets of small items is of independent interest.

Furthermore, we prove Zhang's conjecture (Zhang 2005) on the absolute approximation ratio of the HYBRID FIRST FIT (HFF) algorithm by showing that this ratio is 3.

Organization In Sect. 2, we present our absolute 2-approximation algorithm for bin packing with rotations. In Sect. 3, we analyze the algorithm HFF and prove that it has absolute approximation guarantee 3.

2 An absolute 2-approximation algorithm for bin packing with rotations

We start our presentation in Sect. 2.1 with the introduction of notations and two algorithms for strip packing that we will use as subroutines for our rectangle packing algorithm: the algorithm of Steinberg and NEXT FIT DECREASING HEIGHT. We show that Steinberg's algorithm (Steinberg 1997) yields an absolute 2-approximation for strip packing with rotations and an absolute 4-approximation for rectangle packing with rotations. After that, we show in Sect. 2.2 that a first approach based on an algorithm of Jansen and Solis-Oba (2007) does not lead to the desired approximation ratio. Our main result is presented in Sect. 2.3. The algorithm is based on our main lemma that we prove in Sect. 2.4.

2.1 Steinberg's algorithm and next fit decreasing height

We assume that all items are rotated such that $w_i \geq h_i$. Denote the total area of a given set T of items by $\mathcal{A}(T) = \sum_{i\in T} w_i h_i$ and let $w_{\max} := \max_{r_i \in T} w_i$ and $h_{\max} := \max_{r_i\in T} h_i$. Steinberg (1997) showed the following theorem.

Theorem 1 (Steinberg's algorithm, Steinberg 1997) *If the following inequalities hold,*

$$w_{\max} \leq a, \qquad h_{\max} \leq b,$$

$$2\mathcal{A}(T) \leq ab - (2w_{\max} - a)_+(2h_{\max} - b)_+$$

where $x_+ = \max(x, 0)$, *then it is possible to pack all items from* T *into* $R = (a, b)$ *in time* $\mathcal{O}((n \log^2 n)/\log\log n)$.

In our algorithm, we will repeatedly use the following direct corollary of this theorem.

Corollary 1 *If* $w_{\max} \leq a/2$, $h_{\max} \leq b$ *and* $\mathcal{A}(T) \leq ab/2$, *then it is possible to pack all items from* T *into* $R = (a, b)$ *in time* $\mathcal{O}((n \log^2 n)/\log\log n)$.

The following theorem was already mentioned in (Jansen and Solis-Oba 2007).

Theorem 2 *Steinberg's algorithm gives an absolute 2-approximation for strip packing with rotations.*

Proof Rotate all items $r_i \in I$ such that $w_i \geq h_i$ and let $b := \max(2h_{\max}, 2\mathcal{A}(I))$. Use Steinberg's algorithm to pack I into the rectangle $(1, b)$. This is possible since $2\mathcal{A}(I) \leq b$ and $(2h_{\max} - b)_+ = 0$. The claim on the approximation ratio follows from $\mathrm{OPT} \geq \max(h_{\max}, \mathcal{A}(I)) = b/2$. □

It is well-known that a strip packing algorithm with an approximation ratio of δ directly yields a rectangle packing algorithm with an approximation ratio of 2δ. To see this, cut the strip packing of height h into slices of height 1 so as to get $\lceil h\rceil$ bins of the required size. The rectangles that are split between two bins can be packed into $\lfloor h\rfloor$ additional bins. The strip packing gives a lower bound for rectangle packing. Thus, if $h \leq \delta\,\mathrm{OPT}_{\text{strip}}$, then $\lceil h\rceil + \lfloor h\rfloor \leq 2\delta\,\mathrm{OPT}_{\text{bin}}$. Accordingly, we get the following theorem.

Theorem 3 *Steinberg's algorithm yields an absolute* 4*-approximation algorithm for rectangle packing with rotations*.

Jansen and Zhang (2007) showed a corollary of Steinberg's theorem, which reads as follows if $w_i \geq h_i$ for all items.

Corollary 2 (Jansen and Zhang 2007) *If the total area of a set T of items is at most $1/2$ and there is at most one item of height $h_i > 1/2$, then the items of T can be packed into a bin of unit size in time $\mathcal{O}((n \log^2 n)/\log\log n)$.*

The Next Fit Decreasing Height (NFDH) algorithm was introduced for squares by Meir and Moser (1968) and generalized to rectangles by Coffman et al. (1980). It is given as follows. Sort the items by non-increasing order of height. Pack the items one by one into layers. The height of a layer is defined by its first item, further items are added left-aligned until an item does not fit. In this case this item opens a new layer. The algorithm stops if it runs out of items or a new layer does not fit into the designated area. The running time of the algorithm is $\mathcal{O}(n \log n)$. The following lemma is an easy generalization of the result from Meir and Moser.

Lemma 1 *If a given set T of items is packed into a rectangle $R = (a, b)$ by* NFDH, *then either a total area of at least $(a - w_{\max})(b - h_{\max})$ is packed or the algorithm runs out of items, i.e., all items are packed.*

2.2 First approach

We started our investigation on the problem with rotations with an algorithm of Jansen and Solis-Oba (2007) that finds a packing of profit $(1-\delta)\,\mathrm{OPT}(I)$ into a bin of size $(1, 1+\delta)$, where $\mathrm{OPT}(I)$ denotes the optimum for packing into a unit bin. Using the area of the items as their profit gives an algorithm that packs almost everything into an δ-augmented bin. The algorithm can easily be generalized to a constant number of bins.

An immediate idea to transform such a packing to a packing into $2\,\mathrm{OPT}(I)$ bins is to remove all items that intersect a strip of height δ at the top or bottom of each bin. These items and the items that were not packed by the algorithm would have to be packed separately. In Fig. 1, we present an instance where it is not immediately clear how the removed items can be packed separately. Let A_{top} be the set of items that intersect $y = 1$, A_{bottom} be the set of items that intersect $y = \delta$, and R be the set of remaining items. As shown in Fig. 1, the sets A_{top} and A_{bottom} can both have total area arbitrary close to or even larger than $1/2$ (as both sets are not necessarily disjoint). Thus, adding the additional items R and packing everything with Steinberg's algorithm (see Theorem 1) is not necessarily possible. Furthermore, it is not obvious how to rearrange A_{top} or A_{bottom} such that there is suitable free space to pack R and the items that are above A_{top} or below A_{bottom}.

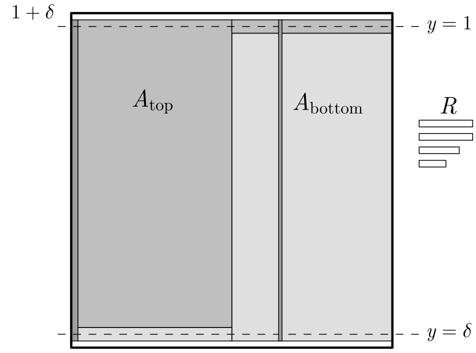

Fig. 1 Packing of Jansen and Solis-Oba's algorithm where it is not immediately clear how to derive a packing into 2 unit bins. The blocks in the packing might consist of several items and might contain small free spaces or items that are not in A_{top} or A_{bottom}. Furthermore, there might be items (printed in *dark grey*) that are in A_{top} and in A_{bottom}

2.3 Our algorithm: overview

As the asymptotic approximation ratio of the algorithm from Bansal et al. (2006a) is less than 2, there exists a constant k such that for any instance with optimal value larger than k the asymptotic algorithm gives a solution of value at most $2\,\mathrm{OPT}(I)$. We address the problem of approximating rectangle packing with rotations within an absolute factor of 2, provided that the optimal value of the given instance is less than k. Combined with the algorithm from Bansal et al. (2006a) we get an overall algorithm with an absolute approximation ratio of 2.

We begin by applying the asymptotic algorithm from Bansal et al. (2006a). Since we do not know whether $\mathrm{OPT}(I) > k$, we apply a second algorithm that is described in the remainder of this section. If $\mathrm{OPT}(I) > k$, then this algorithm might fail as the asymptotic algorithm outputs a solution of value $k' \leq 2\,\mathrm{OPT}(I)$.

Let $\varepsilon := 1/68$. We separate the given input according to the area of the items, so we get a set of large items $L = \{r_i \in I \mid w_i h_i \geq \varepsilon\}$ and a set of small items $S = \{r_i \in I \mid w_i h_i < \varepsilon\}$. If $\mathcal{A}(L) > k$ then $\mathrm{OPT}(I) > k$ and the algorithm halts. Otherwise, we can enumerate all possible packings of the large items since the number of large items in each bin is bounded by $1/\varepsilon$ and their total area is at most k. Take an arbitrary packing of the large items into a minimum number $\ell \leq k$ of bins. If no such packing exists then the asymptotic algorithm from Bansal et al. (2006a) finds a suitable solution and our algorithm halts.

If there are bins that contain items with a total area less than $1/2 - \varepsilon$, we greedily add small items such that the total area of items assigned to each of these bins is in $(1/2-\varepsilon, 1/2]$. We use the method of Jansen and Zhang (2007) to repack these bins including the newly assigned small items. This is possible by Corollary 2. There is at most one item of height $h_i > 1/2$ since otherwise the total area exceeds $1/2$, because $w_i \geq h_i$. If we ran out of items in this step, we found an optimal solution. Assume that there are still small items left and each bin used so far contains items of a total area of at least $1/2-\varepsilon$. The following crucial lemma shows that we can pack the remaining small items well enough to achieve an absolute approximation ratio of 2.

Lemma 2 *Let $0 < \varepsilon \leq 1/68$ and let T be a set of items that all have area at most ε such that for all $r \in T$ the total area of $T \setminus \{r\}$ is less than $1/2+\varepsilon$. We can find a packing of T into a unit bin in time $\mathcal{O}((n \log^2 n)/\log\log n)$.*

The lemma is proved in the next section. To apply Lemma 2 we consider the following partition of the remaining items.

Let $r_1, \dots, r_m$ be the list of remaining small items, sorted by non-increasing order of size. Partition these small items into sets $S_1 = \{r_{t_1}, \dots, r_{t_2-1}\}, S_2 = \{r_{t_2}, \dots, r_{t_3-1}\}, \dots, S_s = \{r_{t_s}, \dots, r_{t_{s+1}-1}\}$ with $t_1 = 1$ and $t_{s+1} = m+1$ such that

$$\mathcal{A}(S_j \setminus \{r_{t_{j+1}-1}\}) < \frac{1}{2}+\varepsilon \quad \text{and} \quad \mathcal{A}(S_j) \geq \frac{1}{2}+\varepsilon$$

for $j = 1, \dots, s-1$. Obviously, each set S_i satisfies the precondition of Lemma 2 and can therefore be packed into a single bin. Only S_s might have a total area of less than $1/2+\varepsilon$. The overall algorithm is given in Algorithm 1.

Note that if no packing of L into at most k bins exists, then $\mathrm{OPT}(I) \geq k$ and thus $k' \leq 2\,\mathrm{OPT}(I)$ by definition of k.

2.4 Packing sets of small items

In this section, we prove Lemma 2. We will use the following partition of a set T of items of area at most ε in the remainder of this section. Let

$$T_1 := \{r_i \in T \mid 2/3 < w_i\},$$
$$T_2 := \{r_i \in T \mid 1/2 < w_i \leq 2/3\},$$
$$T_3 := \{r_i \in T \mid 1/3 < w_i \leq 1/2\},$$
$$T_4 := \{r_i \in T \mid w_i \leq 1/3\}.$$

Since $w_i h_i \leq \varepsilon$ and $w_i \geq h_i$, the heights of the items in each set are bounded as follows.

$$\begin{aligned} h_i &\leq 3/2 \cdot \varepsilon && \text{for } r_i \in T_1,\\ h_i &\leq 2 \cdot \varepsilon && \text{for } r_i \in T_2,\\ h_i &\leq 3 \cdot \varepsilon && \text{for } r_i \in T_3,\\ h_i &\leq \sqrt{\varepsilon} && \text{for } r_i \in T_4. \end{aligned}$$

It turns out that packing the items in T_2 involves the most difficulties. We will therefore consider different cases for packing items in T_2, according to the total height of these items. For all the cases, we need to pack $T_1 \cup T_3 \cup T_4$ afterwards, using the following lemma.

Lemma 3 *Let R be a rectangle of size $(1, H)$ and let T be a set of items that all have area at most ε such that $T_2 = \emptyset$.*

Algorithm 1 Approximate rectangle packing with rotations

apply the asymptotic algorithm from (Bansal et al. 2006a) to derive a packing P' into k' bins
let $\varepsilon := 1/68$
partition I into $L = \{r_i \in I \mid w_i h_i \geq \varepsilon\}$ and $S = \{r_i \in I \mid w_i h_i < \varepsilon\}$
if $\mathcal{A}(L) > k$ or L cannot be packed in k or less bins **then**
  return P'
else
  find a packing of L into $\ell \leq k$ bins, where ℓ is minimal.
  while there exists a bin containing items of total area $< 1/2-\varepsilon$ **do**
    assign small items to this bin until the total area exceeds $1/2-\varepsilon$
    use Steinberg's algorithm (Corollary 2) to repack the bin
  order the remaining small items by non-increasing size
  greedily partition the remaining items into sets $S_1, \dots, S_s$ such that

$$\mathcal{A}(S_j \setminus \{r_{t_{j+1}-1}\}) < \frac{1}{2}+\varepsilon \quad \text{and} \quad \mathcal{A}(S_j) \geq \frac{1}{2}+\varepsilon \quad \text{for } j = 1, \dots, s-1$$

  use the method described in the proof of Lemma 2 to pack each set S_i into a bin
  let P be the resulting packing into $\ell + s$ bins
return the packing from P, P' that uses the least amount of bins

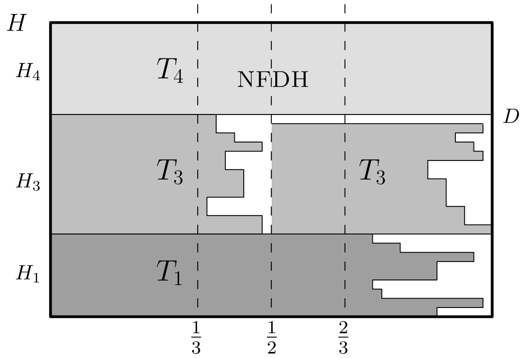

Fig. 2 Packing the sets T_1, T_3 and T_4 into a bin of width 1 and height H. The difference in height between the stacks of T_3 is denoted by D

We can find a packing of a selection $T' \subseteq T$ into R in time $\mathcal{O}(n \log n)$ such that $T' = T$ or

$$\mathcal{A}(T') \geq \frac{2}{3}(H - \sqrt{\varepsilon}) - \varepsilon.$$

Proof See Fig. 2 for an illustration of the following packing. Stack the items of T_1 left-justified into the lower left corner of R. Stop if there is not sufficient space to accommodate the next item. In this case, a total area of at least $\mathcal{A}(T_1') \geq 2/3\,(H - 3/2 \cdot \varepsilon)$ is packed since $w_i > 2/3$ and $h_i \leq 3/2 \cdot \varepsilon$ for items in T_1.

Thus, assume all items from T_1 are packed. Denote the height of the stack by H_1. Obviously, $\mathcal{A}(T_1) \geq 2/3 \cdot H_1$.

Create two stacks of items from T_3 next to each other directly above the stack for T_1 by repeatedly assigning each item to the lower stack. Stop if an item does not fit into the rectangle. In this case, both stacks have a height of at least $H - H_1 - 3\varepsilon$ as otherwise a further item could be packed. Therefore, $\mathcal{A}(T_1 \cup T_3') \geq 2/3(H - 3\varepsilon) \geq 2/3 \cdot (H - \sqrt{\varepsilon})$ since $3\varepsilon \leq \sqrt{\varepsilon}$.

Otherwise, denote the height of the higher stack by H_3 and the height difference by D. The total area of T_3 is at least $\mathcal{A}(T_3) \geq 2/3(H_3 - D) + 1/3 \cdot D \geq 2/3 \cdot H_3 - 1/3 \cdot D \geq 2/3 \cdot H_3 - \varepsilon$ since $w_i \geq 1/3$ and $h_i \leq 3\varepsilon$ for $r_i \in T_3$.

Finally, let $H_4 := H - H_1 - H_3$ and add the items of T_4 by NFDH into the remaining rectangle of size $(1, H_4)$. Lemma 1 yields that either all items are packed, i.e., $T' = T$, or items $T_4' \subseteq T_4$ of total area at least $\mathcal{A}(T_4') \geq 2/3(H_4 - \sqrt{\varepsilon})$ are packed. Thus, the total area of the packed items T' is $\mathcal{A}(T') \geq 2/3 \cdot H_1 + 2/3 \cdot H_3 - \varepsilon + 2/3(H_4 - \sqrt{\varepsilon}) \geq 2/3(H - \sqrt{\varepsilon}) - \varepsilon$.

The running time is dominated by the application of NFDH. □

If $T_4 = \emptyset$ then the last packing step is obsolete and the analysis above yields the following corollary.

Corollary 3 *Let R be a rectangle of size $(1, H)$ and let T be a set of items that all have area at most ε such that $T_2 \cup T_4 = \emptyset$. We can find a packing of a selection $T' \subseteq T$ into R in time $\mathcal{O}(n)$ such that $T' = T$ or*

$$\mathcal{A}(T') \geq \frac{2}{3}H - 2\varepsilon.$$

The above packings are very efficient if there are no items of width within 1/2 and 2/3 as they essentially yield a width guarantee of 2/3 for the whole height, except for some wasted height that is suitably bounded. In order to pack items of T_2, we have to consider both possible orientations to achieve a total area of more than 1/2 in a packing. We are now ready to prove Lemma 2 that we already presented in the previous section. It shows how sets of items including items of width within 1/2 and 2/3 are processed.

Proof of Lemma 2 Let H_2 be the total height of items in T_2. We present three methods for packing T depending on H_2. For each method we give a lower bound on the total area of items that are packed. Afterwards we show that there cannot be any item that remains unpacked. Throughout the proof, we assume that we do not run out of items while packing the items in T. This will eventually lead to a contradiction in all three cases. Let A be the area of the packed items for which we want to derive lower bounds.

Case 1: $H_2 \leq 1/3$. Stack the items of T_2 left-justified into the lower left corner of the bin. Use Lemma 3 to pack $T_1 \cup T_3 \cup T_4$ into the rectangle $(1, 1 - H_2)$ above the stack—see Fig. 3. We get an overall packed area of

$$\begin{aligned} A &\geq \frac{H_2}{2} + \frac{2}{3}(1 - H_2 - \sqrt{\varepsilon}) - \varepsilon \\ &= \frac{2}{3} - \frac{H_2}{6} - \varepsilon - \frac{2}{3}\sqrt{\varepsilon} \\ &\geq \frac{11}{18} - \varepsilon - \frac{2}{3}\sqrt{\varepsilon} \quad \left(\text{since } H_2 \leq \frac{1}{3}\right). \end{aligned}$$

Case 2: $H_2 \in (1/3, 2/3]$. Stack the items of T_2 left-justified into the lower left corner of the bin. Let $B = (1/3, H_2)$ be the free space to the right of the stack. We are going to pack items from $X = \{r_i \in T_3 \cup T_4 \mid w_i \leq H_2\}$ into B. Take an item from X and add it to an initially empty set X' as long as X is non-empty and $\mathcal{A}(X') \leq H_2/6 - \varepsilon$. Rotate the items in X' and use Steinberg's algorithm (Corollary 1) to pack them into B. This is possible since the area of B is $H_2/3$, $\mathcal{A}(X') \leq H_2/6$, and $h_i \leq H_2$ and $w_i \leq \sqrt{\varepsilon} \leq 1/6$ for $r_i \in X'$ (w_i and h_i are the rotated lengths of r_i). Use Lemma 3 to pack $(T_1 \cup T_3 \cup T_4) \setminus X'$ into the rectangle $(1, 1 - H_2)$ above the stack—see Fig. 3. We distinguish two

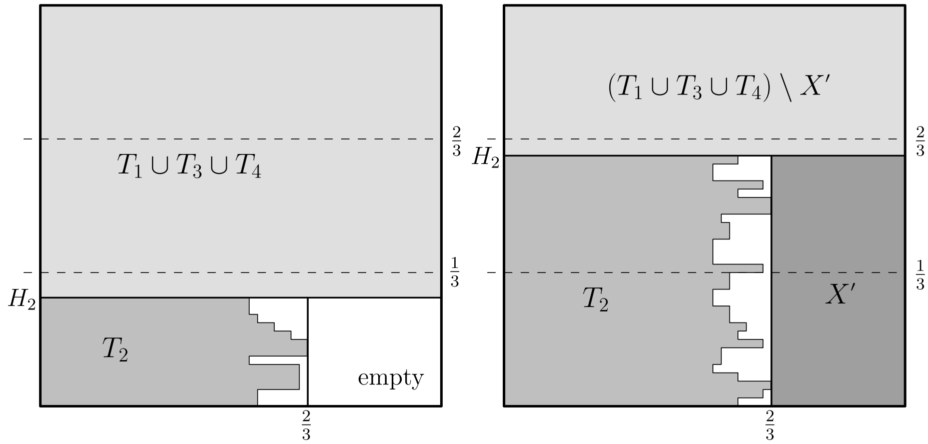

Fig. 3 Packing in Case 1 ($H_2 \le 1/3$) and Case 2 ($1/3 < H_2 \le 2/3$)

cases. If $\mathcal{A}(X') \ge H_2/6 - \varepsilon$, then

$$A \ge \overbrace{\frac{H_2}{2}}^{T_2} + \overbrace{\frac{H_2}{6} - \varepsilon}^{X'} + \overbrace{\frac{2}{3}(1 - H_2 - \sqrt{\varepsilon}) - \varepsilon}^{(T_1 \cup T_3 \cup T_4)\setminus X'}$$
$$= \frac{2}{3} - 2\varepsilon - \frac{2}{3}\sqrt{\varepsilon}.$$

Otherwise, $\mathcal{A}(X') < H_2/6 - \varepsilon$, and since no further item was added to X' we have $X' = X$. As $H_2 > 1/3$, we have $T_4 \subseteq X$ and we can apply Corollary 3 to get a total area of

$$A \ge \frac{H_2}{2} + \frac{2}{3}(1 - H_2) - 2\varepsilon$$
$$= \frac{2}{3} - \frac{H_2}{6} - 2\varepsilon$$
$$\ge \frac{5}{9} - 2\varepsilon \quad \left(\text{since } H_2 \le \frac{2}{3}\right).$$

Case 3: $H_2 \in (2/3, 1 + 4\varepsilon]$. See Fig. 4 for an illustration of the following packing and the notations. Order the items of T_2 in non-increasing order of width. Stack the items left-justified into the lower left corner of the bin while the current height H is less or equal to the width of the last item that was packed. In other words, the top right corner of the last item of this stack is above the line from $(1/2, 1/2)$ to $(2/3, 2/3)$, whereas the top right corners of all other items in the stack are below this line. Denote the height of the stack by H and the set of items that is packed into this stack by X_1. Let $r' = (w', h')$ be the last item on the stack. Clearly, $w_i \le H$ for all items $r_i \in T_2 \setminus X_1$.

Consider the free space $B = (1/3, H)$ to the right of the stack. Rotate the items in $T_2 \setminus X_1$ and stack them horizontally, bottom-aligned into B. Stop if an item does not fit. We denote the items that are packed into B by X_2. Rotate the remaining items $T_2 \setminus (X_1 \cup X_2)$ back into their original orientation and stack them on top of the first stack X_1. Let this set of items be X_3 and let the total height of the stack

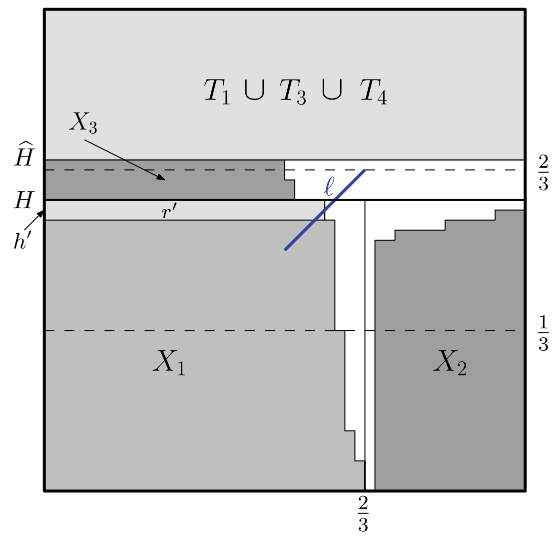

Fig. 4 Packing in Case 3 ($2/3 < H_2 \le 1 + 4\varepsilon$). Item r' of height h' is depicted larger than $\varepsilon \le 1/68$ for the sake of visibility. The diagonal line ℓ shows the threshold at which the stack X_1 is discontinued

$X_1 \cup X_3$ be $\widehat{H}$. Use Lemma 3 to pack $T_1 \cup T_3 \cup T_4$ into the rectangle $(1, 1 - \widehat{H})$ above the stack $X_1 \cup X_3$.

Since $w_i \ge H - h'$ for $r_i \in X_1 \setminus \{r'\}$, we have $\mathcal{A}(X_1) \ge (H - h')^2 + h'/2$. Again we distinguish two cases for the analysis. If $X_3 = \emptyset$ (or, equivalently, $\widehat{H} = H$), then $\mathcal{A}(X_2) \ge (H_2 - H)/2$, and therefore

$$A \ge \overbrace{(H - h')^2 + \frac{h'}{2}}^{X_1} + \overbrace{\frac{H_2 - H}{2}}^{X_2} + \overbrace{\frac{2}{3}(1 - H - \sqrt{\varepsilon}) - \varepsilon}^{T_1 \cup T_3 \cup T_4}$$
$$> (H - h')^2 + \frac{h'}{2} + \frac{1}{3} - \frac{h}{2} + \frac{2}{3}(1 - H - \sqrt{\varepsilon}) - \varepsilon =: A_1$$
$$\left(\text{since } H_2 > \frac{2}{3}\right).$$

To find a lower bound for the total packed area we consider the partial derivative of A_1 to h', which is $\frac{\partial A_1}{\partial h'} = 2h' - 2H +$

$1/2$. Since $2h' - 2H + 1/2 < 0$ for $h' \leq 2\varepsilon$ and $H \geq 1/2$, the total packed area is minimized for the maximal value $h' = 2\varepsilon$ for any H in the domain. After inserting this value for h' we get $A_1 = (h - 2\varepsilon)^2 + \varepsilon + 1/3 - h/2 + 2/3(1 - h - \sqrt{\varepsilon}) - \varepsilon$ and $\frac{\partial A_1}{\partial H} = 2H - 7/6 - 4\varepsilon$. Thus, the minimum is acquired for $H = 7/12 + 2\varepsilon$. We get

$$A_1 \geq \left(\frac{7}{12}\right)^2 + \varepsilon + \frac{1}{3} - \frac{7}{24} - \varepsilon + \frac{2}{3}\left(\frac{5}{12} - 2\varepsilon - \sqrt{\varepsilon}\right) - \varepsilon$$
$$= \frac{95}{144} - \frac{7}{3}\varepsilon - \frac{2}{3}\sqrt{\varepsilon}.$$

Otherwise, $X_3 \neq \emptyset$ (or, equivalently, $\widehat{H} > H$) and thus $\mathcal{A}(X_2) \geq 1/2(1/3 - 2\varepsilon)$ as the stack X_2 leaves at most a width of 2ε of B unpacked. Furthermore, $\widehat{H} \leq 2/3 + 6\varepsilon$ since $H_2 \leq 1 + 4\varepsilon$ and a width of at least $1/3 - 2\varepsilon$ is packed into B. Since $\mathcal{A}(X_3) \geq (\widehat{H} - H)/2$ and $\widehat{H} \leq 2/3 + 6\varepsilon$, we get

$$A \geq \overbrace{(H - h')^2 + \frac{h'}{2}}^{X_1} + \overbrace{\frac{1}{2}\left(\frac{1}{3} - 2\varepsilon\right)}^{X_2}$$
$$+ \overbrace{\frac{\widehat{H} - H}{2}}^{X_3} + \overbrace{\frac{2}{3}(1 - \widehat{H} - \sqrt{\varepsilon}) - \varepsilon}^{T_1 \cup T_3 \cup T_4}$$
$$\geq (H - h')^2 + \frac{h'}{2} + \frac{1}{2}\left(\frac{1}{3} - 2\varepsilon\right)$$
$$- \frac{1}{9} - \varepsilon - \frac{H}{2} + \frac{2}{3}(1 - \sqrt{\varepsilon}) - \varepsilon := A_2.$$

With a similar analysis as before we see that A_2 is minimal for $H = 1/2$ and $h' = 2\varepsilon$. We get

$$A_2 \geq \left(\frac{1}{2} - 2\varepsilon\right)^2 + \varepsilon + \frac{1}{6} - \varepsilon - \frac{1}{9} - \varepsilon$$
$$- \frac{1}{4} + \frac{2}{3}(1 - \sqrt{\varepsilon}) - \varepsilon$$
$$\geq \frac{13}{18} + 4\varepsilon^2 - 4\varepsilon - \frac{2}{3}\sqrt{\varepsilon}.$$

If $H_2 > 1 + 4\varepsilon$ then $A(T_2) \geq 1/2 \cdot H_2 > 1/2 + 2\varepsilon$, which is a contradiction to the assumption of the lemma. Therefore, the three cases cover all possibilities.

It is easy to verify that for $0 < \varepsilon \leq 1/68$ the following inequalities hold:

$11/18 - \varepsilon - 2/3\sqrt{\varepsilon} \geq 1/2 + \varepsilon,$

$2/3 - 2\varepsilon - 2/3\sqrt{\varepsilon} \geq 1/2 + \varepsilon,$

$5/9 - 2\varepsilon \geq 1/2 + \varepsilon,$

$95/144 - 7/3\varepsilon - 2/3\sqrt{\varepsilon} \geq 1/2 + \varepsilon,$

$13/18 + 4\varepsilon^2 - 4\varepsilon - 2/3\sqrt{\varepsilon} \geq 1/2 + \varepsilon.$

Now let us assume that we do not run out of items while packing a set T with the appropriate method above. Then the packed area is at least $1/2 + \varepsilon$ as the inequalities above show. The contradiction follows from the precondition that removing an arbitrary item from T yields a remaining total area of less than $1/2 + \varepsilon$. Thus, all items are packed.

The running time is dominated by the application of Steinberg's algorithm (Steinberg 1997). □

2.5 The approximation ratio

Theorem 4 *Our algorithm is an approximation algorithm for rectangle packing with rotations with an absolute worst case ratio of* 2.

Proof Recall that we denote the number of bins used for an optimal packing of the large items by ℓ. Obviously, $\ell \leq \mathrm{OPT}(I)$. Let s be the number of bins used for packing only small items. If $s \leq \ell$, then the total number of bins is $\ell + s \leq 2\ell \leq 2\,\mathrm{OPT}(I)$. If $s > \ell$, then at least one bin is used for small items and thus all bins for large items contain items with a total area of at least $1/2 - \varepsilon$. According to the partition of the remaining small items, all but the last bin for the small items contain items with a total area of at least $1/2 + \varepsilon$. Let $f > 0$ be the area of the items contained in the last bin. Then

$$\mathrm{OPT}(I) \geq \mathcal{A}(I) \geq \ell \cdot \left(\frac{1}{2} - \varepsilon\right) + (s - 1) \cdot \left(\frac{1}{2} + \varepsilon\right) + f$$
$$> (s + \ell - 1) \cdot \frac{1}{2}.$$

Thus, $s + \ell < 2\,\mathrm{OPT}(I) + 1$, and we get $s + \ell \leq 2\,\mathrm{OPT}(I)$, which proves the theorem. □

3 The approximation ratio of hybrid first fit is 3

In this section, we prove Zhang's conjecture (Zhang 2005) on the absolute approximation ratio of the HYBRID FIRST FIT (HFF) algorithm by showing that this ratio is 3.

We start our presentation in Sect. 3.1 with a description of the HFF algorithm (Chung et al. 1982). A lower bound on the approximation ratio is presented in Sect. 3.2. Note that the instance that we give in this section also holds as a lower bound for Zhang's algorithm (Zhang 2005) and for HYBRID FIRST FIT BY WIDTH that we describe in Sect. 3.1. Thus, it does not suffice to combine all three algorithms in order to derive a better approximation ratio. Finally, we give the proof of the upper bound in Sect. 3.3.

3.1 The hybrid first fit algorithm

HFF is based on the one-dimensional FIRST FIT DECREASING (FFD) bin packing algorithm and on the FIRST FIT DECREASING HEIGHT (FFDH) strip packing algorithm. The latter algorithm is a layer-based strip packing algorithm similar to NFDH that we introduced in Sect. 2.1. It was considered for the first time by Coffman et al. (1980) and is given as follows. Sort the items by non-increasing order of height. Pack the items one by one into layers. The height of a layer is defined by its first item, further items are added left-aligned into the lowest layer with sufficient space. If an item does not fit into any layer opened so far this item opens a new layer. HFF now considers the layers of a FFDH packing one after the other and packs each layer into the first bin with sufficient space. Since the layers are ordered by non-increasing height, this corresponds to a one-dimensional FFD packing. See Fig. 5 for an illustration of HFF.

A simple variant of HFF, that we denote as HYBRID FIRST FIT BY WIDTH packs the strip in the direction of the width. This means that the items are ordered by non-increasing width for the strip packing and FFD is later applied on the width of the resulting layers. This algorithm can also be seen as HFF applied on the instance $J = \{(h_i, w_i) \mid (w_i, h_i) \in I\}$, where each item of I is rotated by 90 degrees. Afterwards the packing is rotated back.

Using an appropriate data structure from (Johnson 1973) the running time of HFF is $\mathcal{O}(n \log n)$. This is slightly faster than the running time of Zhang's algorithm (Zhang 2005), which is dominated by Steinbergs' algorithm (Steinberg 1997) and thus runs in $\mathcal{O}((n \log^2 n)/\log\log n)$—see Theorem 1.

We denote the layers from FFDH by $L_1, \ldots, L_m$ with heights $H_1 \geq H_2 \geq \cdots \geq H_m$ and total widths $W_1, \ldots, W_m$. Note that there is no particular order on the widths of the layers. For the sake of simplicity, we refer to both the layer and the set of items in the layer by L_i. We again denote the total area of a set I of items by $\mathcal{A}(I) = \sum_{r_i \in I} w_i h_i$.

3.2 Lower bound

Let $0 < \delta < 1/34$, such that $1/\delta$ is integer, and consider the instance $I = A_1 \cup A_2 \cup B_1 \cup B_2 \cup C_1 \cup C_2$ consisting of the following sets (see Table 1).

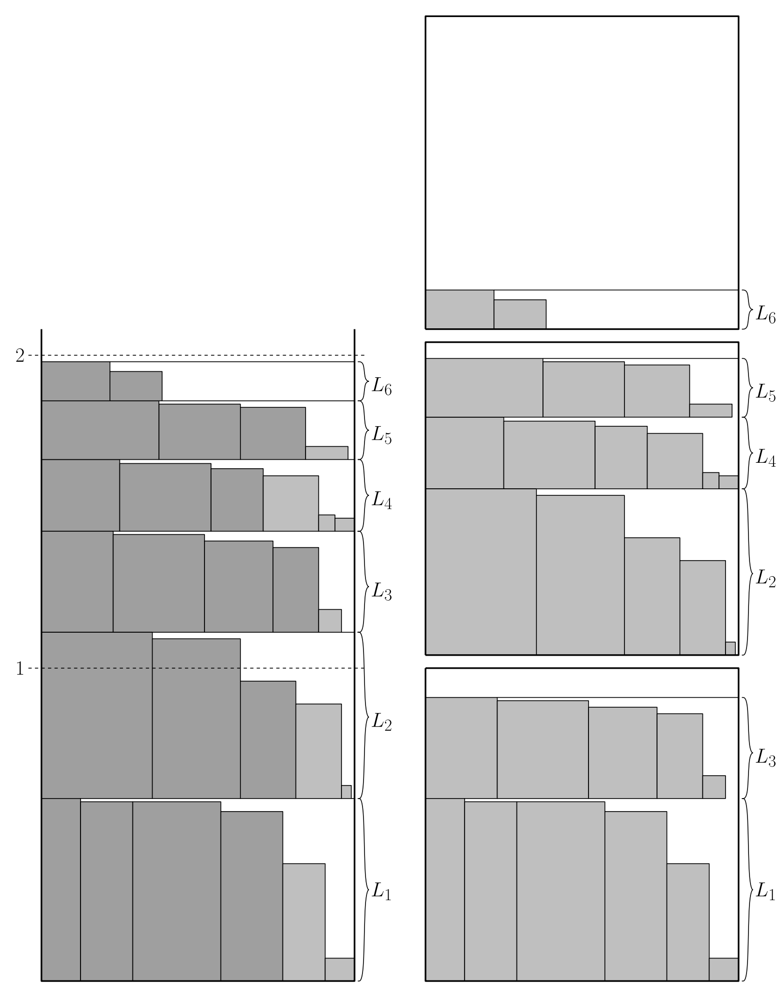

Fig. 5 The HYBRID FIRST FIT algorithm. The layers of a strip packing with FFDH are packed into bins with FFD. Items that are placed into a layer after a new layer was opened are shown in *light grey*

In Fig. 6, we show that $\mathrm{OPT}(I) = 1$ and we give the FFDH packing of I. We assume that the item in A_2 comes before the items in B_2 in the non-increasing order of height. Then FFD packs the item in A_2 together with the first layer into a bin. Since $\delta < \frac{1}{34}$, the total height of all remaining layers is

$$H = \overbrace{\frac{1}{6} + \frac{1}{3}\delta}^{C_1} + \overbrace{\left(\frac{1}{\delta} - 6\right)\delta}^{B_2} = \frac{7}{6} - \frac{17}{3}\delta > 1.$$

Table 1 A lower bound instance for HFF

Set	High items
A_1	1 item of size $(\delta, 1-\delta)$
B_1	$\frac{1}{\delta} - 6$ items of size $(\delta, \frac{1}{2} + \delta)$
C_1	3 item of size $(2\delta, \frac{1}{6} + \frac{1}{3}\delta)$
Set	Wide items
A_2	1 item of size $(1-\delta, \delta)$
B_2	$\frac{1}{\delta} - 6$ items of size $(\frac{1}{2} + \delta, \delta)$
C_2	3 items of size $(\frac{1}{6} + \frac{1}{3}\delta, 2\delta)$

Thus, 3 bins are needed to pack the resulting layers with FFD.

Note that the instance that we describe does not change under rotation by 90 degrees. Thus HYBRID FIRST FIT BY WIDTH outputs the same packing. We refer the reader to Zhang (2005) to verify that Zhang's algorithm uses 3 bins as well since the total area of all high and small items is $\mathcal{A}(A_1 \cup B_1 \cup C_1 \cup C_2) > 1/2$ for $\delta > 0$.

3.3 Upper bound

Before we start with the main proof of the upper bound, we introduce the following important lemma.

Lemma 4 *Let L_i, L_j be two layers with $i < j$. Then*

$$\mathcal{A}(L_i \cup L_j) \geq H_j + (W_i + W_j - 1)h_{\min},$$

where $h_{\min} = \min_{r_k \in L_i \cup L_j} h_k$ is the smallest height of the items in $L_i \cup L_j$.

See Fig. 7 for an illustration of the following proof.

Proof First consider all items that are packed up to and including the first item in layer L_j (dark items in Fig. 7). These

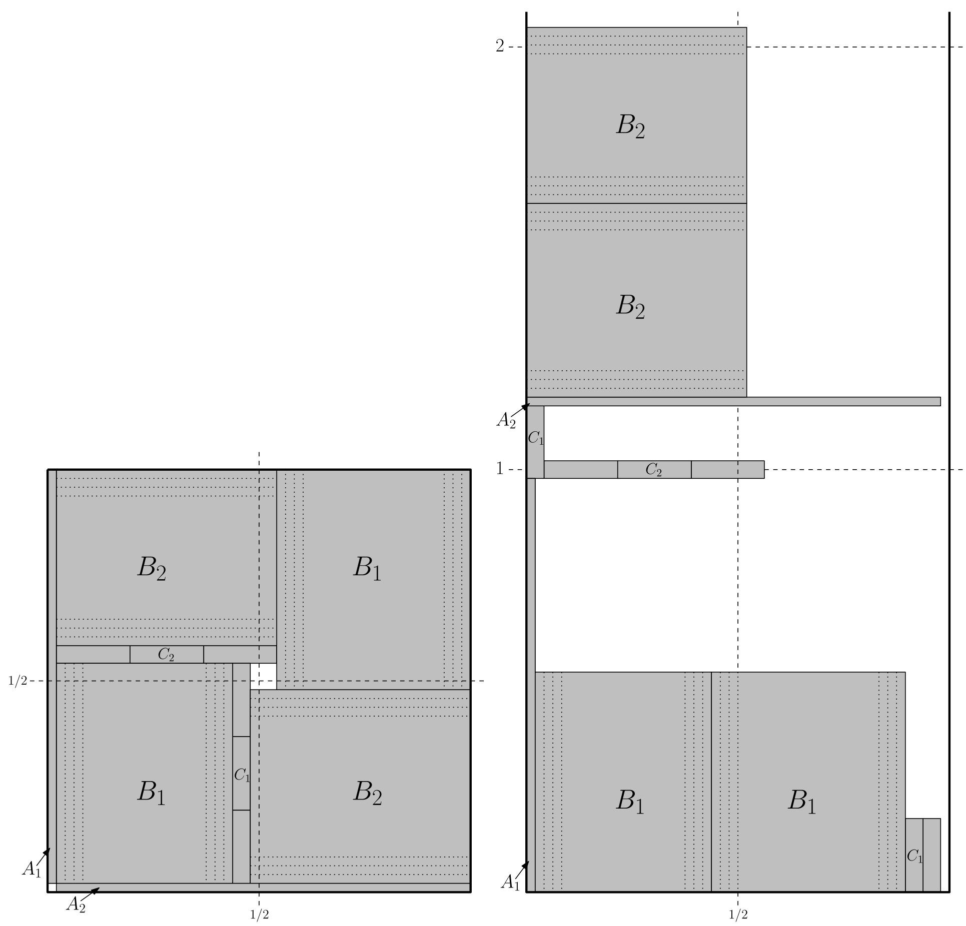

Fig. 6 A lower bound instance for HFF. The left side shows that $\mathrm{OPT}(I) = 1$, whereas the *right side* shows the FFDH packing. FFD packs A_2 together with the first layer into a bin. The remaining layers have total height greater than 1 and thus do not fit into a second bin

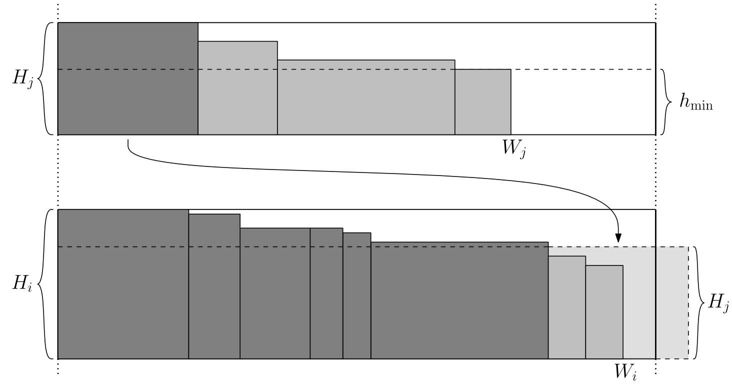

Fig. 7 Deriving a bound for the volume in two layers. All items that are packed up to and including the first item in L_j are dark and have a height of at least H_j. The total width of these items is greater than 1. All other items have height at least $h_{\min}$

items all have height at least H_j. Let W' be the total width of these items. We have $W' > 1$ since the first item of layer L_j did not fit into layer L_i. Thus, we get a total area of at least $W' H_j$ for these items. Since all items in both layers have height at least $h_{\min}$, the remaining width of $W_i + W_j - W'$ makes up an additional area of at least $(W_i + W_j - W')h_{\min}$. Thus, the total area is

$$\begin{aligned}\mathcal{A}(L_i \cup L_j) &\geq W' H_j + (W_i + W_j - W')h_{\min} \\ &\geq H_j + (W_i + W_j - 1)h_{\min}.\end{aligned}$$

Note that $W_i + W_j - 1 \geq 0$ since otherwise the first item of L_j fits into L_i. □

With the previous lemma we are able to derive bounds for the total area of the items that are packed. In addition to the most intuitive lower bound

$$\mathrm{OPT}(I) \geq \mathcal{A}(I), \tag{1}$$

we use the following two bounds. Let S be the set of layers that contain exactly one item. Since it is not possible to pack two of these items next to each other (otherwise, it would have been done by the algorithm) the total height of these layers form the lower bound

$$\mathrm{OPT}(I) \geq \sum_{L_i \in S} H_i. \tag{2}$$

Finally, the set $T = \{r_i = (w_i, h_i) \mid h_i > 1/2\}$ of items of height greater than $1/2$ provides the last lower bound

$$\mathrm{OPT}(I) \geq \sum_{r_i \in T} w_i \tag{3}$$

that we use.

Assume for the sake of contradiction that HFF uses more than $3\,\mathrm{OPT}$ bins. Let L_k be the first layer from FFDH that is packed into bin number $3\,\mathrm{OPT} + 1$ and let $r^* = (w^*, h^*)$ be the first item in L_k. Discard all items that are considered after r^* by FFDH. Note that the packing that remains is exactly the packing that HFF produces on $I' = (I \setminus \bigcup_{i \geq k} L_i) \cup \{r^*\}$, i.e., the reduced set of items. Therefore, we argue about this reduced instance in the remainder of this section.

Lemma 5 *We have $h^* \leq 1/3$.*

Proof Suppose that $h^* > 1/3$. Then all bins contain either one or two layers and are filled up to a height of more than $1 - h^*$. Layers that are alone in a bin have height greater than $1 - h^*$. Thus, if Lemma 4 is applied on two such layers, say L_i, L_j $(i < j)$, we get a combined area of at least $H_j \geq 1 - h^*$. Applied on both layers of a bin that contains two layers we get a combined area of at least h^*,which is a lower bound for the height of the smaller layer in the bin.

Let m be the number of bins that contain exactly one layer (except the bin that contains r^*). These are layers $L_1, \ldots, L_m$. Then the other $3\,\mathrm{OPT} - m$ bins contain two layers each. Note that $H_1 \geq \cdots \geq H_m > 1/2$ as otherwise a second layer would fit into the bin. Let W_i' be the width of items of height $> 1/2$ in layer L_i. Thus, with the lower bound (3) we get

$$\mathrm{OPT}(I') \geq \sum_{i=1}^{m} W_i' \geq \sum_{i=1}^{\lfloor m/2 \rfloor} (W_{2i-1}' + W_{2i}') > \left\lfloor \frac{m}{2} \right\rfloor \tag{4}$$

since any two layers have cumulative width $W_i' + W_j' > 1$.

If m is even, we get

$$\begin{aligned}\mathcal{A}(I') &\geq \overbrace{\frac{m}{2}(1 - h^*)}^{\text{bins with one layer}} + \overbrace{\big(3\,\mathrm{OPT}(I') - m\big)h^*}^{\text{bins with two layers}} \\ &= \frac{m}{2} + \left(3\,\mathrm{OPT}(I') - \frac{3}{2}m\right)h^* \\ &> \frac{m}{2} + \mathrm{OPT}(I') - \frac{m}{2} \quad \text{using } h^* > 1/3 \text{ and (4)} \\ &= \mathrm{OPT}(I').\end{aligned}$$

For odd values of m, say $m = 2n + 1$, we can apply Lemma 4 on the first $2n$ layers as before and on layer L_m together with layer L_k consisting of r^*. This given another area of h^* and we get

$$\begin{aligned}\mathcal{A}(I') &\geq \overbrace{n(1-h^*)+h^*}^{\text{bins with one layer}} + \overbrace{\big(3,\ \mathrm{OPT}(I')-(2n+1)\big)h^*}^{\text{bins with two layers}} \\ &= n + \big(3\,\mathrm{OPT}(I') - 3n\big)h^* \\ &> n + \mathrm{OPT}(I') - n \quad \text{using } h^* > 1/3 \text{ and (4)} \\ &= \mathrm{OPT}(I').\end{aligned}$$

In both cases, we get a contradiction, and thus $h^* \leq 1/3$. □

In the following step, we will use Lemma 4 on pairs of consecutive layers in order to derive a lower bound on the total area of the items.

As r^* is packed into a new bin, all previous bins contain layers of total height greater than $1-h^*$. Thus, we get the following bound for the total height H of the first $k-1$ layers:

$$H = \sum_{i=1}^{k-1} H_i > 3\,\mathrm{OPT}(I')\big(1-h^*\big).$$

We need a slightly different bound which follows immediately since the first bin contains at least layer L_1 of height H_1 and all other bins contain layers of total height greater than $1-h^*$:

$$H = \sum_{i=1}^{k-1} H_i > \big(3\,\mathrm{OPT}(I')-1\big)\big(1-h^*\big) + H_1. \tag{5}$$

Applying Lemma 4 on pairs of consecutive layers L_{2i-1}, L_{2i} and adding the area of r^*, we get

$$\begin{aligned}\mathcal{A}(I') &\geq \sum_{i=1}^{\lfloor (k-1)/2 \rfloor} \mathcal{A}(L_{2i-1} \cup L_{2i}) + w^*h^* \\ &\geq \sum_{i=1}^{\lfloor (k-1)/2 \rfloor} \big(H_{2i} + (W_{2i-1}+W_{2i}-1)h^*\big) + w^*h^* \\ &\quad \text{by Lemma 4} \\ &\geq \sum_{i=1}^{\lfloor (k-1)/2 \rfloor} H_{2i} + \sum_{i=1}^{k-1}\left(W_i - \frac{1}{2}\right)h^* + w^*h^*. \end{aligned} \tag{6}$$

We first derive a lower bound for the first part of the previous inequality. With $h_{2i} \geq h_{2i+1}$ we get

$$2\sum_{i=1}^{\lfloor (k-1)/2 \rfloor} H_{2i} \geq \sum_{i=2}^{k-1} H_i = H - H_1, \tag{7}$$

and thus

$$\begin{aligned}\sum_{i=1}^{\lfloor (k-1)/2 \rfloor} H_{2i} &\geq \frac{H-H_1}{2} \quad \text{from (7)} \\ &> \frac{(3\,\mathrm{OPT}(I')-1)(1-h^*)}{2} \quad \text{from (5)} \\ &\geq \mathrm{OPT}(I')\big(1-h^*\big) + \frac{\mathrm{OPT}(I')-1}{2}\big(1-h^*\big) \\ &\geq \mathrm{OPT}(I') - \mathrm{OPT}(I')h^* + \frac{\mathrm{OPT}(I')-1}{2}2h^* \\ &\quad \text{as } h^* \leq 1/3 \\ &\geq \mathrm{OPT}(I') - h^*. \end{aligned} \tag{8}$$

To simplify the presentation let $A = \sum_{i=1}^{k-1}(W_i - \frac{1}{2})h^* + w^*h^*$. Inequalities (6) and (8) lead to the lower bound

$$\mathcal{A}(I') \geq \mathrm{OPT}(I') - h^* + A.$$

We need the following observation before we can derive a contradiction.

Observation 1 *There are at least 6 layers, and if* $\mathrm{OPT}(I') = 1$ *then the three largest layers are packed into the first two bins.*

The observation is obvious for $\mathrm{OPT}(I') > 1$ as we assume that $3\,\mathrm{OPT}(I') + 1$ bins are used. If $\mathrm{OPT}(I') = 1$ then the height of layer L_2 cannot be greater than $1/2$ as otherwise in contradiction to (3):

$$\sum_{\substack{r_i=(w_i,h_i)\\ h_i>1/2}} w_i > 1 = \mathrm{OPT}(I').$$

Thus, the three largest layers are packed into the first two bins, and each but the first bin contains at least 2 layers. We are now ready to prove Theorem 5.

Theorem 5 *The approximation ratio of* HYBRID FIRST FIT *is* 3.

Proof The lower bound was already given in Sect. 3.2. We consider two different cases to derive a contradiction. In the first case, we show that $A \geq h^*$, which leads to $\mathcal{A}(I') > \mathrm{OPT}(I')$ as a contradiction to the lower bound (1). The tricky part in this case is to consider that there might be a layer of width $W_i < 1/2$ which would result in a negative term in the sum of A. But there can be at most one such layer as otherwise both layers would fit together. In the second case, we use the lower bound (2) that is given by the total height of layers that consist of exactly one item.

Case 1. Assume that there are 3 or more layers with width at least $2/3$. Let L_u, L_v, L_w be the layers with greatest total width, and let L_t be the layer with the smallest total width. Since there are more than 4 layers, we can assume w.l.o.g. that $t \neq \{u, v, w\}$. Then

$$\begin{aligned} A &= \sum_{i=1}^{k-1}\left(W_i - \frac{1}{2}\right)h^* + w^*h^* \\ &\geq 3\left(\frac{2}{3} - \frac{1}{2}\right)h^* + \sum_{\substack{i=1 \\ i\neq t,u,v,w}}^{k-1}\left(W_i - \frac{1}{2}\right)h^* + \left(W_t - \frac{1}{2}\right)h^* \\ &\quad + w^*h^* \\ &> \frac{1}{2}h^* + \sum_{\substack{i=1 \\ i\neq t,u,v,w}}^{k-1}\left(W_i - \frac{1}{2}\right)h^* + \frac{1}{2}h^*. \end{aligned}$$

The last step is due to $W_t > 1 - w^*$ as otherwise r^* fits into layer L_t. Since there is at most one layer with width $W_j < 1/2$ (and this would be L_t), the sum in the middle in non-negative. Thus, $A \geq h^*$, which gives a contradiction to the lower bound (1).

Case 2. Now assume that there are less than 3 layers with width at least $2/3$. Let L_ℓ be the first layer (lowest indexed layer) with width $W_\ell < 2/3$. Consider an item $r_j = (w_j, h_j)$ in layer L_i with $i > \ell$. Then $w_j > 1/3$ since otherwise r_j fits into layer L_ℓ. Thus, $W_i > 2/3$ or L_i contains exactly one item. Let S be the set of layers that contain exactly one item and consider the lower bound (2):

$$\begin{aligned} \mathrm{OPT}(I') &\geq \sum_{L_i \in S} H_i \\ &\geq \sum_{i=1}^{k} H_i - (H_1 + H_2 + H_3), \end{aligned}$$

as there are at most three layers that contain more than one item and these three layers have total height at most $H_1 + H_2 + H_3$. For $\mathrm{OPT}(I') = 1$, Observation 1 implies the contradiction

$$\mathrm{OPT}(I') \geq \sum_{i=1}^{k} H_i - (H_1 + H_2 + H_3) > 1.$$

For $\mathrm{OPT}(I') \geq 2$, at most 3 bins contain layers with more than one item. Thus, we get

$$\begin{aligned} \mathrm{OPT}(I') &\geq \sum_{L_i \in S} H_i \\ &> \big(3\,\mathrm{OPT}(I') - 3\big)\big(1 - h^*\big) + h^* \\ &\geq \mathrm{OPT}(I'). \end{aligned}$$

The last step follows from $(3\,\mathrm{OPT}(I') - 3)(1 - h^*) + h^* \geq 2\,\mathrm{OPT}(I') - 2$ since $h^* \leq 1/3$. □

4 Conclusion and future work

The algorithm we presented in Sect. 2 depends on the asymptotic approximation algorithm from Bansal et al. (2006a), in particular, the constant k that follows from this algorithm. It would be interesting to design an approximation algorithm for rectangle packing with rotations with asymptotic approximation ratio strictly less than 2 and small additive term. This could also improve the efficiency of our algorithm.

We conjecture that every set of items of height at most $1/2$ and total area at most $5/9$ can be packed into a unit bin using rotations. This would again improve the efficiency of our algorithm and might be useful for other packing problems as well. Other interesting open questions for further investigation include the following.

1. Does there exist an approximation algorithm for rectangle packing *without rotations* with an absolute worst case ratio of 2?
2. Does there exist an approximation algorithm for strip packing with or without rotations with an absolute worst case ratio less than 2? An answer to this question for strip packing without rotations would narrow the gap between the lower bound of 3/2 (as strip packing without rotations is a generalization of one-dimensional bin packing) and the upper bound of 2 from Steinberg's algorithm (Steinberg 1997) and Schiermeyer's REVERSE FIT algorithm (Schiermeyer 1994).

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