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ORIGINAL PAPER

Unusual Sequence of the Critical Magnetic Fields H_{c1}, H_{c2}, and H_c in Multicomponent Superconductors

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Abstract

All superconductors in a magnetic field are characterized by three critical magnetic fields: lower critical H_{c1}, upper critical H_{c2} and thermodynamic critical field H_c. Only two sets of inequalities $H_{c2} > H_c > H_{c1}$ or $H_{c1} > H_c > H_{c2}$ are possible in a single-component superconductor. Here, we report our study of the critical fields in multicomponent superconductors with two superconducting components in the framework of the Ginzburg-Landau functional. We derive the relationship between the phases of the components of the superconducting complex order parameter from the charge conservation law in explicit form and insert it into the Ginzburg-Landau functional. Using the modified Ginzburg-Landau equation, we acquire the single vortex state including the analytical expression for asymptotics. Also, we obtain the analytical form for the state in the upper critical field. We find that in some cases an unusual sequence of critical fields $H_{c1}, H_{c2} > H_c$ can be realized in multicomponent superconductors.

Keywords Multiband superconductors · Magnetic critical field

1 Introduction

The lower critical magnetic field H_{c1} together with the upper critical field H_{c2} and the thermodynamic critical field H_c are the fundamental characteristics of superconductors, which describe the thermodynamics of a superconductor in an external magnetic field [1–4]. For one-component superconductors only two cases are possible: $H_{c1} > H_c > H_{c2}$ or $H_{c1} < H_c < H_{c2}$. The superconductors, in which the first inequality is satisfied, are called superconductors of the first kind. Correspondingly, if the second inequality is satisfied, superconductors are of the second kind. Recently, it was found that many superconductors such as Fe-based superconductors [5–8], MgB_2 [9–13], Sr_2RuO_4 [14, 15], heavy fermion superconductors [16, 17], superconductivity at the interface between $LaAlO_3$ and $SrTiO_3$ [18] can not be described by a single-component order parameter. In this connection, a natural question arises, whether these two sequences of the inequalities exhaust all the possibilities in the case of multicomponent superconductors. This article aims to fill this gap.

Here, we show that a different sequence of critical magnetic fields can also be realized in a multicomponent superconductor. We use the conditional variation of the Ginzburg-Landau functional, i.e., the variation under the constraint proposed in [19]. In the presence of topological defects and some other cases, e.g., calculation of H_{c2}, the conditions $\delta F/\delta \phi_i = 0$ cannot be used for the derivation of a closed system of equations. Therefore the continuity equation div $\mathbf{j} = 0$, which follows from the gradient in-variance of the Ginzburg-Landau functional, is used as an independent equation [20]. Resolving the continuity equation one gets a relation between $\{\phi_i\}$ [20]. As a result only $N - 1$ phase differences $\{\mu_k = \phi_1 - \phi_k\}$ can be considered as independent variables with one restriction mentioned above.

In this article, we imply the proposed scheme for a two-component superconductor. It allows to set up a closed system of equations for a state with a single vortex. For this state, we find analytically the asymptomatic behavior of the solutions at short and long distances from the vortex core and numerically at intermediate distances. We also obtain with the perturbation theory the equations for H_{c2} for the two-component superconductor and compare the critical magnetic fields.

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2 The Functional

We start with a Ginzburg-Landau (GL) functional of a two-component superconductor in the form, in which the kinetic energy term is positively defined and diagonalized:

$$\mathcal{F} = \int d^3r \left\{ \sum_{i=1}^{2} \frac{\hbar^2}{4m_i} \left| \left(\frac{\partial}{\partial \mathbf{r}} - \frac{2ie}{\hbar c}\mathbf{A} \right) \Psi_i \right|^2 - \left(U\hat{\Psi}\right)^{\dagger} \hat{D} \left(U\hat{\Psi}\right) + \left(U_1\hat{\Psi}^2\right)^{\dagger} \hat{D}_1 \left(U_1\hat{\Psi}^2\right) \right\} + \frac{1}{8\pi} \int d^3r \left(rot\mathbf{A} - \mathbf{H}_0\right)^2. \quad (1)$$

Here $\{\hat{D}, \hat{D}_1\}$ are diagonal matrices:

$$\hat{D} = \begin{pmatrix} \ln\left(\frac{T_{c1}}{T}\right) & 0 \\ 0 & \ln\left(\frac{T_{c2}}{T}\right) \end{pmatrix}, \quad \hat{D}_1 = \frac{1}{2}\begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix} \quad (2)$$

and $\{U, U_1\}$ are the Euler rotation matrices:

$$U = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad U_1 = \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{pmatrix} \quad (3)$$

with free parameters in the GL functional $\{\theta, \theta_1\}$ and wave functions

$$\hat{\Psi} = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}, \quad \hat{\Psi}^2 = \begin{pmatrix} \Psi_1^2 \\ \Psi_2^2 \end{pmatrix}. \quad (4)$$

A multi-component superconductor may possess a phase shift between the components of the order parameter, which is different from $\{0, \pi\}$ already in a zero external magnetic field. In a such superconductor, the time-reversal symmetry is broken. Superconductors of this kind will be referred to in the text as superconductors with broken time-reversal symmetry (BTRS) or BTRS superconductors (for classification of classes of superconductors see [17, 21]). Both of the cases, with time-reversal symmetry and with broken time-reversal symmetry can be described in the framework of the Ginzburg-Landau functional. For considering below a two-component superconductor it means that two modulus of the order parameters, phase difference, and the vector potential $\mathbf{A}$ can be considered as independent variables. Variation of the Ginzburg-Landau functional in these variables leads to a set of four differential equations. The solution of these equations gives the state of the superconductor in an external magnetic field.

Since the system with a single vortex is a rotational invariant, it is convenient to use the cylindrical system of coordinates ($\mathbf{r} = (\rho\cos\phi, \rho\sin\phi, z)$). Then, we take the components of the wave function $\Psi_i = \Psi_i(\rho, \phi)$ in the form:

$$\Psi_i = |\Psi_i| e^{i\chi_i}, \quad \chi_i = \phi + \tilde{\phi}_i, \quad (5)$$

where ϕ is the polar angle and $\tilde{\phi}_i = \tilde{\phi}_i(\rho)$ are functions depending on ρ. From Eq. (5) one gets

$$\partial\_\Psi_i = \mathbf{e}_\rho \left\{ \frac{\partial|\Psi_i|}{\partial\rho} + i|\Psi_i| \left(\frac{\partial\tilde{\phi}_i}{\partial\rho} - \frac{2e}{\hbar c}A_\rho \right) \right\} e^{i\chi_i} + \mathbf{e}_\phi i|\Psi_i| \left\{ \frac{1}{\rho} - \frac{2e}{\hbar c}A_\phi \right\} e^{i\chi_i}, \quad (6)$$

where

$$\mathbf{A} = \mathbf{e}_\rho A_\rho + \mathbf{e}_\phi A_\phi, \frac{2e}{\hbar c}A_\rho = \frac{\partial\Phi}{\partial\rho}, \quad \mathbf{e}_\rho = (\cos\phi, \sin\phi), \mathbf{e}_\phi = (-\sin\phi, \cos\phi). \quad (7)$$

The current density in the single vortex state is

$$\mathbf{j} = e\hbar \sum_{i=1}^{2} \frac{|\Psi_i|^2}{m_i} \left[\mathbf{e}_\rho \frac{\partial(\tilde{\phi}_i - \Phi)}{\partial\rho} + \mathbf{e}_\phi \left(\frac{1}{\rho} - \frac{2e}{\hbar c}A_\phi \right) \right]. \quad (8)$$

From the symmetry considerations, the radial part of the current vanishes. Hence, from Eq. (8), we get

$$\frac{1}{m_1}|\Psi_1|^2 \frac{\partial(\tilde{\phi}_1 - \Phi)}{\partial\rho} + \frac{1}{m_2}|\Psi_2|^2 \frac{\partial(\tilde{\phi}_2 - \Phi)}{\partial\rho} = 0. \quad (9)$$

To resolve Eq. (9), we introduce a new function $\mu(\rho)$:

$$\mu(\rho) = \tilde{\phi}_1 - \tilde{\phi}_2. \quad (10)$$

with $\mu(\rho)$ being a solution of

$$\frac{\partial\mu}{\partial\rho} = \left(1 + \frac{m_2}{m_1} \frac{|\Psi_1|^2}{|\Psi_2|^2} \right) \frac{\partial}{\partial\rho}(\tilde{\phi}_1 - \Phi). \quad (11)$$

Here, we would like to note that the equations obtained by variations of the functional over $\tilde{\phi}_i$ cannot be used as independent equations to determine $\tilde{\phi}_i$ anymore due to the above constraint. Resoling Eq. (11), we get

$$\frac{\partial(\tilde{\phi}_1 - \Phi)}{\partial\rho} = \frac{\partial\mu}{\partial\rho}\Gamma, \quad \frac{\partial(\tilde{\phi}_2 - \Phi)}{\partial\rho} = \frac{\partial\mu}{\partial\rho}(\Gamma - 1). \quad (12)$$

These equations are the key point of the solution to the problem under consideration. Now, we can rewrite the functional Eq. (1) in the form:

$$\tilde{\mathcal{F}} = \int d^3r \Bigg\{ \frac{\hbar^2}{4m_1}\left[\left(\frac{\partial|\Psi_1|}{\partial\rho}\right)^2 + |\Psi_1|^2\left(\left(\frac{\partial(\tilde{\phi}_1-\Phi)}{\partial\rho}\right)^2 + \left(\frac{1}{\rho}-\frac{2e}{\hbar c}A_\phi\right)^2\right)\right]$$
$$+\frac{\hbar^2}{4m_2}\left[\left(\frac{\partial|\Psi_2|}{\partial\rho}\right)^2 + |\Psi_2|^2\left(\left(\frac{\partial(\tilde{\phi}_2-\Phi)}{\partial\rho}\right)^2 + \left(\frac{1}{\rho}-\frac{2e}{\hbar c}A_\phi\right)^2\right)\right]$$
$$+\left(U_1\begin{pmatrix}|\Psi_1|^2e^{2i\tilde{\phi}_1}\\ |\Psi_2|^2e^{2i\tilde{\phi}_2}\end{pmatrix}\right)^\dagger \hat{D}_1\left(U_1\begin{pmatrix}|\Psi_1|^2e^{2i\tilde{\phi}_1}\\ |\Psi_2|^2e^{2i\tilde{\phi}_2}\end{pmatrix}\right) - \left(U\begin{pmatrix}|\Psi_1|e^{i\tilde{\phi}_1}\\ |\Psi_2|e^{i\tilde{\phi}_2}\end{pmatrix}\right)^\dagger \hat{D}\left(U\begin{pmatrix}|\Psi_1|e^{i\tilde{\phi}_1}\\ |\Psi_2|e^{i\tilde{\phi}_2}\end{pmatrix}\right)\Bigg\}$$
$$+\frac{1}{8\pi}\int d^3r(rot(\mathbf{e}_\phi A_\phi)-H_0)^2 \tag{13}$$

If Eqs. (9, 10 and 11) are satisfied, minimization of functional $\tilde{\mathcal{F}}$ produces for functions $\{|\Psi_1|, |\Psi_2|, A_\phi, \mu\}$ four equations. Minimizing the functional Eq. (13), we find the equations for $\{|\Psi_1|, |\Psi_2|\}$:

$$\frac{\hbar^2}{2m_1}\left[-\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right)|\Psi_1| + \left(\Gamma^2\left(\frac{\partial\mu}{\partial\rho}\right)^2 + \left(\frac{1}{\rho}-\frac{2e}{\hbar c}A_\phi\right)^2\right)|\Psi_1|\right]$$
$$+2|\Psi_1|^3(b_1\cos^2\theta_1 + b_2\sin^2\theta_1) - \sin(2\theta_1)|\Psi_1||\Psi_2|^2(b_1-b_2)\cos(2\mu)$$
$$-2|\Psi_1|\left(\cos^2\theta\ln\left(\frac{T_{c1}}{T}\right)+\sin^2\theta\ln\left(\frac{T_{c2}}{T}\right)\right)+\sin(2\theta)|\Psi_2|\ln\left(\frac{T_{c1}}{T_{c2}}\right)\cos\mu = 0 \tag{14}$$

and

$$\frac{\hbar^2}{2m_2}\left[-\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right)|\Psi_2| + \left((\Gamma-1)^2\left(\frac{\partial\mu}{\partial\rho}\right)^2 + \left(\frac{1}{\rho}-\frac{2e}{\hbar c}A_\phi\right)^2\right)|\Psi_2|\right]$$
$$+2|\Psi_2|^3(b_1\sin^2\theta_1 + b_2\cos^2\theta_1) - \sin(2\theta_1)|\Psi_1|^2|\Psi_2|(b_1-b_2)\cos(2\mu)$$
$$-2|\Psi_2|\left(\sin^2\theta\ln\left(\frac{T_{c1}}{T}\right)+\cos^2\theta\ln\left(\frac{T_{c2}}{T}\right)\right)+\sin(2\theta)|\Psi_1|\ln\left(\frac{T_{c1}}{T_{c2}}\right)\cos\mu = 0, \tag{15}$$

where

$$\Gamma = \left(1+\frac{m_2}{m_1}\frac{|\Psi_1|^2}{|\Psi_2|^2}\right)^{-1}. \tag{16}$$

Further, variation of $\tilde{\mathcal{F}}$ with respect to μ gives

$$\frac{\hbar^2}{2m_1}\left(-\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho|\Psi_1|^2\Gamma^2\frac{\partial\mu}{\partial\rho}\right)\right) + \frac{\hbar^2}{2m_2}\left(-\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho|\Psi_2|^2(\Gamma-1)^2\frac{\partial\mu}{\partial\rho}\right)\right)$$
$$+\sin(2\theta_1)|\Psi_1|^2|\Psi_2|^2(b_1-b_2)\sin(2\mu) - \sin(2\theta)|\Psi_1||\Psi_2|\ln\left(\frac{T_{c1}}{T_{c2}}\right)\sin\mu = 0. \tag{17}$$

The gauge is determined by the Maxwell equation for the vector potential A_ϕ:

$$-\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial A_\phi}{\partial\rho}\right) + \frac{8\pi e^2}{c^2}\left(\frac{1}{m_1}|\Psi_1|^2+\frac{1}{m_2}|\Psi_2|^2\right)A_\phi + \frac{1}{\rho^2}A_\phi = \frac{4\pi e\hbar}{c}\left(\frac{1}{m_1}|\Psi_1|^2+\frac{1}{m_2}|\Psi_2|^2\right)\frac{1}{\rho}. \tag{18}$$

and the boundary conditions. At $\rho \to \infty$ vector potential A_ϕ tends to

$$A_\phi \to \frac{\hbar c}{2e}\frac{1}{\rho}. \tag{19}$$

As a result, we obtain the following quantization rule for one flux:

$$\int d^2r\mathbf{H}(\rho) = \frac{\pi\hbar c}{e} = \Phi_0, \tag{20}$$

where Φ_0 is the flux quantum. The effective penetration depth is

$$\lambda^{-2} = \frac{8\pi e^2}{c^2}\left(\frac{1}{m_1}|\Psi_1|^2 + \frac{1}{m_2}|\Psi_2|^2\right)_{\rho\to\infty}. \tag{21}$$

Using Eq. (20), we can obtain the next expression for the first magnetic critical field H_{c1}.

$$\frac{H_{c1}}{4\pi}\Phi_0 = \int d^2r\frac{H^2(\rho)}{8\pi} + \int d^2r\left(f_1^{(1)} - f_1^{(0)}\right). \tag{22}$$

Here $f_1^{(0,1)}$ are the density of the condensate energy in the ground state and in the state with a single vortex:

and

$$\begin{aligned} f_1^{(1)} = {} & \frac{\hbar^2}{4m_1}\left[\left(\frac{\partial|\Psi_1|}{\partial\rho}\right)^2 + |\Psi_1|^2\left(\Gamma^2\left(\frac{\partial\mu}{\partial\rho}\right)^2 + \left(\frac{1}{\rho} - \frac{2e}{\hbar c}A_\phi\right)^2\right)\right] \\ & + \frac{\hbar^2}{4m_2}\left[\left(\frac{\partial|\Psi_2|}{\partial\rho}\right)^2 + |\Psi_2|^2\left((1-\Gamma)^2\left(\frac{\partial\mu}{\partial\rho}\right)^2 + \left(\frac{1}{\rho} - \frac{2e}{\hbar c}A_\phi\right)^2\right)\right] \\ & + \frac{1}{2}[|\Psi_1|^4(b_1 - \sin^2\theta_1(b_1-b_2)) + |\Psi_2|^4(b_1 - \cos^2\theta_1(b_1-b_2)) - \sin(2\theta_1)|\Psi_1|^2|\Psi_2|^2(b_1-b_2)\cos(2\mu)] \\ & - |\Psi_1|^2\left(\cos^2\theta\ln\left(\frac{T_{c1}}{T}\right) + \sin^2\theta\ln\left(\frac{T_{c2}}{T}\right)\right) - |\Psi_2|^2\left(\sin^2\theta\ln\left(\frac{T_{c1}}{T}\right) + \cos^2\theta\ln\left(\frac{T_{c2}}{T}\right)\right) \\ & + \sin(2\theta)|\Psi_1||\Psi_2|\ln\left(\frac{T_{c1}}{T_{c2}}\right)\cos\mu \end{aligned} \tag{23}$$

$$\begin{aligned} f_1^{(0)} = {} & \frac{1}{2}[|\Psi_1^{(0)}|^4(b_1 - \sin^2\theta_1(b_1-b_2)) + |\Psi_2^{(0)}|^4(b_1 - \cos^2\theta_1(b_1-b_2)) \\ & - \sin(2\theta_1)|\Psi_1^{(0)}|^2|\Psi_2^{(0)}|^2(b_1-b_2)\cos(2(\phi_2^{(0)} - \phi_1^{(0)}))] \\ & - \left[|\Psi_1^{(0)}|^2\left(\cos^2\theta\ln\left(\frac{T_{c1}}{T}\right) + \sin^2\theta\ln\left(\frac{T_{c2}}{T}\right)\right) + |\Psi_2^{(0)}|^2\left(\sin^2\theta\ln\left(\frac{T_{c1}}{T}\right) + \cos^2\theta\ln\left(\frac{T_{c2}}{T}\right)\right)\right] \\ & + \sin(2\theta)|\Psi_1^{(0)}||\Psi_2^{(0)}|\ln\left(\frac{T_{c1}}{T_{c2}}\right)\cos(\phi_1^{(0)} - \phi_2^{(0)}), \end{aligned} \tag{24}$$

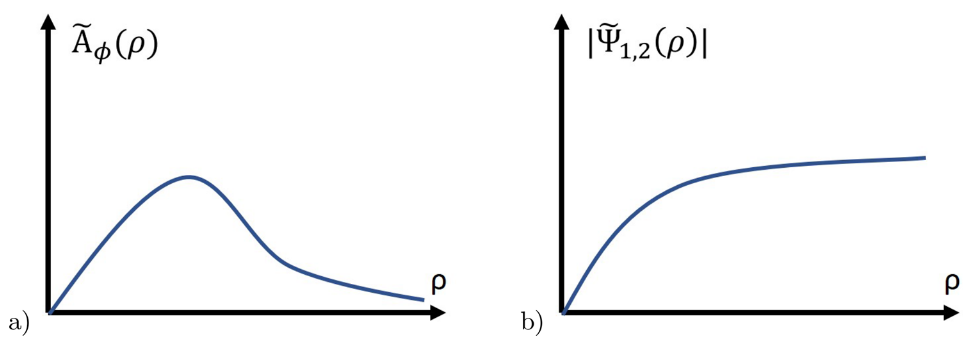

Fig. 1 Schematic ρ-dependence of $\tilde{A}_\phi(\rho)$ and $\tilde{\Phi}_{\{}1,2\}(\rho)$

where the functions $\Psi_{1,2}^{(0)}$ are the values of the correspondent functions in the ground state.

In the dimensionless variables, we obtain (see Appendix A):

$$\begin{aligned}
\tilde{H}_{c1} &= \frac{1}{2}\int_0^{+\infty} dt_0 t_0 \tilde{H}^2 \\
&+ \left(\frac{4\pi e^2\gamma^2}{m_1 c^2}|\Psi_1|^2_{inf}\right)\int_0^{+\infty} dt_0 t_0 \Bigg\{ \left(\frac{\partial|\tilde{\Psi}_1|}{\partial t_0}\right)^2 + |\tilde{\Psi}_1|^2\left(\Gamma^2\left(\frac{\partial \mu}{\partial t_0}\right)^2 + \frac{1}{t_0^2}(1-\tilde{A}t_0)^2\right) \\
&+ \frac{m_1}{m_2}\frac{|\Psi_2|^2_{inf}}{|\Psi_1|^2_{inf}}\left[\left(\frac{\partial|\tilde{\Psi}_2|}{\partial t_0}\right)^2 + |\tilde{\Psi}_2|^2\left((1-\Gamma)^2\left(\frac{\partial \mu}{\partial t_0}\right)^2 + \frac{1}{t_0^2}(1-\tilde{A}t_0)^2\right)\right]\Bigg\} \\
&+ \left(\frac{4\pi e^2\gamma^2}{m_1 c^2}|\Psi_1|^2_{inf}\right)(b_1|\Psi_1|^2_{inf})\int_0^{+\infty} dt_0 t_0 \Bigg\{ (|\tilde{\Psi}_1|^4 - 1)\left(\cos^2\theta_1 + \frac{b_2}{b_1}\sin^2\theta_1\right) \\
&+ \frac{|\Psi_2|^4_{inf}}{|\Psi_1|^4_{inf}}(|\tilde{\Psi}_2|^4 - 1)\left(\sin^2\theta_1 + \frac{b_2}{b_1}\cos^2\theta_1\right)\Bigg\} \\
&- \left(\frac{4\pi e^2\gamma^2}{m_1 c^2}|\Psi_1|^2_{inf}\right)(b_1|\Psi_2|^2_{inf})\left(1-\frac{b_2}{b_1}\right)\sin(2\theta_1)\int_0^{+\infty} dt_0 t_0 \{|\tilde{\Psi}_1|^2|\tilde{\Psi}_2|^2\cos(2\mu) - \cos(2\mu_{inf})\} \\
&- \left(\frac{8\pi e^2\gamma^2}{m_1 c^2}|\Psi_1|^2_{inf}\right)\int_0^{+\infty} dt_0 t_0 \Bigg\{ \left(\cos^2\theta \ln\left(\frac{T_{c1}}{T}\right) + \sin^2\theta \ln\left(\frac{T_{c2}}{T}\right)\right)(|\tilde{\Psi}_1|^2 - 1) \\
&+ \frac{|\Psi_2|^2_{inf}}{|\Psi_1|^2_{inf}}\left(\sin^2\theta \ln\left(\frac{T_{c1}}{T}\right) + \cos^2\theta \ln\left(\frac{T_{c2}}{T}\right)\right)(|\tilde{\Psi}_2|^2 - 1) \\
&- \sin(2\theta)\frac{|\Psi_2|_{inf}}{|\Psi_1|_{inf}}\ln\left(\frac{T_{c1}}{T_{c2}}\right)(|\tilde{\Psi}_1||\tilde{\Psi}_2|\cos(\mu) - \cos(\mu_{inf}))\Bigg\}
\end{aligned} \tag{25}$$

The results of the numerical calculations of the first and second critical magnetic fields H_{c1}, H_{c2}, and also the thermodynamic critical field $\tilde{H}_c$ are given in Table 1. Note, that dependence of $\tilde{H}_c$ from θ is weak. An increase of θ leads to the evolution of the superconductivity so that H_{c1} and H_c cross with the formation of a nontrivial transition region.

The ground state without vortices can be of two types. The first type is with preserved time-reversal symmetry $\sin(\tilde{\phi}_1 - \tilde{\phi}_2) = 0$. The second type is the state with broken time-reversal symmetry, which has the solution with $\sin(\tilde{\phi}_1 - \tilde{\phi}_2) \neq 0$. The first case is trivial. In the second case a separate point can exist $\{\rho = \rho_0\}$ (see Fig. 2). Below this point in the single vortex solution, $|\Psi_{1,2}|$ depend on ρ, but $\tilde{\phi}_1 - \tilde{\phi}_2 = \{0, \pi\}$. As a result Eqs. (13, 14, 16 and 18) shrink to three equations for $\{|\Psi_1|, |\Psi_2|, A_\phi\}$ as in the case with preserved time-reversal symmetry.

Solving the set of equations, one gets the asymptotics:

$$A_\phi = \frac{H(0)}{2}\rho \text{ at } \rho \ll \lambda, \text{ and } A_\phi = \frac{\hbar c}{2e}\frac{1}{\rho} \text{ at } \rho \gg \lambda, \tag{26}$$

where $H(0)$ is the value of the magnetic field at the center of the vortex core. The functions $\{|\Psi_{1,2}|\}$ are proportional to ρ at the distances smaller than the correlation length and approaches with an exponential decay to a constant at large ρ. Qualitative ρ-dependence of A_ϕ,$\tilde{\phi}_1 - \tilde{\phi}_2$ and $|\Psi_{1,2}|$ are presented in Fig. 1a and b.

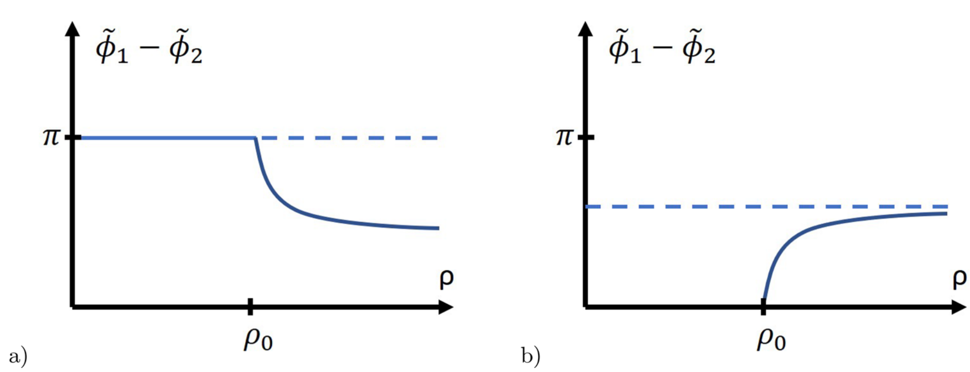

Fig. 2 Two possible ρ dependencies of $\tilde{\phi}_1 - \tilde{\phi}_2$. More details see in the text

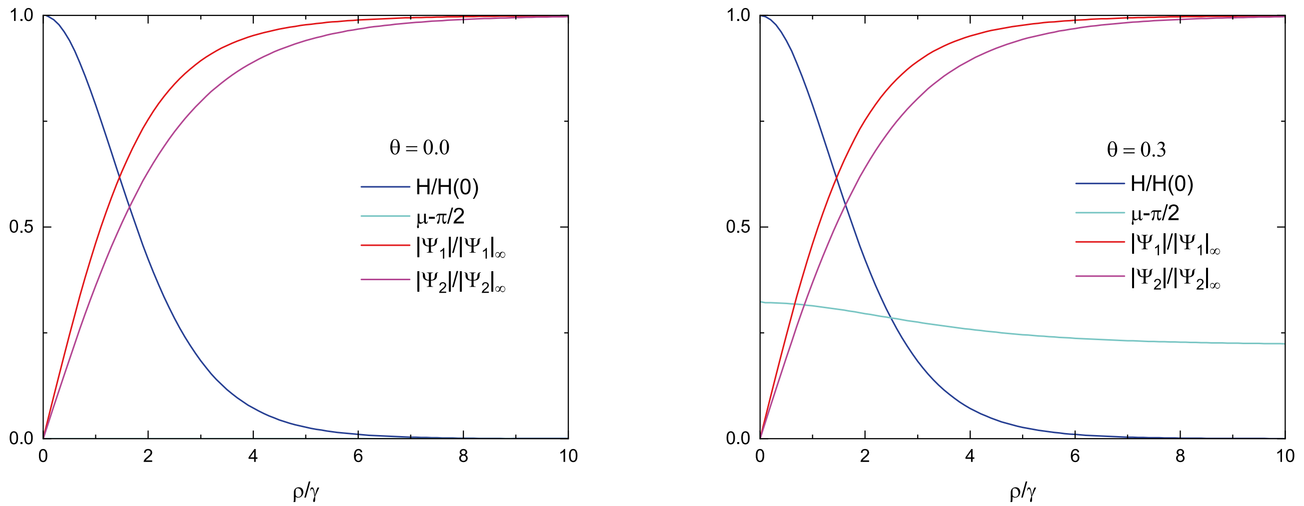

Fig. 3 Normalized magnetic field H/H_0, phase ϕ, and wave functions $|\Psi_1|/|\Psi_1|\infty$. as function of ρ/ρ_0. The parameters are $\gamma^2 = \frac{\hbar^2}{2m_1}$, $\mu = \tilde{\phi}_1 - \tilde{\phi}_2$

Using Eq. (17) one can estimate the value of parameter ρ_0:

$$\left\{ \frac{m_2}{m_1}\frac{\partial}{\partial \rho}\left(\rho|\Psi_1|^2\Gamma^2\right) + \frac{\partial}{\partial \rho}\left(\rho|\Psi_2|^2(1-\Gamma)^2\right) \right\}_{\rho=(\rho_0)_+} = 0 \tag{27}$$

The value of the slope $\left(\frac{\partial \mu}{\partial \rho}\right)_{\rho=(\rho)_+}$ is a free parameter. Its value is fixed by the boundary conditions at infinity. As a result, we get a weak singularity in the functions $\{|\Psi_1|, |\Psi_2|\}$ since the functions themselves and their first derivatives continue at this point.

At large subspace of the intrinsic parameters, the value of ρ_0 is located in the nonphysical region ($\rho < 0$). The intrinsic parameters, used by us for numerical calculations belong to such subspace. The simplest situation for calculations arises for $\theta = 0$. In such case the solution of Eq. (17) is

$$\mu(\rho) = \pm\frac{\pi}{2}. \tag{28}$$

For parameters:

$$m_1 = 2m_2, b_2 = 2b_1, \quad b_1 = 1.5 \cdot 10^{-5}G^{-2}, \quad T_{c1}/T = 1.2, \quad T_{c2}/T = 1.1 \tag{29}$$

and

$$\frac{\hbar^2}{4m_1} = 2.7773 \cdot 10^{-11}cm^2,$$

$$\theta_1 = 0.5, \theta = \{0, 0.1, 0.3\}$$

the dependencies $|\tilde{\Psi}_{1,2}|, \tilde{B}(\rho)$ and $(\tilde{\phi}_1 - \tilde{\phi}_2)_\rho$ for $\theta = 0$ and $\theta = 0.3$ are given in Fig. 2a. For the numerical calculations, we have used dimensionless equations. The details of the numerical calculations are presented in Appendices A-G.

From Eqs. (13-15), we obtain the next values of $\{|\Psi_1|, |\Psi_2|, \tilde{\phi}_1 - \tilde{\phi}_2\}$ at $\rho \to \infty$ in the state with broken time-reversal symmetry:

$$\cos(\tilde{\phi}_1 - \tilde{\phi}_2)|_\infty = \frac{\sin(2\theta)\ln(T_{c1}/T_{c2})}{2|\Psi_1||\Psi_2|\sin(2\theta_1)(b_1 - b_2)}, \tag{30}$$

and

$$|\Psi_1|^2(b_1\cos^2\theta_1 + b_2\sin^2\theta_1) + \frac{1}{2}\sin(2\theta_1)|\Psi_2|^2(b_1 - b_2) = \left(\cos^2\theta\ln\left(\frac{T_{c1}}{T}\right) + \sin^2\theta\ln\left(\frac{T_{c2}}{T}\right)\right) \tag{31}$$

$$|\Psi_2|^2(b_2\cos^2\theta_1 + b_1\sin^2\theta_1) + \frac{1}{2}\sin(2\theta_1)|\Psi_1|^2(b_1 - b_2) = \left(\cos^2\theta\ln\left(\frac{T_{c2}}{T}\right) + \sin^2\theta\ln\left(\frac{T_{c1}}{T}\right)\right) \tag{32}$$

The considered state corresponds to the minima of the free energy functional provided the following inequality is satisfied:

$$\frac{\sin^2(2\theta)\ln^2(T_{c1}/T_{c2})}{4\sin^2(2\theta_1)(b_1 - b_2)^2} < |\Psi_1|^2|\Psi_2|^2.$$

Obviously, for this case, Eqs. (30)-(32) give a single solution and, therefore, they describe the global minimum.

In this case, the vector potential $(A_\phi - (\hbar c)/(2e\rho))$ decays exponentially at infinity as $\propto \exp(-\rho/\lambda)/\sqrt{\rho}$, where the parameter λ is given by the Eq. (21). The three quantity $\{\delta\mu, \delta|\Psi_1|, \delta|\Psi_2|\}$ of the difference of the correspondent values from that at $\rho \to \infty$ decay exponentially at large distances as well:

$$\begin{pmatrix} \delta\mu \\ \delta|\Psi_1| \\ \delta|\Psi_2| \end{pmatrix} = C_1 \exp(-\kappa_1^{(1)}\rho)\frac{1}{\sqrt{\rho}}\mathbf{f}_1 + C_2 \exp(-\kappa_1^{(2)}\rho)\frac{1}{\sqrt{\rho}}\mathbf{f}_2 + C_3 \exp(-\kappa_1^{(3)}\rho)\frac{1}{\sqrt{\rho}}\mathbf{f}_3, \tag{33}$$

where the C_i with $i = 1, 2, 3$ are some coefficients, while $\kappa_1^{(i)}$ and $\mathbf{f}_i$ are eigenvalues and eigenvectors of the next system:

$$\tilde{\tilde{D}}\begin{pmatrix} \delta\mu \\ \delta|\Psi_1| \\ \delta|\Psi_2| \end{pmatrix} = 0. \tag{34}$$

Here $\tilde{\tilde{D}}$ is a Hermitian operator with the following elements:

$$\tilde{\tilde{D}}_{11} = -\kappa_1^2\left(\frac{\hbar^2|\Psi_1^2|\Gamma^2}{2m_1} + \frac{\hbar^2|\Psi_2^2|(1-\Gamma)^2}{2m_2}\right) - \sin(2\theta)|\Psi_1||\Psi_2|\ln\left(\frac{T_{c1}}{T_{c2}}\right)\cos\mu + 2\sin(2\theta_1)|\Psi_1|^2|\Psi_2|^2(b_1-b_2)\cos(2\mu)$$
$$\tilde{\tilde{D}}_{22} = -\kappa_1^2\frac{\hbar^2}{2m_1} + 6|\Psi_1|^2(b_1\cos^2\theta_1 + b_2\sin^2\theta_1) - \sin(2\theta_1)(b_1-b_2)|\Psi_2|^2\cos(2\mu) - 2\left(\cos^2\theta\ln\left(\frac{T_{c1}}{T}\right) + \sin^2\theta\ln\left(\frac{T_{c2}}{T}\right)\right),$$
$$\tilde{\tilde{D}}_{33} = -\kappa_1^2\frac{\hbar^2}{2m_2} + 6|\Psi_2|^2(b_2\cos^2\theta_1 + b_1\sin^2\theta_1) - \sin(2\theta_1)(b_1-b_2)|\Psi_1|^2\cos(2\mu) - 2\left(\sin^2\theta\ln\left(\frac{T_{c1}}{T}\right) + \cos^2\theta\ln\left(\frac{T_{c2}}{T}\right)\right), \tag{35}$$
$$\tilde{\tilde{D}}_{12} = \tilde{\tilde{D}}_{21} = -\sin(2\theta)|\Psi_2|\sin\mu\ln\left(\frac{T_{c1}}{T_{c2}}\right) + 2\sin(2\theta_1)(b_1-b_2)|\Psi_1||\Psi_2|^2\sin(2\mu)$$
$$\tilde{\tilde{D}}_{13} = \tilde{\tilde{D}}_{31} = -\sin(2\theta)|\Psi_1|\sin\mu\ln\left(\frac{T_{c1}}{T_{c2}}\right) + 2\sin(2\theta_1)(b_1-b_2)|\Psi_1|^2|\Psi_2|\sin(2\mu)$$
$$\tilde{\tilde{D}}_{23} = \tilde{\tilde{D}}_{32} = -2\sin(2\theta_1)|\Psi_1||\Psi_2|(b_1-b_2)\cos(2\mu) + \sin(2\theta)\ln\left(\frac{T_{c1}}{T_{c2}}\right)\cos\mu$$

By the correct boundary conditions, the solution at large distances tends to the that given by Eq. (33). The correspondent free parameters for Eqs. (14, 15 and 18) are the slopes at $\rho = 0$ of $|\tilde{\Psi}_1|$, $|\tilde{\Psi}_2|$ and $\tilde{A}_\phi$. For $\tilde{\mu}$ at $\rho = 0$ the initial condition is $\mu(0)$ if ρ_0 does not exists, and Eq. (27) otherwise. At this point, we note that at large distance $|\Psi_1|$, $|\Psi_2|$ and μ decay with the same exponent due to the coupling between the components.

3 Critical Field H_{c2}

At the critical point H_{c2} the order parameters can be found with the following Ansatz:

$$\begin{pmatrix} |\Psi_1| \\ |\Psi_2| \end{pmatrix} = \Psi\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \tag{36}$$

where Ψ is the solution of the equation [1]:

$$-\partial_-^2\Psi = \eta\Psi, \mathbf{A} = (0, Hx, 0), \mathbf{H} = (0, 0, H) \tag{37}$$

and C_1 and C_2 are constants. The solution of Eq. (37) is

$$\Psi = \exp\left\{-\frac{eH}{\hbar c}(x-x_0)^2 + \frac{2ieH}{\hbar c}x_0 y\right\} \tag{38}$$

with $\eta = 2eH/\hbar c$ and x_0 being a free parameter.

For η, we obtain the following quadratic equation

$$\det\begin{pmatrix} \frac{\hbar^2}{4m_1}\eta - \left(\cos^2\theta\ln\left(\frac{T_{c1}}{T}\right) + \sin^2\theta\ln\left(\frac{T_{c2}}{T}\right)\right) & \frac{1}{2}\ln\left(\frac{T_{c1}}{T_{c2}}\right)\sin(2\theta) \\ \frac{1}{2}\ln\left(\frac{T_{c1}}{T_{c2}}\right)\sin(2\theta) & \frac{\hbar^2}{4m_2}\eta - \left(\sin^2\theta\ln\left(\frac{T_{c1}}{T}\right) + \cos^2\theta\ln\left(\frac{T_{c2}}{T}\right)\right) \end{pmatrix} = 0. \tag{39}$$

Solving these equations, we get H_{c2}:

$$\frac{\hbar e}{c}H_{c2} = m_1\left(\cos^2\theta\ln\frac{T_{c1}}{T} + \sin^2\theta\ln\frac{T_{c2}}{T}\right) + m_2\left(\sin^2\theta\ln\frac{T_{c1}}{T} + \cos^2\theta\ln\frac{T_{c2}}{T}\right) + \left[\left(m_1\left(\cos^2\theta\ln\frac{T_{c1}}{T} + \sin^2\theta\ln\frac{T_{c2}}{T}\right) - m_2\left(\sin^2\theta\ln\frac{T_{c1}}{T} + \cos^2\theta\ln\frac{T_{c2}}{T}\right)\right)^2 + m_1 m_2\ln^2\left(\frac{T_{c1}}{T_{c2}}\right)\sin^2(2\theta)\right]^{1/2} \tag{40}$$

The numerical results are

$$H_{c2} = \frac{\hbar c}{2e\gamma^2}\tilde{H}_{c2} \tag{41}$$

and

$$\theta = 0 : \tilde{H}_{c2} = 2\ln 1.2 = 0.36464 \quad (42)$$

$$\theta = 0.3 : \tilde{H}_{c2} = 0.3542472. \quad (43)$$

In both cases, we obtain that the critical fields H_{c1} and H_{c2} are larger than the thermodynamic H_c. Hence, the transition to the vortex state takes place at the external field equal H_{c2}. However, the transition to the homogeneous case happens at $H = H_c$ as a transition of the first order accompanied by a jump in the magnetic moment value. In the region $H_{c2} > H > H_c$ a cascade of transitions with change of the structure of the vortex state is possible [22].

4 Conclusions

We considered a single vortex state and the first critical magnetic field H_{c1} in a multicomponent superconductor with N components in the framework of the Ginzburg-Landau functional. It has been shown that the problem can be reduced to solving a system of $2N-1$ ordinary differential equations if in the ground state, the phase shift between the component of the complex order parameter is 0 or π at zero external magnetic field. Otherwise, it consists of $2N$ equations. At $\rho \to \infty$ the phase difference between the components of the order parameter $\mu_k = \phi_1 - \phi_k$ does not tend to $0, \pi$. And the μ can reach the values $0, \pi$ only at finite $\rho = \rho_0$ and for $\rho < \rho_0$ the solution $\mu = 0, \pi$ is realized (see Fig. 2).

In a single-component superconductor in a magnetic field, the state is determined by the Ginzburg-Landau parameter $\kappa^2 = H_{c2}/H_{cm}$. (The introduced by Ginzburg and Landau in the original work is $\kappa_{GL} = \kappa/\sqrt{2}$). In the approximation of the Ginzburg-Landau functional, κ is temperature independent. For $\kappa = 1$ all three critical fields H_{c1}, H_{c2} and H_{cm} coincide. Multi-component superconductors may show much more broad spectrum of states in an external magnetic field. Magnetic fields H_{cm} and H_{c2} are quite easy to calculate. However, in order to identify the state in an external magnetic field, we need to find also H_{c1}. As a result, in addition to the unusual sequence of the critical fields, the possibility of overscreening can be realized. In this case, it becomes possible for the jump-like transition between different solutions of the Abrikosov lattices. The calculation of the critical field H_{c1} is again given by the solution of the set of Eqs. (69-74) which explicitly take into account the relation between the phases $\tilde{\phi}_1$ and $\tilde{\phi}_2$, which is imposed by the equation $div\mathbf{j} = 0$. Instead of one singular point $\kappa = 1$ existing in single-component superconductors, in a multicomponent superconductor in any case four parameters $(\theta, \theta_1, (b_1 - b_2), ln(T_{c1}/T_{c2}))$ form basis for criterion set of singular "surfaces." Investigation of physical states with parameters close to this set present special large interest and can be made inside presented method.

Appendix A: Dimensionless Form of the Equations

For numerical calculations it is convenient to bring the equations to a dimensionless form. For these purposes, we use the following substitutions:

$$\rho = \gamma t_0 \text{ with } \gamma^2 = \frac{\hbar^2}{2m_1}, \quad |\Psi_1| = |\tilde{\Psi}_1||\Psi_1|_{inf},$$
$$|\Psi_2| = |\tilde{\Psi}_2||\Psi_2|_{inf}, \quad A_\phi = \frac{\hbar c}{2e\gamma}\tilde{A}.$$

Here $|\Psi_1|_{inf}$ and $|\Psi_2|_{inf}$ are values of $|\Psi_1|$ and $|\Psi_2|$ at $\rho \to \infty$. They can be obtained from Eqs. (30-32):

$$|\Psi_1|^2_{inf} = \frac{1}{b_1 b_2}\Big\{(b_1\sin^2\theta_1 + b_2\cos^2\theta_2)\Big[\cos^2\theta\ln\Big(\frac{T_{c1}}{T}\Big) + \sin^2\theta\ln\Big(\frac{T_{c2}}{T}\Big)\Big] - \frac{1}{2}\sin(2\theta_1)(b_1 - b_2)\Big[\sin^2\theta\ln\Big(\frac{T_{c1}}{T}\Big) + \cos^2\theta\ln\Big(\frac{T_{c2}}{T}\Big)\Big]\Big\}, \quad (44)$$

$$|\Psi_2|^2_{inf} = \frac{1}{b_1 b_2}\Big\{(b_1\cos^2\theta_1 + b_2\sin^2\theta_2)\Big[\sin^2\theta\ln\Big(\frac{T_{c1}}{T}\Big) + \cos^2\theta\ln\Big(\frac{T_{c2}}{T}\Big)\Big] - \frac{1}{2}\sin(2\theta_1)(b_1 - b_2)\Big[\cos^2\theta\ln\Big(\frac{T_{c1}}{T}\Big) + \sin^2\theta\ln\Big(\frac{T_{c2}}{T}\Big)\Big]\Big\} \quad (45)$$

and

$$cos(\mu_{inf}) = \frac{\sin(2\theta)\ln\left(T_{c1}/T_{c2}\right)}{2|\Psi_1|_{inf}|\Psi_2|_{inf}\sin(2\theta_1)(b_1 - b_2)}. \quad (46)$$

Then, Eqs. (14, 15) take the following form:

$$-\left(\frac{1}{t_0}\frac{\partial|\tilde{\Psi}_1|}{\partial t_0} + \frac{\partial^2|\tilde{\Psi}_1|}{\partial t_0^2}\right) + \left(\Gamma^2\left(\frac{\partial\mu}{\partial t_0}\right)^2 + \frac{1}{t_0^2}(1 - \tilde{A}t_0)^2\right)|\tilde{\Psi}_1|$$
$$+ 2|\Psi_1|^2_{inf}(b_1\cos^2\theta_1 + b_2\sin^2\theta_2)|\tilde{\Psi}_1|^3 - |\tilde{\Psi}_1||\tilde{\Psi}_2|^2$$
$$\sin(2\theta_1)|\Psi_2|^2_{inf}(b_1 - b_2)\cos(2\mu) \quad (47)$$
$$- 2|\tilde{\Psi}_1|\left(\cos^2\theta\ln\left(\frac{T_{c1}}{T}\right) + \sin^2\theta\ln\left(\frac{T_{c2}}{T}\right)\right)$$
$$+ \sin(2\theta)\frac{|\Psi_2|_{inf}}{|\Psi_1|_{inf}}|\tilde{\Psi}_2|\ln\left(\frac{T_{c1}}{T_{c2}}\right)\cos\mu = 0$$

and

$$
\begin{aligned}
&-\frac{m_1}{m_2}\left(\frac{1}{t_0}\frac{\partial|\tilde{\Psi}_2|}{\partial t_0}+\frac{\partial^2|\tilde{\Psi}_2|}{\partial t_0^2}\right)+\frac{m_1}{m_2}\left((\Gamma-1)^2\left(\frac{\partial\mu}{\partial t_0}\right)^2+\frac{1}{t_0^2}(1-\tilde{A}t_0)^2\right)|\tilde{\Psi}_2|\\
&+2|\Psi_2|^2_{inf}(b_1\sin^2\theta_1+b_2\cos^2\theta_2)|\tilde{\Psi}_2|^3-|\tilde{\Psi}_2||\tilde{\Psi}_1|^2\sin(2\theta_1)|\Psi_1|^2_{inf}(b_1-b_2)\cos(2\mu)\\
&-2|\tilde{\Psi}_2|\left(\sin^2\theta\ln\left(\frac{T_{c1}}{T}\right)+\cos^2\theta\ln\left(\frac{T_{c2}}{T}\right)\right)+\sin(2\theta)\frac{|\Psi_1|_{inf}}{|\Psi_2|_{inf}}|\tilde{\Psi}_1|\ln\left(\frac{T_{c1}}{T_{c2}}\right)\cos\mu=0.
\end{aligned}
\tag{48}
$$

Here Γ is:

$$
\Gamma=\left(1+\frac{m_2}{m_1}\frac{|\tilde{\Psi}_1|^2|\Psi_1|^2_{inf}}{|\tilde{\Psi}_2|^2|\Psi_2|^2_{inf}}\right)^{-1}. \tag{49}
$$

The Maxwell equation in the dimensionless variable has the following form:

$$
-\left(\frac{1}{t_0}\frac{\partial\tilde{A}}{\partial t_0}+\frac{\partial^2\tilde{A}}{\partial t_0^2}\right)+\frac{8\pi e^2\gamma^2}{m_1c^2}|\Psi_1|^2_{inf}\left(|\tilde{\Psi}_1|^2+\frac{m_1}{m_2}\frac{(|\Psi_2|_{inf})^2}{(|\Psi_1|_{inf})^2}|\tilde{\Psi}_2|^2\right)\left(\tilde{A}-\frac{1}{t_0}\right)+\frac{1}{t_0^2}\tilde{A}=0 \tag{50}
$$

The equation for μ is

$$
\begin{aligned}
&-\frac{1}{t_0}\frac{\partial}{\partial t_0}\left(t_0|\tilde{\Psi}_1|^2\Gamma^2\frac{\partial\mu}{\partial t_0}\right)-\frac{m_1}{m_2}\frac{|\Psi_2|^2_{inf}}{|\Psi_1|^2_{inf}}\frac{1}{t_0}\frac{\partial}{\partial t_0}\left(t_0|\tilde{\Psi}_2|^2(1-\Gamma)^2\frac{\partial\mu}{\partial t_0}\right)\\
&+\sin(2\theta_1)|\Psi_2|^2_{inf}|\tilde{\Psi}_1|^2|\tilde{\Psi}_2|^2(b_1-b_2)\sin(2\mu)-\sin(2\theta)\frac{|\Psi_2|_{inf}}{|\Psi_1|_{inf}}|\tilde{\Psi}_1||\tilde{\Psi}_2|\ln\left(\frac{T_{c1}}{T_{c2}}\right)\sin\mu=0,
\end{aligned}
\tag{51}
$$

The magnetic field H is equal to

$$
H=\frac{\hbar c}{2e\gamma^2}\tilde{H},\tilde{H}=\frac{\partial\tilde{A}}{\partial t_0}+\frac{\tilde{A}}{t_0}.
$$

Appendix B: Approximation in the Range $t_0 \ll 1$.

In the range $t_0 \ll 1$ the functions $|\tilde{\Psi}_1|, |\tilde{\Psi}_2|$ are odd functions of t_0, while the function μ is an even function of t_0. They can be expanded in this range as:

$$
|\tilde{\Psi}_1|=\alpha_1t_0-\alpha_3t_0^3+\alpha_5t_0^5+\ldots|\tilde{\Psi}_2|=\beta_1t_0-\beta_3t_0^3+\beta_5t_0^5+\ldots \tag{52}
$$

$$
\tilde{A}=a_1t_0-a_3t_0^3+a_5t_0^5+\ldots\mu=c_0-c_2t_0^2+c_4t_0^4+\ldots \tag{53}
$$

The coefficients $\{\alpha_1, \beta_1, a_1, c_0\}$ can be found from the boundary conditions at $t_0 \to \infty$ see Eqs. (44-46). Inserting the expansions Eqs. (52 and 53) into Eqs. (47-51) we obtain the next expression for the coefficients $\{\alpha_3, \alpha_5, \beta_3, \beta_5, a_3, a_5, c_2, c_4, \Gamma_0\}, \Gamma = \Gamma_0 + \Gamma_1 t_0^2$:

$$
\Gamma_0=\left(1+\frac{m_2}{m_1}\frac{\alpha_1^2}{\beta_1^2}\frac{|\Psi_1|^2_{inf}}{|\Psi_2|^2_{inf}}\right)^{-1}
$$

$$
\Gamma_1=2\frac{m_2}{m_1}\frac{\alpha_1^2}{\beta_1^2}\frac{|\Psi_1|^2_{inf}}{|\Psi_2|^2_{inf}}\left(\frac{\alpha_3}{\alpha_1}-\frac{\beta_3}{\beta_1}\right)\Gamma_0^2
$$

Correspondingly, we get the following set of equations for the expansion coefficients from Eq. (47):

$$
\begin{aligned}
&8\alpha_3-2\alpha_1a_1-2\alpha_1\left(\cos^2\theta\ln\left(\frac{T_{c1}}{T}\right)+\sin^2\theta\ln\left(\frac{T_{c2}}{T}\right)\right)\\
&\quad+\beta_1\sin(2\theta)\frac{|\Psi_2|_{inf}}{|\Psi_1|_{inf}}\ln\left(\frac{T_{c1}}{T_{c2}}\right)\cos c_0=0
\end{aligned}
\tag{54}
$$

and

$$-24\alpha_5+4\alpha_1c_2^2\Gamma_0^2+2(\alpha_3a_1+\alpha_1a_3)+\alpha_1a_1^2+2|\Psi_1|^2_{inf}(b_1\cos^2\theta_1+b_2\sin^2\theta_1)\alpha_1^3$$
$$+\sin(2\theta_1)\alpha_1\beta_1^2|\Psi_2|^2_{inf}(b_1-b_2)(1-2\cos^2(c_0))+2\alpha_3\left(\cos^2\theta\ln\left(\frac{T_{c1}}{T}\right)+\sin^2\theta\ln\left(\frac{T_{c2}}{T}\right)\right) \quad (55)$$
$$+\sin(2\theta)\frac{|\Psi_2|_{inf}}{|\Psi_1|_{inf}}\ln\left(\frac{T_{c1}}{T_{c2}}\right)(-\beta_3\cos(c_0)+\beta_1c_2\sin(c_0))=0,$$

and from Eq. (48):

$$\frac{m_1}{m_2}(8\beta_3-2\beta_1a_1)-2\beta_1\left(\cos^2\theta\ln\left(\frac{T_{c2}}{T}\right)+\sin^2\theta\ln\left(\frac{T_{c1}}{T}\right)\right)+\alpha_1\sin(2\theta)\frac{|\Psi_1|_{inf}}{|\Psi_2|_{inf}}\ln\left(\frac{T_{c1}}{T_{c2}}\right)\cos(c_0)=0 \quad (56)$$

and

$$\frac{m_1}{m_2}[-24\beta_5+4\beta_1c_2^2(1-\Gamma_0)^2+2(\beta_1a_3+\beta_3a_1)+\beta_1a_1^2]+2|\Psi_2|^2_{inf}(b_1\sin^2\theta_1+b_2\cos^2\theta_1)\beta_1^3$$
$$+\sin(2\theta_1)\beta_1\alpha_1^2|\Psi_1|^2_{inf}(b_1-b_2)(1-2\cos^2c_0)+2\beta_3\left(\cos^2\theta\ln\left(\frac{T_{c2}}{T}\right)+\sin^2\theta\ln\left(\frac{T_{c1}}{T}\right)\right) \quad (57)$$
$$+\sin(2\theta)\frac{|\Psi_1|_{inf}}{|\Psi_2|_{inf}}\ln\left(\frac{T_{c1}}{T_{c2}}\right)(\alpha_1c_2\sin(c_0)-\alpha_3\cos(c_0))=0.$$

Further, the Maxwell equation gives the following set of equations:

$$8a_3-\frac{8\pi e^2\gamma^2}{m_1c^2}|\Psi_1|^2_{inf}\left(\alpha_1^2+\frac{m_1}{m_2}\frac{|\Psi_2|^2_{inf}}{|\Psi_1|^2_{inf}}\beta_1^2\right)=0 \quad (58)$$

and

$$-24a_5+\frac{8\pi e^2\gamma^2}{m_1c^2}|\Psi_1|^2_{inf}\left[a_1\left(\alpha_1^2+\frac{m_1}{m_2}\frac{|\Psi_2|^2_{inf}}{|\Psi_1|^2_{inf}}\beta_1^2\right)+2\left(\alpha_1\alpha_3+\frac{m_1}{m_2}\frac{|\Psi_2|^2_{inf}}{|\Psi_1|^2_{inf}}\beta_1\beta_3\right)\right]=0. \quad (59)$$

And the equation for μ yields:

$$8\,\alpha_1^2\Gamma_0^2c_2+8\frac{m_1}{m_2}\frac{|\Psi_2|^2_{inf}}{|\Psi_1|^2_{inf}}\beta_1^2(1-\Gamma_0)^2c_2-\sin(2\theta)\frac{|\Psi_2|_{inf}}{|\Psi_1|_{inf}}\ln\left(\frac{T_{c1}}{T_{c2}}\right)\alpha_1\beta_1\sin(c_0)=0 \quad (60)$$

$$-24\left[\alpha_1\alpha_3\Gamma_0^2c_2-\alpha_1^2\Gamma_0\Gamma_1c_2+\alpha_1^2\Gamma_0^2c_4\right]-24\frac{m_1}{m_2}\frac{|\Psi_2|^2_{inf}}{|\Psi_1|^2_{inf}}(\beta_1\beta_3(1-\Gamma_0)^2c_2+(1-\Gamma_0)\Gamma_1\beta_1^2c_2+c_4\beta_1^2(1-\Gamma_0)^2)$$
$$+\sin(2\theta_1)|\Psi_2|^2_{inf}\alpha_1^2\beta_1^2(b_1-b_2)\sin(2c_0)+\sin(2\theta)\frac{|\Psi_2|_{inf}}{|\Psi_1|_{inf}}\ln\left(\frac{T_{c1}}{T_{c2}}\right)\times((\alpha_3\beta_1+\alpha_1\beta_3)\sin(c_0)+c_2\cos(c_0)\alpha_1\beta_1)=0 \quad (61)$$

Appendix C. Numerical Solution at $\rho=\infty$.

For parameter values, given by Eq. (28), we obtain the next values for quantities $\{|\Psi_1|^2_{inf}, |\Psi_2|^2_{inf}, \cos(\mu_{inf})\}$.

$$|\Psi_1|^2_{inf}=\left\{\begin{array}{l}1.209457\cdot10^4;\ \theta=0\\1.205556\cdot10^4;\ \theta=0.1\\1.175276\cdot10^4;\ \theta=0.3\end{array}\right|_{Gauss^2} \quad (62)$$

$$|\Psi_2|^2_{inf}=\left\{\begin{array}{l}6.464209\cdot10^3;\ \theta=0\\6.487598\cdot10^3;\ \theta=0.1\\6.669154\cdot10^3;\ \theta=0.3\end{array}\right|_{Gauss^2} \quad (63)$$

$$\cos(\mu_{inf})=\left\{\begin{array}{ll}0; & \theta=0\\-0.0774303; & \theta=0.1\\-0.2198284; & \theta=0.3\end{array}\right. \quad (64)$$

Table 1 Resulting values of the expansion at $t_0 \ll 1$ for α_1, β_1, a_1 and c_0 and the critical magnetic fields H_{c1}, H_{c2} and H_c

	α_1	β_1	a_1	c_0	$\tilde{H}(0)$	$\tilde{H}_{c1}$	$\tilde{H}_c$	H_{c2}
$\theta = 0$	0.502624	0.381376	0.178348	$\pm\pi/2$	0.356697	0.381862	0.31753	0.36464
$\theta = 0.3$	0.500653	0.390095	0.179133	$\pi/2 + 0.321145$	0.358285	0.383238	0.318328	0.354247
$\theta = 0.6$	0.491193	0.406449	0.179380	$\pi/2 + 0.528922$	0.359888	0.385749	0.317728	0.325411
$\theta = 0.9$	0.468440	0.416228	0.178354	$\pi/2 + 0.560478$	0.356357	0.386459	0.311181	0.284332

Further, for the numerical calculations, we will use

$$\frac{8\pi e^2\gamma^2}{m_1 c^2} = 3.573832 \cdot 10^{-5}(Gauss)^{-2} \tag{65}$$

Appendix D. Numerical Solution for $\theta = 0$

In the range of parameter $t_0 \gg 1$, we have the following asymptotic behavior of $\tilde{A}$ and $\tilde{H}$:

$$\tilde{A} \approx \frac{1}{t_0} + \frac{R}{\sqrt{t_0}} e^{-t_0/\lambda}\left(1 + \frac{3\lambda}{8t_0} - \frac{15\lambda^2}{128t_0^2}\right) \tag{66}$$

and

$$\tilde{H} \approx -\frac{R}{\lambda\sqrt{t_0}}\left(1 - \frac{\lambda}{8t_0} + \frac{9\lambda}{128t_0^2}\right)e^{-t_0/\tilde{\lambda}}$$

with

$$\tilde{\lambda}^{-2} = \frac{8\pi e^2\gamma^2}{c^2 m_1}\left(|\Psi_1|^2_{inf} + |\Psi_2|^2_{inf}\frac{m_1}{m_2}\right).$$

For $\theta = 0$, we get $\tilde{\lambda}^{-2} = 0.894279$. The asymptotics for $|\tilde{\Psi}_1|$ and $|\tilde{\Psi}_2|$ have the form:

$$\begin{aligned}|\tilde{\Psi}_1| = 1 &- \frac{S_{11}}{\sqrt{t_0}}\exp(-\kappa_1 t_0)\left(1 - \frac{1}{8\kappa_1 t_0}\right)\\ &- \frac{S_{12}}{\sqrt{t_0}}\exp(-\kappa_2 t_0)\left(1 - \frac{1}{8\kappa_2 t_0}\right)\end{aligned} \tag{67}$$

$$\begin{aligned}|\tilde{\Psi}_2| = 1 &- \frac{S_{21}}{\sqrt{t_0}}\exp(-\kappa_1 t_0)\left(1 - \frac{1}{8\kappa_1 t_0}\right)\\ &- \frac{S_{22}}{\sqrt{t_0}}\exp(-\kappa_2 t_0)\left(1 - \frac{1}{8\kappa_2 t_0}\right)\end{aligned} \tag{68}$$

with $S_{21}/S_{11} = 3.62365$, $S_{22}/S_{12} = -0.258166$. Here quantities $\{R, S_{11}, S_{12}\}$ are some constants, which can be found by solving the full set of the differential equations. In the range $t_0 \ll 1$, we obtain from Eqs. (58-61).

$$\alpha_3 = 0.25\alpha_1 a_1 + 0.04558\alpha_1 \tag{69}$$

$$\beta_3 = 0.25\beta_1 a_1 + 0.0119138\beta_1 \tag{70}$$

$$a_3 = 0.05403(\alpha_1^2 + 1.068944\beta_1^2) \tag{71}$$

$$a_5 = 0.01801\{a_1(\alpha_1^2 + 1.06894\beta_1^2) + 2(\alpha_1\alpha_3 + 1.068944\beta_1\beta_3)\} \tag{72}$$

$$\alpha_5 = \frac{1}{24}\left\{2(\alpha_1\alpha_3 + \alpha_1 a_3) + \alpha_1 a_1^2 + 0.446235\alpha_1^3 - 0.0815917\alpha_1\beta_1^2 + 0.364643\alpha_3\right\} \tag{73}$$

$$\beta_5 = \frac{1}{24}\left\{2(\beta_1 a_3 + \beta_3 a_1) + \beta_1 a_1^2 + 0.171639\beta_1^3 - 0.0763292\beta_1\alpha_1^2 + 0.09531\beta_3\right\}. \tag{74}$$

For small θ the function μ can be presented in the form $\mu = \pi/2 + \delta$ with $\delta \propto \theta \ll 1$. So, for $\theta = 0.1$ Eq. (A.8) in the first order of perturbation theory over θ can be presented in the form:

$$\begin{aligned}&-\frac{1}{t_0}\frac{\partial}{\partial t_0}\left(t_0|\tilde{\Psi}_1|^2\Gamma^2\frac{\partial\delta}{\partial t_0}\right) - 1.07628\frac{1}{t_0}\frac{\partial}{\partial t_0}\left(t_0|\tilde{\Psi}_2|^2(1-\Gamma)^2\frac{\partial\delta}{\partial t_0}\right)\\ &+ 0.163774|\tilde{\Psi}_1|^2|\tilde{\Psi}_2|^2\delta = 1.268103\cdot 10^{-2}|\tilde{\Psi}_1||\tilde{\Psi}_2|\end{aligned} \tag{75}$$

Corrections to quantities $\{|\tilde{\Psi}_1|, |\tilde{\Psi}_2|\}$ are of the second order by θ. Hence, in the leading approximation, we can use the values of function $\{|\tilde{\Psi}_1|, |\tilde{\Psi}_2|\}$ at the point $\theta = 0$.

Appendix E. Numerical Solution of the Eqs. $\theta = 0$

It is follows from Eq. (46) the point $\theta = 0$ is singular. It this point $\mu = \pm\pi/2$. As the result the equation system from four equations Eqs. (47-50) reduces to the system of three equations. The solution of its has a special interest, since the solution is more simple in such case and can be easy spread on a large region over θ. Solving Eqs. (44-46) on estimates the four parameters $\{\alpha_1, \beta_1, a_1, c_0\}$. Their values are presented in the table.

At $\theta = 0$, we have the next equation for $\{|\tilde{\Psi}_1|, |\tilde{\Psi}_2|, \tilde{A}\}$

$$-\left(\frac{1}{t_0}\frac{\partial|\tilde{\Psi}_1|}{\partial t_0} + \frac{\partial^2|\tilde{\Psi}_1|}{\partial t_0^2}\right) + \frac{1}{t_0^2}(1-\tilde{A}t_0)^2|\tilde{\Psi}_1| + 0.446235|\Psi_1|^3 - 0.0815917|\tilde{\Psi}_1||\tilde{\Psi}_2|^2 - 0.364643|\tilde{\Psi}_1| = 0 \quad (76)$$

$$-2\left(\frac{1}{t_0}\frac{\partial|\tilde{\Psi}_2|}{\partial t_0} + \frac{\partial^2|\tilde{\Psi}_2|}{\partial t_0^2}\right) + \frac{2}{t_0^2}(1-\tilde{A}t_0)^2|\tilde{\Psi}_2| + 0.343279|\Psi_2|^3 - 0.152658|\tilde{\Psi}_2||\tilde{\Psi}_1|^2 - 0.1906204|\tilde{\Psi}_2| = 0 \quad (77)$$

$$-\left(\frac{1}{t_0}\frac{\partial\tilde{A}}{\partial t_0} + \frac{\partial^2\tilde{A}}{\partial t_0^2}\right) + 0.43224(|\tilde{\Psi}_1|^2 + 1.068944|\tilde{\Psi}_2|^2)\left(\tilde{A} - \frac{1}{t_0}\right) + \frac{1}{t_0^2}\tilde{A} = 0 \quad (78)$$

From the numerical solution, we find the coefficients R, S_{11} and S_{22} in asymptotics presented by Eqs. (66-68):

$$R = -4.89675, S_{11} = 3.32825, S_{22} = 2.33331. \quad (79)$$

Appendix F. Small θ Values, Correction to the Phase Difference

We obtain the next equation for the function $\delta(t_0)$ in the region $t_0 \ll 1$.

$$\delta = \delta_0 - \delta^{(2)}t_0^2 + \delta^{(4)}t_0^4 \quad (80)$$

where

$$\alpha_3 = 5.15 \cdot 10^{-2}, \beta_3 = 2.60634 \cdot 10^{-2},$$
$$\Gamma_0 = 0.392089, \Gamma_1 = 1.58942 \cdot 10^{-2},$$

$$\delta^{(2)} = 1.585135 \cdot 10^{-3}$$

$$\frac{\alpha_1\beta_1}{\alpha_1^2\Gamma_0^2 + 1.0762888\beta_1^2(1-\Gamma_0)^2} = 3.129603 \cdot 10^{-3} \quad (81)$$

$$\delta^{(4)} = -7.180505 \cdot 10^{-5} + 2.982978 \cdot 10^{-3}\delta_0$$

Numerical calculations for $\theta = 0.1$ give

$$\delta_{inf} = 0.077430178, \delta_0 = 0.111235. \quad (82)$$

For $t_0 \gg 1$, we have the following asymptotic $\delta \approx \delta_{inf} + \frac{0.2917}{\sqrt{t_0}}e^{-0.5620824t_0}$. The phase difference in full range of t_0 for $\theta = 0.1$ is presented at Fig. 3.

Appendix G. Case $\theta = 0.3$

Consider now the case of $\theta = 0.3$. Parameters $\{\alpha_1, \beta_1, a_1, c_0\}$ are free parameters and for quantities $\{\alpha_3, \beta_3, a_3, \alpha_5, \beta_5, a_5, c_2, c_4\}$, we obtain from Eqs. (52-57) in the region $t_0 \ll 1$ the following values:

$$\alpha_3 = 0.25\alpha_1 a_1 + 4.3680665 \cdot 10^{-2}\alpha_1 - 4.744229 \cdot 10^{-3}\beta_1\cos(c_0)$$

$$\beta_3 = 0.25\beta_1 a_1 + 1.286363 \cdot 10^{-2}\beta_1 - 3.974876 \cdot 10^{-3}\alpha_1\cos(c_0)$$

$$\Gamma_0 = (1 + 0.837834 \cdot \alpha_1^2/\beta_1^2)^{-1}, \Gamma_1 = \frac{1.675668\alpha_1^2}{\beta_1^2}\Gamma_0^2\left(\frac{\alpha_3}{\alpha_1} - \frac{\beta_3}{\beta_1}\right)$$

$$c_2 = \frac{4.744229 \cdot 10^{-3}\alpha_1\beta_1\sin(c_0)}{\alpha_1\Gamma_0^2 + 1.193554\beta_1^2(1-\Gamma_0)^2}$$

$$a_3 = 4.992322 \cdot 10^{-2}(\alpha_1^2 + 1.193554\beta_1^2)$$

$$\alpha_5 = \frac{1}{24}\{4\Gamma_0^2\alpha_1 c_2^2 + 2(\alpha_1 a_3 + \alpha_3 a_1) + \alpha_1 a_1^2 + 0.412317\alpha_1^3 - 8.417844 \cdot 10^{-2}\alpha_1\beta_1^2(1 - 2\cos^2(c_0)) + 0.349445\alpha_3 + 3.79538 \cdot 10^{-2}(-\beta_3\cos(c_0) + \beta_1 c_2\sin(c_0))\}$$

$$\beta_5 = \frac{1}{24}\{4(1-\Gamma_0)^2\beta_1 c_1^2 + 2(\beta_1 a_3 + \beta_3 a_1) + \beta_1 a_1^2 + 0.177081\beta_1^3 - 7.052755 \cdot 10^{-2}\alpha_1^2\beta_1(1 - 2\cos^2(c_0)) + 0.10290903\beta_3 + 3.1799 \cdot (-\alpha_3\cos(c_0) + \alpha_1 c_2\sin(c_0))\}$$

$$a_5 = 1.664107 \cdot 10^{-2}\{a_1(\alpha_1^2 + 1.193554\beta_1^2) + 2(\alpha_1\alpha_3 + 1.193554\beta_1\beta_3)\}$$

$$c_4 = \frac{1}{\alpha_1^2\Gamma_0^2 + 1.193554\beta_1^2(1-\Gamma_0)^2}\{(\alpha_1^2\Gamma_0\Gamma_1 - \alpha_1\alpha_3\Gamma_0^2)c_2 - 1.193554(\beta_1\beta_3(1-\Gamma_0)^2 + (1-\Gamma_0)\Gamma_1\beta_1^2)c_2 - 3.507435 \cdot 10^{-3}\alpha_1^2\beta_1^2\sin(2c_0) + 1.58141 \cdot 10^{-3}(\alpha_3\beta_1\sin c_0 + \alpha_1\beta_3\sin c_0 + c_2\alpha_1\beta_1\cos c_0)\}$$

At $t_0 \to \infty$ the variable tends to $\mu_{inf} \to \pi/2 + 0.2216385$ and $\Gamma_{\text{inf}} \to 0.531596$.

For $\theta = 0.3$, we obtain the following system of differential equations for quantities $\{|\tilde{\Psi}_1|, |\tilde{\Psi}_2|, \tilde{A}, \mu\}$:

$$- \left(\frac{1}{t_0} \frac{\partial|\tilde{\Psi}_1|}{\partial t_0} + \frac{\partial^2|\tilde{\Psi}_1|}{\partial t_0} \right) + \left(\Gamma^2 \left(\frac{\partial \mu}{\partial t_0} \right)^2 + \frac{1}{t_0^2}(1 - \tilde{A}t_0)^2 \right) |\tilde{\Psi}_1| + B_{13}|\tilde{\Psi}_1|^3 + B_{11}|\tilde{\Psi}_1| + C_{13}|\tilde{\Psi}_1||\tilde{\Psi}_2|^2(1 - 2\cos^2\mu) + C_{11}|\tilde{\Psi}_2|\cos\mu = 0 \tag{83}$$

$$- \left(\frac{1}{t_0} \frac{\partial|\tilde{\Psi}_2|}{\partial t_0} + \frac{\partial^2|\tilde{\Psi}_1|}{\partial t_0} \right) + \left((1-\Gamma)^2 \left(\frac{\partial \mu}{\partial t_0} \right)^2 + \frac{1}{t_0^2}(1 - \tilde{A}t_0)^2 \right) |\tilde{\Psi}_2| + B_{23}|\tilde{\Psi}_2|^3 + B_{21}|\tilde{\Psi}_2| + C_{23}|\tilde{\Psi}_2||\tilde{\Psi}_1|^2(1 - 2\cos^2\mu) + C_{21}|\tilde{\Psi}_1|\cos\mu = 0 \tag{84}$$

$$-\frac{1}{t_0}\frac{\partial}{\partial t_0}\left(t_0 \frac{\partial \tilde{A}}{\partial t_0} \right) + \left(F_1|\tilde{\Psi}_1|^2 + F_2|\tilde{\Psi}_2|^2 \right)\left(\tilde{A} - \frac{1}{t_0} \right) + \frac{1}{t_0^2}\tilde{A} = 0 \tag{85}$$

$$- \frac{1}{t_0}\frac{\partial}{\partial t_0}\left(t_0|\tilde{\Psi}_1|^2\Gamma^2\frac{\partial \mu}{\partial t_0} \right) - G_0\frac{1}{t_0}\frac{\partial}{\partial t_0}\left(t_0|\tilde{\Psi}_2|^2(1-\Gamma)^2\frac{\partial \mu}{\partial t_0} \right) - C_{13}|\tilde{\Psi}_1|^2|\tilde{\Psi}_2|^2\sin(2\mu) - C_{11}\cdot|\tilde{\Psi}_1||\tilde{\Psi}_2|\sin\mu = 0, \tag{86}$$

where

$$\Gamma = (1 + 0.881129|\tilde{\Psi}_1|^2/|\tilde{\Psi}_2|^2)^{-1} \tag{87}$$

and

$$B_{11} = -0.349445,\ B_{13} = 0.433624,\ C_{13} = -8.417844\cdot 10^{-2},\ C_{11} = 3.700964\cdot 10^{-2},$$

$$B_{21} = -0.102909,\ B_{23} = 0.177081,\ C_{23} = -7.4172086\cdot 10^{-2},\ C_{21} = 3.261003\cdot 10^{-2},$$

$$F_1 = 0.420024,\ F_2 = 0.476689,\ G_0 = 1.134908.$$

At $\theta = 0.3$ at large distances $t_0 \gg 1$, we get the following asymptotic expression for the magnetic field:

$$\tilde{H}(t_0) = -\frac{2.82979}{\sqrt{t_0}}\left(1 - \frac{0.1320029}{t_0} \right)\exp\left(-0.9469491 t_0 \right) \tag{88}$$

In both numerical investigated cases, the superconductor turns out unusual state. The value of H_{c1} and H_{c2} are larger that H_c.

The three correlation length can be estimated from the system of equations Eqs. (84-86):

$$\begin{pmatrix} 0.875383 - \kappa^2 & -0.1602211 & 3.61043\times 10^{-2} \\ -0.141176 & 0.361331 - \kappa^2 & 3.18117\times 10^{-2} \\ 6.791168\times 10^{-2} & 6.79117\times 10^{-2} & 0.301396 - \kappa^2 \end{pmatrix}\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = 0 \tag{89}$$

The solution of the Eqs. (89) is

$$\begin{aligned} \kappa_1 &= 0.50125, \mathbf{f}_1 = (0.196846, 0.541458, -1) \\ \kappa_2 &= 0.60715, \mathbf{f}_2 = (0.183665, 0.806245, 1) \\ \kappa_3 &= 0.95824, \mathbf{f}_3 = (1, -0.248778, 8.27139\times 10^{-2}) \end{aligned} \tag{90}$$

The numerical calculations of Eqs. (83-85) yields the following asymptotic expression for $\{|\tilde{\Psi}_1|, |\tilde{\Psi}_1|, \mu\}$:

$$\begin{pmatrix} |\tilde{\Psi}_1| \\ |\tilde{\Psi}_2| \\ \mu \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \frac{\pi}{2} + 0.2216385 \end{pmatrix} - \frac{1.87027}{\sqrt{t_0}}\left(1 - \frac{0.24938}{t_0}\right)\mathbf{f}_1 \exp(-0.501254 t_0) - \frac{1.80889}{\sqrt{t_0}}\left(1 - \frac{0.20588}{t_0}\right)\mathbf{f}_2 \exp(-0.6071449 t_0) - \frac{3.6315}{\sqrt{t_0}}\left(1 - \frac{0.13045}{t_0}\right)\mathbf{f}_3 \exp(-0.958238 t_0) \tag{91}$$

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